On representations and structures of infinite-dimensional Lie algebras

SHUSHMA RANI

A thesis submitted for the partial fulfillment of

the degree of Doctor of Philosophy

Department of Mathematical Sciences

Indian Institute of Science Education and Research Mohali

Knowledge city, Sector 81, SAS Nagar, Manauli PO, Mohali 140306, Punjab, India.

June 2023

Declaration

The work presented in this thesis has been carried out by me under the guidance of Dr. Tanusree Khandai at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Shushma Rani

In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Tanusree Khandai

(Supervisor)

Dedicated

to

My Parents

Acknowledgements

It gives me immense pleasure to acknowledge with appreciation all those who have supported and encouraged me at various stages.

I want to start by expressing my sincere gratitude to my supervisor, Dr. Tanusree Khandai. She always stands by me, no matter what. Without her commitment and great pleasure, PhD's life would not be possible. She taught me a lot of things.

Without friends, life can be very challenging. The time I got to spend with my friends Ursashi, Niranjan, Suneel, Bandna, and Ravi was really valuable to me. Ursashi, who is always by my side in every circumstance, has my gratitude. Together, we went through a lot of good and bad times. I want to express my gratitude to Niranjan for assisting me in stressful conditions. He has been by my side throughout every phase of my PhD. I'd also like to thank him for our helpful mathematical conversations. I want to express my gratitude to Suneel for always inspiring me and making things simpler for me. Through his initiative, I was able to learn Python, which has been immensely useful to me in my professional life. I also want to express my gratitude to Ravi and Bandna for always being there for me. I'm grateful for their constant cheer.

I want to express my gratitude to Manpreet for always assisting me in making crucial decisions. He is always there for me whenever I need assistance with my work.

I am also grateful to my juniors Tejbir, Priya, Vinay, Rakesh Halder, Mandeep and Sheenam for all the enjoyable times we've spent together laughing and having a great time. Raman and

Priyanka, two of my university friends, deserve my gratitude for consistently encouraging me.

I want to thank Dr. Anuj Jakhar for supporting me throughout my early PhD years and inspiring me each time. Together, we have a lot of fun. Every step of the way, he's been there to advise me. I'd like to express my gratitude to Dr. Arunkumar Ganesan, with whom I worked on my initial PhD project. He helped me learn math and was a constant source of inspiration. I am appreciative of Dr. Tushar Kanta Naik's encouragement and assistance in getting my first published paper. I really got a lot out of our conversations.

I also want to thank Dr. Amit Kulshrestha and Dr. Varadharaj R. Srinivasan for serving on my mentoring committee and for their objective evaluation.

I'd like to express my heartfelt gratitude to Dr. P. Visakhi and the rest of the library staff for their outstanding library services. When I started my PhD, one of my favourite places to study was the IISER Mohali Library.

My teachers Dr. Shalini Gupta and Dr. Navpreet Singh Noorie helped me make the decision to pursue a PhD, and I am grateful for their encouragement and inspiration. My math teachers Mrs. Saroj Bala and Mr. Pawan Kohli, who made the subject seem so simple, are to thank for igniting my interest in the subject. I want to thank Savitri Ma'am for always being an inspiration to me in her aerobics class. I thoroughly enjoyed my time with her.

I spent some time at Plaksha University in Mohali during the final year of my PhD, where I met Ankita, Vishal, Viraj, Vijay, and Abhishek. They spent a lot of time with me and helped me out in many ways, and I am sincerely grateful for that.

I want to thank my juniors, Divya and Nidhi, for their encouragement and assistance.

Last but not least, I would like to express my sincere gratitude to my family, especially to my mother, my brothers Jatinder and Bharat, my late father, and my sister Manisha, for their unwavering support and for not bothering me about my PhD. Only because of my doting grandmother was I able to get to the point where I could pursue a doctorate degree. I owe her a great debt of gratitude because she made the most important decision of my life by insisting that I major in mathematics during my twelfth grade. I appreciate the support I have always received from my cousin Neeraj, Sonia, my uncle, and my aunt. Additionally, I appreciate Pankaj Jiju's unwavering support of my every choice.

I want to express my gratitude to everyone who has sent me good wishes.

Abstract

In this thesis, we study two aspects of infinite dimensional Lie algebras.

In the first part, we study the fusion product modules for current Lie algebras of type A_2. Fusion products of finite-dimensional cyclic modules, that were defined in [23], form an important class of graded representations of current Lie algebras. In [16], a family of finite-dimensional indecomposable graded representations of the current Lie algebra called the Chari-Venkatesh(CV) modules, were introduced via generators and relations, and it was shown that these modules are related to fusion products. We study a class of CV modules for current Lie algebras of type A_2. By constructing a series of short exact sequences, we obtain a graded decomposition for them and show that they are isomorphic to fusion products of two finite-dimensional irreducible modules for current Lie algebras of $\mathfrak{sl}_3$. Further, using the graded character of these CV-modules, we obtain an algebraic characterization of the Littlewood-Richardson coefficients that appear in the decomposition of tensor products of irreducible $\mathfrak{sl}_3(\mathbb{C})$-modules.

In the second part, we study the free root spaces of Borcherds-Kac-Moody Lie superalgebras. Let $\mathfrak{L}$ be a Borcherds-Kac-Moody Lie superalgebra (BKM superalgebra in short) with the associated graph G. Any such $\mathfrak{L}$ is constructed from a free Lie superalgebra by introducing three different sets of relations on the generators: (1) Chevalley relations, (2) Serre relations, and (3) Commutation relations coming from the graph G. By Chevalley relations we get a triangular decomposition $\mathfrak{L} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ and each roots space $\mathfrak{L}_\alpha$ is either contained in $\mathfrak{n}_+$ or $\mathfrak{n}_-$. In particular, each $\mathfrak{L}_\alpha$ involves only the relations (2) and (3). We study the root

spaces of $\mathfrak{L}$ which are independent of the Serre relations. We call these roots the free roots of $\mathfrak{L}$. Since these root spaces involve only commutation relations coming from the graph, G we can study them combinatorially.We construct two different bases for these root spaces of, $\mathfrak{L}$ using combinatorics of Lyndon heaps and super Lyndon words. Finally, we relate the $\mathbf{k}$-chromatic polynomial with root multiplicities of BKM superalgebras.

Table of contents

Chapter 1

Introduction

In this thesis, we study two aspects of infinite dimensional Lie algebras. The affine Kac-Moody Lie algebras (in short, KMLA) and the Borcherds Kac-Moody Lie Superalgebras (in short, BKM superalgebras) are two important classes of infinite-dimensional Lie algebras. The current algebras of simple Lie algebras are natural parabolic subalgebras of the affine Lie algebras. The first part is based on our paper [40], where we study certain finite-dimensional representations of current Lie algebras. The second part is based on paper [49], where we study the structures of BKM Lie superalgebras.

Let $\mathfrak{g}$ be the simple Lie algebra over field $\mathbb{C}$ and $\mathfrak{g}[t] := \mathfrak{g} \otimes \mathbb{C}[t]$ be the corresponding current Lie algebra which inherits a grading coming from the natural grading on $\mathbb{C}[t]$. Let $\mathcal{C}$ be a category of finite dimensional graded $\mathfrak{g}[t]$- modules. Since the last couple of decades, researchers have been taking a keen interest in the category $\mathcal{C}$ due to its applications in Quantum affine Lie algebra, mathematical physics and combinatorics. It was shown in [15], that the graded limit of irreducible finite dimensional representations of quantum affine Lie algebras are indecomposable (not necessarily irreducible) representations of current algebras and these play an important role in the study of the tensor product modules of quantum affine Lie algebra.

Graded tensor products of finite-dimensional cyclic representations of of current Lie algebras called fusion product modules were introduced by Feigin and Loktev in [23]. It was shown in [14], that as conjectured in [23], the graded character of the fusion modules can be written in terms of Kostka polynomials. Subsequently, it has been shown that these modules have connections with symmetric Macdonald polynomials [6, 12, 36], Schur positivity [25, 29], mock theta functions [5, 7] etc, indicating the importance of these representations in combinatorics and number theory. The category $\mathcal{C}$ is also studied in mathematical physics due to its connection with problems arising there, for example for $X = M$ conjecture [1, 28, 47].

In the first part of the thesis, we study a class of finite-dimensional representations of the current Lie algebras known as the fusion product modules. It is well known that the set of irreducible $\mathfrak{g}$-modules of simple Lie algebra are parameterized by the set of dominant integral weights, P^+. For $\lambda \in P^+$, let $V(\lambda)$ denote the irreducible finite-dimensional $\mathfrak{g}$-module with the highest weight λ and highest weight vector v_λ. Given a k-tuple of dominant integral weights $\boldsymbol{\lambda} := (\lambda_1, \cdots, \lambda_k)$ of $\mathfrak{g}$ and $\mathbf{a} = (a_1, a_2, \cdots, a_k) \in \mathbb{C}^k$, the tensor product of the irreducible $\mathfrak{g}$-modules $V(\lambda_1), V(\lambda_2), \cdots, V(\lambda_k)$, acquires the structure of a $\mathfrak{g}[t]$-module, called an *evaluation module* as follows:

$$x \otimes P(t).v_1 \otimes v_2 \otimes \cdots \otimes v_k = \sum_{i=1}^{k} P(a_i) v_1 \otimes \cdots v_{i-1} \otimes x.v_i \otimes v_{i+1} \otimes \cdots v_k.$$

We denote this $\mathfrak{g}[t]$-module by $V(\boldsymbol{\lambda}, \mathbf{a})$. When these a_i's are all distinct, $V(\boldsymbol{\lambda}, \mathbf{a})$ is an irreducible $\mathfrak{g}[t]$-module. Constructing a $\mathfrak{g}$-equivariant filtration on $V(\boldsymbol{\lambda}, \mathbf{a})$, a $\mathbb{N}$-graded, the highest weight, cyclic $\mathfrak{g}[t]$-module $V(\lambda_1)^{a_1} * \cdots * V(\lambda_k)^{a_k}$, called the fusion product module, was introduced in [23]. These modules were proposed as a way of constructing the generalized versions of the Kostka polynomials. In 1999, Feigin and Loktev gave the following conjecture, which claimed that the fusion product modules are independent of the evaluation parameters.

Conjecture 1.0.1. *[23] Let $\mathfrak{g}$ be a simple Lie algebra, $V_1, V_2, \ldots, V_k$ be cyclic $\mathfrak{g}$-modules. Then for arbitrary k-tuples $(z_1, \ldots, z_k)$, $(a_1, \ldots, a_k) \in \mathbb{C}^k$ of distinct complex numbers,*

$$V_1^{z_1} * \ldots * V_k^{z_k} \cong V_1^{a_1} * \ldots * V_k^{a_k}$$

as $\mathfrak{g}[t]$-modules.

□

While the conjecture in its complete generality is not yet proved, it is known to hold in certain special cases [14, 24, 16, 46, 27, 13, 4]. It is well understood that the fusion product modules are quotients of a class of finite-dimensional highest weight integrable modules called the local Weyl modules [15] and as $\mathfrak{g}$-module, $V(\lambda_1)^{z_1} * \ldots * V(\lambda_k)^{z_k}$ is isomorphic to $V(\lambda_1) \otimes \cdots \otimes V(\lambda_k)$.

We study the structure of the fusion product of two irreducible $\mathfrak{sl}_3$-modules and, in this case, give a new proof of the conjecture. We must mention, that the conjecture has already been resolved for this case in [4]. However, the methods used to establish it are different. Our work is motivated by the study of the construction of the monomial basis of the local Weyl modules of $\mathfrak{sl}_{n+1}$ in [14] and its relation with the presentations of the fusion product modules given in [16]. We generalize the approach taken in the latter and obtain a graded decomposition of the fusion product of two irreducible $\mathfrak{sl}_3$-modules. Though, like in [4], we settle the conjecture by considering the graded decomposition of the corresponding fusion product modules, we do not resort to the convex polytopes associated with the Littlewood-Richardson coefficients. Our method is purely representation theoretic. We prove our results using a set of recurrence relations on the dimensions of a set of CV modules and deduce the Littlewood-Richardson coefficients in type A_2 as a consequence of the methods that we use.

We now describe our results in some details. Given a pair of dominant integral weights $\boldsymbol{\mu} = (\mu_1, \mu_2)$, one can canonically associate with it a family of partitions $\boldsymbol{\xi}_{\boldsymbol{\mu}}$ indexed by the positive roots of $\mathfrak{g}$. It can be easily shown that every fusion product module of the form $V(\mu_1)^{a_1} * V(\mu_2)^{a_2}$ is a quotient of the Chari Venkatesh module $V(\boldsymbol{\xi}_{\boldsymbol{\mu}})$. By associating a series of short exact sequences of $\mathfrak{sl}_3[t]$-modules with $V(\boldsymbol{\xi}_{\boldsymbol{\mu}})$, we obtain a sequence of recurrence formulae on the dimensions of such modules. Using these, induction and the fact that the dimension of $V(\boldsymbol{\xi}_{\boldsymbol{\mu}})$ is greater than equal to the product of the dimensions of $V(\mu_1)$ and $V(\mu_2)$, we compute the dimension of the module $V(\boldsymbol{\xi}_{\boldsymbol{\mu}})$. This helps us to prove that $V(\boldsymbol{\xi}_{\boldsymbol{\mu}})$ is isomorphic to the fusion product of two irreducible evaluation modules for $\mathfrak{sl}_3[t]$ with the highest weights μ_1 and μ_2 respectively. Our methods help us obtain the graded character of the modules $V(\boldsymbol{\xi}_{\boldsymbol{\mu}})$. We use them to define a set of polynomials in $\mathbb{C}[q]$, which in the limiting case $q \to 1$ give the Littlewood-Richardson coefficients associated with the decomposition of the $\mathfrak{sl}_3$-module $V(\mu_1) \otimes V(\mu_2)$. Additionally, we give an algebraic characterization of the Littlewood-Richardson coefficients that appear in the decomposition of tensor products of irreducible $\mathfrak{sl}_3(\mathbb{C})$-modules.

In the second part of the thesis, we study the free root spaces of Borcherds Kac-Moody Lie superalgebras. Borcherds Kac-Moody Lie superalgebras (in short, BKM superalgebras) [50, 51, 48, 42] are natural generalizations of two important classes of Lie algebras namely Borcherds algebras (Generalized Kac-Moody algebras) [9, 37, 38, 3] and the Kac-Moody Lie superalgebras [39, 18, 41, 21, 22]. Due to their application in mathematical physics, [10, 30, 34], in particular, in the theory of supersymmetry, chiral supergravity, and Gauge theory, [17, 31], there has been a lot of interest in the study of the structure and representation theory of these Lie algebras.

Let $\mathfrak{L}$ be the Borcherds Kac-Moody Lie superalgebra associated with a Borcherds-Cartan matrix (A, Ψ) and G be the quasi-Dynkin diagram of $\mathfrak{L}$. It is well known that such a BKM Lie superalgebra $\mathfrak{L}$ is constructed from a free Lie superalgebra by introducing three different sets of relations, namely the Chevalley relations, Serre relations, and the commutation relations that are obtained from the graph G. The roots of $\mathfrak{L}$ that are independent of the Serre relations are called the free roots.

Heaps of pieces is a combinatorial tool that was introduced by Xavier Viennot in [55]. In [43], Lalonde introduced a special class of heaps, namely, Lyndon heaps and showed that there exists a bijective correspondence between set of Lyndon heaps and basis of free partially commutative Lie algebras. We observe that these bases can be used to obtain a basis of the free root spaces of BKM Lie algebra and show that given a supergraph (G,Ψ) one can extend the notion of Lyndon heaps to super Lyndon heaps and use them to obtain a basis of free partially commutative Lie superalgebras. In particular we show that for a fixed a tuple of non-negative integers $\mathbf{k} = (k_i : i \in I)$, if $\eta(\mathbf{k})$ is a free root, then there exists a natural vector space isomorphism between the free root space $\mathfrak{g}_{\eta(\mathbf{k})}$ and the $\mathbf{k}$ grade space of free partially commutative Lie superalgebras. This gives us a Lyndon heaps basis for the free root spaces of BKM Lie superalgebras.

Graph polynomials are important graph invariants that carry useful pieces of information about the associated graphs. Among others, the chromatic polynomials are the most celebrated ones. Chromatic polynomials were introduced by Birkhoff as an attempt to solve the four-color conjecture. In [53, Propositions 1 and 2], a connection between the characters of integrable representations of a Kac-Moody Lie algebra and linear coefficients of the chromatic polynomial of the associated Dynkin diagram was established. In [54], as a continuation, an expression for the chromatic polynomial of a graph G in terms of root multiplicities of the associated Kac-Moody Lie algebra was obtained and in [3], this connection was extended to the level of Borcherds algebras and the generalized $\mathbf{k}$-chromatic polynomials. As an application of this connection, a basis for certain root spaces were constructed using Lyndon words. We construct a similar basis for free root spaces of BKM Lie superalgebras using super Lyndon words. We call this new basis LLN basis(Lyndon-Left-Normed basis).

R. Stanley introduced a symmetric function generalization of chromatic polynomials, which are called the chromatic symmetric functions. In [2], a connection between the root multiplicities of Borcherds algebras and the chromatic symmetric functions of the associated quasi-Dynkin diagrams was discussed. We extend the connection between root multiplicities of Borcherds algebras and the chromatic polynomial of the associated quasi-Dynkin diagrams to the case of Borcherds-Kac-Moody Lie superalgebras and explore the combinatorial applications of this connection.

The thesis is organized as follows. While the results on the first part of the thesis are given in Chapters 2-4 , the results second part of the thesis are given in chapters 5-7. The thesis is structured in the following manner in detail.

- In Chapter 2, we establish the notations and recall the definitions of local Weyl modules, fusion product module and the Chari-Venkatesh modules. We then show that given a k-tuple of dominant integral weights $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_k)$, one can canonically associate a Chari-Venkatesh $V(\boldsymbol{\xi_\lambda})$ module with it. For $k = 2$, such an association is unique. We also list the properties of these modules from [16, 13].

- In Chapter 3, we state and prove the main results of the thesis. Beginning with the statement of the main theorem Theorem 3.1.2, we study the Chari-Venkatesh modules associated to a pair of dominant integral weights of $\mathfrak{sl}_3$. We show that a series of short exact sequences of $\mathfrak{sl}_3[t]$-modules can be associated with such a module. Further, we study these short sequences and, using dimension arguments on the corresponding modules, complete the proof of the main result which helps establish Conjecture 1.0.1 in the case considered.

- In Chapter 4, we conclude the first part of the thesis, by giving the graded character of fusion product modules. Additionally, from the graded character of the fusion product modules, we are able to prove an analog of the Schur positivity conjecture in this case and also obtain a set of polynomials in $\mathbb{C}[q]$, which in the limiting case $q \to 1$ give the Littlewood-Richardson coefficients, Theorem 4.4.1.

- In Chapter 5, we recall the definitions of BKM Lie superalgebras, free root spaces, denominator identity, free partially commutative Lie superalgebras and heaps monoid.

- In Chapter 6, we recall the definition of Lyndon heaps and obtain the Lyndon heaps basis of Free partially commutative Lie superalgebra. Then by identifying the root spaces with grade spaces of free partially commutative Lie superalgebras, we obtain Lyndon heaps basis for free root spaces of BKM Lie superalgebra. By using the notion of Lyndon words, we construct LLN basis for free root spaces of BKM Lie superalgebra.

- In Chapter 7, we study some combinatorial properties of free root spaces. The number of ways a graph G can be $\mathbf{k}$–multicolored using q colors is a polynomial in q, called the generalized $\mathbf{k}$-chromatic polynomial. We relate the $\mathbf{k}$-chromatic polynomial with root multiplicities of BKM superalgebras. As a corollary, we obtain a combinatorial formula for the multiplicities of free roots.

Part-I

Chapter 2

Preliminaries

In this chapter, we set the notations that will be used throughout. We shall denote by $\mathbb{C}$ the field of complex numbers, by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$, the set of integers, the set of non-negative integers and set of positive integers respectively and by V^* the dual of finite dimensional vector space V over $\mathbb{C}$. We will recall all definitions using notations from [35].

2.1 Simple Lie algebra

Let $\mathfrak{g}$ be a simple Lie algebra of rank n, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and $I = \{1, \ldots, n\}$ be the indexing set. Let $\Delta = \{\alpha_i : i \in I\}$ be the set of simple roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, R^+ be the set of positive roots, $R^- = -R^+$ and $R = R^+ \cup R^-$ be the set of roots. Let $\{\omega_i : i \in I\}$ be a set of fundamental weights. Let $P = \sum\limits_{i \in I} \mathbb{Z}\omega_i$ (resp. $Q = \sum\limits_{i=n} \mathbb{Z}\alpha_i$) be the weight lattice (resp. root lattice) of $\mathfrak{g}$ and $P^+ = \sum\limits_{i \in I} \mathbb{Z}_+\omega_i$ (resp. $Q^+ = \sum\limits_{i \in I} \mathbb{Z}_+\alpha_i$) be the set of dominant integral weights (resp. positive root lattice) of $\mathfrak{g}$. For $\alpha \in R$, let $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : h.x = \alpha(h)x, \forall\, h \in \mathfrak{h}\}$. Fix a Chevalley basis $\{x^\pm_\alpha, h_i : \alpha \in R^+, 1 \leq i \leq n\}$ of $\mathfrak{g}$ with $x^\pm_\alpha \in \mathfrak{g}_{\pm\alpha}$. Let $\mathfrak{n}^\pm = \bigoplus\limits_{\alpha \in R^+} \mathbb{C}x^\pm_\alpha$.

Then $\mathfrak{g}$ has a triangular decomposition given by

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-.$$

For $i \in I$, $h_i = [x^+_{\alpha_i}, x^-_{\alpha_i}] \in \mathfrak{h}$ and be the elements of the Cartan subalgebra $\mathfrak{h}$. For any $\alpha = \sum n_i \alpha_i \in R^+$, $h_\alpha = \sum d_i n_i h_i$ where $d_i = \frac{2}{(\alpha_i, \alpha_i)}$, for $i \in I$. It is well known that $\mathbb{C}$-span of the set $\{h_\alpha, x^\pm_\alpha\}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ and we shall denote this subalgebra by $\mathfrak{sl}_2(\alpha)$.

2.1.1 Irreducible modules for simple Lie algebras

By Weyl's Theorem [35, Theorem 6.3], every finite dimensional representation of a simple Lie algebra is completely reducible. It is well known that the set of irreducible $\mathfrak{g}$-modules of simple Lie algebra are parameterized by the set of dominant integral weights. Given $\lambda = \sum\limits_{i=1}^{n} m_i \omega_i \in P^+$, let $V(\lambda)$ denote the irreducible finite-dimensional $\mathfrak{g}$-module with the highest weight λ and highest weight vector v_λ. Then $V(\lambda) = \mathbf{U}(\mathfrak{n}^-)v_\lambda$ and

(i) $x^+_\alpha.v_\lambda = 0, \forall\, \alpha \in R^+$.

(ii) $h.v_\lambda = \lambda(h)v_\lambda, \forall\, h \in \mathfrak{h}$.

(iii) $x_\alpha^{-\lambda(h_\alpha)+1}.v_\lambda = 0, \forall\, \alpha \in R^+$.

2.1.2 Weyl group of simple Lie algebra

For each root $\alpha \in R$, we have a reflection s_α, defined as $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$s_\alpha(\lambda) = \lambda - \lambda(h_\alpha)\alpha.$$

A reflection s_α is said to be a simple reflection if α is simple root. The group $W = \langle s_\alpha : \alpha \in \Delta \rangle$ generated by simple reflections is called Weyl group of $\mathfrak{g}$.

2.2 Affine Lie algebra

With each finite dimensional Lie algebra, one can associate an infinite dimensional Lie algebra, known as the loop algebra. The underlying vector space of a loop algebra is given as

$$\mathfrak{g}[t,t^{-1}] := \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]$$

where $\mathbb{C}[t,t^{-1}]$ is the ring of Laurent polynomials in the indeterminate t and the Lie bracket is given by

$$[x \otimes f, y \otimes g] = [x,y] \otimes fg$$

for all $x,y \in \mathfrak{g}$ and $f,g \in \mathbb{C}[t,t^{-1}]$.

A non-twisted affine Kac-Moody Lie algebra $\widehat{\mathfrak{g}}$ is the semi-direct product of the universal central extension of $\mathfrak{g}[t,t^{-1}]$ with a derivation d of $\mathfrak{g}[t,t^{-1}]$. As a vector space

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

where c is central element and d is derivation, with the Lie bracket defined as

$$[x \otimes t^m, y \otimes t^n] = [x,y] \otimes t^{m+n} + \delta_{m,n}(x|y)c$$

and

$$[d, x \otimes t^m] = m(x \otimes t^m)$$

for all $x,y \in \mathfrak{g}$, $m,n \in \mathbb{Z}$.

The subalgebras $\widehat{\mathfrak{n}}^{\pm}$ and the Cartan subalgebra $\widehat{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}$ are defined as follows:

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

$$\widehat{\mathfrak{n}}^{\pm} = \mathfrak{n}^{\pm} \otimes \mathbb{C}[t] \oplus (\mathfrak{n}^{\mp} \oplus \mathfrak{h}) \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}].$$

By defining $\lambda(c) = \lambda(d) = 0$ for $\lambda \in \mathfrak{h}^*$, every element of $\mathfrak{h}^*$ can be considered as an element of dual space $\widehat{\mathfrak{h}}^*$. Let $\delta, \Lambda_0 \in \widehat{\mathfrak{h}}^*$ be given by

$$\delta(c) = 0 = \delta(\mathfrak{h}), \delta(d) = 1,$$

$$\Lambda_0(d) = 0 = \Lambda_0(\mathfrak{h}), \Lambda(c) = 1.$$

Let $\{\alpha_i : 0 \leq i \leq n, \alpha_0 = \delta - \theta\}$ denote the set of simple roots of $\widehat{\mathfrak{g}}$. Let $\{e_i, f_i : 0 \leq i \leq n\}$ be a set of Chevalley generators of $\widehat{\mathfrak{g}}$ where

$$e_0 = f_\theta \otimes t,\ f_0 = e_\theta \otimes t^{-1},\ e_i = e_{\alpha_i} \otimes 1,\ f_i = f_{\alpha_i} \otimes 1,\ 1 \leq i \leq n$$

Set

$$\widehat{R}^+ := \{\alpha + n\delta : \alpha \in R, n \in \mathbb{N}\} \cup R^+ \cup \{n\delta : n \in \mathbb{N}\}$$

$$\widehat{R}^- := \{\alpha - n\delta : \alpha \in R, n \in \mathbb{N}\} \cup R^- \cup \{-n\delta : n \in \mathbb{N}\}$$

$$\widehat{R}_{re} = \{\alpha + n\delta : \alpha \in R, n \in \mathbb{Z}\}$$

$$\widehat{R}_{im} = \{n\delta : n \in \mathbb{Z} \setminus \{0\}\}$$

Then $\widehat{R}^+$, $\widehat{R}^-$, $\widehat{R}_{re}$, $\widehat{R}_{im}$ are the set of positive, negative, real and imaginary roots of $\widehat{\mathfrak{g}}$ with respect to $\widehat{\mathfrak{h}}$ and $\widehat{R} = \widehat{R}^+ \cup \widehat{R}^- = \widehat{R}_{re} \cup \widehat{R}_{im}$ is the set of roots of $\widehat{\mathfrak{g}}$. The root space decomposition of $\widehat{\mathfrak{g}}$ is given by

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{h}} \oplus \bigoplus_{\beta \in \widehat{R}} \widehat{\mathfrak{g}}_\beta$$

where $\widehat{\mathfrak{g}}_\beta = \{x \in \widehat{\mathfrak{g}} : [h,x] = \beta(h)x, \forall\ h \in \widehat{\mathfrak{h}}\}$. For $\beta \in \widehat{R}_{re}$, $\dim \widehat{\mathfrak{g}}_\beta = 1$ and for $\beta \in \widehat{R}_{im}$, $\dim \widehat{\mathfrak{g}}_\beta = \dim \mathfrak{h}$. Let $\widehat{Q} = \sum_{i=0}^{n} \mathbb{Z}\alpha_i$ (resp.$\widehat{Q}^+ = \sum_{i=0}^{n} \mathbb{Z}_+\alpha_i$) be the root lattice (resp. positive root lattice) and $\widehat{P} = \{\lambda \in \widehat{\mathfrak{h}}^* : \lambda(h_{\alpha_i}) \in \mathbb{Z}, \forall\ i, 0 \le i \le n\}$ (resp. $\widehat{P}^+ = \{\lambda \in \widehat{\mathfrak{h}}^* : \lambda(h_{\alpha_i}) \in \mathbb{Z}_+, \forall\ i, 0 \le i \le n\}$) denote the weight lattice (resp. dominant weight lattice) of $\widehat{\mathfrak{g}}$.

Let

$$\widehat{\mathfrak{n}}^+ = \bigoplus_{\beta \in \widehat{R}^+} \widehat{\mathfrak{g}}_\beta$$

then $\widehat{\mathfrak{b}} = \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}^+$ is a Borel subalgebra of $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{b}} \oplus (\widehat{\mathfrak{n}}^- \otimes \mathbb{C}[t])$ is a maximal parabolic subalgebra of $\widehat{\mathfrak{g}}$.

2.3 Current Lie algebra

Let

$$\widehat{\mathfrak{n}}^+ = \bigoplus_{\beta \in \widehat{R}^+} \widehat{\mathfrak{g}}_\beta$$

and

$$\widehat{\mathfrak{b}} = \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}^+.$$

Then $\widehat{\mathfrak{b}}$ is a Borel subalgebra of $\widehat{\mathfrak{g}}$ and $\mathfrak{g}[t] := \mathfrak{h} \oplus \widehat{\mathfrak{n}}^+ \oplus \mathfrak{n}^-$ is a maximal parabolic subalgebra of the loop algebra $\mathfrak{g}[t,t^{-1}]$ of $\widehat{\mathfrak{g}}$, known as the current Lie algebra.

As a vector space $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$, where $\mathbb{C}[t]$ denotes the polynomial ring in indeterminate t. Clearly, $\mathfrak{g}[t]$ is a Lie algebra with Lie bracket given by

$$[a \otimes f, b \otimes g] = [a,b] \otimes fg$$

for all $a,b \in \mathfrak{g}$ and $f,g \in \mathbb{C}[t]$. $\mathbb{C}[t]$ has a natural $\mathbb{Z}_+$ grading, determined by the degree of the polynomial. Using the triangular decomposition of $\mathfrak{g}$, we have the following decomposition for current algebra $\mathfrak{g}[t]$ as $\mathfrak{n}^+[t] \oplus \mathfrak{h}[t] \oplus \mathfrak{n}^-[t]$, where for a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, $\mathfrak{a}[t] := \mathfrak{a} \otimes \mathbb{C}[t]$.

2.3.1 Universal enveloping algebra of $\mathfrak{g}[t]$

Let $\mathbf{U}(\mathfrak{g}[t])$ be the universal enveloping algebra of $\mathfrak{g}[t]$. With the $\mathbb{Z}_+$-grading inherited from $\mathfrak{g}[t]$, $\mathbf{U}(\mathfrak{g}[t])$ is $\mathbb{Z}_+$-graded. We say an element $X \in \mathbf{U}(\mathfrak{g}[t])$ has grade $r_1 + r_2 + \cdots + r_p$ if X is of the form $(x_1 \otimes t^{r_1})(x_2 \otimes t^{r_2}) \cdots (x_p \otimes t^{r_p})$. For a positive integer s, we denote by $\mathbf{U}(\mathfrak{g}[t])[s]$ the subspace of $\mathbf{U}(\mathfrak{g}[t])$ spanned by elements of grade s.

For every $\alpha \in R^+$, define the power series in the indeterminate u.

$$X_\alpha^-(u) = \sum_{m=0}^{\infty} x_\alpha^- \otimes t^m \, u^{m+1}, \qquad H_\alpha(u) = \exp\left(-\sum_{r=1}^{\infty} \frac{h_\alpha \otimes t^r}{r} u^r\right),$$

and for $r,s \in \mathbb{N}$, set

$$S(r,s) = \{(b_p)_{p\geq 0} : b_p \in \mathbb{Z}_+, \sum_{p\geq 0} b_p = r, \sum_{p\geq 0} p b_p = s\}, \quad x(r,s) = \sum_{(b_p)_{p\geq 0} \in S(r,s)} \prod_{i=0}^{s} (x \otimes t^i)^{(b_i)}.$$

The following result was proved in [32, Lemma 7.5] and formulated in its present form in [15, Lemma 1.3].

Lemma 2.3.1. *Given $s \in \mathbb{N}$, $r \in \mathbb{Z}_+$ and $\alpha \in R^+$, we have*

$$(x_\alpha^+ \otimes t)^{(s)} (x_\alpha^- \otimes 1)^{(r+s)} - (-1)^s \Big(\sum_{k\geq 0} x_\alpha^-(r, r+s-k) P_\alpha(u)_k\Big) \in U(\mathfrak{g}[t])\mathfrak{n}^+[t]. \tag{2.3.1}$$

where $P_\alpha(u)_k$ denotes the coefficient of u^k is the power series $H_\alpha(u)$ and $(y)^{(s)} := \frac{y^s}{s!}$ for $y \in \mathfrak{g}[t]$.

2.4 Representations of current Lie algebra $\mathfrak{g}[t]$

2.4.1 Graded representations

A graded representation of $\mathfrak{g}[t]$ is a $\mathbb{Z}_+$-graded vector space V such that

$$V = \bigoplus_{r \in \mathbb{Z}_+} V[r]$$

$$\mathbf{U}(\mathfrak{g}[t])[s].V[r] \subseteq V[r+s], \quad \forall\ r,s \in \mathbb{Z}_+.$$

If U,V are two graded representations of $\mathfrak{g}[t]$, then we say $\psi : U \to V$ is a morphism of graded $\mathfrak{g}[t]$-modules if $\psi(U[r]) \subseteq V[r]$ for all $r \in \mathbb{Z}_+$. For $s \in \mathbb{Z}$, let τ_s be the grade-shifting operator given by

$$\tau_s(V)[k] = V[k-s]$$

for all $k \in \mathbb{Z}_+$, and graded representations V of $\mathfrak{g}[t]$. Given $z \in \mathbb{C}$ and a $\mathfrak{g}$-module U, let $\mathrm{ev}_z(U)$ denote the corresponding evaluation module for $\mathfrak{g}[t]$. Clearly, for $z = 0$ $\mathrm{ev}_0(U)$ is a graded representation and $\mathrm{ev}_0(U)[0] = U$.

2.4.2 Local Weyl modules

The notion of local Weyl modules was introduced by Chari and Pressley in [15].

Definition 2.4.1. For $\lambda \in P^+$, **the local Weyl module** $W_{loc}(\lambda)$, is the $\mathfrak{g}[t]$-module generated by an element w_λ with the following defining relations:

- $\mathfrak{n}^+ \otimes \mathbb{C}[t] w_\lambda = 0$, $(h \otimes t^s) w_\lambda = \lambda(h)\delta_{s,0} w_\lambda, \forall h \in \mathfrak{h}$
- $(x^-_{\alpha_i} \otimes 1)^{\lambda(h_i)+1} w_\lambda = 0, \forall\ i \in I.$

$W_{loc}(\lambda)$ is a graded $\mathfrak{g}[t]$-module. Any finite dimensional, cyclic, integrable module is a quotient of $W_{loc}(\lambda)$.

2.4.3 Evaluation module

Definition 2.4.2. Given a finite-dimensional irreducible $\mathfrak{g}$-module V and $z \in \mathbb{C}$, we define an action of $\mathfrak{g}[t]$ on V as follows:

$$(x \otimes t^r)w = z^r x.w, x \in \mathfrak{g}, w \in V, r \in \mathbb{Z}_+.$$

We denote the $\mathfrak{g}[t]$-module thus obtained by $ev_z(V)$ and refer to it as the *evaluation module*.

2.4.4 Fusion modules

The fusion product of $\mathfrak{g}[t]$-modules was defined in [23]. Here we recall the definition in the case that is of interest to us.

Definition 2.4.3. Given $\mathbf{z} = (z_1, \cdots, z_s)$, a s-tuple of distinct complex numbers and finitely many evaluation modules $ev_{z_1}(V_1), \cdots, ev_{z_s}(V_s)$, it was proved in [11] that the tensor product $\mathbf{V}(\mathbf{z}) = ev_{z_1}(V_1) \otimes \cdots \otimes ev_{z_s}(V_s)$ is an irreducible $\mathfrak{g}[t]$ module. In this case, the $\mathbb{N}$-grading in $\mathbf{U}(\mathfrak{g}[t])$ induces a $\mathfrak{g}$-equivariant grading on $\mathbf{V}(\mathbf{z})$ given by

$$\mathbf{V}(\mathbf{z})[k] = \bigoplus_{0 \leq r \leq k} \mathbf{U}(\mathfrak{g}[t][r].v_1 \otimes \cdots \otimes v_s,$$

where v_i is the generator of V_i for $1 \leq i \leq s$. Then the associated graded $\mathfrak{g}[t]$-module

$$\bigoplus_{k \in \mathbb{N}} \mathbf{V}(\mathbf{z})[k]/\mathbf{V}(\mathbf{z})[k-1]$$

is called the *fusion product* of $V_1, \cdots, V_s$ at $\mathbf{z}$ and we denote it by $V_1^{z_1} * \cdots * V_s^{z_s}$.

Remark 2.4.4. Given a k-tuple of dominant integral weights $\lambda_1, \dots, \lambda_k$, and a tuple $\mathbf{z} = (z_1, \cdots, z_k)$, of pairwise distinct complex numbers, the fusion product module $V(\lambda_1)^{z_1} * \dots * V(\lambda_k)^{z_k}$ is a cyclic, finite dimensional, highest weight module with highest weight $\sum_{i=1}^{k} \lambda_i$. Hence, by the universal property of local Weyl modules, it is a quotient of $W_{loc}(\sum_{i=1}^{k} \lambda_i)$.

Remark 2.4.5. It was observed in [23] that as $\mathfrak{g}$-modules, $V(\lambda_1) \otimes \cdots \otimes V(\lambda_k)$ is isomorphic to $V(\lambda_1)^{z_1} * \dots * V(\lambda_k)^{z_k}$. Hence, the dimension of fusion product $V(\lambda_1)^{z_1} * \dots * V(\lambda_k)^{z_k}$ is equal to $\prod_{i=1}^{k} \dim V(\lambda_i)$.

In this context, the following lemmas are useful:

Lemma 2.4.6. *Given an irreducible $\mathfrak{g}[t]$-module $\mathbf{V}(\mathbf{z})$, for $v \in \mathbf{V}(\mathbf{z})$ let $\bar{v}$ denote its image in $V^{z_1} * \cdots * V^{z_s}$. Then*

$$x \otimes t^p . \bar{v} = x \otimes (t - a_1) \cdots (t - a_p) . \bar{v}, \quad \forall\, x \in \mathfrak{g}, \text{ and } a_1, \cdots, a_p \in \mathbb{C}.$$

2.5 Conjecture on Fusion modules

In 1999, Feigin and Loktev gave the following conjecture, which claims that fusion product is independent of the evaluation parameters.

Conjecture 2.5.1. *[[23]] Let $\mathfrak{g}$ be a simple Lie algebra, $V_1, V_2, \dots, V_k$, a set of finitely many cyclic $\mathfrak{g}$-modules. Then for any k-tuple of distinct complex numbers, $(z_1, \dots, z_k), (a_1, \dots, a_k) \in \mathbb{C}^k$, there exists a $\mathfrak{g}[t]$-module isomorphism between $V_1^{z_1} * \dots * V_k^{z_k}$ and $V_1^{a_1} * \dots * V_k^{a_k}$.*

In 2006, Chari and Loktev gave a basis for a Local Weyl module and showed that when $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\lambda = \sum_{i=1}^{n} m_i \omega_i$ is a dominant integral weight of $\mathfrak{g}$, then the dimension of $W_{loc}(\lambda)$ is equal to $\prod_{i=1}^{n} \binom{n+1}{i}^{m_i}$ in [14]. Along with the remark 2.4.4, 2.4.5, this showed that the local Weyl module, $W_{loc}(\lambda)$, is isomorphic to the fusion product of the fundamental representations,

$\underset{i}{*}V(\omega_i)^{*m_i}$. Following independent approaches, Fourier and Littlemann proved the same result for simply laced current Lie algebras in [26] and Naoi proved it for non simply laced current Lie algebras in [46].

In 2015, Chari and Venkatesh introduced a new family of quotients of the local Weyl modules. Using these modules, they reproved the conjecture 2.5.1 when $\mathfrak{g}$ is of type A_1. Using similar methods, it was shown in [24] and [4] that the conjecture holds in some particular cases for $k = 2$.

2.6 CV modules

The new family of quotients of local Weyl modules introduced by Chari and Venkatesh in [16] are referred to as CV- modules. We now recall their definition.

Definition 2.6.1. For $\lambda \in P^+$, a $|R^+|$-tuple of partitions $\boldsymbol{\xi} = (\xi^\alpha)_{\alpha \in R^+}$ is said to be λ-compatible if,

$$\xi^\alpha = (\xi_1^\alpha \geq \xi_2^\alpha \geq \cdots), \quad \text{and} \quad \sum_{i \geq 1} \xi_i^\alpha = \lambda(h_\alpha), \ \ \forall\, \alpha \in R^+.$$

Given a λ-compatible R^+-tuple of partitions $\boldsymbol{\xi} = (\xi_\alpha)_{\alpha \in R^+}$, the Chari-Venkatesh module $V(\boldsymbol{\xi})$ is defined as the cyclic $\mathfrak{g}[t]$-module generated by a non-zero vector $v_{\boldsymbol{\xi}}$ with the following defining relations:

$$(\mathfrak{n}^+ \otimes \mathbb{C}[t])v_{\boldsymbol{\xi}} = 0, \ (h \otimes t^s)v_{\boldsymbol{\xi}} = \lambda(h)\delta_{s,0}v_{\boldsymbol{\xi}}, \ \ \forall h \in \mathfrak{h}, \tag{2.6.1}$$

$$(x_\alpha^- \otimes 1)^{\lambda(h_\alpha)+1}v_{\boldsymbol{\xi}} = 0, \tag{2.6.2}$$

$$(x_\alpha^+ \otimes t)^{(s)}(x_\alpha^- \otimes 1)^{(r+s)}v_{\boldsymbol{\xi}} = 0, \quad \alpha \in R^+, r,s \in \mathbb{N}, s+r \geq 1 + rk + \sum_{j \geq k+1} \xi_j^\alpha, \text{ for } k \in \mathbb{N}. \tag{2.6.3}$$

Note that by definition CV-module $V(\boldsymbol{\xi})$ is the graded quotient of local Weyl module $W_{loc}(\lambda)$ by the submodule generated by graded elements

$$\{(x_\alpha^+\otimes t)^{(s)}(x_\alpha^-\otimes 1)^{(r+s)}w_\lambda : \alpha\in R^+, r,s\in\mathbb{N}, s+r\geq 1+rk+\sum_{j\geq k+1}\xi_j^\alpha,\ \text{for } k\in\mathbb{N}\}.$$

2.6.1 Properties of CV-module

For $k\in\mathbb{Z}_+$ and $r,s\in\mathbb{N}$, set

$$\begin{aligned}
{}_kS(r,s) &= \{(b_p)_{p\geq 0}\in S(r,s) : b_p=0, \text{for } p<k\},\\
S(r,s)_k &= \{(b_p)_{p\geq 0}\in S(r,s) : b_p=0, \text{for } p\geq k\},\\
{}_kx(r,s) &= \sum_{(b_p)_{p\geq 0}\in {}_kS(r,s)} \textstyle\prod_{i=0}^s (x\otimes t^i)^{(b_i)},\\
x(r,s)_k &= \sum_{(b_p)_{p\geq 0}\in S(r,s)_k} \textstyle\prod_{i=0}^s (x\otimes t^i)^{(b_i)}
\end{aligned}$$

The following lemma was proved in [16, Section 2].

Lemma 2.6.2. *Let $r,s,k\in\mathbb{N}$ and $L\in\mathbb{Z}_+$ such that $r+s\geq kr+L$. Then the following holds.*

(i) For $\alpha\in R^+$,

$$x_\alpha^-(r,s) = {}_kx_\alpha^-(r,s)+\sum_{(r',s')} x_\alpha^-(r-r',s-s')_k\ {}_kx_\alpha^-(r',s'),$$

where sum is taken over all pairs $r',s'\in\mathbb{N}$ such that $r'<r, s'<s$ and $r'+s'\geq r'k+L$.

(ii) If V is a $\mathfrak{g}[t]$-module and $v\in V$, then for $\alpha\in R^+$,

$$x_\alpha^-(r,s)v=0, \quad \text{if and only if} \quad {}_kx_\alpha^-(r,s).v=0.$$

The following lemma is deduced from Lemma 2.3.1, [16, Proposition 2.7].

Lemma 2.6.3. *Given a R^+-tuple of partitions $\boldsymbol{\xi} = (\xi^\alpha)_{\alpha\in R^+}$, let $V(\boldsymbol{\xi})$ be the associated Chari-Ventakesh module with generator $v_{\boldsymbol{\xi}}$. Then for all $\alpha \in R^+$, we have*

$$(x_\alpha^+ \otimes t)^{(r)}(x_\alpha^- \otimes 1)^{(r+s)}.v_{\boldsymbol{\xi}} = x_\alpha^-(r,s).v_{\boldsymbol{\xi}}, \tag{2.6.4}$$

$$(x_\alpha^-(r,s) - {}_1x_\alpha^-(r,s))v_{\boldsymbol{\xi}} \in \sum_{r'<r} U(\mathfrak{sl}_3(\mathbb{C}))x_\alpha^-(r',s)v_{\boldsymbol{\xi}}, \tag{2.6.5}$$

Proof. By part(2) of above Lemma 2.6.2, for $r,s,k \in \mathbb{N}$ and $L \in \mathbb{Z}_+$,

$$x_\alpha^-(r,s)v_{\boldsymbol{\xi}} = 0, \quad \text{if and only if } \ {}_kx_\alpha^-(r,s).v_{\boldsymbol{\xi}} = 0.$$

By [16, Proposition 2.7], $V(\boldsymbol{\xi})$ be the Chari-Ventakesh module is generated by $v_{\boldsymbol{\xi}}$ with defining relations of $W_{loc}(\lambda)$ and ${}_kx_\alpha^-(r,s).v_{\boldsymbol{\xi}} = 0$. Therefore by definition of CV-module,

$$(x_\alpha^+ \otimes t)^{(r)}(x_\alpha^- \otimes 1)^{(r+s)}.v_{\boldsymbol{\xi}} = x_\alpha^-(r,s).v_{\boldsymbol{\xi}}.$$

Again using part(1) of Lemma 2.6.2 for $k = 1$, we have $s = s'$ and

$$x_\alpha^-(r,s) = {}_1x_\alpha^-(r,s) + \sum_{r'} x_\alpha^-(r-r',0)_1 \ {}_1x_\alpha^-(r',s),$$

where sum is taken over $r' \in \mathbb{N}$ such that $r' < r$. Since $x_\alpha^-(r-r',0)_1 \in U(\mathfrak{sl}_3)$. Thus,

$$(x_\alpha^-(r,s) - {}_1x_\alpha^-(r,s))v_{\boldsymbol{\xi}} \in \sum_{r'<r} U(\mathfrak{sl}_3(\mathbb{C}))x_\alpha^-(r',s)v_{\boldsymbol{\xi}}.$$

□

2.7 Relation between CV-modules and Fusion modules

2.7.1 The set $P^+(\lambda,k)$ and module $\mathcal{F}_{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda} \in P^+(\lambda,k)$

Given $\lambda \in P^+$ and a positive integer k, let

$$P^+(\lambda,k) = \{\boldsymbol{\lambda} = (\lambda_1,\cdots,\lambda_k) \in (P^+)^{\times k} : \sum_{i=1}^{k} \lambda_i = \lambda\}.$$

For $\alpha \in R^+$, let $\boldsymbol{\lambda}_\alpha = (\lambda_1(h_\alpha),\ldots,\lambda_k(h_\alpha))$ and let $\boldsymbol{\lambda}_\alpha^\downarrow = (\lambda_1(h_\alpha)^\downarrow \geq \ldots \geq \lambda_k(h_\alpha)^\downarrow) \in \mathbb{Z}^k$ be the k-tuple of integers obtained by rearranging the $\lambda_j(h_\alpha)$'s in non-increasing order. Observe that for each $\alpha \in R^+$, $\boldsymbol{\lambda}_\alpha^\downarrow$ is a partition of $\lambda(h_\alpha)$. Hence, $\boldsymbol{\xi}_{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}_\alpha^\downarrow)_{\alpha\in R^+}$ is a λ-compatible R^+-tuple of partitions. For $\boldsymbol{\lambda} \in P^+$, we denote the CV module associated to the R^+-tuple of partitions $\boldsymbol{\xi}_{\boldsymbol{\lambda}}$ by $\mathcal{F}_{\boldsymbol{\lambda}}$.

We define an ordering on $P^+(\lambda,k)$ as follows. Given $\boldsymbol{\lambda} = (\lambda_1,\cdots,\lambda_k) \in P^+(\lambda,k)$ and $\boldsymbol{\mu} = (\mu_1,\cdots,\mu_k) \in P^+(\lambda,k)$ we say $\boldsymbol{\lambda}$ majorizes $\boldsymbol{\mu}$ and write $\boldsymbol{\lambda} \succeq \boldsymbol{\mu}$ if for every $\alpha \in R^+$,

$$\sum_{j=i}^{k} \lambda_j(h_\alpha)^\downarrow \geq \sum_{j=i}^{k} \mu_j(h_\alpha)^\downarrow, \qquad \text{for all } 1 \leq i \leq k.$$

Denoting the image of w_λ in $\mathcal{F}_{\boldsymbol{\lambda}}$ by $v_{\boldsymbol{\lambda}}$, we see that $\mathcal{F}_{\boldsymbol{\lambda}}$ is a graded $\mathfrak{g}[t]$ module generated by $v_{\boldsymbol{\lambda}}$ with defining relations:

$$(\mathfrak{n}^- \otimes \mathbb{C}[t])v_{\boldsymbol{\lambda}} = 0,\ (h \otimes t^s)w_{\boldsymbol{\lambda}} = \lambda(h)\delta_{s,0}v_{\boldsymbol{\lambda}},\ \text{ for all } h \in \mathfrak{h}, \tag{2.7.1}$$

$$(x_i^- \otimes 1)^{\lambda(h_i)+1}v_{\boldsymbol{\lambda}} = 0, \tag{2.7.2}$$

$$(x_\alpha^+ \otimes t)^s(x_\alpha^- \otimes 1)^{r+s}v_{\boldsymbol{\lambda}} = 0,\ \ s+r \geq 1 + r\ell + \sum_{j\geq \ell+1} \lambda_j(h_\alpha)^\downarrow,\ \forall\, \ell \in \mathbb{N}. \tag{2.7.3}$$

Observe that for $k > 2$, there can exist $\boldsymbol{\lambda},\boldsymbol{\mu} \in P^+(\lambda,k)$ such that $\mathcal{F}_{\boldsymbol{\mu}} = \mathcal{F}_{\boldsymbol{\lambda}}$. For example, consider the case when $\mathfrak{g}$ is of type $\mathfrak{sl}_3(\mathbb{C})$, $\lambda = 3\omega_1 + 3\omega_2$ and $\boldsymbol{\mu} = (2\omega_2, \omega_1, 2\omega_1 + \omega_2)$ and $\boldsymbol{\lambda} = (2\omega_2 + \omega_1, \omega_2, 2\omega_1)$ are elements of $P^+(\lambda,3)$.

2.7.2 Properties of $\mathcal{F}_{\boldsymbol{\lambda}}$-module

The following Lemma will be used in the proof of the main theorem.

Lemma 2.7.1. *Let* $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_k), \boldsymbol{\mu} = (\mu_1,\dots,\mu_k) \in P^+(\lambda,k)$. *Then*

(i) $\mathcal{F}_{\boldsymbol{\mu}}$ *is a quotient of* $\mathcal{F}_{\boldsymbol{\lambda}}$ *whenever* $\boldsymbol{\lambda} \succeq \boldsymbol{\mu}$. *In particular, the zero graded module* $V(\lambda)[0] \cong \mathrm{ev}_0(V(\lambda))$ *is the unique irreducible quotient of* $\mathcal{F}_{\boldsymbol{\lambda}}$ *for all* $\boldsymbol{\lambda} \in P^+(\lambda,k)$. .

(ii) $\mathfrak{g} \otimes t^k\mathbb{C}[t].\mathcal{F}_{\boldsymbol{\lambda}} = 0$ *and hence* $\mathcal{F}_{\boldsymbol{\lambda}}$ *is a module for Lie algebra* $\mathfrak{g} \otimes \mathbb{C}[t]/(t^k)$.

(iii) The graded $\mathfrak{g}[t]$*-module,* $V(\lambda_1)^{z_1} * \cdots * V(\lambda_k)^{z_k}$ *is a quotient of* $\mathcal{F}_{\boldsymbol{\lambda}}$, *for any set of distinct scalars* $z_1,\cdots,z_k$.

Proof. Given $\lambda \in P^+$ and $\boldsymbol{\lambda},\boldsymbol{\mu} \in P^+(\lambda,k)$, let $v_{\boldsymbol{\lambda}}$, $v_{\boldsymbol{\mu}}$ be the image of w_λ in $\mathcal{F}_{\boldsymbol{\lambda}}$ and $\mathcal{F}_{\boldsymbol{\mu}}$ respectively.

(i) Using definition, for all $\alpha \in R^+$,

$$(x_\alpha^+ \otimes t)^{(s)}(x_\alpha^- \otimes 1)^{(r+s)}.v_{\boldsymbol{\mu}} = 0, \text{ whenever } r,s,q \in \mathbb{N}, \text{ satisfy } r+s \geq 1 + qr + \sum_{j=q+1}^{k} \mu_j(h_\alpha)^{\downarrow}.$$

Since $\boldsymbol{\lambda} \succeq \boldsymbol{\mu}$, $\sum_{j=q+1}^{k} \lambda_j(h_\alpha)^{\downarrow} \geq \sum_{j=q+1}^{k} \mu_j(h_\alpha)^{\downarrow}$, for all $0 \leq q \leq k-1$, $\alpha \in R^+$, for $r',s',q \in \mathbb{N}$ are such that $r'+s' \geq 1 + r'q + \sum_{j=q+1}^{k} \lambda_j(h_\alpha)^{\downarrow}$, we have

$$(x_\alpha^+ \otimes t)^{(s')}(x_\alpha^- \otimes 1)^{(r'+s')} v_{\boldsymbol{\mu}} = 0, \text{ whenever } r',s',q \in \mathbb{N}, \text{ satisfy } r'+s' \geq 1 + qr' + \sum_{j=q+1}^{k} \lambda_j(h_\alpha)^{\downarrow}.$$

Hence there exists a surjective $\mathfrak{g}[t]$-module homomorphism $\Psi^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}} : \mathcal{F}_{\boldsymbol{\lambda}} \to \mathcal{F}_{\boldsymbol{\mu}}$, such that $\Psi^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(v_{\boldsymbol{\lambda}}) = v_{\boldsymbol{\mu}}$. Since $\boldsymbol{\lambda} \succeq (\lambda, 0, \cdots, 0)$, for all $\boldsymbol{\lambda} \in P^+(\lambda, k)$, $V(\lambda)[0]$ is a quotient of $\mathcal{F}_{\boldsymbol{\lambda}}$. Further, as $V(\lambda)[0]$ is irreducible as a $\mathfrak{g}$-module, it is irreducible as a $\mathfrak{g}[t]$-module. Thus $V(\lambda)[0]$ is the unique irreducible quotient of $\mathcal{F}_{\boldsymbol{\lambda}}$ for all $\boldsymbol{\lambda} \in P^+(\lambda, k)$.

(ii) Given $\boldsymbol{\lambda} \in P^+(\lambda, k)$, by definition of $\mathcal{F}_{\boldsymbol{\lambda}}$, we know that for all $\alpha \in R^+$, $\boldsymbol{\lambda}^{\downarrow}_{\alpha}$ is a partition of $\lambda(h_\alpha)$ with at most k parts. Hence,

$$(x^+_\alpha \otimes t)^{(s)}(x^-_\alpha \otimes 1)^{(r+s)}.v_{\boldsymbol{\lambda}} = 0, \quad r+s \geq 1+kr.$$

In particular for $r = 1$, we have $x^-_\alpha \otimes t^k.v_{\boldsymbol{\lambda}} = 0$ for all $\alpha \in R^+$, which implies that

$$\mathfrak{g} \otimes t^k \mathbb{C}[t].v_{\boldsymbol{\lambda}} = 0.$$

(iii) We know $ev_{z_i}V(\lambda_i)$ is a quotient of the local Weyl module $W_{loc}(\lambda_i)$ for $1 \leq i \leq k$. If v_i is the image of w_{λ_i} in $ev_{z_i}(V(\lambda_i))$, then $V(\lambda_1)^{z_1} * \cdots * V(\lambda_k)^{z_k}$ is a integrable $\mathfrak{g}[t]$-module of highest weight $\sum_{i=1}^k \lambda_i = \lambda$, generated by the vector $v_1 * \cdots * v_k$. Hence $V(\lambda_1)^{z_1} * \cdots * V(\lambda_k)^{z_k}$ is a quotient of $W_{loc}(\lambda)$, we have

$$(\mathfrak{n}^+ \otimes \mathbb{C}[t])v_1 * \cdots * v_k = 0, \quad (h_i \otimes t^s).\lambda(h_i)\delta_{s,0}v_1 * \cdots * v_k = 0$$

$$(x^-_{\alpha_i} \otimes 1)^{\lambda(h_i)+1}.v_1 * \cdots * v_k = 0.$$

For $\alpha \in R^+$, if σ is an element of the symmetric group $\mathcal{S}_k$ such that

$$\boldsymbol{\lambda}^{\downarrow}_{\alpha} = (\lambda_{\sigma(1)}(h_\alpha) \geq \cdots \geq \lambda_{\sigma(k)}(h_\alpha)),$$

then using the fact that

$$x_\alpha^- \otimes (t-z_{\sigma(1)})(t-z_{\sigma(2)})\cdots(t-z_{\sigma(\ell)}).ev_{z_{\sigma(1)}}V(\lambda_{\sigma(1)})\otimes\cdots\otimes ev_{z_{\sigma(\ell)}}V(\lambda_{\sigma(\ell)}) = 0,$$

and

$$(x_\alpha^+\otimes t)^s(x_\alpha^-\otimes 1)^{r+s}v_{\sigma(\ell+1)}\otimes\cdots\otimes v_{\sigma(k)} = 0 \ \ \forall\, r+s\geq 1+r\ell+\sum_{j=\ell+1}^{k}\lambda_{\sigma(j)}(h_\alpha),$$

the same proof as [16, Proposition 6.8] shows that for each $\alpha\in R^+$

$${}_\ell x_\alpha^-(r,s).v_1 * \cdots * v_k = 0$$

whenever $r,s,\ell\in\mathbb{N}$ are such that $r+s\geq 1+r\ell+\sum_{j\geq\ell+1}\lambda_j(h_\alpha)^\downarrow$. Since $V(\lambda_1)^{z_1}*\cdots*V(\lambda_k)^{z_k}$ is quotient of local Weyl module $W_{loc}\sum_{i=1}^{k}\lambda_i)$, it follows that $V(\lambda_1)^{z_1}*\cdots*V(\lambda_k)^{z_k}$ is a quotient of $\mathcal{F}_{\boldsymbol{\lambda}}$. □

Remark 2.7.2. It follows from part (iii) of Lemma 2.7.1 and Remark 2.4.5 that,

$$\dim\mathcal{F}_{\boldsymbol{\lambda}} \geq \prod_{i=1}^{k}\dim V(\lambda_i).$$

□

Chapter 3

CV modules and Fusion modules for current Lie algebra of type A_2

In this chapter, we state and prove our main results on fusion product modules. We study the CV modules and Fusion modules for $\mathfrak{sl}_3[t]$ and prove that the fusion product of two finite-dimensional irreducible $\mathfrak{sl}_3[t]$-module is isomorphic to a CV-module. Since the CV modules are defined by generators and relations, this isomorphism helps to establish the conjecture 2.5.1 in the case when $k = 2$ and $\mathfrak{g}$ is of type A_2. While we prove our result using a representation-theoretic approach, in [8], Barth and Kus establish it using combinatorial tools.

3.1 Main Results

For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, using a series of canonical short exact sequences, it had been proved in [16] that, given a partition ξ of a positive integer n, the CV-module, $V(\xi)$, is isomorphic to the fusion product of evaluation modules for $\mathfrak{sl}_2[t]$. Besides proving the conjecture 2.5.1, this helped obtain an explicit monomial basis for the modules $V(\xi)$. In this section, we extend this method in the case when $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and $\boldsymbol{\lambda} \in P^+(\lambda, 2)$ for $\lambda \in P^+$. For $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 \in P^+$, let $|\lambda| := \lambda_1 + \lambda_2$.

Definition 3.1.1. Let $\mathfrak{g} := \mathfrak{sl}_3(\mathbb{C})$. Given $\nu \in P^+$ and $(\lambda, \mu) \in P^+(\nu, 2)$ with $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$ and $\mu = \mu_1\omega_1 + \mu_2\omega_2$, we say:

i. (λ, μ) is a partition of ν of first kind if $|\lambda| \geq |\mu|$, $\lambda_i \geq \mu_i$ for $i = 1, 2$;

ii. (λ, μ) is a partition of ν of second kind if $|\lambda| \geq |\mu|$, $\lambda_1 \geq \mu_1$ and $\mu_2 > \lambda_2$.

iii. (λ, μ) is a partition of ν of third kind if $|\lambda| \geq |\mu|$, $\lambda_2 \geq \mu_2$ and $\mu_1 > \lambda_1$.

Notice that if (λ, μ) is a partition of third kind, then $(\hat{\lambda}, \hat{\mu}) := (\lambda_2\omega_1 + \lambda_1\omega_2, \mu_2\omega_1 + \mu_1\omega_2)$ is a partition of second kind and the Dynkin diagram automorphism of $\mathfrak{sl}_3(\mathbb{C})$ that maps α_1 to α_2 establishes a $\mathfrak{g}[t]$-module isomorphism between $\mathcal{F}_{\lambda,\mu}$ and $\mathcal{F}_{\hat{\lambda},\hat{\mu}}$. In the rest of the chapter, we shall therefore study the modules $\mathcal{F}_{\lambda,\mu}$ for partitions (λ, μ) of $\lambda + \mu$ of first and second kind only.

Given dominant integral weights, $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$, $\mu = \mu_1\omega_1 + \mu_2\omega_2$ such that $|\lambda| \geq |\mu|$ and $\lambda_1 \geq \mu_1$, set

$$\mathcal{F}^+_{\lambda,\mu} = \begin{cases} \mathcal{F}_{\lambda+\omega_2, \mu-\omega_2} & \text{if } \lambda_2 \geq \mu_2 > 0 \\ \mathcal{F}_{\lambda+\omega_1, \mu-\omega_1} & \text{if } \mu_1 > 0, \mu_2 = 0 \\ \mathcal{F}_{\lambda+(\mu_2-\lambda_2)\omega_2, \mu-(\mu_2-\lambda_2)\omega_2} & \text{if } \lambda_2 < \mu_2 \end{cases}$$

Theorem 3.1.2. *Given $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$, $\mu = \mu_1\omega_1 + \mu_2\omega_2 \in P^+$ with $|\lambda| \geq |\mu|$ and $\lambda_1 \geq \mu_1$.*

(i) Then there exists a short exact sequence

$$0 \to \ker(\lambda, \mu) \to \mathcal{F}_{\lambda,\mu} \xrightarrow{\phi(\lambda,\mu)} \mathcal{F}^+_{\lambda,\mu} \to 0$$

where the kernel, $\ker(\lambda,\mu)$ *admits a filtration whose successive quotients are the direct sum of finitely many Chari-Venkatesh modules.*

(ii) For any pair of distinct complex numbers $(z_1,z_2)\in\mathbb{C}^2$*, there exists a* $\mathfrak{sl}_3[t]$*-module isomorphism between* $\mathcal{F}_{\lambda,\mu}$ *and the fusion product* $V^{z_1}(\lambda)*V^{z_2}(\mu)$ *.*

3.2 Relations in $\mathcal{F}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$

Proposition 3.2.1. *Given* $\lambda,\mu\in P^+$*, let* $\mathcal{F}_{\lambda,\mu},\mathcal{F}^+_{\lambda,\mu}$ *be as described in Theorem 3.1.2. Let* $v_{\lambda,\mu}$ *and* $v^+_{\lambda,\mu}$ *be the generators of* $\mathcal{F}_{\lambda,\mu}$ *and* $\mathcal{F}^+_{\lambda,\mu}$ *respectively. Set*

$$\mathcal{K}_{\lambda,\mu}=\begin{cases}\{(x^-_{\alpha_2}\otimes t)^{\mu_2}v_{\lambda,\mu},(x^-_{\alpha_{12}}\otimes t)^{\mu_1+\mu_2}v_{\lambda,\mu}\}, & \text{if } \lambda_2\geq\mu_2>0,\\ \{(x^-_{\alpha_1}\otimes t)^{\mu_1}v_{\lambda,\mu},(x^-_{\alpha_{12}}\otimes t)^{\mu_1}v_{\lambda,\mu}\}, & \text{if } \mu_1>0,\ \mu_2=0,\\ \{(x^-_{\alpha_{12}}\otimes t)^{\mu_1+\lambda_2+s}v_{\lambda,\mu}:0<s\leq\mu_2-\lambda_2\}, & \text{if } \lambda_2<\mu_2\end{cases}$$

Then there exists a surjective $\mathfrak{g}[t]$*-module homomorphism* $\phi(\lambda,\mu):\mathcal{F}_{\lambda,\mu}\to\mathcal{F}^+_{\lambda,\mu}$ *such that* $\phi(\lambda,\mu)v_{\lambda,\mu}=v^+_{\lambda,\mu}$ *and the kernel* $\ker\phi(\lambda,\mu)$ *is the* $\mathfrak{g}[t]$*-module generated by the set* $\mathcal{K}_{\lambda,\mu}$*.*

Proof. Given $(\lambda,\mu)\in P^+(\lambda+\mu,2)$, such that $\lambda(h_{\alpha_{12}})=|\lambda|\geq|\mu|=\mu(h_{\alpha_{12}})$ and $\lambda(h_{\alpha_1})\geq\mu(h_{\alpha_1})$, note that

$$(\lambda,\mu)\succeq\begin{cases}(\lambda+\omega_2,\mu-\omega_2) & \text{if } \lambda_2\geq\mu_2>0\\ (\lambda+\omega_1,\mu-\omega_1) & \text{if } \lambda_1\geq\mu_1>0,\mu_2=0\\ (\lambda+(\mu_2-\lambda_2)\omega_2,\mu-(\mu_2-\lambda_2)\omega_2) & \text{if } \lambda_2<\mu_2\end{cases}$$

Hence by Lemma 2.7.1(iii) there exists is a surjective homomorphism $\phi(\lambda,\mu):\mathcal{F}_{\lambda,\mu}\to\mathcal{F}^+_{\lambda,\mu}$ such that $\phi(\lambda,\mu)(v_{\lambda,\mu})=v^+_{\lambda,\mu}$.

We now analyse the generators of $\ker\phi(\lambda,\mu)$ case by case.

Case 1. Suppose $0<\mu_2<\lambda_2$. By definition, we have $\mathcal{F}^+_{\lambda,\mu}=\mathcal{F}_{\lambda+\omega_2,\mu-\omega_2}$. Hence,

$$\begin{aligned}
x^-_{\alpha_1}(r,s).v^+_{\lambda,\mu}&=0, \quad \forall\, r,s\in\mathbb{N}, \text{ with } r+s\geq 1+r+\mu_1\\
x^-_{\alpha_2}(r,s).v^+_{\lambda,\mu}&=0, \quad \forall\, r,s\in\mathbb{N}, \text{ with } r+s\geq 1+r+\mu_2-1\\
x^-_{\alpha_{12}}(r,s).v^+_{\lambda,\mu}&=0, \quad \forall\, r,s\in\mathbb{N}, \text{ with } r+s\geq 1+r+\mu_1+\mu_2-1
\end{aligned}$$

whereas,

$$\begin{aligned}
x^-_{\alpha_i}(r,s).v_{\lambda,\mu}&=0, \quad \forall\, r,s\in\mathbb{N}, \text{ with } r+s\geq 1+r+\mu_i, \text{ for } i=1,2\\
x^-_{\alpha_{12}}(r,s).v_{\lambda,\mu}&=0, \quad \forall\, r,s\in\mathbb{N}, \text{ with } r+s\geq 1+r+\mu_1+\mu_2
\end{aligned}$$

Since by Lemma 2.7.1, $(x^-_\alpha\otimes t^2).v_{\lambda,\mu}=0$ for all $\alpha\in R^+$, it follows that in this case,

$$x^-_{\alpha_2}(r,s).v_{\lambda,\mu},\ x^-_{\alpha_{12}}(r',s').v_{\lambda,\mu}\in\ker\phi(\lambda,\mu)$$

for $r=s=\mu_2$ and $r'=s'=\mu_1+\mu_2$, i.e.,

$$(x^-_{\alpha_2}\otimes t)^{\mu_2}.v_{\lambda,\mu},\ (x^-_{\alpha_{12}}\otimes t)^{\mu_1+\mu_2}.v_{\lambda,\mu}\in\ker\phi(\lambda,\mu).$$

Case 2. Suppose $\mu_2=0$ and $\lambda_1\geq\mu_1>0$. By definition, we have $\mathcal{F}^+_{\lambda,\mu}=\mathcal{F}_{\lambda+\omega_1,\mu-\omega_1}$. Following similar arguments, it is easy to see that

$$(x^-_{\alpha_1}\otimes t)^{\mu_1}.v_{\lambda,\mu},\ (x^-_{\alpha_{12}}\otimes t)^{\mu_1}.v_{\lambda,\mu}\in\ker\phi(\lambda,\mu).$$

Case 3. Suppose $\mu_2 > \lambda_2 \geq 0$. By definition, we have $\mathcal{F}^+_{\lambda,\mu} = \mathcal{F}_{\lambda+(\mu_2-\lambda_2)\omega_2,\mu-(\mu_2-\lambda_2)\omega_2}$. Hence,

$$\begin{aligned} x^-_{\alpha_i}(r,s).v^+_{\lambda,\mu} &= 0, \quad \forall\, r,s \in \mathbb{N}, \text{ with } r+s \geq 1+r+\mu_i \text{ for } i=1,2,\\ x^-_{\alpha_{12}}(r,s).v^+_{\lambda,\mu} &= 0, \quad \forall\, r,s \in \mathbb{N}, \text{ with } r+s \geq 1+r+\mu_1+\lambda_2 \end{aligned}$$

whereas,

$$\begin{aligned} x^-_{\alpha_i}(r,s).v_{\lambda,\mu} &= 0, \quad \forall\, r,s \in \mathbb{N}, \text{ with } r+s \geq 1+r+\mu_i, \text{ for } i=1,2\\ x^-_{\alpha_{12}}(r,s).v_{\lambda,\mu} &= 0, \quad \forall\, r,s \in \mathbb{N}, \text{ with } r+s \geq 1+r+\mu_1+\mu_2 \end{aligned}$$

Since $(x^-_{\alpha_{12}} \otimes t^2).v_{\lambda,\mu} = 0$, we see that $x^-_{\alpha_{12}}(r,s).v_{\lambda,\mu} \in \ker\phi(\lambda,\mu)$ for $\mu_1+\lambda_2 < r = s \leq \mu_2+\mu_1$, i.e.,

$$\{(x^-_{\alpha_{12}} \otimes t)^{\mu_1+\lambda_2+p}.v_{\lambda,\mu} : 0 < p \leq \mu_2-\lambda_2\} \subset \ker\phi(\lambda,\mu).$$

Conversely, if $X \in \mathbf{U}(\mathfrak{g}[t])$ is such that $X.v_{\lambda,\mu} \in \ker\phi(\lambda,\mu)$, then $X.v^+_{\lambda,\mu} = 0$ and hence X can be written as $X = Y+Z$ where Y is in the left ideal of $\mathbf{U}(\mathfrak{g}[t])$ generated by the set

$$I_Y(\lambda,\mu) = \left\{x^+_\alpha \otimes t^q,\, (x^-_\alpha \otimes 1)^{(\lambda+\mu)(h_\alpha)+1}, (h\otimes t^q) - \delta_{q,0}(\lambda+\mu)(h).1 : q \in \mathbb{Z}_+, \alpha \in R^+, h \in \mathfrak{h}\right\},$$

and Z is in the left ideal of $\mathbf{U}(\mathfrak{g}[t])$ generated by the set $I^+_Z(\lambda,\mu)$ where,

(i). For $0 < \mu_2 \leq \lambda_2$,

$$I^+_Z(\lambda,\mu) := \begin{array}{l} \{x^-_{\alpha_1}(r,s) : r,s \in \mathbb{N},\, r+s \geq 1+r+\mu_1\}\\ \cup\ \left\{x^-_{\alpha_2}(r,s), x^-_{\alpha_{12}}(r',s') : r,s,r',s' \in \mathbb{N},\, r+s \geq r+\mu_2,\, r'+s' \geq r'+\mu_1+\mu_2\right\}. \end{array}$$

(ii). For $0 < \mu_1 \leq \lambda_1$, $\mu_2 = 0$,

$$I_Z^+(\lambda,\mu) := \{x_{\alpha_1}^-(r,s), x_{\alpha_{12}}^-(r',s') : r,s,r',s' \in \mathbb{N},\, r+s \geq r+\mu_1,\, r'+s' \geq r'+\mu_1\}.$$

(iii). For $\mu_2 > \lambda_2$,

$$I_Z^+(\lambda,\mu) := \begin{array}{l} \left\{x_{\alpha_1}^-(r,s), x_{\alpha_2}^-(r',s') : r,s,r',s' \in \mathbb{N},\, r+s \geq 1+r+\mu_1, r'+s' \geq 1+r'+\lambda_2\right\} \\ \cup\{x_{\alpha_{12}}^-(r,s) : r,s \in \mathbb{N},\, r+s \geq 1+r+\mu_1+\lambda_2\}. \end{array}$$

Since $Y.v_{\lambda,\mu} = 0$ for all $Y \in I_Y(\lambda,\mu)$ and $Z.v_{\lambda,\mu} = 0$, for all $Z \in \{x_{\alpha_1}^-(r,s) : r,s \in \mathbb{N},\, r+s \geq 1+r+\mu_1\}$ when $0 < \mu_2 \leq \lambda_2$ and $Z \in \{x_{\alpha_1}^-(r,s), x_{\alpha_2}^-(r',s') : r,s,r',s' \in \mathbb{N},\, r+s \geq 1+r+\mu_1,\, r'+s' > 1+r'+\lambda_2\}$ when $\mu_2 > \lambda_2$, it is clear that $\ker\phi(\lambda,\mu)$ is generated by elements of $\mathcal{K}_{\lambda,\mu}$. Hence, the proposition. $\square$

Lemma 3.2.2. *Given $a,b,p \in \mathbb{Z}_+$, we have,*

(i). $[x_{\alpha_1}^+ \otimes t^a, (x_{\alpha_{12}}^- \otimes t^b)^{(p)}] = -(x_{\alpha_{12}}^- \otimes t^b)^{(p-1)}(x_{\alpha_2}^- \otimes t^{a+b})$,

(ii). $[x_{\alpha_2}^+ \otimes t^a, (x_{\alpha_{12}}^- \otimes t^b)^{(p)}] = (x_{\alpha_{12}}^- \otimes t^b)^{(p-1)}(x_{\alpha_1}^- \otimes t^{a+b})$,

(iii). $[x_{\alpha_1}^- \otimes t^a, (x_{\alpha_{12}}^+ \otimes t^b)^{(p)}] = (x_{\alpha_{12}}^+ \otimes t^b)^{(p-1)}(x_{\alpha_2}^+ \otimes t^{a+b})$,

(iv). $[x_{\alpha_2}^- \otimes t^a, (x_{\alpha_{12}}^+ \otimes t^b)^{(p)}] = -(x_{\alpha_{12}}^+ \otimes t^b)^{(p-1)}(x_{\alpha_1}^+ \otimes t^{a+b})$,

The following is an easy corollary of the lemma.

Corollary 3.2.3. *Given $a_1,a_2,b \in \mathbb{Z}_+$ we have the following:*

(i) $[x_{\alpha_1}^+, (x_{\alpha_1}^- \otimes t)^{(a_1)}(x_{\alpha_{12}}^- \otimes t)^{(b)}(x_{\alpha_2}^- \otimes t)^{(a_2)}] + (x_{\alpha_1}^- \otimes t)^{(a_1)}(x_{\alpha_{12}}^- \otimes t)^{(b-1)}(x_{\alpha_2}^- \otimes t)^{(a_2+1)}$ *is contained in* $U(\mathfrak{g}[t])\mathfrak{h}t\mathbb{C}[t] \oplus U(\mathfrak{g}[t])\mathfrak{n}^- t^2\mathbb{C}[t]$.

(ii) $[x_{\alpha_2}^+, (x_{\alpha_1}^- \otimes t)^{(a_1)}(x_{\alpha_{12}}^- \otimes t)^{(b)}(x_{\alpha_2}^- \otimes t)^{(a_2)}] - (x_{\alpha_1}^- \otimes t)^{(a_1+1)}(x_{\alpha_{12}}^- \otimes t)^{(b-1)}(x_{\alpha_2}^- \otimes t)^{(a_2)}$ *is contained in* $U(\mathfrak{g}[t])\mathfrak{h}t\mathbb{C}[t] \oplus U(\mathfrak{g}[t])\mathfrak{n}^- t^2\mathbb{C}[t]$.

We shall use the following notation in the rest of the chapter. For $s, r \in \mathbb{N}$ and $\alpha \in R^+$, set

$$\mathbf{X}_\alpha(r,s) = (x_\alpha^+ \otimes t)^{(s)}(x_\alpha^- \otimes 1)^{(r+s)}$$

Further, we denote $\mathbf{X}_{\alpha_i}(r,s)$ by $\mathbf{X}_i(r,s)$ and $\mathbf{X}_{\alpha_{12}}(r,s)$ by $\mathbf{X}_{12}(r,s)$.

Lemma 3.2.4. *Suppose V is a $\mathfrak{sl}_3[t]$-module and for all $\alpha \in R^+$, $v \in V$ satisfies*

$$\mathfrak{n}^+ \otimes \mathbb{C}[t].v = 0, \quad \mathfrak{h} \otimes t\mathbb{C}[t].v = 0, \quad x_\alpha^- \otimes t^2.v = 0, \quad (x_\alpha^- \otimes 1)^l.v = 0, \forall\, l \geq L_\alpha^{(0)},$$

$$\mathbf{X}_\alpha(r,s)v = 0 \quad \textit{for } r,s \in \mathbb{N},\ r+s \geq rk + L_\alpha^{(k)}, \quad \textit{where } L_\alpha^{(k)} \in \mathbb{Z}_+ \textit{ for each } k \geq 1,$$

then we have,

$$\mathbf{X}_\alpha(r,s)(x_\alpha^- \otimes t)^d v = 0, \qquad \forall\, r+s \geq rk + L_\alpha^{(k)} - 2d,\ \alpha \in R^+, \tag{3.2.1}$$

$$\mathbf{X}_{\alpha_{12}}(r,s)(x_{\alpha_i}^- \otimes t)^d v = 0, \qquad \forall\, r+s \geq rk + L_{\alpha_{12}}^{(k)} - d, \textit{for some } k \geq 1 \quad i = 1,2, \tag{3.2.2}$$

$$\mathbf{X}_1(r,s)(x_{\alpha_2}^- \otimes t)^d.v = 0 \quad \textit{when } r+s \geq rk + L_{\alpha_1}^{(k)} \textit{ for some } k \geq 1, \tag{3.2.3}$$

$$(x_{\alpha_1}^- \otimes 1)^l (x_{\alpha_2}^- \otimes t)^d.v = 0 \quad \textit{when } l \geq L_{\alpha_1}^{(0)} + d \tag{3.2.4}$$

$$\sum_{d=0}^{l} \binom{l}{d} \mathbf{X}_1(r-d,s)(x_{\alpha_{12}}^- \otimes t)^d (x_{\alpha_2}^- \otimes t)^{l-d} v = 0, \quad \forall\, r+s \geq rk + L_{\alpha_1}^{(0)} \tag{3.2.5}$$

$$\begin{aligned}
\mathbf{X}_2(r,s)(x_{\alpha_1}^- \otimes t)^d.v &= 0, && \textit{when } r+s \geq rk + L_{\alpha_2}^{(k)} \textit{ for some } k \geq 1,\\
(x_{\alpha_2}^- \otimes 1)^l (x_{\alpha_1}^- \otimes t)^d.v &= 0, && \textit{when } l \geq L_{\alpha_2}^{(0)} + d\\
\sum_{d=0}^{l} \tbinom{l}{d} \mathbf{X}_2(r-d,s)(x_{\alpha_{12}}^- \otimes t)^d (x_{\alpha_1}^- \otimes t)^{l-d} v &= 0, && \forall\, r+s \geq rk + L_{\alpha_2}^{(0)}
\end{aligned} \tag{3.2.6}$$

Proof. The relation (3.2.1) follows from [16, Corollary 6.6]. Using Lemma 3.2.2, we have

$$x_{\alpha_2}^-.\mathbf{X}_{12}(r,s).v = \mathbf{X}_{12}(r,s)x_{\alpha_2}^-.v + \mathbf{X}_{12}(r,s-1)(x_{\alpha_2}^- \otimes t).v.$$

Using (2.3.1), and the relations satisfied by v, it follows that,

$$\begin{aligned} x^-_{\alpha_2}.\mathbf{X}_{12}(r,s).v &= (-1)^s\left(x^-_{\alpha_{12}}(r,s)x^-_{\alpha_2}.v + x^-_{\alpha_{12}}(r,s-1)x^-_{\alpha_2}\otimes t.v\right) + \mathbf{X}_{12}(r,s-1).(x^-_{\alpha_2}\otimes t).v. \\ &= (-1)^s\left(x^-_{\alpha_2}.x^-_{\alpha_{12}}(r,s).v + x^-_{\alpha_{12}}(r,s-1)x^-_{\alpha_2}\otimes t.v\right) + \mathbf{X}_{12}(r,s-1).(x^-_{\alpha_2}\otimes t).v. \end{aligned}$$

Therefore, when $r+s \geq rk + L^{(k)}_{\alpha_{12}}$ for some $k \in \mathbb{N}$, using the relations satisfied by v, commutativity of $x^-_{\alpha_2}$ and $x^-_{\alpha_{12}}\otimes t^s$ and Lemma 2.6.2 we get,

$$\mathbf{X}_{12}(r,s-1).(x^-_{\alpha_2}\otimes t).v = (-1)^{s-1}x^-_{\alpha_{12}}(r,s-1).(x^-_{\alpha_2}\otimes t).v. \tag{3.2.7}$$

Since,

$$(x^-_{\alpha_{12}})^{(r-s)}(x^-_{\alpha_{12}}\otimes t)^{(s)}.v = 0, \quad \text{when } r+s \geq rk + L^{(k)}_{\alpha_{12}}, \text{for some } k \in \mathbb{N} \tag{3.2.8}$$

applying $x^+_{\alpha_1}$ to (3.2.8) we get,

$$(x^-_{\alpha_{12}})^{(r-s)}(x^-_{\alpha_{12}}\otimes t)^{(s-1)}(x^-_{\alpha_2}\otimes t).v + (x^-_{\alpha_{12}})^{(r-s-1)}(x^-_{\alpha_{12}}\otimes t)^{(s)}.v = 0, \tag{3.2.9}$$

when $r+s \geq rk + L^{(k)}_{\alpha_{12}}$ for some $k \in \mathbb{N}$. On the other hand, if $r+s \geq rk + L^{(k)}_{\alpha_{12}}$, then for all $k \geq 0$, $(r-1)+s \geq rk + L^{(k)}_{\alpha_{12}} - 1 \geq (r-1)k + L^{(k)}_{\alpha_{12}}$. Hence,

$$(x^-_{\alpha_{12}})^{(r-1-s)}.(x^-_{\alpha_{12}}\otimes t)^{(s)}.v = 0, \quad \text{when } r+s \geq rk + L^{(k)}_{\alpha_{12}}, \text{ for some } k \in \mathbb{N}.$$

It thus follows from (3.2.7) and (3.2.9) that,

$$\mathbf{X}_{12}(r,s-1).(x^-_{\alpha_2}\otimes t).v = (-1)^{s-1}x^-_{\alpha_{12}}(r,s-1).(x^-_{\alpha_2}\otimes t).v = 0,$$

whenever $r+s \geq rk + L^{(k)}_{\alpha_{12}}$, for some $k \in \mathbb{N}$. This shows that

$$\mathbf{X}_{12}(r,s).(x^-_{\alpha_2}\otimes t).v = 0, \quad \text{when } r+s \geq rk + L^{(k)}_{\alpha_{12}} - 1, \text{ for some } k \in \mathbb{N}.$$

For any $q > 0$,

$$x_{\alpha}^{+}.(x_{\alpha_2} \otimes t)^q.v = 0, \quad h \otimes t^s.(x_{\alpha_2} \otimes t)^q.v = 0, \; s > 0$$

by repeating the above arguments we see that (3.2.2) holds.

Using Lemma 3.2.2 we have,

$$\begin{aligned}(x_{\alpha_2}^{-} \otimes t)\mathbf{X}_1(r,s).v &= \mathbf{X}_1(r,s)(x_{\alpha_2}^{-} \otimes t) + \mathbf{X}_1(r-1,s).(x_{\alpha_{12}}^{-} \otimes t).v \\ &= \mathbf{X}_1(r,s)(x_{\alpha_2}^{-} \otimes t) + (x_{\alpha_{12}}^{-} \otimes t).\mathbf{X}_1(r-1,s).v\end{aligned} \tag{3.2.10}$$

If $r+s \geq rk + L_{\alpha_1}^{(k)}$, then $r-1+s \geq (r-1)k + L_1^{(k)}$ for $k \geq 1$. Therefore, using the relations satisfied by v, we see that

$$\mathbf{X}_1(r,s).(x_{\alpha_2}^{-} \otimes t).v = 0, \quad \text{when } r+s \geq rk + L_{\alpha_1}^{(k)} \text{ for some } \; k \geq 1,$$

Further applying $x_{\alpha_2}^{-} \otimes t$ to the relation $(x_{\alpha_1}^{-} \otimes 1)^l.v = 0$ for $l \geq L_{\alpha_1}^{(0)}$, we get

$$(x_{\alpha_1}^{-} \otimes 1)^l.x_{\alpha_2}^{-} \otimes t.v = 0, \quad \text{when } r+s \geq L_{\alpha_1}^{(0)} + 1.$$

As above, by repeating the above arguments we see that (3.2.4) holds.

Finally, applying $(x_{\alpha_2}^{-} \otimes t)^l$ on both sides of (3.2.10) and using the relations satisfied by v, Lemma 3.2.2 and Lemma 2.3.1 repeatedly, we get

$$\sum_{d=0}^{l} \binom{l}{d} \mathbf{X}_1(r-d,s)(x_{\alpha_{12}}^{-} \otimes t)^d (x_{\alpha_2}^{-} \otimes t)^{l-d} v = 0, \quad \forall \; r+s \geq rk + L_{\alpha_1}^{(0)}.$$

This shows that (3.2.5) holds. Similarly by applying $x_{\alpha_1}^{-}$ on $\mathbf{X}_{12}(r,s)v$ and using Lemma 2.3.1 one can show that (3.2.6) holds. □

3.3 Structure of $\ker\phi(\lambda,\mu)$

We have seen that $\ker\phi(\lambda,\mu)$ is a graded submodule of $\mathcal{F}_{\lambda,\mu}$. Note that for each $m\in\mathbb{Z}_+$, $\ker\phi(\lambda,\mu)[m]$ is a finite dimensional $\mathfrak{sl}_3$-module. In this section we give a filtration of $\ker\phi(\lambda,\mu)$, then using induction and dimension argument obtain a $\mathfrak{sl}_3$- module decompositions of the associated graded subspaces of $\ker(\lambda,\mu)$.

By a case by case study, we now give a filtration of the $\mathfrak{sl}_3[t]$- submodule $\ker\phi(\lambda,\mu)$ of $\mathcal{F}_{\lambda,\mu}$. Throughout this section, we assume that for given a pair $(\lambda,\mu)\in P^+(\lambda+\mu,2)$, $|\lambda|\geq|\mu|$ and $\lambda_1\geq\mu_1$.

3.3.1 Case 1: $\lambda_2\geq\mu_2>0$

Under the given conditions on λ,μ, set

$$\mathbb{S}_{ninv}(\lambda,\mu)=\left\{(a_1,a_2)\in\mathbb{Z}^2_{\geq 0}:\begin{array}{l}0\leq a_i\leq\mu_i,\, i=\{1,2\}\\ \mu_2-\lambda_1\leq a_2-a_1\leq\lambda_2-\mu_1\end{array}\right\}. \tag{3.3.1}$$

For $0\leq j\leq\mu_1+\mu_2$, let

$$\mathbb{S}_{ninv}(\lambda,\mu)[j]=\{(a_1,a_2)\in\mathbb{S}_{ninv}(\lambda,\mu): a_1+a_2=j\};$$

$$V_j=\begin{cases}\sum\limits_{(a_1,a_2)\in\mathbb{S}_{ninv}(\lambda,\mu)[j]}\mathbf{U}(\mathfrak{sl}_3[t])(x^-_{\alpha_1}\otimes t)^{(a_1)}(x^-_{\alpha_{12}}\otimes t)^{(|\mu|-a_1-a_2)}(x^-_{\alpha_2}\otimes t)^{(a_2)}.v_{\lambda,\mu}+V_{|\mu|}, & \text{if } 0\leq j\leq|\mu|-1\\ \mathbf{U}(\mathfrak{sl}_3[t])(x^-_{\alpha_2}\otimes t)^{(\mu_2)}.v_{\lambda,\mu} & \text{if } j=|\mu|\end{cases}$$

Using Proposition 3.2.1, we have the following theorem which gives us a filtration of $\ker(\lambda,\mu)$.

Proposition 3.3.1. *Given $\lambda,\mu\in P^+$ with $|\lambda|\geq|\mu|$, $\lambda_i\geq\mu_i$ for $i=1,2$ and $\mu_2>0$, let $\mathbb{S}_{ninv}(\lambda,\mu)$, and $\mathbb{S}_{ninv}(\lambda,\mu)[j]$, V_j for $0\leq j\leq|\mu|$ be defined as above. Then $0\subset V_{|\mu|}\subset$*

$\cdots \subset V_1 \subset V_0$ *gives a filtration of* $\ker\phi(\lambda,\mu)$. *Further, there exists a surjective* $\mathfrak{sl}_3[t]$-*homomorphism*

$$\phi_j^{(\lambda,\mu)} : \bigoplus_{(a_1,a_2)\in\mathbb{S}_{ninv}(\lambda,\mu)[j]} \tau_{|\mu|}^* V(\lambda-(\mu_2+a_1-2a_2)\omega_1-(\mu_1+a_2-2a_1)\omega_2) \to V_j/V_{j+1},$$

for $0 \le j < |\mu|$, *and a surjective homomorphism*

$$\phi_{|\mu|}^{(\lambda,\mu)} : \tau_{|\mu|}^*(\mathcal{F}_{\lambda+\mu_2(\omega_1-\omega_2),\mu_1\omega_1}) \to V_{|\mu|}.$$

Proof. Using Lemma 3.2.4 with $v = v_{\lambda,\mu}$ and,

$$L_\alpha^{(k)} = \begin{cases} (\lambda+\mu)(h_\alpha)+1, & k=0 \\ \mu(h_\alpha)+1, & k=1 \\ 1 & k>1 \end{cases} \quad \forall\ \alpha \in R^+.$$

By Proposition 3.2.1, $\ker\phi(\lambda,\mu) = U(\mathfrak{sl}_3[t]).(x_{\alpha_{12}}^-\otimes t)^{(|\mu|)}.v_{\lambda,\mu} + U(\mathfrak{sl}_3[t])(x_{\alpha_2}^-\otimes t)^{(\mu_2)}v_{\lambda,\mu}$. By definition of $\mathcal{F}_{\lambda,\mu}$,

$$(x_{\alpha_{12}}^-\otimes t)^k.v_{\lambda,\mu} = 0, \text{ for } k \ge |\mu|, \quad (x_{\alpha_i}^-\otimes t)^\ell.v_{\lambda,\mu} = 0, \text{ for } \ell \ge \mu_i,\ i=1,2.$$

Applying $(x_{\alpha_2}^+\otimes t)^{a_1}(x_{\alpha_1}^+\otimes t)^{a_2}$ to the relation $(x_{\alpha_{12}}^-\otimes t)^k.v_{\lambda,\mu} = 0$ for $k > |\mu|$, using (2.3.1) and the relation that $x\otimes t^2.v_{\lambda,\mu} = 0$ for all $x \in \mathfrak{g}$, it is easy to see that

$$(x_{\alpha_1}^-\otimes t)^{(a_1)}(x_{\alpha_{12}}^-\otimes t)^{(a_{12})}(x_{\alpha_2}^-\otimes t)^{(a_2)}.v_{\lambda,\mu} = 0, \quad a_1+a_{12}+a_2 > |\mu|.$$

Hence $(x\otimes t).(x_{\alpha_{12}}^-\otimes t)^{(|\mu|)}v_{\lambda,\mu} = 0$, and $(x\otimes t).(x_{\alpha_2}^-\otimes t)^{(\mu_2)}v_{\lambda,\mu} = 0$, for all $x \in \mathfrak{g}$.

Now observe that for $a_i \leq \mu_i$, $i = 1,2$,

$$(x^+_{\alpha_2} \otimes 1)^{a_1}(x^+_{\alpha_1} \otimes 1)^{a_2}(x^-_{\alpha_{12}} \otimes t)^{(|\mu|)}.v_{\lambda,\mu}$$
$$= (x^-_{\alpha_1} \otimes t)^{(a_1)}(x^-_{\alpha_{12}} \otimes t)^{(|\mu|-a_1-a_2)}(x^-_{\alpha_2} \otimes t)^{(a_2)}.v_{\lambda,\mu} \in \ker\phi(\lambda,\mu).$$

However, since $v_{\lambda,\mu}$ satisfies the conditions of Lemma 3.2.4, putting $r = |\mu| - a_2$, $s = a_1$ and $l = |\mu| - a_1$ in (3.2.5), for $r + s = |\mu| - a_2 + a_1 > \lambda_1 + \mu_1$ we get,

$$\sum_{d=0}^{|\mu|-a_1} \binom{|\mu|-a_1}{d} \mathbf{X}_1(|\mu|-a_2-d,a_1)(x^-_{\alpha_{12}} \otimes t)^d (x^-_{\alpha_2} \otimes t)^{|\mu|-a_1-d} v_{\lambda,\mu} = 0.$$

As $(x^-_{\alpha_2} \otimes t)^{|\mu|-a_1-d} v_{\lambda,\mu} = 0$, for $|\mu| - a_1 - d > \mu_2$,

$$\sum_{i=0}^{\mu_2} \binom{|\mu|-a_1}{\mu_1 - a_1 + i} \mathbf{X}_1(\mu_2 - a_2 + a_1 - i, a_1)(x^-_{\alpha_{12}} \otimes t)^{\mu_1 - a_1 + i}(x^-_{\alpha_2} \otimes t)^{\mu_2 - i} v_{\lambda,\mu} = 0,$$

whenever $\mu_2 - a_2 + a_1 > \lambda_1$. Consequently, we have,

$$\sum_{i=0}^{\mu_2-a_2-1} \tbinom{|\mu|-a_1}{\mu_1-a_1+i} \mathbf{X}_1(\mu_2 - a_2 + a_1 - i, a_1)(x^-_{\alpha_{12}} \otimes t)^{\mu_1-a_1+i}(x^-_{\alpha_2} \otimes t)^{\mu_2-i} v_{\lambda,\mu}$$
$$+ \tbinom{|\mu|-a_1}{\mu_1-a_1+\mu_2-a_2} \mathbf{X}_1(a_1,a_1)(x^-_{\alpha_{12}} \otimes t)^{\mu_1-a_1+\mu_2-a_2}(x^-_{\alpha_2} \otimes t)^{a_2} v_{\lambda,\mu}$$
$$+ \sum_{i=\mu_2-a_2+1}^{\mu_2} \tbinom{|\mu|-a_1}{\mu_1-a_1+i} \mathbf{X}_1(\mu_2 - a_2 + a_1 - i, a_1)(x^-_{\alpha_{12}} \otimes t)^{\mu_1-a_1+i}(x^-_{\alpha_2} \otimes t)^{\mu_2-i} v_{\lambda,\mu} = 0,$$

for $\mu_2 - a_2 + a_1 > \lambda_1$. Using the relation $x^-_\alpha \otimes t^s.v_{\lambda,\mu} = 0$ for $s \geq 2$ and $\alpha \in R^+$, we see that

$$(x^-_{\alpha_1} \otimes t)^{(a_1)}(x^-_{\alpha_{12}} \otimes t)^{(|\mu|-a_1-a_2)}(x^-_{\alpha_2} \otimes t)^{(a_2)} =$$
$$- \left(\sum_{i=0}^{\mu_2-a_2-1} \binom{|\mu|-a_1}{\mu_1 - a_1 + i} \mathbf{X}_1(\mu_2 - a_2 + a_1 - i, a_1)(x^-_{\alpha_{12}} \otimes t)^{\mu_1-a_1+i}(x^-_{\alpha_2} \otimes t)^{\mu_2-i} v_{\lambda,\mu} \right) \tag{3.3.2}$$

for $\mu_2 - a_2 + a_1 > \lambda_1$, i.e, for $(a_1,a_2) \in \mathbb{Z}^2_{>0}$ with $a_1 \leq \mu_1$ and $a_2 \leq \mu_2$

$$(x^-_{\alpha_1}\otimes t)^{(a_1)}(x^-_{\alpha_{12}}\otimes t)^{(|\mu|-a_1-a_2)}(x^-_{\alpha_2}\otimes t)^{(a_2)} \subseteq$$
$$-\left(\sum_{i=0}^{\mu_2-a_2-1} U(\mathfrak{g})(x^-_{\alpha_1}\otimes t)^{a_1}(x^-_{\alpha_{12}}\otimes t)^{\mu_1-a_1}(x^-_{\alpha_{12}}\otimes t)^{\mu_1-a_1+i}(x^-_{\alpha_2}\otimes t)^{\mu_2-i}v_{\lambda,\mu}\right),$$

whenever $\mu_2 - a_2 + a_1 > \lambda_1$. Applying the above relations repeatedly, we see that

$$\begin{aligned}\ker\phi(\lambda,\mu) \subseteq & \sum_{(a_1,a_2)\in\mathbb{S}_{ninv}(\lambda,\mu)} U(\mathfrak{g})(x^-_{\alpha_1}\otimes t)^{(a_1)}(x^-_{\alpha_{12}}\otimes t)^{(|\mu|-a_1-a_2)}(x^-_{\alpha_2}\otimes t)^{(a_2)}.v_{\lambda,\mu} \\ & + U(\mathfrak{g})(x^-_{\alpha_2}\otimes t)^{(\mu_2)}.v_{\lambda,\mu} \\ = & \sum_{j=0}^{|\mu|} V_j\end{aligned}.$$

Let $0 \leq j < |\mu|$. For a fixed $(a_1,a_2) \in \mathbb{S}_{ninv}(\lambda,\mu)[j]$, set

$$V_{j,a_1} = U(\mathfrak{sl}_3[t])(x^-_{\alpha_1}\otimes t)^{(a_1)}(x^-_{\alpha_{12}}\otimes t)^{(|\mu|-a_1-a_2)}(x^-_{\alpha_2}\otimes t)^{(j-a_1)}.v_{\lambda,\mu}.$$

Since $a_1 + a_2 = j$ for $(a_1,a_2) \in \mathbb{S}_{ninv}(\lambda,\mu)[j]$, it follows from (3.2.3) that

$$x^+_{\alpha_i}.V_{j,a_1} \subset V_{j+1,a_1} + V_{j+1,a_1+1} \subseteq V_{j+1}, \quad \text{for } i = 1,2.$$

Further, as $h\otimes t^k.v_{\lambda,\mu} = 0$ and $x^-_\alpha \otimes t^\ell = 0$ for $k \geq 1$ and $\ell \geq 2$, we see that,

$$h\otimes t^k.(x^-_{\alpha_1}\otimes t)^{(a_1)}(x^-_{\alpha_{12}}\otimes t)^{(|\mu|-a_1-a_2)}(x^-_{\alpha_2}\otimes t)^{(j-a_1)}.v_{\lambda,\mu}. = 0, \quad \forall\, h \in \mathfrak{h},\ k \geq 1.$$

Using Lemma 3.2.4 with $L^{(k)}_{\alpha_{12}}, L^{(k)}_i$ as given above, it follows from (3.2.2) that for $i = 1,2$,

$$(x^-_{\alpha_i}\otimes t).\left((x^-_{\alpha_1}\otimes t)^{(a_1)}(x^-_{\alpha_{12}}\otimes t)^{(\mu_1+\mu_1-a_1-a_2)}(x^-_{\alpha_2}\otimes t)^{(j-a_1)}.v_{\lambda,\mu}\right)$$
$$= x^+_{\alpha_i}.(x^+_{\alpha_1})^{(a_2)}.(x^+_{\alpha_2})^{(a_1)}(x^-_{\alpha_{12}}\otimes t)^{(|\mu|+1)}.v_{\lambda,\mu} = 0,$$

and

$$x^-_{\alpha_{12}} \otimes t. \quad \left((x^-_{\alpha_1} \otimes t)^{(a_1)}(x^-_{\alpha_{12}} \otimes t)^{(|\mu|-a_1-a_2)}(x^-_{\alpha_2} \otimes t)^{(j-a_1)}.v_{\lambda,\mu}\right)$$

$$= (x^+_{\alpha_1})^{(a_2)}.(x^+_{\alpha_2})^{(a_1)}(x^-_{\alpha_{12}} \otimes t)^{|\mu|+1}.v_{\lambda,\mu} = 0,$$

Hence, $V_{j,a_1}/V_{j+1} \cap V_{j,a_1}$ is a quotient of the CV-module $\tau^*_{|\mu|}V(\boldsymbol{\xi}_{j,a_1})$ where $\boldsymbol{\xi}_{j,a_1} = (\xi^{\alpha}_{j,a_1})_{\alpha \in R^+}$ with

$$\xi^{\alpha_1}_{j,a_1} = \lambda_1 - \mu_2 - a_1 + 2a_2,$$
$$\xi^{\alpha_2}_{j,a_1} = \lambda_2 - \mu_1 - a_2 + 2a_1,$$
$$\xi^{\alpha_{12}}_{j,a_1} = |\lambda| + a_1 + a_2 - \mu_1 - \mu_2.$$

As $(a_1, a_2) \in \mathbb{S}_{ninv}(\lambda, \mu)$, we see that $\xi^{\alpha_1}_{j,a_1} \geq 0$, $\xi^{\alpha_2}_{j,a_1} \geq 0$ and hence $\xi^{\alpha_{12}}_{j,a_1} = \xi^{\alpha_1}_{j,a_1} + \xi^{\alpha_2}_{j,a_1} \geq 0$. This means for each pair (j, a_1), $V(\boldsymbol{\xi}_{j,a_1})$ is isomorphic to the irreducible $\mathfrak{sl}_3$-module $V(\lambda - (\mu_2 + a_1 - 2a_2)\omega_1 - (\mu_1 + a_2 - 2a_1)\omega_2)$. Since, for $1 \leq j < |\mu|$,

$$V_j = \sum_{0 \leq a_1 \leq \mu_1,} V_{j,a_1},$$

and $a_1's$ are all distinct, it follows that $V_{j+1} \subset V_j$ and V_j/V_{j+1} is a quotient of

$$\bigoplus_{(a_1,a_2) \in \mathbb{S}(\lambda,\mu)[j]} \tau^*_{|\mu|}V(\lambda - (\mu_2 + a_1 - 2a_2)\omega_1 - (\mu_1 + a_2 - 2a_1)\omega_2), \quad \forall\ 1 \leq j < |\mu|.$$

To see that $\phi^{(\lambda,\mu)}_{|\mu|}$ is a surjective homomorphism, observe that using Lemma 3.2.2 and the relations satisfied by $v_{\lambda,\mu}$,

$$x^+_{\alpha_i}.(x^-_{\alpha_2} \otimes t)^{(\mu_2)}v_{\lambda,\mu} = 0, \text{ for } i = 1,2; \quad h \otimes t^k.(x^-_{\alpha_2} \otimes t)^{(\mu_2)}v_{\lambda,\mu} = 0 \text{ for } k > 1 \ \cdot$$

Further, using Lemma 3.2.4 with $L^{(k)}_{\alpha_{12}}, L^{(k)}_i$ as given above, we see that from (3.2.1), –,(3.2.4), it follows that $V_{|\mu|}$ is a quotient of the CV-module $\tau^*_{|\mu|}V(\boldsymbol{\xi})$ where $\boldsymbol{\xi} = (\xi^{\alpha})_{\alpha \in R^+}$ with

$$\xi^{\alpha_1} = (\lambda_1 + \mu_2 \geq \mu_1 \geq 0), \quad \xi^{\alpha_2} = (\lambda_2 - \mu_2 \geq 0), \quad \xi^{\alpha_{12}} = (|\lambda| \geq \mu_1 \geq 0).$$

Clearly, $V(\xi)$ is isomorphic to $\mathcal{F}_{\lambda+\mu_2(\omega_1-\omega_2),\mu_1\omega_1}$. This completes the proof of the proposition. $\square$

3.3.2 Case 2: $\lambda_2 \geq \mu_2$, $\mu_2 = 0$

In the Proposition 3.3.1, notice that if $\mu_1 = 0$, applying $(x^-_{\alpha_2}\otimes t)^{\mu_2}$ to the relation

$$(x^-_{\alpha_1}\otimes 1)^k v_{\lambda,\mu} = 0, \quad \forall\, k \geq \lambda_1+1,$$

we get, $\sum\limits_{d=0}^{\mu_2}\binom{\mu_2}{d}(x_{\alpha_1}\otimes 1)^{k-d}(x^-_{\alpha_{12}}\otimes t)^d(x^-_{\alpha_2}\otimes t)^{\mu_2-d}v = 0, \quad \forall\, k \geq \lambda_1+1.$

When $\mu_2 > \lambda_1$ and $k = \lambda_1+1$, the above relation reduces to

$$\sum_{d=0}^{\lambda_1+1}\binom{\mu_2}{d}(x_{\alpha_1}\otimes 1)^{\lambda_1+1-d}(x^-_{\alpha_{12}}\otimes t)^d(x^-_{\alpha_2}\otimes t)^{\mu_2-d}v = 0.$$

Hence we have,

$$(x^-_{\alpha_{12}}\otimes t)^{\lambda_1+1}(x^-_{\alpha_2}\otimes t)^{\mu_2-\lambda_1-1}v_{\lambda,\mu} = \sum_{d=0}^{\lambda_1}\binom{\mu_2}{d}(x_{\alpha_1}\otimes 1)^{\lambda_1+1-d}(x^-_{\alpha_{12}}\otimes t)^d(x^-_{\alpha_2}\otimes t)^{\mu_1-d}v_{\lambda,\mu}. \tag{3.3.3}$$

Since, by Proposition 3.2.1, there is a surjective homomorphism

$$\phi(\lambda,\mu_2\omega_2) : \mathcal{F}_{\lambda,\mu_2\omega_2} \to \mathcal{F}_{\lambda+\omega_2,(\mu_2-1)\omega_2}$$

whose kernel is generated as a $\mathfrak{sl}_3[t]$-module by $(x^-_{\alpha_{12}}\otimes t)^{\mu_2}v_{\lambda,\mu}$ and $(x^-_{\alpha_2}\otimes t)^{\mu_2}v_{\lambda,\mu}$, following similar arguments as in Proposition 3.3.1 and using (3.3.3) repeatedly, we see that

$$\ker\phi(\lambda,\mu_2\omega_2) \subset \sum_{(0,a_2)\in\mathbb{S}_{ninv}(\lambda,\mu_2\omega_2)} U(\mathfrak{g})(x^-_{\alpha_{12}}\otimes t)^{(\mu_2-a_2)}(x^-_{\alpha_2}\otimes t)^{(a_2)}.v_{\lambda,\mu},$$

where, $\quad \mathbb{S}_{ninv}(\lambda,\mu_2\omega_1) = \{(0,a_2)\in\mathbb{Z}^2_+ : \max\{0,\mu_2-\lambda_1\} \leq a_2 \leq \mu_2\}.$

Let $M_2 = \max\{0, \mu_2 - \lambda_1\}$;

$$V_j = \begin{cases} U(\mathfrak{sl}_3[t])(x^-_{\alpha_{12}} \otimes t)^{(\mu_2 - j)}(x^-_{\alpha_2} \otimes t)^{(j)}.v_{\lambda,\mu} + U(\mathfrak{sl}_3[t])(x^-_{\alpha_2} \otimes t)^{(\mu_2)}v_{\lambda,\mu}, & M_2 \leq j \leq \mu_2 - 1, \\ U(\mathfrak{sl}_3[t])(x^-_{\alpha_2} \otimes t)^{(\mu_2)}v_{\lambda,\mu}, & j = \mu_2 \end{cases}$$

Similar arguments as in Proposition 3.3.1 show that

$$\ker\phi(\lambda, \mu_2\omega_2) = U(\mathfrak{sl}_3[t]).(x^-_{\alpha_{12}} \otimes t)^{(\mu_1 - M_2)}(x_{\alpha_2} \otimes t)^{(M_2)}.v_{\lambda,\mu} + U(\mathfrak{sl}_3[t])(x^-_{\alpha_2} \otimes t)^{(\mu_2)}v_{\lambda,\mu},$$

and $0 \subset V_{\mu_2} \subset \cdots\cdots V_{M_2} = \ker\phi(\lambda, \mu_2\omega_2)$ is a filtration of $\ker\phi(\lambda, \mu_2\omega_2)$. Further using the relation

$$(x^+_{\alpha_1} \otimes 1)((x^-_{\alpha_{12}} \otimes t)^{(\mu_2 - j)}.(x^-_{\alpha_2} \otimes t)^{(j)}.v_{\lambda,\mu}) = (x^-_{\alpha_{12}} \otimes t)^{(\mu_2 - j - 1)}.(x^-_{\alpha_2} \otimes t)^{(j+1)}.v_{\lambda,\mu},$$

we see that in this case V_j/V_{j+1} is a quotient of $\tau^*_{\mu_2}V(\lambda - (\mu_2 - 2j)\omega_1 - j\omega_2)$ for $M_2 \leq j \leq \mu_2$. This shows that Proposition 3.3.1 holds even when, $\mu_1 = 0$ and in this case we have the following result:

Proposition 3.3.2. *Let $\lambda, \mu \in P^+$ with $|\lambda| \geq |\mu|$, $\lambda_i \geq \mu_i$ for $i = 1, 2$ and $\mu_1 = 0$. Then*

$$\dim\ker(\lambda, \mu) \leq \sum_{i=M_2}^{\mu_2} \dim V(\lambda - (\mu_2 - 2i)\omega_1 - i\omega_2).$$

By taking, μ_1, α_1, λ_2 and ω_1 in place of μ_2, α_2, λ_1 and ω_1 resp., the following proposition can be deduced which gives the filtration for $\ker\phi(\lambda, \mu)$ in the case when $\mu = \mu_1\omega_1$.

Proposition 3.3.3. *Let $\lambda, \mu \in P^+$ with $|\lambda| \geq |\mu|$, $\lambda_i \geq \mu_i$ for $i = 1, 2$ and $\mu_2 = 0$. Then there exists a surjective homomorphism*

$$\phi(\lambda, \mu_1\omega_1) : \mathcal{F}_{\lambda,\mu_1\omega_1} \to \mathcal{F}_{\lambda,(\mu_1 - 1)\omega_1},$$

and $\ker\phi(\lambda,\mu_1\omega_1)$ *have a filtration* $0\subset V_{\mu_1}\subset\cdots V_{M_1}=\ker\phi(\lambda,\mu_1\omega_1)$ *where* $M_1=\max\{0,\mu_1-\lambda_2\}$,

$$V_j=U(\mathfrak{sl}_3[t])(x^-_{\alpha_1}\otimes t)^{(j)}(x^-_{\alpha_{12}}\otimes t)^{(\mu_1-j)}..v_{\lambda,\mu}+U(\mathfrak{sl}_3[t])(x^-_{\alpha_1}\otimes t)^{(\mu_1)}v_{\lambda,\mu},\quad \textit{for } M_1\le j\le\mu_1,$$

and V_j/V_{j+1} *is a quotient of* $\tau^*_{\mu_1}V(\lambda-j\omega_1-(\mu_1-2j)\omega_2)$ *for* $M_1\le j\le\mu_1$.

Hence, in this case, we also have the corresponding analog of Proposition 3.3.2:

Proposition 3.3.4. *Let* $\lambda=\lambda_1\omega_1+\lambda_2\omega_2\in P^+$ *and* $\mu=\mu_1\omega_1$. *Then*

$$\dim\ker(\lambda,\mu)\le\sum_{i=\max\{0,\mu_1-\lambda_2\}}^{\mu_1}\dim V(\lambda-i\omega_1-(\mu_1-2i)\omega_2).$$

3.3.3 Case 3: $\mu_2\ge\lambda_2$

Under the given conditions, for $1\le\ell\le\mu_2-\lambda_2$, set,

$${}_\ell\mathbb{S}_{inv}(\lambda,\mu)=\left\{(a_1,a_2)\in\mathbb{Z}^2_{\ge0}:\begin{array}{c}0\le a_1\le\mu_1,\ 0\le a_2\le\lambda_2,\\ \mu_1-\mu_2+\ell\le a_1-a_2\le\lambda_1-\lambda_2-\ell\end{array}\right\}; \tag{3.3.4}$$

$${}_\ell\mathbb{S}_{inv}(\lambda,\mu)[j_\ell]=\{(a_1,a_2)\in{}_\ell\mathbb{S}_{inv}(\lambda,\mu):a_1+a_2=j_\ell\},\quad \text{for } 1\le j_\ell\le\mu_1+\lambda_2;$$

$$V_{\ell,0}=\quad U(\mathfrak{sl}_3[t]).(x^-_{\alpha_{12}}\otimes t)^{(\mu_1+\lambda_2+\ell)}.v_{\lambda,\mu},\qquad \text{for } 1\le\ell\le\mu_2-\lambda_2,$$

$$\begin{aligned}V_{\ell,j_\ell}=&\sum_{\substack{(a_1,a_2)\in{}_\ell\mathbb{S}_{inv}(\lambda,\mu),\\ a_1+a_2=j_\ell}}U(\mathfrak{sl}_3[t]).(x^-_{\alpha_1}\otimes t)^{(a_1)}(x^-_{\alpha_{12}}\otimes t)^{(\mu_1+\lambda_2+\ell-a_1-a_2)}(x^-_{\alpha_2}\otimes t)^{(a_2)}.v_{\lambda,\mu}\\ &+\ U(\mathfrak{sl}_3[t]).(x^-_{\alpha_{12}}\otimes t)^{(\mu_1+\lambda_2+\ell+1)}.v_{\lambda,\mu},\quad \text{for } 1\le j_\ell\le\mu_1+\lambda_2.\end{aligned} \tag{3.3.5}$$

Proposition 3.3.5. *Let $\lambda,\mu\in P^+$ with $|\lambda|\geq|\mu|$, $\lambda_1\geq\mu_1$ and $\mu_2\geq\lambda_2$. For $1\leq\ell\leq\mu_2-\lambda_2$, $0\leq j_\ell\leq\mu_1+\lambda_2$, let ${}_\ell\mathbb{S}_{inv}(\lambda,\mu)$, and ${}_\ell\mathbb{S}_{inv}(\lambda,\mu)[j_\ell]$, V_{ℓ,j_ℓ} be defined as above. Then*

$$0\subset V_{\mu_2-\lambda_2,\mu_1+\lambda_2}\subset\cdots\subset V_{2,\mu_1+\lambda_2}\subset V_{2,2}\cdots V_{2,1}\subset V_{2,0}\subset V_{1,\mu_1+\lambda_2}\subset\cdots V_{1,1}\subset V_{1,0}=\ker\phi(\lambda,\mu), \tag{3.3.6}$$

gives a filtration of $\ker\phi(\lambda,\mu)$. *Further, for $1\leq\ell\leq\lambda_2-\mu_2$, $0<j_\ell\leq\mu_1+\lambda_2$, there exists $\mathfrak{sl}_3[t]$-epimorphisms*

$$\phi_{\ell,j_\ell}^{(\lambda,\mu)}:\bigoplus_{(a_1,a_2)\in{}_\ell\mathbb{S}_{inv}(\lambda,\mu)[j_\ell]}\tau^*_{\mu_1+\lambda_2+\ell}V(\lambda_1\omega_1+\mu_2\omega_2-(\lambda_2+a_1-2a_2+\ell)\omega_1-(\mu_1+a_2-2a_1+\ell)\omega_2)$$
$$\longrightarrow V_{\ell,j_\ell}/V_{\ell,j_\ell+1}$$

and

$$\phi_{\ell,0}^{(\lambda,\mu)}:\tau^*_{\mu_1+\lambda_2+\ell}V(\lambda+\mu-\ell(\omega_1+\omega_2))\to V_{\ell,0}/V_{\ell,1}.$$

Proof. We prove this proposition in the same way as Proposition 3.3.1, by repeatedly using Lemma 3.2.4 with $v=v_{\lambda,\mu}$,

$$L_\alpha^{(k)}=\begin{cases}(\lambda+\mu)(h_\alpha)+1, & k=0\\ \min\{\mu(h_\alpha),\lambda(h_\alpha)\}+1, & k=1\\ 1 & k>1\end{cases}\qquad\forall\,\alpha\in R^+.$$

By Proposition 3.2.1, in this case $\ker\phi(\lambda,\mu)$ is generated as a $\mathfrak{sl}_3[t]$-module by the set of vectors $\{(x_{\alpha_{12}^-}\otimes t)^{(\mu_1+\lambda_2+\ell)}v_{\lambda,\mu}:1\leq\ell\leq\mu_2-\lambda_2\}$. As $(x^-_{\alpha_{12}}\otimes t)^p.v_{\lambda,\mu}=0$ for $p>|\mu|$ the same arguments as in Proposition 3.3.1 show that

$$(x^-_{\alpha_1}\otimes t)^{a_1}(x^-_{\alpha_{12}}\otimes t)^{a_{12}}(x^-_{\alpha_2}\otimes t)^{a_2}.v_{\lambda,\mu}=0,\quad\text{for } a_1+a_{12}+a_2\geq|\mu|+1$$

and for each $1 \le \ell \le \mu_2 - \lambda_2$,

$$(x^-_{\alpha_1}\otimes t)^{a_1}(x^-_{\alpha_{12}}\otimes t)^{(\mu_1+\lambda_2+\ell-a_1-a_2)}(x^-_{\alpha_2}\otimes t)^{a_2}.v_{\lambda,\mu}\in\ker\phi(\lambda,\mu),\quad \text{for } a_1\le\mu_1 \text{ and } a_2\le\lambda_2.$$

However, since $v_{\lambda,\mu}$ satisfies the conditions of Lemma 3.2.4, putting $r=\mu_1+\lambda_2-a_2+\ell$, $s=a_1$ and $l=\mu_1+\lambda_2+\ell-a_1$ in (3.2.5), for $r+s=\mu_1+\lambda_2+\ell-a_2+a_1>\lambda_1+\mu_1$, we get,

$$\sum_{d=0}^{\mu_1+\lambda_2+\ell-a_1}\binom{\mu_1+\lambda_2+\ell-a_1}{d}\mathbf{X}_1(\mu_1+\lambda_2+\ell-a_2-d,a_1)(x^-_{\alpha_{12}}\otimes t)^d(x^-_{\alpha_2}\otimes t)^{\mu_1+\lambda_2+\ell-a_1-d}=0.$$

Using $(x^-_{\alpha_2}\otimes t)^{\mu_1+\lambda_2+\ell-a_1-d}v_{\lambda,\mu}=0$ for $\mu_1+\lambda_2+\ell-a_1-d>\lambda_2$, we get,

$$\sum_{i=0}^{\lambda_2}\binom{\mu_1+\lambda_2+\ell-a_1}{\mu_1+\ell-a_1+i}\mathbf{X}_1(\lambda_2-a_2+a_1-i,a_1)(x^-_{\alpha_{12}}\otimes t)^{\mu_1+\ell-a_1+i}(x^-_{\alpha_2}\otimes t)^{\lambda_2-i}=0,$$

and hence,

$$\begin{aligned}&(x^-_{\alpha_1}\otimes t)^{(a_1)}(x^-_{\alpha_{12}}\otimes t)^{(\mu_1+\lambda_2-a_1-a_2+\ell)}(x^-_{\alpha_2}\otimes t)^{(a_2)}\\ &=-\Big(\sum_{i=0}^{\lambda_2-a_2-1}\tbinom{\mu_1+\lambda_2+\ell-a_1}{\mu_1+\ell-a_1+i}\mathbf{X}_1(\lambda_2-a_2+a_1-i,a_1)(x^-_{\alpha_{12}}\otimes t)^{\mu_1+\ell-a_1+i}(x^-_{\alpha_2}\otimes t)^{\lambda_2-i}\Big).\end{aligned}$$

The same arguments as Proposition 3.3.1 show that in this case

$$\begin{aligned}\ker\phi(\lambda,\mu) &\subseteq \sum_{\ell=1}^{\mu_2-\lambda_2}\sum_{(a_1,a_2)\in{}_\ell\mathbb{S}_{inv}(\lambda,\mu)}U(\mathfrak{g})(x^-_{\alpha_1}\otimes t)^{(a_1)}(x^-_{\alpha_{12}}\otimes t)^{(|\mu|+\ell-a_1-a_2)}(x^-_{\alpha_2}\otimes t)^{(a_2)}.v_{\lambda,\mu},\\ &= \sum_{\ell=1}^{\mu_2-\lambda_2}\sum_{j_\ell=0}^{\mu_1+\lambda_2+\ell}V_{\ell,j_\ell}.\end{aligned}$$

For a fixed $1 \leq \ell \leq \mu_2 - \lambda_2$, and $(a_1, a_2) \in {}_{\ell}\mathbb{S}_{inv}(\lambda, \mu)[j_\ell]$, set

$$V^{(\ell)}_{j_\ell, a_1} = U(\mathfrak{sl}_3[t])(x^-_{\alpha_1} \otimes t)^{(a_1)}(x^-_{\alpha_{12}} \otimes t)^{(\mu_1+\lambda_2+\ell-j_\ell)}(x^-_{\alpha_2} \otimes t)^{(j_\ell - a_1)}.v_{\lambda,\mu}.$$

By definition, $V^{(\ell)}_{j_\ell, a_1} \subset V_{\ell, j_\ell}$. The same arguments as in Proposition 3.3.1, show that for $1 \leq \ell \leq \mu_2 - \lambda_2$,

$$x^+_{\alpha_i}.V^{(\ell)}_{j_\ell, a_1} \subset V^{(\ell)}_{j_\ell+1, a_1} + V^{(\ell)}_{j_\ell+1, a_1+1} \subset V_{j_\ell+1, \ell}$$

$$h \otimes t^m.(x^-_{\alpha_1} \otimes t)^{(a_1)}(x^-_{\alpha_{12}} \otimes t)^{(\mu_1+\lambda_2+\ell-j_\ell)}(x^-_{\alpha_2} \otimes t)^{(j_\ell-a_1)}.v_{\lambda,\mu}. = 0, \text{ for } m > 0,$$

$$\begin{aligned}(x^-_{\alpha_{12}} \otimes t)(x^-_{\alpha_1} \otimes t)^{(a_1)}(x^-_{\alpha_{12}} \otimes t)^{(\mu_1+\lambda_2+\ell-j_\ell)}(x^-_{\alpha_2} \otimes t)^{(j_\ell-a_1)}.v_{\lambda,\mu} \\ \in U(\mathfrak{sl}_3[t])(x^-_{\alpha_{12}} \otimes t)^{(\mu_1+\lambda_2+\ell-j_\ell+1)}, \\ (x^-_{\alpha_i} \otimes t)(x^-_{\alpha_1} \otimes t)^{(a_1)}(x^-_{\alpha_{12}} \otimes t)^{(\mu_1+\lambda_2+\ell-j_\ell)}(x^-_{\alpha_2} \otimes t)^{(j_\ell-a_1)}.v_{\lambda,\mu} \\ \in U(\mathfrak{sl}_3[t])(x^-_{\alpha_{12}} \otimes t)^{(\mu_1+\lambda_2+\ell-j_\ell+1)}.\end{aligned}$$

As

$$U(\mathfrak{sl}_3[t])(x^-_{\alpha_{12}} \otimes t)^{(\mu_1+\lambda_2+\ell+1)} \subset V_{\ell, j_\ell+1}, \quad \text{for } j_\ell < \mu_1 + \lambda_2,$$

$$U(\mathfrak{sl}_3[t])(x^-_{\alpha_{12}} \otimes t)^{(\mu_1+\lambda_2+\ell+1)} \subset V_{\ell+1, 0}, \quad \text{for } j_\ell = \mu_1 + \lambda_2,$$

we see that for $1 \leq \ell \leq \mu_2 - \lambda_2$, $0 \leq j_\ell < \mu_1 + \lambda_2$ (respectively $j_\ell = \mu_1 + \lambda_2$), $V^{(\ell)}_{j_\ell, a_1}/(V^{(\ell)}_{j_\ell, a_1} \cap V_{\ell, j_\ell+1})$ (respectively, $V^{(\ell)}_{\mu_1+\lambda_2, a_1}/(V^{(\ell)}_{\mu_1+\lambda_2, a_1} \cap V_{\ell+1,0})$) is a quotient of the Chari Venkatesh module $V(\boldsymbol{\xi}_{j_\ell, a_1})$ where $\boldsymbol{\xi}_{j_\ell, a_1} = (\xi^{\alpha}_{j_\ell, a_1})_{\alpha \in R^+}$ with

$$\begin{aligned}\xi^{\alpha_1}_{j_\ell, a_1} &= \lambda_1 - \lambda_2 - a_1 + 2a_2 - \ell \geq 0, \\ \xi^{\alpha_2}_{j_\ell, a_1} &= \mu_2 - \mu_1 - a_2 - \ell + 2a_1 \geq 0, \\ \xi^{\alpha_{12}} &= (\lambda_1 - \lambda_2) + (\mu_2 - \mu_1) + a_1 + a_2 - 2\ell \geq 0,\end{aligned}$$

which is isomorphic to the irreducible $\mathfrak{sl}_3$-module

$$\tau^*_{\mu_1+\lambda_2+\ell}V((\lambda_1-\lambda_2-a_1+2a_2-\ell)\omega_1+(\mu_2-\mu_1-a_2+2a_1-\ell)\omega_2).$$

Since these $a_1's$ are all distinct, it follows from (3.3.5), (3.3.6), that $V_{\ell,j_\ell}/V_{\ell,j_\ell+1}$ (respectively, $V_{\ell,\mu_1+\lambda_2}/V_{\ell+1,0}$) is a quotient of

$$\bigoplus_{(a_1,a_2)\in{}_\ell\mathbb{S}_{inv}(\lambda,\mu)[j_\ell]} \tau^*_{\mu_1+\lambda_2+\ell}V(\lambda_1\omega_1+\mu_2\omega_2-(\lambda_2+a_2-2a_2+\ell)\omega_1-(\mu_1+a_2-2a_1+\ell)\omega_2)$$

and this completes the proof of the proposition. □

3.4 Proof of Theorem 3.1.2

In this section, we complete the proof of Theorem 3.1.2. Recall from Remark 2.7.2 that for given $(\lambda,\mu)\in P^+(\lambda+\mu,2)$

$$\dim\mathcal{F}_{\lambda,\mu}\geq\dim V(\lambda).\dim V(\mu). \tag{3.4.1}$$

We obtain the reverse inequality in the following proposition by deducing, from the results of Section 3.3, a set of recurrence relations on the dimension of CV-modules and using induction on them. Throughout this section, we assume for ν, $\gamma\in P$,

$\dim\mathcal{F}_{\nu,\gamma}=0$, whenever either ν or γ is non-zero, non-dominant weight and
$\dim\mathcal{F}_{\nu,\gamma}=\dim V(\nu)$ whenever $\gamma=0$, and $\nu\in P^+$.

Proposition 3.4.1. *Let $\lambda,\mu\in P^+$ with $|\lambda|\geq|\mu|$ and $\lambda_1\geq\mu_1$.*

(i) If (λ,μ) is a partition of $\lambda+\mu$ of first kind with $\mu_2>0$ and $\mu_1=0$, then the surjective homomorphisms $\{\phi_j^{(\lambda,\mu)} : M_2\le j\le|\mu|\}$ given in Proposition 3.3.2 are isomorphisms. (Here, $M_2=\max\{0,\mu_2-\lambda_1\}$.)

(ii) If (λ,μ) is a partition of $\lambda+\mu$ of first kind with $\mu_2>0$ and $\mu_1>0$, then the surjective homomorphisms $\{\phi_j^{(\lambda,\mu)} : 0\le j\le|\mu|\}$ given in Proposition 3.3.1 are isomorphisms.

(iii) If (λ,μ) is a partition of $\lambda+\mu$ of second kind, then the surjective homomorphism $\phi_{\ell,j_\ell}^{(\lambda,\mu)}$ given in Proposition 3.3.5 is an isomorphism for every $1\le\ell\le\mu_2-\lambda_2$, $0\le j_\ell\le\mu_1+\lambda_2$.

Consequently, $\dim\mathcal{F}_{\lambda,\mu}=\dim V(\lambda)\dim V(\mu)$.

3.4.1 Proof of Proposition 3.4.1(i).

Subcase 1: Suppose λ and μ are both multiples of ω_1 and $\lambda_1\ge\mu_1$.

By Proposition 3.3.2, $\ker\phi(\lambda_1\omega_1,\mu_1\omega_1)$ is a quotient of $V((\lambda_1-\mu_1)\omega_1+\mu_1\omega_2)$. Hence,

$$\begin{aligned}\dim\mathcal{F}_{\lambda,\mu} &= \dim\mathcal{F}_{(\lambda_1+1)\omega_1,(\mu_1-1)\omega_1}+\dim\ker\phi(\lambda,\mu)\\ &\le \dim\mathcal{F}_{(\lambda_1+1)\omega_1,(\mu_1-1)\omega_1}+\dim V((\lambda_1-\mu_1)\omega_1+\mu_1\omega_2).\end{aligned}\tag{3.4.2}$$

Note that for $\mu_1=1$, $\mathcal{F}_{(\lambda_1+1)\omega_1,(\mu_1-1)\omega_1}\cong V((\lambda_1+1)\omega_1)$. Hence, we have,

$$\begin{aligned}\dim\mathcal{F}_{\lambda,\mu} &\le \dim V(\lambda_1+1)\omega_1)+\dim V((\lambda_1-1)\omega_1+\omega_2)\\ &\le \tfrac{1}{2}[(\lambda_1+2)(\lambda_1+3)+2\lambda_1(\lambda_1+2)]=\tfrac{3}{2}.(\lambda_1+1)(\lambda_1+2)\\ &= \dim V(\lambda_1\omega_1)\dim V(\omega_1).\end{aligned}$$

Along with (3.4.1), this inequality implies that, $\dim\mathcal{F}_{\lambda_1\omega_1,\omega_1}=\dim V(\lambda_1\omega_1)\dim V(\omega_1)$.

Now, by induction hypothesis assume that $\dim \mathcal{F}_{\lambda_1\omega_1,\mu_1\omega_1} = \dim V(\lambda_1\omega_1)\dim V(\mu_1\omega_1)\ \forall\ \mu_1 \in \mathbb{N}$ such that $\mu_1 < n$. Then by using Proposition 3.3.2 we have,

$$\begin{aligned}4\dim \mathcal{F}_{\lambda_1\omega_1,n\omega_1} &\leq 4\dim \mathcal{F}_{(\lambda_1+1)\omega_1,(n-1)\omega_1} + 4\dim V((\lambda_1-n)\omega_1+n\omega_2)\\ &\leq [(\lambda_1+2)(\lambda_1+3)n(n+1)+2(\lambda_1-n+1)(n+1)(\lambda_1+2)]\\ &\leq (\lambda_1+1)(\lambda_1+2)(n+1)(n+2) = 4\dim V(\lambda_1\omega_1)\dim V(n\omega_1).\end{aligned} \tag{3.4.3}$$

Once using (3.4.1), we deduce from 3.4.3 that $\dim \mathcal{F}_{\lambda_1\omega_1,n\omega_1} = \dim V(\lambda_1\omega_1)\dim V(n\omega_1)$. Note that equality of dimension is possible only if $\ker\phi(\lambda_1\omega_1,\mu_1\omega_1)$ is isomorphic to the CV module $\mathcal{F}_{(\lambda_1-\mu_1)\omega_1+\mu_1\omega_2,0} = V((\lambda_1-\mu_1)\omega_1+\mu_1\omega_2,)$. Hence Theorem 3.1.2 holds whenever λ and μ are multiples of a fundamental weight.

Subase 2: Suppose $\lambda,\mu \in P^+$, is such that $\lambda_2 > 0$, $\mu_2 = 0$ and $\lambda_1 > \mu_1 > 0$.

Then by Proposition 3.3.2,

$$\dim\ker(\lambda,\mu_1\omega_1) \leq \sum_{i=\max\{0,\mu_1-\lambda_2\}}^{\mu_1} \dim V(\lambda - i\omega_1 - (\mu_1-2i)\omega_2)) \tag{3.4.4}$$

In particular, this means when $\mu_1 = 1$,

$$\begin{aligned}2\dim \mathcal{F}_{\lambda,\omega_1} \leq\ & 2\dim V(\lambda+\omega_1) + 2\dim V(\lambda+(\omega_2-\omega_1)) + 2\dim V(\lambda-\omega_2)\\ \leq\ & (\lambda_1+2)(\lambda_2+1)(\lambda_1+\lambda_2+3) + \lambda_1(\lambda_2+2)(\lambda_1+\lambda_2+2)\\ & +(\lambda_1+1)(\lambda_2)(\lambda_1+\lambda_2+1)\\ \leq\ & 3(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2) + (\lambda_2+1)(2\lambda_1+\lambda_2+4)\\ & -(\lambda_2-\lambda_1+1)(\lambda_1+\lambda_2+2) - (\lambda_1+1)(\lambda_1+2\lambda_2+2)\\ \leq\ & 3(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2) = 2\dim V(\omega_1).\dim V(\lambda).\end{aligned} \tag{3.4.5}$$

Using the same argument as in Subcase 1, it follows that $\ker\phi(\lambda,\omega_1)$ is isomorphic to the direct sum of the CV modules $\mathcal{F}_{\lambda-(\omega_1-\omega_2),0} \oplus \mathcal{F}_{\lambda-\omega_1,0}$ that is the Theorem 3.1.2 when

$\mu_1 = 1$. By induction hypothesis assume that Theorem 3.1.2 holds for CV module $\mathcal{F}_{\lambda,\mu_1\omega_1}$ whenever $\mu_1 \in \mathbb{N}$ is such that $\mu_1 < n$, i.e,

$$\dim \mathcal{F}_{\lambda,\mu_1\omega_1} = \dim V(\lambda) \dim V(\mu_1\omega_1) \text{ for all } \mu_1 < n.$$

Using induction hypothesis and Proposition 3.3.2, for $\mu_1 = n$ we thus have,

$$\begin{aligned} \dim\ker\phi(\lambda, n\omega_1) \leq\ & \dim V(\lambda + n(\omega_2 - \omega_1)) \\ & + \textstyle\sum_{i=\max\{0,n-\lambda_2\}}^{n-1} \dim V(\lambda - \omega_2 - i\omega_1 - (n-1-2i)\omega_2)) \\ \leq\ & \dim V(\lambda + n(\omega_2 - \omega_1)) + \dim\ker\phi(\lambda - \omega_2, (n-1)\omega_1) \end{aligned} \tag{3.4.6}$$

Since by Proposition 3.3.2, $\dim\ker\phi(\lambda, \mu_1\omega_1) = \dim \mathcal{F}_{\lambda,\mu_1\omega_1} - \dim \mathcal{F}_{\lambda+\omega_1,(\mu_1-1)\omega_1}$, it follows from (3.4.6) that,

$$\begin{aligned} \dim \mathcal{F}_{\lambda,n\omega_1} \leq\ & \dim \mathcal{F}_{\lambda+\omega_1,(n-1)\omega_1} + \dim \mathcal{F}_{\lambda-\omega_2,(n-1)\omega_1} - \dim \mathcal{F}_{\lambda+\omega_1-\omega_2,(n-2)\omega_1} \\ & + \dim V(\lambda + n(\omega_2 - \omega_1)). \end{aligned} \tag{3.4.7}$$

$$\begin{aligned} & 4\dim \mathcal{F}_{\lambda,n\omega_1} \\ \leq\ & \Big((\lambda_1+2)(\lambda_2+1)(\lambda_1+\lambda_2+3) + (\lambda_1+1)(\lambda_2)(\lambda_1+\lambda_2+1)\Big)(n)(n+1) \\ & -(\lambda_1+2)(\lambda_2)(\lambda_1+\lambda_2+2)(n-1)n + 2(\lambda_1-n+1)(\lambda_2+n+1)(\lambda_1+\lambda_2+2) \\ \leq\ & \Big(3(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2) - \lambda_1(\lambda_2+2)(\lambda_1+\lambda_2+2)\Big)n(n+1) \\ & -(\lambda_1+2)(\lambda_2)(\lambda_1+\lambda_2+2)(n-1)n + 2(\lambda_1-n+1)(\lambda_2+n+1)(\lambda_1+\lambda_2+2) \\ & \hspace{8cm} \text{(using (3.4.5))} \\ \leq\ & \Big(3(\lambda_1+1)(\lambda_2+1) - \lambda_1(\lambda_2+2) - \lambda_2(\lambda_1+2)\Big)(\lambda_1+\lambda_2+2)(n+1)n \\ & +2\Big((\lambda_1+2)\lambda_2 n + (\lambda_1-n+1)(\lambda_2+n+1)\Big)(\lambda_1+\lambda_2+2) \\ \leq\ & \Big((\lambda_1+1)(\lambda_2+1)+2\Big)(\lambda_1+\lambda_2+2)n(n+1) \\ & +2\Big(n(\lambda_1\lambda_2+2\lambda_2-\lambda_2+\lambda_1+1) - n(n+1) + (\lambda_1+1)(\lambda_2+1)\Big)(\lambda_1+\lambda_2+2) \\ \leq\ & (\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)n(n+1) + 2[(n+1)(\lambda_1+1)(\lambda_2+1)](\lambda_1+\lambda_2+2) \end{aligned}$$

$$\leq \quad (\lambda_1+\lambda_2+2)(\lambda_1+1)(\lambda_2+1)(n+1)(n+2) = 4\dim V(\lambda).\dim V(\mu_1\omega_1).$$

Along with (3.4.1), using the same arguments as in Subcase 1, we now see that Proposition 3.4.1(i) holds and hence Theorem 3.1.2 holds in this case.

3.4.2 Proof of Proposition 3.4.1(ii)

Given (λ,μ) a partition of $\lambda+\mu$ of first kind with $\mu_1,\mu_2>0$. then using Proposition 3.3.1 we have,

$$\begin{aligned} \dim \mathcal{F}_{\lambda,\mu} = & \ \dim \mathcal{F}_{\lambda+\omega_2,\mu-\omega_2} + \dim\ker(\lambda,\mu) \\ \leq & \ \dim \mathcal{F}_{\lambda+\omega_2,\mu-\omega_2} + \dim \mathcal{F}_{\lambda+\mu_2(\omega_1-\omega_2),\mu_1\omega_1} \\ & + \sum_{(a_1,a_2)\in\mathbb{S}_{ninv}(\lambda,\mu)} \dim V(\lambda-(\mu_2+a_1-2a_2)\omega_1-(\mu_1+a_2-2a_1)\omega_2). \end{aligned} \tag{3.4.8}$$

In particular when $\mu_2=1$, for $(a_1,a_2)\in\mathbb{S}_{ninv}(\lambda,\mu_1\omega_1+\omega_2)$, $0\leq a_2<1$ and $\mu_1-\lambda_2\leq a_1\leq\min\{\mu_1,\lambda_1-1\}$. Hence $a_2=0$ and we have,

$$\begin{aligned} \dim \mathcal{F}_{\lambda,\mu_1\omega_1+\omega_2} \leq & \ \dim \mathcal{F}_{\lambda+\omega_2,\mu_1\omega_1} + \dim \mathcal{F}_{\lambda+(\omega_1-\omega_2),\mu_1\omega_1} \\ & + \sum_{i=\max\{0,\mu_1-\lambda_2\}}^{\min\{\mu_1,\lambda_1-1\}} \dim V(\lambda-(1+i)\omega_1-(\mu_1-2i)\omega_2). \end{aligned} \tag{3.4.9}$$

Further, using Proposition 3.3.2 and Proposition 3.4.1(i), we have

$$\begin{aligned} \dim\ker(\lambda-\theta,\mu-\theta) = & \ \dim\ker(\lambda-\theta,(\mu_1-1)\omega_1) \\ = & \sum_{i=\max\{0,\mu_1-\lambda_2\}}^{\mu_1-1} \dim V(\lambda-\theta-i\omega_1-(\mu_1-1-2i)\omega_2) \end{aligned}$$

Hence (3.4.9) can be rewritten as,

$$\begin{aligned} \dim \mathcal{F}_{\lambda,\mu_1\omega_1+\omega_2} \leq & \ \dim \mathcal{F}_{\lambda+\omega_2,\mu_1\omega_1} + \dim \mathcal{F}_{\lambda+(\omega_1-\omega_2),\mu_1\omega_1} + \dim\ker(\lambda-\theta,(\mu_1-1)\omega_1) \\ & + (1-\delta_{\mu_1,\lambda_1})\dim V(\lambda-\omega_1-\mu_1\omega_1+\mu_1\omega_2) \end{aligned} \tag{3.4.10}$$

When $\lambda_1 = \mu_1$, then the right-hand side of the inequality (3.4.10) is equal to,

$$\begin{aligned}
& 4 \ \Big(\dim \mathcal{F}_{\lambda+\omega_2,\lambda_1\omega_1} + \dim \mathcal{F}_{\lambda+(\omega_1-\omega_2),\lambda_1\omega_1} + \dim \ker(\lambda-\theta,(\lambda_1-1)\omega_1)\Big) \\
= \ & 4\Big(\dim \mathcal{F}_{\lambda+\omega_2,\lambda_1\omega_1} + \dim \mathcal{F}_{\lambda+(\omega_1-\omega_2),\lambda_1\omega_1} + \dim \mathcal{F}_{\lambda-\theta,(\lambda_1-1)\omega_1} - \dim \mathcal{F}_{\lambda-\omega_2,(\lambda_1-2)\omega_1}\Big) \\
= \ & (\lambda_1+1)(\lambda_2+2)(\lambda_1+\lambda_2+3)(\lambda_1+1)(\lambda_1+2)+(\lambda_1+2)(\lambda_2)(\lambda_1+\lambda_2+2)(\lambda_1+1)(\lambda_1+2) \\
& +\lambda_1\lambda_2(\lambda_1+\lambda_2)\lambda_1(\lambda_1+1)-(\lambda_1+1)\lambda_2(\lambda_1+\lambda_2+1)(\lambda_1-1)\lambda_1 \\
= \ & (\lambda_1+1)(\lambda_1+2)\Big((\lambda_1+1)(\lambda_2+2)(\lambda_1+\lambda_2+3)+\lambda_2(\lambda_1+2)(\lambda_1+\lambda_2+2)\Big) \\
& +\lambda_1\lambda_2(\lambda_1+1)\Big(\lambda_1(\lambda_1+\lambda_2)-(\lambda_1-1)(\lambda_1+\lambda_2+1)\Big) \\
= \ & (\lambda_1+1)(\lambda_1+2)\Big[(\lambda_1+1)\Big((\lambda_2+1)(\lambda_1+\lambda_2+2)+\lambda_1+2\lambda_2+4\Big) \\
& +(\lambda_1+\lambda_2+2)\Big((\lambda_2+1)(\lambda_1+1)-\lambda_1-1+\lambda_2\Big)\Big]+\lambda_1\lambda_2(\lambda_1+1)(\lambda_2+1) \\
= \ & 2(\lambda_1+1)^2(\lambda_1+2)(\lambda_2+1)(\lambda_1+\lambda_2+2)+(\lambda_1+1)^2(\lambda_1+2)(\lambda_1+2\lambda_2+4) \\
& +(\lambda_1+1)(\lambda_1+2)\lambda_2(\lambda_1+\lambda_2+2)-(\lambda_1+1)^2(\lambda_1+2)(\lambda_1+\lambda_2+2)+\lambda_1\lambda_2(\lambda_1+1)(\lambda_2+1) \\
= \ & 2(\lambda_1+1)^2(\lambda_1+3)(\lambda_2+1)(\lambda_1+\lambda_2+2)-2(\lambda_1+1)^2(\lambda_2+1)(\lambda_1+\lambda_2+2) \\
& +(\lambda_1+1)^2(\lambda_1+2)(\lambda_2+2)+(\lambda_1+1)(\lambda_1+2)\lambda_2(\lambda_1+\lambda_2+2)+\lambda_1\lambda_2(\lambda_1+1)(\lambda_2+1) \\
= \ & 2(\lambda_1+1)^2(\lambda_1+3)(\lambda_2+1)(\lambda_1+\lambda_2+2) \\
& +(\lambda_1+1)\Big((\lambda_1+\lambda_2+2)(\lambda_1\lambda_2+2\lambda_2-2\lambda_1\lambda_2-2\lambda_1-2\lambda_2-2) \\
& +(\lambda_1+1)(\lambda_1\lambda_2+2(\lambda_1+\lambda_2+2))+\lambda_1\lambda_2(\lambda_2+1)\Big) \\
= \ & 2(\lambda_1+1)^2(\lambda_1+3)(\lambda_2+1)(\lambda_1+\lambda_2+2) = \dim V(\lambda)\dim V(\lambda_1\omega_1+\omega_2)
\end{aligned}$$

When $\lambda_1 > \mu_1$, the right-hand side of the inequality (3.4.10) is equal to,

$$\begin{aligned}
& 4\Big(\dim \mathcal{F}_{\lambda+\omega_2,\mu_1\omega_1} + \dim \mathcal{F}_{\lambda+(\omega_1-\omega_2),\mu_1\omega_1} + \dim \mathcal{F}_{\lambda-\theta,(\mu_1-1)\omega_1} \\
& + \dim V((\lambda_1-\mu_1-1)\omega_1+(\lambda_2+\mu_1)\omega_2) - \dim \mathcal{F}_{(\lambda_1+\mu_2-2)\omega_1+(\lambda_2-\mu_2)\omega_2+\omega_1,(\mu_1-2)\omega_1}\Big) \\
& = \Big((\lambda_1+1)(\lambda_2+2)(\lambda_1+\lambda_2+3)(\mu_1+1)(\mu_1+2)+(\lambda_1+2)(\lambda_2)(\lambda_1+\lambda_2+2)\Big)(\mu_1+1) \\
& \qquad (\mu_1+2)+\Big(\lambda_1\lambda_2(\lambda_1+\lambda_2)\mu_1(\mu_1+1)+(\lambda_1-\mu_1)(\lambda_2+\mu_1+1)(\lambda_1+\lambda_2+1)2 \\
& \qquad -(\lambda_1+1)\lambda_2(\lambda_1+\lambda_2+1)\mu_1(\mu_1-1)\Big) \qquad \text{(Using Proposition 3.4.1.(i))}
\end{aligned}$$

$$= \Big(3(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)-(\lambda_2+1)(\lambda_1)(\lambda_1+\lambda_2+1)\Big)(\mu_1+1)(\mu_1+2)$$
$$+\Big(\lambda_1\lambda_2(\lambda_1+\lambda_2)\mu_1(\mu_1+1)+(\lambda_1-\mu_1)(\lambda_2+\mu_1+1)(\lambda_1+\lambda_2+1)2$$
$$-(\lambda_1+1)\lambda_2(\lambda_1+\lambda_2+1)\mu_1(\mu_1+1-2)\Big)$$
$$= 3(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\mu_1+1)(\mu_1+2)$$
$$-\Big((\lambda_2+1)(\lambda_1)(\lambda_1+\lambda_2+1)-\lambda_1\lambda_2(\lambda_1+\lambda_2)+(\lambda_1+1)\lambda_2(\lambda_1+\lambda_2+1)\Big)(\mu_1+1)\mu_1$$
$$+2(\lambda_1+\lambda_2+1)\Big((\lambda_1-\mu_1)(\lambda_2+\mu_1+1)+(\lambda_1+1)\lambda_2\mu_1-(\lambda_2+1)\lambda_1(\mu_1+1)\Big)$$
$$= 3(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\mu_1+1)(\mu_1+2)$$
$$-\Big((\lambda_1+\lambda_2+\lambda_1\lambda_2)(\lambda_1+\lambda_2+1)+\lambda_1\lambda_2\Big)(\mu_1+1)\mu_1-2(\lambda_1+\lambda_2+1)\mu_1(\mu_1+1)$$
$$= 3(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\mu_1+1)(\mu_1+2)-\mu_1(\mu_1+1)(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)$$
$$= 2(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\mu_1+1)(\mu_1+3) = 4\dim V(\lambda)\dim V(\mu_1\omega_1+\omega_2)$$

Hence, along with it (3.4.1), it follows that in this case, Proposition 3.4.1(ii) holds and $\dim \mathcal{F}_{\lambda,\mu_1\omega_1+\omega_2} = \dim V(\lambda)\dim V(\mu_1\omega_1+\omega_2)$.

By induction hypothesis assume Proposition 3.4.1(ii) holds for $\mu_2 \in \mathbb{N}$ with $|\mu_2| \leq k$. Also note, it follows from the definition of $\mathbb{S}_{ninv}$ that whenever $\lambda_i \geq \mu_i > 0$, for $i = 1,2$,

$$\begin{aligned}\mathbb{S}_{ninv}(\lambda,\mu) \;&= \mathbb{S}_{ninv}(\lambda-\theta,\mu-\theta)\cup\{(\mu_1,b): b\in\mathbb{Z}_+, 0\leq b\leq \mu_2-1,\ |\mu|-b\leq\lambda_1, b\leq\lambda_2\}\\ &\cup\{(a,\mu_2-1): a\in\mathbb{Z}_+, 0\leq a\leq\mu_1-1, \mu_2+a-\mu_2+1\leq\lambda_1,\ |\mu|-1-a\leq\lambda_2\}.\end{aligned} \tag{3.4.11}$$

Hence using Proposition 3.3.1, Proposition 3.3.2, and induction hypothesis on μ_2 we have the following equations :

$$\begin{aligned}&\sum_{(a_1,a_2)\in\mathbb{S}_{ninv}(\lambda-\theta,\mu-\theta)} \dim V(\lambda-\theta-(\mu_2-1+a_1-2a_2)\omega_1-(\mu_1-1+a_2-2a_1)\omega_2)\\ &\quad= \dim\ker(\lambda-\theta,\mu-\theta)-\dim\mathcal{F}_{\lambda-\theta+(\mu_2-1)(\omega_1-\omega_2),(\mu_1-1)\omega_1}\\ &\quad= \dim\mathcal{F}_{\lambda-\theta,\mu-\theta}-\dim\mathcal{F}_{\lambda-\theta+\omega_2,\mu-\theta-\omega_2}-\dim\mathcal{F}_{\lambda-\theta+(\mu_2-1)(\omega_1-\omega_2),(\mu_1-1)\omega_1}.\end{aligned} \tag{3.4.12}$$

$$\begin{aligned}
&\sum_{i=\max\{0,|\mu|-1-\lambda_2\}}^{\mu_2-1} \dim V(\lambda-(\mu_2+\mu_1-2b)\omega_1-(\mu_1+b-2\mu_1)\omega_1)\\
&= \sum_{i=\max\{0,|\mu|-1-\lambda_2\}}^{\mu_2-1} \dim V(\lambda-(\mu_1+1)\omega_1+\mu_1\omega_2-(\mu_2-1-2b)\omega_1-b\omega_1)\\
&= \dim\ker(\lambda',(\mu_2-1)\omega_2 = \dim\mathcal{F}_{\lambda',(\mu_2-1)\omega_2}-\dim\mathcal{F}_{\lambda'+\omega_2,(\mu_2-2)\omega_2},
\end{aligned} \tag{3.4.13}$$

where $\lambda'=(\lambda_1-\mu_1-1)\omega_1+(\lambda_2+\mu_1)\omega_2$, and

$$\begin{aligned}
&\sum_{i=\max\{0,|\mu|-1-\lambda_2\}}^{\mu_1-1} \dim V(\lambda-(\mu_2+a-2(\mu_2-1))\omega_1-(|\mu|-1-2a)\omega_2)\\
&= \sum_{i=\max\{0,|\mu|-1-\lambda_2\}}^{\mu_1-1} \dim V(\lambda+\mu_2(\omega_1-\mu_2)-2\omega_1-a\omega_1-(\mu_1-1-2a)\omega_2)\\
&= \dim\ker(\lambda'',(\mu_1-1)\omega_1)\\
&= \dim\mathcal{F}_{\lambda'',(\mu_1-1)\omega_1}-\dim\mathcal{F}_{\lambda''+\omega_1,(\mu_1-2)\omega_1},
\end{aligned} \tag{3.4.14}$$

where $\lambda''=\lambda-\theta+(\mu_2-1)(\omega_1-\omega_2)$. Using (3.4.11) —(3.4.14), for $\mu_2=k+1$, we can thus rewrite the inequality (3.4.8) as follows,

$$\begin{aligned}
4\dim\mathcal{F}_{\lambda,\mu} \le\ & 4\Big(\dim\mathcal{F}_{\lambda+\omega_2,\mu_1\omega_1+k\omega_2}+\dim\mathcal{F}_{\lambda+(k+1)(\omega_1-\omega_2),\mu_1\omega_1}+\dim\mathcal{F}_{\lambda-\theta,(\mu_1-1)\omega_1+k\omega_2}\\
&-\dim\mathcal{F}_{\lambda-\theta+\omega_2,(\mu_1-1)\omega_1+(k-1)\omega_2}-\dim\mathcal{F}_{\lambda-\theta+k(\omega_1-\omega_2),(\mu_1-1)\omega_1}+\dim\mathcal{F}_{\lambda',k\omega_2}\\
&-\dim\mathcal{F}_{\lambda'+\omega_2,(k-1)\omega_2}+\dim\mathcal{F}_{\lambda'',(\mu_1-1)\omega_1}-\dim\mathcal{F}_{\lambda''+\omega_1,(\mu_1-2)\omega_1}\Big)\\
\le\ & 4\Big(\dim\mathcal{F}_{\lambda+\omega_2,\mu_1\omega_1+k\omega_2}+\dim\mathcal{F}_{\lambda+\mu_2(\omega_1-\omega_2),\mu_1\omega_1}+\dim\mathcal{F}_{\lambda-\theta,(\mu_1-1)\omega_1+k\omega_2}\\
&+\dim\mathcal{F}_{\lambda',k\omega_2}-\dim\mathcal{F}_{\lambda-\omega_1,(\mu_1-1)\omega_1+(k-2)\omega_2}\\
&-\dim\mathcal{F}_{(\lambda_1+k)\omega_1+(\lambda_2-k-1)\omega_2,(\mu_1-2)\omega_1}-\dim\mathcal{F}_{\lambda'+\omega_2,(k-1)\omega_2}\Big)
\end{aligned} \tag{3.4.15}$$

where $\lambda'=(\lambda_1-\mu_1-1)\omega_1+(\lambda_2+\mu_1)\omega_2$ and $\lambda''=(\lambda_1+k-1)\omega_1+(\lambda_2-k-1)\omega_2$. Assuming $\dim\mathcal{F}_{\lambda',\mu'}=0$ whenever λ' or μ' is not a dominant integral weight and $\dim\mathcal{F}_{\lambda',\mu'}=\dim V(\lambda')$ whenever $\lambda'\in P^+$ and $\mu'=0$, and using induction hypothesis, the right hand side of the above inequality is as follows :

$$
\begin{aligned}
= \; & (\lambda_1+1)(\lambda_2+2)(\lambda_1+\lambda_2+3)(\mu_1+1)(k+1)(\mu_1+k+2) \\
& +(\lambda_1+k+2)(\lambda_1+\lambda_2+2)(\lambda_2-k)(\mu_1+1)(\mu_1+2)+\lambda_1\lambda_2(\lambda_1+\lambda_2)\mu_1(k+1)(\mu_1+k+1) \\
& -\lambda_1(\lambda_2+1)(\lambda_1+\lambda_2+1)\mu_1 k(\mu_1+k)+(\lambda_1-\mu_1)(\lambda_2+\mu_1+1)(\lambda_1+\lambda_2+1)(k+1)(k+2) \\
& -(\lambda_1-\mu_1)(\lambda_2+\mu_1+2)(\lambda_1+\lambda_2+2)(k)(k+1) \\
& -(\lambda_1+k+1)(\lambda_2-k)(\lambda_1+\lambda_2+1)(\mu_1-1)\mu_1 \\
= \; & (\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\mu_1+1)(k+1)(\mu_1+k+2) \\
& +(\lambda_1+1)(\lambda_2+2)(\lambda_1+\lambda_2+3)(\mu_1+1)(\mu_1+2k+2) \\
& +(\lambda_2-\lambda_1-2k-1)(\lambda_1+\lambda_2+2)(\mu_1+1)(\mu_1+2) \\
& +\lambda_1\lambda_2(\lambda_1+\lambda_2)\mu_1(\mu_1+2k+1)-\lambda_1(\lambda_2+1)(\lambda_1+\lambda_2+1)\mu_1(\mu_1+2k-1) \\
& +(\lambda_1-\mu_1)(\lambda_2+\mu_1+1)(\lambda_1+\lambda_2+1)(2k+2)-(\lambda_1-\mu_1)(\lambda_2+\mu_1+2)(\lambda_1+\lambda_2+2)2k \\
& -(\lambda_2-\lambda_1-2k)(\lambda_1+\lambda_2+1)(\mu_1-1)\mu_1 \\
= \; & (\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\mu_1+1)(k+1)(\mu_1+k+2) \\
& +(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\mu_1+1)(\mu_1+2k+4) \\
& -2(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\mu_1+1)+(\lambda_1+1)(\lambda_1+2\lambda_2+4)(\mu_1+1)(\mu_1+2k+2) \\
& +2\lambda_1\lambda_2(\lambda_1+\lambda_2)\mu_1-\lambda_1\mu_1(\lambda_1+2\lambda_2+1)(\mu_1+2k-1) \\
& +2(\lambda_1-\mu_1)(\lambda_2+\mu_1+1)(\lambda_1+\lambda_2+1)-(\lambda_1-\mu_1)(\lambda_1+\mu_1+2\lambda_2+3)2k \\
& +(\lambda_1+\lambda_2+1)(\lambda_2-\lambda_1-2k)(4\mu_1+2)-2(\lambda_1+k+1)(\mu_1^2+3\mu_1+2) \\
= \; & (\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\mu_1+1)(k+2)(\mu_1+k+3) \\
& -2(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\mu_1+1)+(\lambda_1+1)(\lambda_1+2\lambda_2+4)(\mu_1+1)(\mu_1+2k+2) \\
& +2\lambda_1\lambda_2(\lambda_1+\lambda_2)\mu_1-(\lambda_1+2\lambda_2+1)\lambda_1\mu_1(\mu_1+2k-1) \\
& +2(\lambda_1-\mu_1)(\lambda_2+\mu_1+1)(\lambda_1+\lambda_2+1)-(\lambda_1-\mu_1)(\lambda_1+\mu_1+2\lambda_2+3)2k \\
& +(\lambda_1+\lambda_2+1)(\lambda_2-\lambda_1-2k)(4\mu_1+2)-2(\lambda_1+k+1)(\mu_1^2+3\mu_1+2)
\end{aligned}
$$

Set

$$
\begin{aligned}
A = \; & -2(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\mu_1+1)+(\lambda_1+1)(\lambda_1+2\lambda_2+4)(\mu_1+1)(\mu_1+2k+2) \\
& +2\lambda_1\lambda_2(\lambda_1+\lambda_2)\mu_1-\lambda_1\mu_1(\lambda_1+2\lambda_2+1)(\mu_1+2k-1) \\
& +2(\lambda_1-\mu_1)(\lambda_2+\mu_1+1)(\lambda_1+\lambda_2+1)-(\lambda_1-\mu_1)(\lambda_1+\mu_1+2\lambda_2+3)2k \\
& +(\lambda_1+\lambda_2+1)(\lambda_2-\lambda_1-2k)(4\mu_1+2)-2(\lambda_1+k+1)(\mu_1^2+3\mu_1+2)
\end{aligned}
$$

Then, $4\dim\mathcal{F}_{\lambda,\mu_1\omega_1+(k+1)\omega_1} \leq (\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\mu_1+1)(k+2)(\mu_1+k+3)+A$.

Observe that the coefficient of $2k$ in A is :

$$\begin{aligned}
&= (\lambda_1+1)(\mu_1+1)(\lambda_1+2\lambda_2+4)-\lambda_1\mu_1(\lambda_1+2\lambda_2+1)-\lambda_1(\lambda_1+\mu_1+2\lambda_2+3)\\
&\quad +\mu_1(\lambda_1+\mu_1+2\lambda_2+3)-(\lambda_1+\lambda_2+1)(4\mu_1+2)-(\mu_1^2+3\mu_1+2)\\
&= \lambda_1\mu_1\Big(\lambda_1+2\lambda_2+4-\lambda_1-2\lambda_2-1-4\Big)+(\lambda_1+\mu_1+1)(\lambda_1+2\lambda_2+4)-(\lambda_1^2-\mu_1^2)\\
&\quad -(\lambda_1-\mu_1)(2\lambda_2+3)2\lambda_1-2(\lambda_2+1)(2\mu_1+1)-(\mu_1^2+3\mu_1+2)\\
&= -\lambda_1\mu_1+\lambda_1\Big(\lambda_1+\mu_1+1+2\lambda_2+4-2\lambda_2-3-2-\lambda_1\Big)\\
&\quad +2\lambda_2+4+\mu_1\Big(2\lambda_2+3+2\lambda_2+4-4\lambda_2-4-3\Big)-2(\lambda_2+1)-2=0
\end{aligned}$$

Using the above relation we see that A reduces to the following:

$$\begin{aligned}
A = &\; -2(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\mu_1+1)+(\lambda_1+1)(\lambda_1+2\lambda_2+4)(\mu_1+1)(\mu_1+2)\\
&\; +2\lambda_1\lambda_2(\lambda_1+\lambda_2)\mu_1-\lambda_1\mu_1(\lambda_1+2\lambda_2+1)(\mu_1-1)+2(\lambda_1-\mu_1)(\lambda_2+\mu_1+1)(\lambda_1+\lambda_2+1)\\
&\; +(\lambda_1+\lambda_2+1)(\lambda_2-\lambda_1)(4\mu_1+2)-2(\lambda_1+1)(\mu_1^2+3\mu_1+2)\\
= &\; (\lambda_1+1)(\mu_1+1)\Big((\mu_1+2)(\lambda_1+\lambda_2+2)+(\mu_1+2)\lambda_2-2(\lambda_2+1)(\lambda_1+\lambda_2+2)\Big)\\
&\; +2\lambda_1\lambda_2(\lambda_1+\lambda_2)\mu_1-\lambda_1\mu_1(1+2\lambda_2+\lambda_1)(\mu_1-1)+2(\lambda_1-\mu_1)(\lambda_2+\mu_1+1)(\lambda_1+\lambda_2+1)\\
&\; +(\lambda_1+\lambda_2+1)(\lambda_2-\lambda_1)(4\mu_1+2)\\
= &\; (\lambda_1+1)(\mu_1+1)\Big((\lambda_1+\lambda_2+2)(\mu_1-2\lambda_2)-\lambda_2(\mu_1+2)\Big)\\
&\; +\lambda_1\mu_1\Big(2\lambda_2(\lambda_1+\lambda_2)-(1+2\lambda_2+\lambda_1).(\mu_1-1)\Big)+2(\lambda_1-\mu_1)(\lambda_2+\mu_1+1)(\lambda_1+\lambda_2+1)\\
&\; +(\lambda_1+\lambda_2+1)(\lambda_2-\lambda_1)(4\mu_1+2)\\
= &\; \lambda_1\mu_1\Big(\mu_1(\lambda_1+\lambda_2)-2\lambda_2(\lambda_1+\lambda_2)-4\lambda_2+\lambda_2\mu_1+2\lambda_2+2\lambda_2(\lambda_1+\lambda_2)-\mu_1(\lambda_1+2\lambda_2)\\
&\; +2\lambda_2\Big)+2\lambda_1\mu_1^2-\lambda_1\mu_1^2+\lambda_1^2\mu_1+\lambda_1\mu_1+(\lambda_1+\mu_1+1)\Big((\lambda_1+\lambda_2+2)(\mu_1-2\lambda_2)\\
&\; +\lambda_2(\mu_1+2)\Big)+2(\lambda_1-\mu_1)(\lambda_2+\mu_1+1)(\lambda_1+\lambda_2+1)+(\lambda_1+\lambda_2+1)(\lambda_2-\lambda_1)(4\mu_1+2)\\
= &\; (\lambda_1+\mu_1+1)\Big((\lambda_1+\lambda_2+2)(\mu_1-2\lambda_2)+\lambda_2(\mu_1+2)+\lambda_1\mu_1\Big)\\
&\; +2(\lambda_1-\mu_1)(\lambda_2+\mu_1+1)(\lambda_1+\lambda_2+1)+(\lambda_1+\lambda_2+1)(\lambda_2-\lambda_1)(4\mu_1+2)\\
= &\; 2(\lambda_1+\lambda_2+1)\Big((\lambda_1+\mu_1+1)(\mu_1-\lambda_2)+(\lambda_1-\mu_1)(\lambda_2+\mu_1+1)+(\lambda_2-\lambda_1)(2\mu_1+1)\Big)\\
= &\; 0
\end{aligned}$$

Hence, $4\dim\mathcal{F}_{\lambda,\mu_1\omega_1+(k+1)\omega_1} \le (\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\mu_1+1)(k+2)(\mu_1+k+3)$, which along with (3.4.1) implies

$$\dim\mathcal{F}_{\lambda,\mu_1\omega_1+(k+1)\omega_1} = \dim V(\lambda)\dim V(\mu_1\omega_1+(k+1)\omega_1),$$

whenever $\dim\mathcal{F}_{\lambda,\mu_1\omega_1+\ell\omega_1} = \dim V(\lambda)\dim V(\mu_1\omega_1+\ell\omega_1)$ for $\ell \le k$. This completes the proof of the proposition in this case.

3.4.3 Proof of Proposition 3.4.1(iii)

Given a partition (λ,μ) of $\lambda+\mu$ of second kind, we begin by setting some notations. Given $(\lambda,\mu)\in(P^+)^2$, set $(\zeta^\lambda_\mu,\zeta^\mu_\lambda)\in(P^+)^2$ such that

$$\zeta^\lambda_\mu = \lambda_1\omega_1+\mu_2\omega_2, \quad \zeta^\mu_\lambda = \mu_1\omega_1+\lambda_2\omega_2,$$

Using Proposition 3.3.5, we have,

$$\dim\mathcal{F}_{\lambda,\mu} = \dim\mathcal{F}_{\zeta^\lambda_\mu,\zeta^\mu_\lambda} + \dim\ker(\lambda,\mu),$$

$$\dim\ker(\lambda,\mu) \le \sum_{(a_1,a_2)\in \bigcup\limits_{1\le\ell\le\mu_2-\lambda_2} {}_\ell\mathbb{S}_{inv}(\lambda,\mu)} \dim V(\zeta^\lambda_\mu - \ell\theta - (\lambda_2+a_1-2a_2)\omega_1 - (\mu_1+a_2-2a_1)\omega_2). \tag{3.4.16}$$

We now prove the result by applying induction on $\mu_2-\lambda_2$. For $\mu_2-\lambda_2=1$, we have

$$\dim\ker(\lambda,\mu) = \dim\mathcal{F}_{\lambda,\mu} - \dim\mathcal{F}_{\zeta^\lambda_\mu,\zeta^\mu_\lambda},$$

$$\dim\ker(\lambda,\mu) \le \sum_{(a_1,a_2)\in {}_1\mathbb{S}_{inv}(\lambda,\mu)} \dim V(\zeta^\lambda_\mu - \theta - (\lambda_2+a_1-2a_2)\omega_1 - (\mu_1+a_2-2a_1)\omega_2).$$

Given $\lambda, \mu \in P^+$ such that $|\lambda| \geq |\mu|$ and $\mu_2 = \lambda_2 + 1$, clearly $\lambda_1 > \mu_1$. Hence $(\lambda - \omega_1, \mu - \omega_2) \in (P^+)^2$ with $\lambda_1 - 1 \geq \mu_1$. Now comparing definitions (3.3.1) and (3.3.4) we see that

$$
\begin{aligned}
&{}_1\mathbb{S}_{inv}(\lambda, \mu_1\omega_1 + (\lambda_2+1)\omega_2) \\
&= \left\{ (a_1, a_2) \in \mathbb{Z}^2_{\geq 0} : \begin{array}{l} 0 \leq a_2 \leq \lambda_2,\ 0 \leq a_1 \leq \mu_1, \\ \lambda_2 - (\lambda_1 - 1) \leq a_2 - a_1 \leq (\mu_2 - 1) - \mu_1 \end{array} \right\} \\
&= \mathbb{S}_{ninv}(\lambda - \omega_1, \mu - \omega_2) \cup \{(a_1, \lambda_2) \in \mathbb{Z}^2_{>0} : 0 \leq a_1 \leq \mu_1, \lambda_2 - (\lambda_1 - 1) \leq \lambda_2 - a_1 \leq \lambda_2 - \mu_1\} \\
&= \mathbb{S}_{ninv}(\lambda - \omega_1, \mu - \omega_2) \cup \{(\mu_1, \lambda_2)\}
\end{aligned}
$$

Further, using Proposition 3.4.1(i)-(ii), we have,

$$
\begin{aligned}
& \sum_{(a_1,a_2) \in {}_1\mathbb{S}_{inv}(\lambda,\mu)} \dim V(\zeta^\lambda_\mu - \theta - (\lambda_2 + a_1 - 2a_2)\omega_1 - (\mu_1 + a_2 - 2a_1)\omega_2) \\
= & \sum_{(a_1,a_2) \in \mathbb{S}_{ninv}(\lambda-\omega_1,\mu-\omega_2)} \dim V((\lambda_1 - 1)\omega_1 + \lambda_2\omega_2 - (\lambda_2 + a_1 - 2a_2)\omega_1 - (\mu_1 + a_2 - 2a_1)\omega_2) \\
& \quad + \quad \dim V((\lambda_1 - \mu_1 + \lambda_2 - 1)\omega_1 + \mu_1\omega_2) \\
= & \quad \dim V((\lambda_1 - \mu_1 + \lambda_2 - 1)\omega_1 + \mu_1\omega_2) + \dim\ker(\lambda - \omega_1, \mu - \omega_2)
\end{aligned}.
$$

Note,

$$
\dim\ker(\lambda - \omega_1, \mu - \omega_2) = \begin{cases} \begin{array}{l} \dim \mathcal{F}_{\lambda-\omega_1,\mu-\omega_2} - \dim \mathcal{F}_{\lambda-\omega_1+\omega_2,\mu-2\omega_2} \\ \qquad - \dim \mathcal{F}_{\lambda-\omega_1+\lambda_2(\omega_1-\omega_2),\mu_1\omega_1}, \end{array} & \text{if } \lambda_2 \geq 1, \\ \\ \dim \mathcal{F}_{\lambda-\omega_1,\mu-\omega_2} - \dim \mathcal{F}_{\lambda-\omega_1+\omega_1,\mu-\omega_2-\omega_1}, & \text{if } \lambda_2 = 0, \end{cases}
$$

and using Proposition 3.4.1(ii), it follows from above that when $\lambda_2 \geq 1$ and $\mu_2 = \lambda_2 + 1$,

$$
\begin{aligned}
\dim \mathcal{F}_{\lambda,\mu} \leq \ & \dim V(\lambda_1\omega_1 + (\lambda_2+1)\omega_2).\dim V(\mu_1\omega_1 + \lambda_2\omega_2) \\
& + \dim V((\lambda_1 - 1)\omega_1 + \lambda_2\omega_2) \dim V(\mu_1\omega_1 + \lambda_2\omega_2) \\
& - \dim V((\lambda_1 - 1)\omega_1 + (\lambda_2+1)\omega_2).\dim V(\mu_1\omega_1 + (\lambda_2 - 1)\omega_2) \\
& - \dim V((\lambda_1 + \lambda_2 - 1)\omega_1) \dim V(\mu_1\omega_1) + \dim V((\lambda_1 + \lambda_2 - \mu_1 - 1)\omega_1 + \mu_1\omega_2)
\end{aligned}
$$

$$\begin{aligned}
&4\dim \mathcal{F}_{\lambda,\mu}\\
\leq\;& \Big((\lambda_1+1)(\lambda_2+2)(\lambda_1+\lambda_2+3)+\lambda_1(\lambda_2+1)(\lambda_1+\lambda_2+1)\Big)(\mu_1+1)(\lambda_2+1)(\mu_1+\lambda_2+2)\\
&-(\lambda_1)(\lambda_2+2)(\lambda_1+\lambda_2+2)(\mu_1+1)(\lambda_2)(\mu_1+\lambda_2+1))\\
&+\Big(2(\lambda_1+\lambda_2-\mu_1)-(\lambda_1+\lambda_2)(\mu_1+2)\Big)(\lambda_1+\lambda_2+1)(\mu_1+1)\\
\leq\;& \Big(3(\lambda_1+1)(\lambda_2+1)-(\lambda_1+2)\lambda_2\Big)(\lambda_1+\lambda_2+2)(\mu_1+1)(\lambda_2+1)(\mu_1+\lambda_2+2)\\
&-(\lambda_1)(\lambda_2+2)(\lambda_1+\lambda_2+2)(\mu_1+1)(\lambda_2)(\mu_1+\lambda_2+1))\\
&-(\lambda_1+\lambda_2+1)(\lambda_1+\lambda_2+2)(\mu_1)(\mu_1+1) \qquad [using(3.4.5)]\\
\leq\;& \Big(3(\lambda_1+1)(\lambda_2+1)-(\lambda_1+2)\lambda_2-\lambda_1(\lambda_2+2)\Big)(\lambda_1+\lambda_2+2)(\mu_1+1)(\lambda_2+1)(\mu_1+\lambda_2+2)\\
&+\lambda_1(\lambda_2+2)(\lambda_1+\lambda_2+2)(\mu_1+1)(2\lambda_2+\mu_1+2)-(\lambda_1+\lambda_2+1)(\lambda_1+\lambda_2+2)\mu_1(\mu_1+1)\\
\leq\;& \Big((\lambda_1+1)(\lambda_2+1)+2\Big)(\lambda_1+\lambda_2+2)(\mu_1+1)(\lambda_2+1)(\mu_1+\lambda_2+2)\\
&+\Big((\lambda_1)(\lambda_2+2)(2\lambda_2+\mu_1+2)-(\lambda_1+\lambda_2+1)\mu_1\Big)(\lambda_1+\lambda_2+2)(\mu_1+1)\\
\leq\;& [(\lambda_1+1)(\lambda_2+1)+2](\lambda_1+\lambda_2+2)(\mu_1+1)(\lambda_2+1)(\mu_1+\lambda_2+2)\\
&+\Big([(\lambda_1+1)(\lambda_2+1)+(\lambda_1-\lambda_2-1)](2\lambda_2+\mu_1+2)-(\lambda_1+\lambda_2+1)\mu_1]\Big)(\lambda_1+\lambda_2+2)(\mu_1+1)\\
\leq\;& (\lambda_1+\lambda_2+2)(\mu_1+1)[(\lambda_1+1)(\lambda_2+1)(\lambda_2+2)(\mu_1+\lambda_2+3)-2(\lambda_1+1)(\lambda_2+1)\\
&+(\lambda_1-\lambda_2-1)(2\lambda_2+\mu_1+2)+2(\lambda_2+1)(\mu_1+\lambda_2+2)-\mu_1(\lambda_1+\lambda_2+1)]\\
\leq\;& (\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\mu_1+1)(\lambda_2+2)(\mu_1+\lambda_2+3)\\
=\;& 4\dim V(\lambda).\dim V(\mu_1\omega_1+(\lambda_2+1)\omega_2)
\end{aligned}$$

Along with (3.4.1), the same argument that is used in Proposition 3.4.1(ii) show that Proposition 3.4.1(iii) holds when $\mu_2=\lambda_2+1$ and $\lambda_2\geq 1$. Similarly, it can be shown that the result holds when $\lambda_2=0$ and $\mu_2=1$.

By induction hypothesis assume that the result holds for all $(\lambda,\mu)\in P^+(\lambda+\mu,2)$ of second kind with $\mu_2-\lambda_2\leq k$. For $\mu_2-\lambda_2=\ell>1$, note that by definition (3.3.4),

$$\begin{aligned}
{}_{\ell+1}\mathbb{S}_{inv}(\lambda,\mu) &= \left\{(a_1,a_2)\in\mathbb{Z}^2_{\geq 0}: \begin{array}{c} 0\leq a_1\leq\mu_1,\, \lambda_2+\ell+1+a_1-a_2\leq\lambda_1,\\ 0\leq a_2\leq\lambda_2,\, \mu_1+\ell+1-a_1+a_2\leq\mu_2\end{array}\right\},\\
&= {}_{\ell}\mathbb{S}_{inv}(\lambda-\omega_1,\mu-\omega_2)
\end{aligned}$$

Since under the given conditions, the pair $(\lambda-\omega_1,\mu-\omega_2)\in(P^+)^2$ satisfies the condition $\mu_2-1-\lambda_2=\ell-1$, applying induction hypothesis we see that for all $\ell\le k$,

$$\begin{aligned}\sum_{(a_1,a_2)\in\bigcup\limits_{2\le\ell\le\mu_2-\lambda_2}{}_{\ell}\mathbb{S}_{inv}(\lambda,\mu)} &\dim V(\zeta^\lambda_\mu-\ell\theta-(\lambda_2+a_1-2a_2)\omega_1-(\mu_1+a_2-2a_1)\omega_2)\\ &= \dim\ker(\lambda-\omega_1,\mu-\omega_2) \quad=\quad \dim\mathcal{F}_{\lambda-\omega_1,\mu-\omega_2}-\dim\mathcal{F}_{\zeta^\lambda_\mu-\theta,\zeta^\mu_\lambda}.\end{aligned}\tag{3.4.17}$$

Since $\lambda_1>\mu_1$ and $\mu_2-\lambda_2\ge 1$, using definitions (3.3.4) and (3.3.1), observe that,

$$\begin{aligned}&{}_1\mathbb{S}_{inv}(\lambda,\mu)\\ &=\{(a_1,a_2)\in\mathbb{Z}^2_{\ge0}:\ 0\le a_1\le\mu_1, 0\le a_2\le\lambda_2, \lambda_2-(\lambda_1-1)\le a_2-a_1\le(\mu_2-1)-\mu_1\ \}\\ &=\begin{cases}\mathbb{S}_{ninv}(\zeta^\lambda_\mu-\theta,\zeta^\mu_\lambda)\\ \cup\{(a_1,\lambda_2):\max\{0,\mu_1+\lambda_2-\mu_2+1\}\le a_1\le\min\{\mu_1,\lambda_1-1\}\}, & \text{if } \lambda_2\ne0,\\ \\ \mathbb{S}_{ninv}(\zeta^\lambda_\mu-\theta,\zeta^\mu_\lambda), & \text{if } \lambda_2=0.\end{cases}\end{aligned}$$

Further, using Proposition 3.3.1 and Proposition 3.4.1(ii), we get,

$$\begin{aligned}\sum_{(a_1,a_2)\in\mathbb{S}_{ninv}(\zeta^\lambda_\mu-\theta,\zeta^\mu_\lambda)} &\dim V(\zeta^\lambda_\mu-\theta-(\lambda_2+a_1-2a_2)\omega_1-(\mu_1+a_2-2a_1)\omega_2)\\ &= \dim\ker(\zeta^\lambda_\mu-\theta,\zeta^\mu_\lambda)-(1-\delta_{\lambda_2,0})\dim\mathcal{F}_{\zeta^\lambda_\mu-\theta+\lambda_2(\omega_1-\omega_2),\mu_1\omega_1}\\ &= \dim\mathcal{F}_{\zeta^\lambda_\mu-\theta,\zeta^\mu_\lambda}-\dim\mathcal{F}_{\zeta^\lambda_\mu-\theta+\omega_2,\zeta^\mu_\lambda-\omega_2}-(1-\delta_{\lambda_2,0})\dim\mathcal{F}_{\zeta^\lambda_\mu-\theta+\lambda_2(\omega_1-\omega_2),\mu_1\omega_1}\end{aligned}\tag{3.4.18}$$

and using Proposition 3.3.2 and Proposition 3.4.1(i), we get

$$\begin{aligned}&\sum_{(a_1,a_2)\in\mathcal{S}}\dim V(\zeta^\lambda_\mu-\theta-(\lambda_2+a_1-2a_2)\omega_1-(\mu_1+a_2-2a_1)\omega_2)\\ &=\sum_{a_1=\max\{0,\mu_1+\lambda_2-\mu_2+1\}}^{\mu_1}\dim V(\zeta^\lambda_\mu-\theta-(\lambda_2+a_1-2\lambda_2)\omega_1-(\mu_1+\lambda_2-2a_1)\omega_2)\\ &= \dim\ker(\zeta^\lambda_\mu-\theta+\lambda_2(\omega_1-\omega_2),\mu_1\omega_1)\\ &= \dim\mathcal{F}_{\zeta^\lambda_\mu-\theta+\lambda_2(\omega_1-\omega_2),\mu_1\omega_1}-\dim\mathcal{F}_{\zeta^\lambda_\mu-\theta+\lambda_2(\omega_1-\omega_2)+\omega_1,(\mu_1-1)\omega_1}\end{aligned}\tag{3.4.19}$$

where $\mathcal{S} = \{(a_1,\lambda_2) : \max\{0,\mu_1+\lambda_2-\mu_2+1\} \le a_1 \le \min\{\mu_1,\lambda_1-1\}\}$. Using (3.4.17)–(3.4.19) in (3.4.16), and applying induction hypothesis, for $1<\ell\le k+1$ we have,

$$\dim\mathcal{F}_{\lambda,\mu} \le \begin{cases} \dim\mathcal{F}_{\zeta^\lambda_\mu,\zeta^\mu_\lambda} + \dim\mathcal{F}_{\lambda-\omega_1,\mu-\omega_2} - \dim\mathcal{F}_{\zeta^\lambda_\mu-\omega_2,(\mu_1-1)\omega_1}, & \text{if } \lambda_2=0,\\ \dim\mathcal{F}_{\zeta^\lambda_\mu,\zeta^\mu_\lambda} + \dim\mathcal{F}_{\lambda-\omega_1,\mu-\omega_2} - \dim\mathcal{F}_{\zeta^\lambda_\mu-\omega_1,(\lambda_2-1)\omega_2}, & \text{if } \mu_1=0,\\ \dim\mathcal{F}_{\zeta^\lambda_\mu,\zeta^\mu_\lambda} + \dim\mathcal{F}_{\lambda-\omega_1,\mu-\omega_2} - \dim\mathcal{F}_{\zeta^\lambda_\mu-\omega_1,\zeta^\mu_\lambda-\omega_2} & \\ \qquad - \dim\mathcal{F}_{\zeta^\lambda_\mu-\lambda_2(\omega_2-\omega_1)-\omega_2,(\mu_1-1)\omega_1}, & \text{if } \mu_1,\lambda_2\neq 0. \end{cases} \tag{3.4.20}$$

Note that the cases when $\lambda_2=0$ and $\mu_1=0$ can be obtained from the case when $\mu_1,\lambda_2\neq 0$ using appropriate substitutions. By induction hypothesis assume that the result holds when $\mu_2-\lambda_2\le k$. We then prove that the result in the case when $\mu_1\lambda_2\neq 0$ and $\mu_2-\lambda_2=k+1$. Using Proposition 3.4.1(ii), it follows from (3.4.20) that

$$\begin{aligned} 4\dim\mathcal{F}_{\lambda,\mu} \le\ & 4\Big(\dim V(\lambda_1\omega_1+(\lambda_2+k+1)\omega_2)\dim V(\mu_1\omega_1+\lambda_2\omega_2)\\ & +\dim V((\lambda_1-1)\omega_1+\lambda_2\omega_2)\dim V(\mu_1\omega_1+(\lambda_2+k)\omega_2)\\ & -\dim V((\lambda_1-1)\omega_1+(\lambda_2+k+1)\omega_2)\dim V(\mu_1\omega_1+(\lambda_2-1)\omega_2)\\ & -\dim V((\lambda_1+\lambda_2)\omega_1+k\omega_2)\dim V((\mu_1-1)\omega_1)\Big)\\ \le\ & (\lambda_1+1)(\lambda_2+k+2)(\lambda_1+\lambda_2+k+3)(\mu_1+1)(\lambda_2+1)(\mu_1+\lambda_2+2)\\ & +\lambda_1(\mu_1+1)\Big((\lambda_2+1)(\lambda_1+\lambda_2+1)(\lambda_2+k+1)(\mu_1+\lambda_2+k+2)\\ & -(\lambda_2+k+2)(\lambda_1\lambda_2+k+2)\lambda_2(\mu_1+\lambda_2+1)\Big)\\ & -(\lambda_1+\lambda_2+1)(k+1)(\lambda_1+\lambda_2+k+2)\mu_1(\mu_1+1) \end{aligned},$$

$$\begin{aligned} 4\ \ & \dim\mathcal{F}_{\lambda,\mu}\\ \le\ \ & (\lambda_1+1)(\mu_1+1)(\lambda_2+1)(\mu_1+\lambda_2+2)\Big((\lambda_2+k+1)(\lambda_1+\lambda_2+k+2)+\lambda_1+2\lambda_2+2k+4\Big)\\ & +\lambda_1(\lambda_2+1)(\lambda_1+\lambda_2+1)(\mu_1+1)\Big((\lambda_2+k)(\mu_1+\lambda_2+k+1)+\mu_1+2\lambda_2+2k+2\Big)\\ & -\lambda_1(\mu_1+1)\lambda_2(\mu_1+\lambda_2+1)\Big((\lambda_2+k+1)(\lambda_1+\lambda_2+k+1)+\lambda_1+2\lambda_2+2k+3\Big)\\ & -(\lambda_1+\lambda_2+1)\mu_1(\mu_1+1)\Big(k(\lambda_1+\lambda_2+k+1)+\lambda_1+\lambda_2+2k+2\Big) \end{aligned} \tag{3.4.21}$$

Since for $\mu_2 = \lambda_2 + k$, by induction hypothesis,

$$\begin{aligned}\dim \mathcal{F}_{\lambda,\mu-\omega_2} = \;& \dim V(\lambda)\dim V(\mu-\omega_2)\\ =\;& \dim V(\lambda_1\omega_1 + (\lambda_2+k)\omega_2)\dim V(\mu_1\omega_1+\lambda_2\omega_2)\\ &+\dim V((\lambda_1-1)\omega_1+\lambda_2\omega_2)\dim V(\mu_1\omega_1+(\lambda_2+k-1)\omega_2)\\ &-\dim V((\lambda_1-1)\omega_1+(\lambda_2+k)\omega_2).\dim V(\mu_1\omega_1+(\lambda_2-1)\omega_2)\\ &-\dim V((\lambda_1+\lambda_2)\omega_1+(k-1)\omega_2)\dim V((\mu_1-1)\omega_1)\end{aligned}$$

and $2\dim V(\mu) = 2\dim V(\mu-\omega_2) + (\mu_1+1)(\mu_1+2\lambda_2+2k+4)$, rearranging the coefficients in inequality (3.4.21), we get:

$$\begin{aligned}4 \;& \dim \mathcal{F}_{\lambda,\mu}\\ \leq\;& 4\dim V(\lambda)\dim V(\mu-\omega_2)\\ &+(\lambda_1+1)(\lambda_2+1)(\mu_1+1)\Big((\lambda_1+\lambda_2+2)(\mu_1+2\lambda_2+2k+4)-(\lambda_1-\mu_1)(\lambda_2+2k+2)\Big)\\ &-\lambda_1\lambda_2\Big((\lambda_1+\lambda_2+1)(\mu_1+2\lambda_2+2k+3)-(\lambda_1-\mu_1)(\lambda_2+2k+2)\Big)(\mu_1+1)\\ &+(\lambda_1+\lambda_2+1)\Big(\lambda_1(\lambda_2+1)(\mu_1+2\lambda_2+2k+2)-\mu_1(\lambda_1+\lambda_2+2k+2)\Big)(\mu_1+1)\\ \leq\;& 4\dim V(\lambda)\dim V(\mu)-\Big((\lambda_1+1)(\lambda_2+1)-\lambda_1\lambda_2\Big)(\lambda_1-\mu_1)(\lambda_2+2k+2)(\mu_1+1)\\ &-\lambda_1\lambda_2(\lambda_1+\lambda_2+1)(\mu_1+2\lambda_2+2k+3-\mu_1-2\lambda_2-2k-2)(\mu_1+1)\\ &+(\lambda_1+\lambda_2+1)[\lambda_1(\mu_1+2\lambda_2+2k+2)-\mu_1(\lambda_1+\lambda_2+2k+2)](\mu_1+1)\\ \leq\;& 4\dim V(\lambda)\dim V(\mu)-\Big((\lambda_1+1)(\lambda_2+1)-\lambda_1\lambda_2\Big)(\lambda_1-\mu_1)(\lambda_2+2k+2)(\mu_1+1)\\ &-\lambda_1\lambda_2(\lambda_1+\lambda_2+1)(\mu_1+2\lambda_2+2k+3-\mu_1-2\lambda_2-2k-2)(\mu_1+1)\\ &+(\lambda_1+\lambda_2+1)\Big((\lambda_1-\mu_1)(\lambda_2+2k+2)+\lambda_1\lambda_2\Big)(\mu_1+1)\\ =\;& 4\dim V(\lambda)\dim V(\mu).\end{aligned}$$

Along with (3.4.1), the above inequality shows that the results holds when $\mu_2 = \lambda_2 + k + 1$ and $\lambda_2 \geq 1$ and this completes the proof of Proposition 3.4.1(iii).

Proof of Theorem 3.1.2 (i) Observe that part(i) of the theorem follows from Proposition 3.2.1, Proposition 3.3.1-Proposition 3.3.5 and Proposition 3.4.1.

(ii). By Lemma 2.7.1, for $(\lambda, \mu) \in P^+(\lambda + \mu, 2)$, the fusion product $V(\lambda)^{z_1} * V(\mu)^{z_2}$ is a $\mathfrak{sl}_3[t]$-quotient of $\mathcal{F}_{\lambda,\mu}$ and by Proposition 3.4.1, $\dim \mathcal{F}_{\lambda,\mu} = \dim V(\lambda) \dim V(\mu)$. So using, Remark 2.7.1 we conclude that for any distinct pair of complex numbers (z_1, z_2), $\mathcal{F}_{\lambda,\mu}$ is isomorphic to $V(\lambda)^{z_1} * V(\mu)^{z_2}$ as a $\mathfrak{sl}_3[t]$-module. □

3.5 Discussion on the case $k > 2$.

There is a natural question that arises here. Why are we restricting ourselves to the case $k = 2$?

To answer this, we observe that when $\mathfrak{g}$ is of type A_2, there exists a one-one correspondence between elements of $P^+(\nu, 2)$ and R^+-tuple of ν-compatible partitions with number of parts less than equal to 2. However this fails when we consider elements of $P^+(\nu, k)$ for $k \geq 3$. Via an example, we show that for $k \geq 3$, there exist fusion product modules that are proper quotients of CV modules. In future, we intend to study such modules further.

For any $\lambda \in P^+$, $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3) \in P^+(\lambda, 3)$, and $\mathbf{z} = (z_1, z_2, z_3)$, distinct triplet of complex numbers, set,

$$V^*(\boldsymbol{\nu}, \mathbf{z}) := V(\nu_1)^{z_1} * V(\nu_2)^{z_2} * V(\nu_3)^{z_3}.$$

In this case, consider $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, $\lambda = 3\omega_1 + 3\omega_2$ and $\boldsymbol{\mu} = (2\omega_2, \omega_1, 2\omega_1 + \omega_2)$ and $\boldsymbol{\lambda} = (2\omega_2 + \omega_1, \omega_2, 2\omega_1)$ are elements of $P^+(\lambda, 3)$. Observe that $\mathcal{F}_{\boldsymbol{\lambda}} = \mathcal{F}_{\boldsymbol{\mu}}$. But $V^*(\boldsymbol{\lambda}, \mathbf{z}) \ncong V^*(\boldsymbol{\mu}, \mathbf{z})$, as even $\mathfrak{g}$ module decomposition of both is different. Both are proper quotients of $\mathcal{F}_{\boldsymbol{\lambda}}$ follows from following arguments.

We have the following exact sequence,

$$0 \longrightarrow Ker(\boldsymbol{\lambda}) \longrightarrow \mathcal{F}_{\boldsymbol{\lambda}} \twoheadrightarrow \mathcal{F}_{2\omega_1+\omega_2, \omega_1+2\omega_2} \longrightarrow 0$$

with $\mathrm{Ker}(\boldsymbol{\lambda}) = \mathbf{U}(\mathfrak{g}[t])(x^-_{\alpha_{12}} \otimes t^2)v_{\boldsymbol{\lambda}}$ having filtration $0 \subset V_3 + V_4 \subset V_2 \subset V_1 = \mathrm{Ker}(\boldsymbol{\lambda})$ where

$$V_2 = \mathbf{U}(\mathfrak{g}[t])(x^-_{\alpha_{12}} \otimes t)(x^-_{\alpha_{12}} \otimes t^2)v_{\boldsymbol{\lambda}}$$
$$V_3 = \mathbf{U}(\mathfrak{g}[t])(x^-_{\alpha_{11}} \otimes t)(x^-_{\alpha_{12}} \otimes t^2)v_{\boldsymbol{\lambda}}$$
$$V_4 = \mathbf{U}(\mathfrak{g}[t])(x^-_{\alpha_{22}} \otimes t)(x^-_{\alpha_{12}} \otimes t^2)v_{\boldsymbol{\lambda}}$$

such that V_1/V_2 is quotient of $V(2\omega_1 + 2\omega_2)$, hence isomorphic to $V(2\omega_1 + 2\omega_2)$ as V_1 being generated by a non-zero vector. $V_2/(V_3 + V_4)$ generated by non-zero vector of weight $\omega_1 + \omega_2$, is quotient of $V(\omega_1 + \omega_2)$, hence isomorphic to this. $V_3 + V_4$ is a quotient of $V(3\omega_1) \oplus V(3\omega_2)$. Hence,

$$\dim \ker(\boldsymbol{\lambda}) \leq \dim V(2\omega_1 + 2\omega_2) + \dim V(\omega_1 + \omega_2) + \dim V(3\omega_1) + \dim V(3\omega_2)$$

But $\dim \mathcal{F}_{2\omega_1+\omega_2,\omega_1+2\omega_2} + \dim V(2\omega_1 + 2\omega_2) + \dim V(\omega_1 + \omega_2) + \dim V(3\omega_1) = \dim V^*(\boldsymbol{\mu}, \mathbf{z})$ and $\dim \mathcal{F}_{2\omega_1+\omega_2,\omega_1+2\omega_2} + \dim V(2\omega_1 + 2\omega_2) + \dim V(\omega_1 + \omega_2) + \dim V(3\omega_2) = \dim V^*(\boldsymbol{\lambda}, \mathbf{z})$. Hence

$$\begin{aligned}
\dim \mathcal{F}_{\boldsymbol{\lambda}} = & \dim \mathcal{F}_{2\omega_1+\omega_2,\omega_1+2\omega_2} + \dim \ker(\boldsymbol{\lambda}) \\
\leq & \dim \mathcal{F}_{2\omega_1+\omega_2,\omega_1+2\omega_2} + \dim V(2\omega_1 + 2\omega_2) + \dim V(\omega_1 + \omega_2) + \dim V(3\omega_1) \\
& + \dim V(3\omega_2) \\
= & \dim V^*(\boldsymbol{\lambda}, \mathbf{z}) + \dim V(3\omega_1) \\
= & \dim V^*(\boldsymbol{\mu}, \mathbf{z}) + \dim V(3\omega_2)
\end{aligned}$$

But $V^*(\boldsymbol{\mu}, \mathbf{z})$ (resp. $V^*(\boldsymbol{\lambda}, \mathbf{z})$) is the quotient of $\mathcal{F}_{\boldsymbol{\lambda}}$ and $V(3\omega_1)$ (resp. $V(3\omega_2)$) occurs in $\mathfrak{g}$-module decomposition of $V^*(\boldsymbol{\mu}, \mathbf{z})$ (resp. $V^*(\boldsymbol{\lambda}, \mathbf{z})$) but not in $\mathcal{F}_{2\omega_1+\omega_2,\omega_1+2\omega_2}$, so $V(3\omega_1)$ (resp. $V(3\omega_2)$) occurs in $\mathfrak{g}$-module decomposition of $\mathrm{Ker}(\boldsymbol{\lambda})$. Thus,

$$\dim \ker(\boldsymbol{\lambda}) \geq \dim V(2\omega_1 + 2\omega_2) + \dim V(\omega_1 + \omega_2) + \dim V(3\omega_1) + \dim V(3\omega_2).$$

Hence, $\dim \mathcal{F}_{\boldsymbol{\lambda}} = \dim V^*(\boldsymbol{\lambda}, \mathbf{z}) + \dim V(3\omega_1) = \dim V^*(\boldsymbol{\mu}, \mathbf{z}) + \dim V(3\omega_2)$. Therefore,

$$V^*(\boldsymbol{\mu}, \mathbf{z}) = \frac{\mathcal{F}_{\boldsymbol{\lambda}}}{< (x_{\alpha_1}^- \otimes t)(x_{\alpha_{12}}^- \otimes t^2) v_{\boldsymbol{\lambda}} >} \quad \text{and} \quad V^*(\boldsymbol{\lambda}, \mathbf{z}) = \frac{\mathcal{F}_{\boldsymbol{\lambda}}}{< (x_{\alpha_2}^- \otimes t)(x_{\alpha_{12}}^- \otimes t^2) v_{\boldsymbol{\lambda}} >}.$$

□

Chapter 4

Graded character of module $\mathcal{F}_{\lambda,\mu}$

For $\lambda, \mu \in P^+$ the CV-module $\mathcal{F}_{\lambda,\mu}$ is a $\mathbb{Z}_+$-graded vector space and for each $s > 0$, the subspace $\mathcal{F}_{\lambda,\mu}[s]$ is a finite-dimensional $\mathfrak{g}$-module on which the action of $\mathfrak{h}$ is semisimple, i.e.,

$$\mathcal{F}_{\lambda,\mu} = \bigoplus_{(\eta,s)\in P\times\mathbb{Z}_+} \mathcal{F}_{\lambda,\mu}[s]_\eta,$$

where $\mathcal{F}_{\lambda,\mu}[s]_\eta = \{u \in \mathcal{F}_{\lambda,\mu}[s] : hu = \eta(h)u, \forall\, h \in \mathfrak{h}\}$. The graded character of $\mathcal{F}_{\lambda,\mu}$ is the polynomial in indeterminate q with coefficient in $\mathbb{Z}[P]$ given by

$$\mathrm{ch}_{gr}\, \mathcal{F}_{\lambda,\mu} = \sum_{(\eta,s)\in P\times\mathbb{Z}_+} \dim \mathcal{F}_{\lambda,\mu}[s]_\eta \; e(\eta) q^s.$$

Since for each $s \in \mathbb{Z}_+$, $\mathcal{F}_{\lambda,\mu}[s]$ is a finite-dimensional $\mathfrak{g}$-module, using Weyl's theorem, $\mathcal{F}_{\lambda,\mu}[s]$ can be written as the direct sum of $\tau_s^* V(\nu)$, with $\nu \in P^+$. Hence,

$$\mathrm{ch}_{gr}\, \mathcal{F}_{\lambda,\mu} = \sum_{s\in\mathbb{Z}_+} \mathrm{ch}_{\mathfrak{g}}\, \mathcal{F}_{\lambda,\mu}[s]\, q^s = \sum_{(\nu,s)\in P^+\times\mathbb{Z}_+} \mathrm{ch}_{\mathfrak{g}}(\tau_s^* V(\nu))\, q^s.$$

Define a polynomial in indeterminate q by,

$$[\mathcal{F}_{\lambda,\mu} : V(\nu)]_q = \sum_{p\geq 0} [\mathcal{F}_{\lambda,\mu} : \tau_p^*(V(\nu))]\, q^p,$$

where $[\mathcal{F}_{\lambda,\mu} : \tau_p^* V(\nu)]$ is the multiplicity of $\tau_p^* V(\nu)$ in a given filtration of the module $\mathcal{F}_{\lambda,\mu}$. The polynomial $[\mathcal{F}_{\lambda,\mu} : V(\nu)]_q$ is called the graded multiplicity of $V(\nu)$ in $\mathcal{F}_{\lambda,\mu}$ and, at q = 1, it gives the numerical multiplicity of $V(\nu)$ in the $\mathfrak{g}$-module $V(\lambda)\otimes V(\mu)$.

4.1 Let $(\lambda,\mu) \in P^+(\lambda+\mu,2)$ be a partition of first kind with $\mu_2 = 0$

We know by Proposition 3.3.2 and Proposition 3.4.1(i) that

$$\mathcal{F}_{\lambda,\mu_1\omega_1} = \mathcal{F}_{\lambda+\omega_1,(\mu_1-1)\omega_1} \oplus \ker\phi(\lambda,\mu_1\omega_1),$$

$$\ker\phi(\lambda,\mu_1\omega_1) \cong_{\mathfrak{sl}_3[t]} \bigoplus_{a=\max\{0,\mu_1-\lambda_2\}}^{\mu_1} \tau_{\mu_1}^* V(\lambda - a\omega_1 - (\mu_1 - 2a)\omega_2)$$

Using the $\mathfrak{sl}_3[t]$-module decomposition for the successive quotients, $\mathcal{F}_{\lambda+j\omega_1,(\mu_1-j)\omega_1}$, $1 \leq j \leq \mu_1$, we have

$$\mathcal{F}_{\lambda,\mu} \cong_{\mathfrak{sl}_3[t]} \bigoplus_{j=0}^{\mu_1} \bigoplus_{b_j=\max\{0,\mu_1-j-\lambda_2\}}^{\mu_1-j} \tau_{\mu_1-j}^* V(\lambda + (j-b_j)\omega_1 - (\mu_1 - j - 2b_j)\omega_2).$$

As a consequence the graded character of $\mathcal{F}_{\lambda,\mu_1\omega_1}$ is given as follows: Comparing the coefficients of ω_1 and ω_2, for given pairs of integers (j,b_j) and (k,b_k) we see,

$$\lambda + (j-b_j)\omega_1 - (\mu_1 - j - 2b_j)\omega_2 = \lambda + (k-b_k)\omega_1 - (\mu_1 - k - 2b_k)\omega_2,$$

only if $j=k$ and $b_j=b_k$. Hence, when $\lambda_1\geq\mu_1$, each $\mathfrak{sl}_3[t]$-irreducible component of $\mathcal{F}_{\lambda,\mu_1\omega_1}$ is multiplicity free. Let

$$\begin{aligned} \nu_{(a,p)} :=&\ \lambda+(\mu_1-p-a)\omega_1-(p-2a)\omega_2, \quad \forall\,(a,p)\in\mathbb{Z}_+^2, \\ P_{ninv}(\lambda,\mu_1\omega_1) =&\ \{\nu_{(a,p)} : a\in[p-\lambda_2,p]\cap\mathbf{Z}_+,\ p\in[0,\mu_1]\cap\mathbb{Z}_+\}. \end{aligned}$$

Then, $\quad [\mathcal{F}_{\lambda,\mu}:V(\nu)]_q=\begin{cases} q^p, & \text{if } \nu=\nu_{(a,p)}\in P_{ninv}(\lambda,\mu_1\omega_1), \\ 0, & \text{otherwise.}\end{cases}$

4.2 Let $(\lambda,\mu)\in P^+(\lambda+\mu,2)$ be a partition of first kind with $\mu_1,\mu_2>0$

By Proposition 3.3.1 and Proposition 3.4.1(ii), we have,

$$\mathcal{F}_{\lambda,\mu}=\mathcal{F}_{\lambda+\omega_2,\mu-\omega_2}\oplus\ker\phi(\lambda,\mu),$$

$$\begin{aligned} \ker\phi(\lambda,\mu)\cong_{\mathfrak{sl}_3[t]}&\tau^*_{\mu_2}\mathcal{F}_{\lambda+\mu_2(\omega_1-\omega_2),\mu_1\omega_1} \\ &\oplus\bigoplus_{(a_1,a_2)\in\mathbb{S}_{ninv}(\lambda,\mu)}\tau^*_{|\mu|}V(\lambda-(\mu_2+a_1-2a_2)\omega_1-(\mu_1+a_2-2a_1)\omega_2) \end{aligned}$$

Using the $\mathfrak{sl}_3[t]$-module decomposition for the successive quotients $\mathcal{F}_{\lambda+j\omega_2,\mu-j\omega_2}$, $1\leq j\leq\mu_2$, we have,

$$\begin{aligned} \mathcal{F}(\lambda,\mu)&=\bigoplus_{j=0}^{\mu_2-1}\ker\phi(\lambda+j\omega_2,\mu-j\omega_2)\oplus\mathcal{F}_{\lambda+\mu_2\omega_2,\mu_1\omega_1} \\ &\cong_{\mathfrak{sl}_3[t]}\bigoplus_{j=0}^{\mu_2-1}\tau^*_{\mu_2-j}\mathcal{F}_{\lambda+j\omega_2+(\mu_2-j)(\omega_1-\omega_2),\mu_1\omega_1}\oplus\mathcal{F}_{\lambda+\mu_2\omega_2,\mu_1\omega_1} \\ &\quad\oplus\bigoplus_{j=0}^{\mu_2}\left(\bigoplus_{(a_{j_1},a_{j_2})\in\mathbb{S}_{ninv}(\lambda+j\omega_2,\mu-j\omega_2)}\tau^*_{|\mu|-j}V(\upsilon^j_{(a_{j_1},a_{j_2})})\right) \end{aligned}$$

where $\upsilon^j_{(a_{j_1},a_{j_2})} = \lambda + j\omega_2 - (\mu_2 - j + a_{j_1} - 2a_{j_2})\omega_1 - (\mu_1 + a_{j_2} - 2a_{j_1})\omega_2$. For $(\ell, j) \in \mathbb{Z}^2$, let $M^\lambda_\mu(\ell, j) = \max\{0, |\mu| - \ell - \lambda_2 - 2j\}$. Using 4.1, we have,

$$\begin{aligned}
\mathrm{ch}_{gr}\, \mathcal{F}_{\lambda,\mu} &= \sum_{j=0}^{\mu_2} \mathrm{ch}_{gr}\, \mathcal{F}_{\lambda + j\omega_2 + (\mu_2 - j)(\omega_1 - \omega_2), \mu_1\omega_1}\ q^{\mu_2 - j} \\
&\quad + \sum_{j=0}^{\mu_2 - 1} \left(\sum_{(a_{j_1},a_{j_2}) \in \mathbb{S}_{ninv}(\lambda + j\omega_2, \mu - j\omega_2)} \mathrm{ch}_{\mathfrak{g}}\, \tau^*_{|\mu| - j} V(\upsilon^j_{(a_{j_1},a_{j_2})}) \right) q^{|\mu| - j} \\
&= \sum_{j=0}^{\mu_2} \sum_{\ell=0}^{\mu_1} \left(\sum_{a^j_\ell = M^\lambda_\mu(\ell,j)}^{\mu_1 - \ell} \mathrm{ch}_{\mathfrak{g}}\, \tau^*_{|\mu| - \ell - j} V(\lambda + j\omega_2 + (\mu_2 - j)(\omega_1 - \omega_2) + (\ell - a^j_\ell)\omega_1 - (\mu_1 - \ell - 2a^j_\ell)\omega_2) \right) q^{|\mu| - \ell - j} \\
&\quad + \sum_{j=0}^{\mu_2 - 1} \left(\sum_{(a_{j_1},a_{j_2}) \in \mathbb{S}_{ninv}(\lambda + j\omega_2, \mu - j\omega_2)} \mathrm{ch}_{\mathfrak{g}}\, \tau^*_{|\mu| - j} V(\upsilon^j_{(a_{j_1},a_{j_2})}) \right) q^{|\mu| - j}
\end{aligned}$$

Comparing the coefficients of ω_1 and ω_2 in the irreducible components of $\mathcal{F}_{\lambda,\mu}$, we now determine the polynomials $[\mathcal{F}_{\lambda,\mu} : V(\nu)]_q$.

(i). Given $(a_1, a_2) \in \mathbb{S}_{ninv}(\lambda, \mu)$, and integers ℓ, j such that $0 \le j \le \mu_2$ and $0 \le \ell \le \mu_1$,

$$\begin{aligned}
\lambda + j\omega_2 + (\mu_2 - j)(\omega_1 - \omega_2) + (\ell - a^j_\ell)\omega_1 - (\mu_1 - \ell - 2a^j_\ell)\omega_2 \\
= \lambda - (\mu_2 + a_1 - 2a_2)\omega_1 - (\mu_1 + a_2 - 2a_1)\omega_2
\end{aligned}$$

only if $\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2\mu_2 - j + a^j_\ell \\ 2j - \mu_2 + \ell + 2a^j_\ell \end{pmatrix}$, that is,

$$(a_1, a_2) = (j + \ell + a^j_\ell, \mu_2 + \ell).$$

But by definition, (3.3.1), $a_2 < \mu_2$, therefore this case cannot occur.

(ii). For integers j,ℓ such that $0\le j\le\mu_2$ and $0\le\ell\le\mu_1$ and $(a_{j_1},a_{j_2})\in\mathbb{S}_{ninv}(\lambda+j\omega_2,\mu-j\omega_2)$, $a_\ell\in[\max\{0,|\mu|-\ell-\lambda_2\},\mu_1-\ell]\cap\mathbf{Z}_+$,

$$\lambda+\mu_2(\omega_1-\omega_2)+(\ell-a_\ell)\omega_1-(\mu_1-\ell-2a_\ell)\omega_2$$
$$=\lambda+j\omega_2-(\mu_2-j+a_{j_1}-2a_{j_2})\omega_1-(\mu_1+a_{j_2}-2a_{j_1})\omega_2$$

only if $\begin{pmatrix}-1 & 2\\ 2 & -1\end{pmatrix}\begin{pmatrix}a_{j_1}\\ a_{j_2}\end{pmatrix}=\begin{pmatrix}2\mu_2+\ell-a_\ell-j\\ -\mu_2-j+\ell+2a_\ell\end{pmatrix}$, that is,

$$(a_{j_1},a_{j_2})=(\ell+a_\ell-j,\mu_2-j+\ell).$$

But by definition, (3.3.1), $a_{j_2}<\mu_2-j$, therefore this case cannot occur.

(iii). Given triples of integers, (j,ℓ,a^j_ℓ) and $(j+s,r,a^{j+s}_r)$, with $0\le j\le j+s\le\mu_2$, $\ell,r\in[0,\mu_1]$ and $\max\{0,|\mu|-k-\lambda_2-2s\}\le a^s_k\le\mu_1-k$ for $s\in\{\ell,r\}$ and $k\in\{j,j+s\}$,

$$\lambda+j\omega_2+(\mu_2-j)(\omega_1-\omega_2)+(\ell-a^j_\ell)\omega_1-(\mu_1-\ell-2a^j_\ell)\omega_2$$
$$=\lambda+(j+s)\omega_2+(\mu_2-j-s)(\omega_1-\omega_2)+(r-a^{j+s}_r)\omega_1-(\mu_1-r-2a^{j+s}_r)\omega_2,$$

only if $a^j_\ell-a^{j+s}_r=s+\ell-r$ and $2(a^{j+s}_r-a^j_\ell)=\ell-r-2s$, implying,

$$\ell=r,\quad\text{and}\quad a^j_\ell=a^{j+s}_r+s=a^{j+s}_\ell+s.$$

Observe,

if $a^j_\ell\in[|\mu|-\lambda_2-\ell-2j,\mu_1-\ell]\cap\mathbf{Z}_+$, for some $(\ell,j)\in[0,\mu_1]\times[0,\mu_2]$, then, $a^j_\ell+1\in[|\mu|-\lambda_2-\ell-2(j-1),\mu_1-\ell]\cap\mathbf{Z}_+$, unless $a^j_\ell\in\{\mu_1-\ell,|\mu|-\lambda_2-\ell-2j\}$.

$$\text{Setting,}\quad \nu_{a_\ell^j}=\lambda+(\mu_2+\ell-(a_\ell^j+j))\omega_1-(|\mu|-\ell-2(a_\ell^j+j))\omega_2,$$

$$\begin{aligned}{}_\ell P_{ninv}(\lambda,\mu)=&\{\nu_{a_\ell^0}:a_\ell^0\in[|\mu|-\lambda_2-\ell,\mu_1-\ell]\cap\mathbf{Z}_+\}\\ &\cup\bigcup_{j=1}^{\mu_2}\{\nu_{a_\ell^j}:a_\ell^j\in\{\mu_1-\ell,|\mu|-\lambda_2-\ell-2j\}\cap\mathbb{Z}_+\},\end{aligned}$$

we have,

$$[\mathcal{F}_{\lambda,\mu}:V(\nu_{a_\ell^j})]_q=\sum_{s=0}^{a_\ell^j}q^{|\mu|-\ell-s-j},\ \text{for}\ \nu_{a_\ell^j}\in{}_\ell P_{ninv}(\lambda,\mu),\quad\forall\,(\ell,j)\in[0,\mu_1]\times[0,\mu_2].$$

(iv). Given triples of integers, (j,a_{j_1},a_{j_2}) and $(j+s,a_{j_1'},a_{j_2'})$, with $0\le j\le j+s\le\mu_2-1$, $(a_{j_1},a_{j_2})\in\mathbb{S}_{ninv}(\lambda+j\omega_2,\mu-j\omega_2)$ and $(a_{j_1'},a_{j_2'})\in\mathbb{S}_{ninv}(\lambda+(j+s)\omega_2,\mu-(j+s)\omega_2)$,

$$\begin{aligned}&\lambda+j\omega_2-(\mu_2-j+a_{j_1}-2a_{j_2})\omega_1-(\mu_1+a_{j_2}-2a_{j_1})\omega_2\\ &\qquad=\lambda+(j+s)\omega_2-(\mu_2-j-s+a_{j_1'}-2a_{j_2'})\omega_1-(\mu_1+a_{j_2'}-2a_{j_1'})\omega_2\end{aligned}$$

only if $\begin{pmatrix}1&-2\\-2&1\end{pmatrix}\begin{pmatrix}a_{j_1}\\a_{j_2}\end{pmatrix}=\begin{pmatrix}1&-2\\-2&1\end{pmatrix}\begin{pmatrix}a_{j_1'}\\a_{j_2'}\end{pmatrix}-\begin{pmatrix}s\\s\end{pmatrix}$ that is,

$$(a_{j_1},a_{j_2})=(a_{j_1'}+s,a_{j_2'}+s).$$

From the sets $\mathbb{S}_{ninv}(\lambda+j\omega_2,\mu-j\omega_2)$, for $0\le j\le\mu_2$, (3.3.1), we see that, if $(a,b)\in\mathbb{S}_{ninv}(\lambda+(j+1)\omega_2,\mu-(j+1)\omega_2)$, then $(a+1,b+1)\in\mathbb{S}_{ninv}(\lambda+j\omega_2,\mu-j\omega_2)$. unless $\quad b-a=\lambda_2+j-\mu_1 \text{or} b-a=\mu_2-\lambda_1-j$

$$\text{Setting,}\quad \nu_{(a_j,b_j)}=\lambda+j\omega_2-(\mu_2-j+a_j-2b_j)\omega_1-(\mu_1+b_j-2a_j)\omega_2,$$

$$\begin{aligned}P^1_{ninv}(\lambda,\mu)=\quad&\{\nu_{(a_0,b_0)}:(a_0,b_0)\in S_{ninv}(\lambda,\mu)\}\\ &\bigcup_{j=1}^{\mu_2-1}\left\{\nu_{(a_j,b_j)}:\begin{array}{c}(a_j,b_j)\in\mathbb{S}_{ninv}(\lambda+j\omega_2,\mu-j\omega_2),\\ b_j-a_j\in\{\lambda_2-\mu_1+j,\mu_2-\lambda_1-j\}\end{array}\right\},\quad.\end{aligned}\tag{4.2.1}$$

$$\text{we have,} \quad [\mathcal{F}_{\lambda,\mu} : V(\nu_{(a_j,b_j)})]_q = \sum_{s=0}^{\min\{a_j,b_j\}} q^{|\mu|-s-j}, \quad \forall\, \nu_{(a_j,b_j)} \in P^1_{ninv}(\lambda,\mu).$$

Hence,

$$[\mathcal{F}_{\lambda,\mu} : V(\nu)]_{q=1} = \begin{cases} \min\{a_j,b_j\}+1, & \text{if } \nu = \nu_{a_j,b_j} \in P^1_{ninv}(\lambda,\mu) \\ a^j_\ell + 1, & \text{if } \nu = \nu_{a^j_\ell} \in {}_\ell P_{ninv}(\lambda,\mu) \forall\, (\ell,j) \in [0,\mu_1]\times[0,\mu_2] \\ 0 & \text{otherwise.} \end{cases}$$

4.3 Let $(\lambda,\mu) \in P^+(\lambda+\mu,2)$ is a partition of second kind

It follows from Proposition 3.3.5 and Proposition 3.4.1(iii) that,

$$\mathcal{F}_{\lambda,\mu} = \mathcal{F}_{\zeta^\lambda_\mu,\zeta^\mu_\lambda} \oplus \ker\phi(\lambda,\mu),$$

$$\ker\phi(\lambda,\mu) \cong_{\mathfrak{sl}_3[t]} \bigoplus_{\ell=1}^{\mu_2-\lambda_2} \bigoplus_{(a_1,a_2)\in {}_\ell\mathbb{S}_{inv}(\lambda,\mu)} \tau^*_{\mu_1+\lambda_2+\ell} V(\zeta^\lambda_\mu - (\lambda_2+a_1-2a_2+\ell)\omega_1 - (\mu_1+a_2-2a_1+\ell)\omega_2).$$

For $(\ell,j)\in\mathbb{Z}^2$, let $\widehat{M}^\lambda_\mu(\ell,j) = \max\{0, |\zeta^\mu_\lambda| - \lambda_2 - \ell - 2j\}$. Since for (λ,μ), a partition of second kind, $(\zeta^\lambda_\mu,\zeta^\mu_\lambda)$ is a partition of $\lambda+\mu$ of first kind, using 4.2, we get,

$$\begin{aligned} &\mathrm{ch}_q\mathcal{F}_{\lambda,\mu} \\ &= \sum_{\ell=1}^{\mu_2-\lambda_2} q^{|\zeta^\mu_\lambda|+\ell} \left(\sum_{(a_1,a_2)\in {}_\ell\mathbb{S}_{inv}(\lambda,\mu)} \mathrm{ch}_{\mathfrak{g}} V(\zeta^\lambda_\mu - (\lambda_2+a_1-2a_2+\ell)\omega_1 - (\mu_1+a_2-2a_1+\ell)\omega_2) \right) \\ &\quad + \sum_{j=0}^{\lambda_2} \left(\sum_{\ell=0}^{\mu_1} \left(\sum_{a^j_\ell = \widehat{M}^\lambda_\mu(\ell,j)}^{\mu_1-\ell} \mathrm{ch}_{\mathfrak{g}} V(\eta^{j,\ell}_{a^j_\ell}) \right) \right) q^{|\zeta^\mu_\lambda|-\ell-j} \\ &\quad + \sum_{j=0}^{\lambda_2-1} \left(\sum_{(a_1,a_2)\in\mathbb{S}_{ninv}(\zeta^\lambda_\mu+j\omega_2,\zeta^\mu_\lambda-j\omega_2)} \mathrm{ch}_{\mathfrak{g}} V(\eta^j_{(a_1,a_2)}) \right) q^{|\zeta^\mu_\lambda|-j} \end{aligned}$$

where $\eta^{j,\ell}_{a^j_\ell} = \zeta^\lambda_\mu + j\omega_2 + (\lambda_2-j)(\omega_1-\omega_2) + (\ell-a^j_\ell)\omega_1 - (\mu_1-\ell-2a^j_\ell)\omega_2$ and $\eta^j_{(a_1,a_2)} = \zeta^\lambda_\mu + j\omega_2 - (\lambda_2 - j + a_{j_1} - 2a_{j_2})\omega_1 - (\mu_1 + a_{j_2} - 2a_{j_1})\omega_2$. Now, by comparing the coefficients of ω_1 and ω_2 in the irreducible components of $\mathcal{F}_{\lambda,\mu}$ we determine the polynomials $[\mathcal{F}_{\lambda,\mu};V(\nu)]_q$.

First observe that for $1\leq \ell \leq \mu_2-\lambda_2$,

$$ {}_{\ell}\mathbb{S}_{inv}(\lambda,\mu) = {}_{\mu_2-\lambda_2-r}\mathbb{S}_{inv}(\lambda,\mu) \quad \text{for } r=\mu_2-\lambda_2-\ell. $$

Therefore,

$$ {}_{\mu_2-\lambda_2-r}\mathbb{S}_{inv}(\lambda,\mu) = \left\{ (a_1,a_2)\in\mathbb{Z}^2 : \begin{array}{l} 0\leq a_1\leq \mu_1, 0\leq a_2\leq\lambda_2 \\ \mu_2-\lambda_1-r\leq a_2-a_1\leq \lambda_2-\mu_1+r \end{array} \right\} \forall\, 0\leq r\leq \mu_2-\lambda_2-1; $$

and

$$ \begin{aligned} {}_{\mu_2-\lambda_2-r-1}\mathbb{S}_{inv}(\lambda,\mu) = & \ {}_{\mu_2-\lambda_2-r}\mathbb{S}_{inv}(\lambda,\mu) \\ & \cup \left\{ (a_1,a_2)\in\mathbb{Z}^2 : \begin{array}{l} 0\leq a_1\leq\mu_1,\ 0\leq a_1\leq\lambda_2, \\ a_2-a_1\in\{\mu_2-\lambda_1-r-1, \lambda_2-\mu_1+r+1\} \end{array} \right\}. \end{aligned} \tag{4.3.1} $$

(i). Given triplet of integers (r,a_1,a_2) with $1\leq r\leq \mu_2-\lambda_2$, $(a_1,a_2)\in {}_{r}\mathbb{S}_{inv}(\lambda,\mu)$ and (ℓ,j,a_ℓ^j) with $0\leq \ell\leq\mu_1$, , $0\leq j\leq\lambda_2$, $a_\ell^j\in[|\zeta_\lambda^\mu|-\lambda_2-\ell-2j, \mu_1-\ell]\cap\mathbf{Z}_+$,

$$ \begin{aligned} &\zeta_\mu^\lambda-(\lambda_2+a_1-2a_2+r)\omega_1-(\mu_1+a_2-2a_1+r)\omega_2 \\ &\qquad = \zeta_\mu^\lambda + j\omega_2 + (\lambda_2-j)(\omega_1-\omega_2)+(\ell-a_\ell^j)\omega_1-(\mu_1-\ell-2a_\ell^j)\omega_2 \end{aligned} $$

only if $\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} = \begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}\begin{pmatrix}a_\ell^j\\ \lambda_2\end{pmatrix} + \begin{pmatrix}j-r-\ell\\ 2j+r+\ell\end{pmatrix}$ i.e., when $(a_1,a_2)=(a_\ell^j+j+r+\ell, \lambda_2+\ell+r)$. This cannot happen since $a_2\leq\lambda_2$ and $r\leq 1$.

(ii) Given triplets of integers $(\mu_2-\lambda_2-\ell, a_1, a_2)$ and $(\mu_2-\lambda_2-\ell-s, b_1, b_2)$ with $0 \le \ell \le \ell+s \le \mu_2-\lambda_2-1$, $(a_1,a_2) \in {}_{\mu_2-\lambda_2-\ell}\mathbb{S}_{inv}(\lambda,\mu)$ and $(b_1,b_2) \in {}_{\mu_2-\lambda_2-\ell-s}\mathbb{S}_{inv}(\lambda,\mu)$,

$$\zeta_\mu^\lambda - (\lambda_2+a_1-2a_2+\mu_2-\lambda_2-\ell)\omega_1 - (\mu_1+a_2-2a_1+\mu_2-\lambda_2-\ell)\omega_2$$
$$= \zeta_\mu^\lambda - (\lambda_2+b_1-2b_2+\mu_2-\lambda_2-\ell-s)\omega_1 - (\mu_1+b_2-2b_1+\mu_2-\lambda_2-\ell-s)\omega_2)$$

only if $\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \begin{pmatrix} s \\ s \end{pmatrix}$ i.e., when

$$(a_1,a_2) = (b_1+s, b_2+s). \tag{4.3.2}$$

For $(a_r,b_r) \in {}_{\mu_2-\lambda_2-r}\mathbb{S}_{inv}(\lambda,\mu)$, set

$$\vartheta_{(a_r,b_r)} = \zeta_\mu^\lambda - (\lambda_2+a_r-2b_r+\mu_2-\lambda_2-r)\omega_1 - (\mu_1+b_r-2a_r+\mu_2-\lambda_2-r)\omega_2,$$
$$P_{inv}(\lambda,\mu) = \{\vartheta_{(a_0,b_0)} : (a_0,b_0) \in {}_{\mu_2-\lambda_2}\mathbb{S}_{inv}(\lambda,\mu)\} \cup \bigcup_{r=1}^{\mu_2-\lambda_2-1} \left\{ \vartheta_{(a_r,b_r)} : \begin{array}{c} (a_r,b_r) \in {}_{\mu_2-\lambda_2-r}\mathbb{S}_{inv}(\lambda,\mu), \\ b_r-a_r \in \{\mu_2-\lambda_1-r, \lambda_2-\mu_1+r\} \end{array} \right\}.$$

Then it follows from (4.3.1) and (4.3.2) that the distinct irreducible components of $\mathcal{F}_{\lambda,\mu}$ that are parametrized by the elements of $\bigcup_{\ell=1}^{\mu_2-\lambda_2} {}_\ell\mathbb{S}_{inv}(\lambda,\mu)$ are $V(\vartheta_{(a_r,b_r)})$, with $\vartheta_{(a_r,b_r)} \in P_{inv}(\lambda,\mu)$.

Thus, $\quad [\mathcal{F}_{\lambda,\mu} : V(\vartheta_{(a_r,b_r)})]_q = \sum_{s=0}^{\min\{a_r,b_r\}} q^{|\mu|-r-s}$, for $\vartheta_{(a_r,b_r)} \in P_{inv}(\lambda,\mu)$.

(iii). Given a triplet of integers (ℓ, a_1, a_2) with $1 \le \ell \le \mu_2-\lambda_2$, $(a_1,a_2) \in {}_\ell\mathbb{S}_{inv}(\lambda,\mu)$ and (j, b_1, b_2) with $0 \le j \le \lambda_2-1$, $(b_1,b_2) \in \mathbb{S}_{ninv}(\zeta_\mu^\lambda + j\omega_2, \zeta_\lambda^\mu - j\omega_2)$,

$$\zeta_\mu^\lambda - (\lambda_2+a_1-2a_2+\ell)\omega_1 - (\mu_1+a_2-2a_1+\ell)\omega_2$$
$$= \zeta_\mu^\lambda - (\lambda_2+b_1-2b_2-j)\omega_1 - (\mu_1+b_2-2b_1-j)\omega_2$$

only if $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -\ell - j \\ -\ell - j \end{pmatrix}$, i.e., when

$$(b_1, b_2) = (a_1 - \ell - j, a_2 - \ell - j). \tag{4.3.3}$$

Hence using (4.3.3), we see,

if for some $r = \mu_2 - \lambda_2 - \ell$, $\vartheta_{(a_r,b_r)} = \nu_{(a_j,b_j)}$ with $\vartheta_{(a_r,b_r)} \in P_{inv}(\lambda,\mu)$ and $\nu_{(a_j,b_j)} \in P^1_{ninv}(\zeta^\lambda_\mu, \zeta^\mu_\lambda)$ then $b_r - a_r = b_j - a_j$. However, by (4.2.1), for $j \geq 1$, $\nu_{(a_j,b_j)} \in P^1_{ninv}(\zeta^\lambda_\mu, \zeta^\mu_\lambda)$ only if

$$b_j - a_j = \mu_2 - \mu_1 + j \quad \text{or} \quad \lambda_2 - \lambda_1 - j,$$

whereas, by (3.3.4),

$$\lambda_2 - \lambda_1 + (\mu_2 - \lambda_2 - r) \leq b_r - a_r \leq \mu_2 - \mu_1 - (\mu_2 - \lambda_2 - r),$$

with $0 \leq r \leq \mu_2 - \lambda_2 - 1$. As $\lambda_2 - \lambda_1 - j < \lambda_2 - \lambda_1 + (\mu_2 - \lambda_2 - r)$ and $\mu_2 - \mu_1 - (\mu_2 - \lambda_2 - r) < \mu_2 - \mu_1 + j$ for all $0 \leq r \leq \mu_2 - \lambda_2 - 1$ and $j > 0$, we see that in order to determine the polynomials $[\mathcal{F}_{(\lambda,\mu)} : V(\eta)]_q$ it is sufficient to consider the cases when $\vartheta_{(a_r,b_r)} = \nu_{(a_0,b_0)}$. Observe that

$$\begin{aligned}
&\zeta^\lambda_\mu - (\lambda_2 + a_r - 2b_r + r)\omega_1 - (\mu_1 + b_r - 2a_r + r)\omega_2 \\
&\qquad = \zeta^\lambda_\mu - (\lambda_2 + (a_r - r) - 2(b_r - r))\omega_1 - (\mu_1 + (b_r - r) - 2(a_r - r))\omega_2
\end{aligned}$$

and $[\lambda_2 - \lambda_1 + (\mu_2 - \lambda_2 - r), \mu_2 - \mu_1 - (\mu_2 - \lambda_2 - r)] \subset [\lambda_2 - \lambda_1, \mu_2 - \mu_1]$, for $0 \leq r \leq \mu_2 - \lambda_2 - 1$, and setting,

$$\tilde{P}^1_{ninv}(\zeta^\lambda_\mu, \zeta^\mu_\lambda) = \bigcup_{j=0}^{\mu_2 - 1} \left\{ \nu_{(a_j,b_j)} : \begin{array}{c} (a_j, b_j) \in \mathbb{S}_{ninv}(\zeta^\lambda_\mu + j\omega_2, \zeta^\mu_\lambda - j\omega_2), \\ b_j - a_j \in \{\lambda_2 - \mu_1 + j, \mu_2 - \lambda_1 - j\} \end{array} \right\},$$

$$\text{we get, } [\mathcal{F}_{\lambda,\mu} : V(\nu_{(a_j,b_j)})]_q = \sum_{s=0}^{\min\{a_j,b_j\}} q^{|\mu|-s-j}, \quad \text{for } \nu_{a_j,b_j} \in \tilde{P}^1_{ninv}(\zeta^\lambda_\mu, \zeta^\mu_\lambda).$$

Thus given a partition (λ,μ) of $\lambda+\mu$ of the second kind, we have,

$$[\mathcal{F}_{\lambda,\mu} : V(\nu)]_{q=1} = \begin{cases} \min\{a_j,b_j\}+1, & \text{if } \nu = \nu_{a_j,b_j} \in \tilde{P}^1_{ninv}(\zeta^\lambda_\mu, \zeta^\mu_\lambda), \\ a^j_\ell + 1, & \text{if } \nu = \nu_{a^j_\ell} \in {}_\ell P_{ninv}(\zeta^\lambda_\nu, \zeta^\mu_\lambda), \text{ for } (l,j) \in [0,\mu_1]\times[0,\lambda_2], \\ \min\{a_r,b_r\}+1 & \text{if } \nu = \vartheta_{(a_r,b_r)} \in P_{inv}(\lambda,\mu) \\ 0 & \text{otherwise} \end{cases}$$

4.4 Littlewood-Richardson Coefficients

The following is an important consequence of the results obtained in this section.

Theorem 4.4.1. *For $\lambda,\mu,\eta \in P^+$, let $c^\eta_{\lambda,\mu}$ be the multiplicity of $V(\eta)$ in the $\mathfrak{sl}_3(\mathbb{C})$-module $V(\lambda)\otimes V(\mu)$.*

i. Suppose (λ,μ) is a partition of $\lambda+\mu$ of first kind with $\mu = \mu_1\omega_1$. Let $P_{ninv}(\lambda,\mu_1\omega_1)$ be as defined in Section 4.1. Then

$$c^\eta_{\lambda,\mu} = \begin{cases} 1, & \textit{if } \eta \in P_{ninv}(\lambda,\mu_1\omega_1) \\ 0, & \textit{otherwise}, \end{cases}$$

ii. Suppose (λ,μ) is a partition of $\lambda+\mu$ of first kind with $\mu_i > 0$ for $i = 1,2$. Let $P^1_{ninv}(\lambda,\mu)$ and ${}_\ell P_{ninv}(\lambda,\mu)$ be defined as in Section 4.2. Then

$$c^\eta_{\lambda,\mu} = \begin{cases} \min\{a_j,b_j\}+1, & \textit{when } \eta = \nu_{(a_j,b_j)} \in P^1_{ninv}(\lambda,\mu), \\ a^j_\ell + 1, & \textit{when } \eta = \nu_{a^j_\ell} \in {}_\ell P_{ninv}(\lambda,\mu), \textit{for } (l,j) \in [0,\mu_1]\times[0,\mu_2], \\ 0 & \textit{otherwise} \end{cases}$$

iii. *Suppose* (λ,μ) *is a partition of* $\lambda+\mu$ *of second kind. Let* $P_{inv}(\lambda,\mu), \tilde{P}^1_{ninv}(\zeta^\lambda_\mu, \zeta^\mu_\lambda)$, *and* ${}_\ell P_{ninv}(\zeta^\lambda_\nu, \zeta^\mu_\lambda)$ *be defined as in Section 4.3. Then*

$$c^\eta_{\lambda,\mu} = \begin{cases} \min\{a_r, b_r\}+1, & when\, \eta = \vartheta_{(a_r,b_r)} \in P_{inv}(\lambda,\mu), \\ \min\{a_j, b_j\}+1, & when\, \eta = \nu_{a_j,b_j} \in \tilde{P}^1_{ninv}(\zeta^\lambda_\mu, \zeta^\mu_\lambda), \\ a^j_\ell + 1, & when\, \eta = \nu_{a^j_\ell} \in {}_\ell P_{ninv}(\zeta^\lambda_\nu, \zeta^\mu_\lambda), for\,(l,j) \in [0,\mu_1]\times[0,\lambda_2], \\ 0 & otherwise \end{cases}$$

□

Part-II

Chapter 5

Borcherds Kac-Moody Lie superalgebra

Borcherds Kac-Moody Lie superalgebra (BKM superalgebra in short) is a natural generalization of Kac-Moody Lie superalgebra. It can also be regarded as $\mathbb{Z}_2$-graded generalization of Borcherds Kac-Moody Lie algebra. In this part of thesis, we study the free root spaces of BKM Lie superalgebra and give two types of basis of it using combinatorial tool heaps of pieces.

In this chapter, we set the notations, recall some basic notions and results from [56, 50] that we shall use in the subsequent chapters. In this part of the thesis, we shall denote by I, a countable (possibly infinite) set.

5.1 BKM Supermatrix and associated Lie superalgebra

Definition 5.1.1. A $\mathbb{Z}_2$-graded vector space V is a direct sum, $V_0 \oplus V_1$, of vector spaces. We call elements of V_0 (resp. V_1) even (resp. odd). The non-zero elements of $V_0 \cup V_1$ are all homogeneous. For any homogeneous element $x \in V_i$, $i \in \mathbb{Z}_2$, we set $\bar{x} = i$, the degree of x.

A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{L} = \mathfrak{L}_0 \oplus \mathfrak{L}_1$ with Lie bracket, $[\,,\,] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ satisfying the following:

(i) $[\mathfrak{L}_i, \mathfrak{L}_j] \subseteq \mathfrak{L}_{i+j}$ for $i, j \in \mathbb{Z}_2$,

(ii) $[a,b] = -(-1)^{\bar{a}\bar{b}}[b,a]$,

(iii) $(-1)^{\bar{a}\bar{c}}[a,[b,c]] + (-1)^{\bar{a}\bar{b}}[b,[a,c]] + (-1)^{\bar{b}\bar{c}}[c,[a,b]] = 0$

for all homogeneous elements $a,b,c \in \mathfrak{L}$

Remark 5.1.2. Any Lie superalgebra $\mathfrak{L}$ is a Lie sub-superalgebra of $gl(\mathfrak{L})$ via the adjoint action, where $\mathfrak{L}$ is considered as a $\mathbb{Z}_2$-graded vector space.

Definition 5.1.3. Let Ψ be any subset of I. A real matrix $A = (a_{ij})_{i,j \in I}$ together with a choice of Ψ is said to be a Borcherds-Kac-Moody supermatrix (BKM supermatrix in short) if the following conditions are satisfied: For $i, j \in I$ we have

(i) $a_{ii} = 2$ or $a_{ii} \leq 0$.

(ii) $a_{ij} \leq 0$ if $i \neq j$.

(iii) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

(iv) $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$.

(v) $a_{ij} \in 2\mathbb{Z}$ if $a_{ii} = 2$ and $i \in \Psi$.

Remark 5.1.4. A is said to be **symmetrizable** if there exists a diagonal matrix $D = \text{diag}(d_i)_{i \in I}$ with positive entries such that DA is symmetric.

Definition 5.1.5. An index $i \in I$ is said to be real if $a_{ii} = 2$ and imaginary if $a_{ii} \leq 0$. Denote by

$$I^{\text{re}} = \{i \in I : a_{ii} = 2\},$$
$$\Psi^{re} = \Psi \cap I^{re}, \qquad .$$
$$\Psi_0 = \{i \in \Psi : a_{ii} = 0\}$$

Definition 5.1.6. The BKM Lie superalgebra associated with a BKM supermatrix (A,Ψ) is the Lie superalgebra $\mathfrak{L}(A,\Psi)$ generated by $e_i, f_i, h_i, i \in I$ with the following defining relations:

(i) $[h_i,h_j] = 0$ for $i, j \in I$,

(ii) $[h_i,e_j] = a_{ij}e_j$, $[h_i, f_j] = -a_{ij} f_j$ for $i, j \in I$,

(iii) $[e_i, f_j] = \delta_{ij}h_i$ for $i, j \in I$,

(iv) $\deg h_i = 0, i \in I$,

(v) $\deg e_i = 0 = \deg f_i$ if $i \notin \Psi$, $\deg e_i = 1 = \deg f_i$ if $i \in \Psi$,

(vi) $(\text{ad } e_i)^{1-a_{ij}} e_j = 0 = (\text{ad } f_i)^{1-a_{ij}} f_j$ if $i \in I^{re}$ and $i \neq j$,

(vii) $(\text{ad } e_i)^{1-\frac{a_{ij}}{2}} e_j = 0 = (\text{ad } f_i)^{1-\frac{a_{ij}}{2}} f_j$ if $i \in \Psi^{re}$ and $i \neq j$,

(viii) $(\text{ad } e_i)^{1-\frac{a_{ij}}{2}} e_j = 0 = (\text{ad } f_i)^{1-\frac{a_{ij}}{2}} f_j$ if $i \in \Psi_0$ and $i = j$, i.e., $[e_i,e_i] = 0 = [f_i, f_i]$ for $i \in \Psi_0$,

(ix) $[e_i,e_j] = 0 = [f_i, f_j]$ if $a_{ij} = 0$.

The relations (vi)-(viii) are refered to as Serre relations of $\mathfrak{L}$.

The abelian Lie sub-superalgebra $\mathfrak{h}$ spanned by $\{h_i : i \in I\}$ is called the Cartan subalgebra of $\mathfrak{L}(A,\Psi)$. Let $\{\alpha_i : i \in I\}$ be the set of simple roots of $\mathfrak{L}(A,\Psi)$, $\{\alpha_i : i \in \Psi = I_1\}$ be the set of odd simple roots and $\{\alpha_i : i \in I_0 = I \setminus I_1\}$ be the set of even simple roots of $\mathfrak{L}(A,\Psi)$. If $a_{ii} > 0$ for all $i \in I$ then $\mathfrak{L}(A,\Psi)$ is said to be Kac-Moody Lie superalgebra.

Remark 5.1.7. All finite dimensional semisimple Lie algebras and affine Lie algebras are BKM algebras. However, all finite dimensional and affine Lie superalgebras are not BKM Lie superalgebras. A simple finite dimensional Lie superalgebra $\mathfrak{L}$ is a BKM superalgebra if and only if $\mathfrak{L}$ is of type $A(m,0) = \mathfrak{sl}(m+1,1), A(m,1) = \mathfrak{sl}(m+1,2), B(0,n) =$

$\mathfrak{osp}(1,2n), B(m,1) = \mathfrak{osp}(2m+1,2), C(n) = \mathfrak{osp}(2,2n-2), D(m,1) = \mathfrak{osp}(2m,2), D(2,1;\alpha)$ for $\alpha \neq 0,-1$, $F(4)$, and $G(3)$. An affine BKM Lie superalgebra is either a Kac-Moody Lie superalgebra or has degenerate generalized symmetric Cartan matrix $A = (0)$.

5.2 Quasi Dynkin diagram

Definition 5.2.1. Let G be a countable (possibly infinite) simple graph with vertex set $V = \{\alpha_i : i \in I\}$ and a subset $\Psi \subseteq I$. The vertices in $V_1 = \{\alpha_i : i \in \Psi\}$ (resp. $V_0 = \{\alpha_i : i \in I \backslash \Psi\}$) are called odd (resp. even) vertices of G. Such a graph (G,Ψ) with $\mathbb{Z}_2$-graded vertex set V is called a **supergraph**. If A is the classical adjacency matrix of the graph G then the pair (A,Ψ) is called the adjacency matrix of the supergraph (G,Ψ).

Definition 5.2.2. Let $(A = (a_{ij}),\Psi)$ be a BKM supermatrix and $\mathfrak{L}$ be the associated BKM superalgebra. **The quasi Dynkin diagram** of $\mathfrak{L}$ is the supergraph (G,Ψ) with vertex set V such that two vertices $\alpha_i, \alpha_j \in V$ are connected by an edge if and only if $a_{ij} \neq 0$. We often refer to (G,Ψ) simply as the graph of $\mathfrak{L}$.

In other words, the quasi Dynkin diagram can be obtained from the Dynkin diagram of $\mathfrak{L}$ by replacing all the multi edges with a single edge. For any subset $S \subseteq \Pi$, we denote by $|S|$ the number of elements in S. The subgraph induced by the subset S is denoted by G_S. A subset $S \subseteq \Pi$ is said to be *connected* if the corresponding subgraph G_S is connected, and S is said to be *independent* if G_S is totally disconnected.

5.3 Root System

Definition 5.3.1. The formal root lattice Q is defined as the free abelian group generated by $\alpha_i, i \in I$ with a real valued bilinear form $(\alpha_i, \alpha_j) = a_{ij}$. Let Δ be root system of BKM superalgebra and $\Delta_+ := \Delta \cap Q_+$ the set of positive roots where $Q_+ := \sum_{i\in I} \mathbb{Z}_+ \alpha_i$ is the positive root lattice of $\mathfrak{L}$.

Definition 5.3.2. Define three functions $\mathrm{ht} : \Delta \longrightarrow \mathbb{N}$, $\mathrm{supp} : \Delta \longrightarrow P(I)$, and $\mathrm{wt} : \Delta \longrightarrow \mathbb{Z}_+^I$ where $P(I)$ denotes the power set of I such that for $\alpha = \sum k_i \alpha_i \in \Delta$,

$$\begin{aligned} \mathrm{ht}(\alpha) = &\ \sum k_i \\ \mathrm{supp}(\alpha) = &\ \{i \in I : k_i \neq 0\} \ \cdot \\ \mathrm{wt}(\alpha) := &\ \mathbf{k} = (k_i : i \in I) \end{aligned}$$

For $\alpha = \sum_{j=1}^{n} m_j \alpha_j \in \Delta$, the root space $\mathfrak{L}_\alpha$ is defined as

$$\mathfrak{L}_\alpha = \{x \in \mathfrak{L} : [h,x] = \alpha(h)x \quad \forall \ h \in \mathfrak{h}\}.$$

For $i \in I$, $\mathfrak{L}_{\alpha_i} = \mathbb{C}e_i$ and $\mathfrak{L}_{-\alpha_i} = \mathbb{C}f_i$. The root space $\mathfrak{L}_\alpha$ (resp. $\mathfrak{L}_{-\alpha}$) is generated by the brackets of an element of $\mathfrak{L}_{\alpha_1}$ m_1-times with an element of $\mathfrak{L}_{\alpha_2}$ m_2-times, ..., with an element of $\mathfrak{L}_{\alpha_n}$ m_n-times, i.e., $[e_{i_j}, [\cdots [e_{i_2}, e_{i_1}]]]$ (resp. $[f_{i_j}, [\cdots [f_{i_2}, f_{i_1}]]]$). Such a root α is said to be an odd root if the number of $i_k, 1 \leq k \leq j$, coming from I_1 is odd otherwise it is an even root, denoted as Δ_1 and Δ_0 respectively. So, a root space $\mathfrak{L}_\alpha$ is either contained in the even part $\mathfrak{L}_{\bar{0}}$ or odd part $\mathfrak{L}_{\bar{1}}$ of the BKM superalgebra $\mathfrak{L}$. The dimension of root space $\mathfrak{L}_\alpha$ is called the multiplicity of root α. All root spaces are finite dimensional. Observe that $\dim \mathfrak{L}_{\alpha_i} = 1 = \dim \mathfrak{L}_{-\alpha_i}$, $i \in I$.

Definition 5.3.3. A root $\alpha = \sum_{i\in I} k_i \alpha_i \in Q^+$ having weight $\mathbf{k} = (k_i : i \in I)$ is said to be a **free root**, if $k_i \leq 1$ for $i \in I^{re} \sqcup \Psi_0$. A root $\alpha \in \Delta$ is called real if and only if $(\alpha, \alpha) > 0$ otherwise we call it an imaginary root. The set of real roots is denoted by Δ^{re} and imaginary roots by $\Delta^{im} = \Delta \setminus \Delta^{re}$.

It was shown in [56, Proposition 2.40] that for $i \in I^{im}$, if $\alpha \in \Delta_+ \setminus \{\alpha_i\}$ such that $\mathrm{supp}(\alpha+\alpha_i)$ is connected then, for all $j \in \mathbb{Z}_+$, $\alpha + j\alpha_i \in \Delta_+$. If $\alpha \in \Delta$ then $\mathrm{supp}(\alpha)$ is connected.

Definition 5.3.4. The generalized Cartan decomposition of the BKM superalgebra $\mathfrak{L}$ is defined as

$$\mathfrak{L} \cong \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \text{ where } \mathfrak{n}^{\pm} = \bigoplus_{\alpha \in \pm\Delta_+} \mathfrak{L}_\alpha.$$

5.4 Weyl Group

For $\alpha \in \Delta^{re}$, define the reflection s_α along the hyperplane perpendicular to α by

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha.$$

The Weyl Group W of BKM superalgebra $\mathfrak{L}$ is generated by simple reflections s_α, $\alpha \in \Delta^{re}$. Also, bilinear form on $\mathfrak{L}$ is W-invariant, i.e. $(w(\alpha), w(\beta)) = (\alpha, \beta)$ for all $\alpha, \beta \in \Delta$.

Define the length of $w \in W$ by length of reduced expression of w, denoted by $\ell(w)$. A vector $\rho \in \mathfrak{h}^*$ such that $2(\rho, \alpha_i) = (\alpha_i, \alpha_i)$ for all $i \in I$ is called the Weyl vector. Such a vector exist only if there exists a non-degenerate bilinear form on $\mathfrak{h}^*$.

Remark 5.4.1. The Chevalley automorphism w of order 4 acting on BKM superalgebra $\mathfrak{L}$ by

$$w(e_i) = \begin{cases} f_i, & \text{if } i \in \Psi \\ -f_i, & otherwise \end{cases}$$

5.5 Denominator identity

Let Ω be the set of all $\gamma \in Q_+$ such that

(i) $\gamma = \sum_{j=1}^{r} \alpha_{i_j} + \sum_{k=1}^{s} l_{i_k}\beta_{i_k}$ where the α_{i_j} (resp. β_{i_k}) are distinct even (resp. odd) imaginary simple roots,

(ii) $(\alpha_{i_j}, \alpha_{i_k}) = (\beta_{i_j}, \beta_{i_k}) = 0$ for $j \neq k$; $(\alpha_{i_j}, \beta_{i_k}) = 0$ for all j,k;

(iii) If $l_{i_k} \geq 2$, then $(\beta_{i_k}, \beta_{i_k}) = 0$.

The following denominator identity of BKM superalgebras is proved in [56, Section 2.6]:

$$\sum_{w\in W}\sum_{\gamma\in\Omega} \varepsilon(w)\varepsilon(\gamma)e^{w(\rho-\gamma)-\rho} = \frac{\prod_{\alpha\in\Delta_+^0}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}{\prod_{\alpha\in\Delta_+^1}(1+e^{-\alpha})^{\mathrm{mult}(\alpha)}} \tag{5.5.1}$$

where $\mathrm{mult}(\alpha) = \dim \mathfrak{L}_\alpha$, $\varepsilon(w) = (-1)^{l(w)}$ and $\varepsilon(\gamma) = (-1)^{\mathrm{ht}\gamma}$.

Remark 5.5.1. If Ψ is the empty set then Equation (5.5.1) reduces to the denominator identity of the Borcherds algebras. Further, if I^{im} is also empty, then Equation (5.5.1) reduces to the denominator identity of the Kac-Moody algebras.

5.6 Free partially commutative Lie superalgebras

Let (G,Ψ) be a supergraph with a vertex set $V = \{\alpha_i : i \in I\}$ and edge set $E(G). = V_0 \sqcup V_1$ where $V_j = \{\alpha_i : i \in I_j\}$ for $j \in \{0,1\}$ and edge set $E(G)$. V is totally ordered with respect to the order induced from set I. Let V^* be the **free monoid** generated by V. Note that V^* is totally ordered with respect to the lexicographic order. A word $\mathbf{w} \in V^*$ is called even if the number of alphabets from V_1 in $\mathbf{w}$ is even and odd otherwise. This defines a $\mathbb{Z}_2-$gradation on V^*. First, we define the free Lie superalgebra on a $\mathbb{Z}_2$-graded set $V = V_0 \sqcup V_1$.

Definition 5.6.1. Let $\mathcal{V}$ be the $\mathbb{Z}_2$-graded vector space with basis V and $T(\mathcal{V})$ be the tensor algebra on V. The algebra $T(\mathcal{V})$ has an induced $\mathbb{Z}_2$-gradation, which makes it an associative superalgebra with basis V^*. Clearly, $T(\mathcal{V})$ has a natural Lie superalgebra structure. The **free**

Lie superalgebra on the superset, V denoted by $\mathcal{FLS}(V)$, is defined to be the smallest Lie sub-superalgebra of $T(\mathcal{V})$ containing V.

If V_1 is an empty set, then $\mathcal{FLS}(V)$ is a free Lie algebra on the set V_0.

Definition 5.6.2. Let J be the ideal in $\mathcal{FLS}(V)$ generated by the relations $\{[\alpha_i,\alpha_j] : (\alpha_i,\alpha_j) \notin E(G)\}$. The quotient algebra $\frac{\mathcal{FLS}(V)}{J}$, denoted by $\mathcal{LS}(G,\Psi)$, is the **free partially commutative Lie superalgebra** associated with the supergraph (G,Ψ). When $\Psi = I_1$ is the empty set, $\mathcal{LS}(G,\Psi)$ is the free partially commutative Lie algebra associated with the graph G and is denoted by $\mathcal{L}(G)$. It is well-known that $\mathcal{FLS}(V)$ and hence $\mathcal{LS}(G,\Psi)$ is graded by $\mathbb{Z}_+^I$.

5.7 Free partially commutative super monoid

Let $M(V,G,\Psi) := V^*/\sim$ be the **free partially commutative super monoid** associated with a supergraph (G,Ψ), where $\sim$ is generated by the relations $ab \sim ba,$ if $(a,b) \notin E(G)$. Observe that $M(V,G,\Psi)$ has a natural $\mathbb{Z}_2$-gradation induced from the $\mathbb{Z}_2$-gradation of V^*. When Ψ is empty, $M(V,G,\Psi)$ is called the free partially commutative monoid associated with the graph G and denoted simply by $M(V,G)$.

Associate with each element $[a] \in M(V,G,\Psi)$ a unique element $\tilde{a} \in V^*$ which is the maximal element in $[a]$ with respect to the lexicographic order. This element is called the **standard word** of the class $[a]$ and it is denoted by $\mathrm{st}([a])$. A total order on $M(V,G,\Psi)$ is then given as follows:

$$[a] < [b] \Leftrightarrow \mathrm{st}\,[a] < \mathrm{st}\,[b]. \qquad (5.7.1)$$

5.8 Heaps monoid

We now recall essential definitions from the theory of heaps of pieces to define pyramids and Lyndon heaps from [44].

5.8.1 Heaps of pieces

Let (G,Ψ) be a supergraph with a vertex set $V = \{\alpha_i : i \in I\}$ and edge set $E(G)$. V is totally ordered with respect to order induced from set I. Define a relation ζ on the set, V such that $a\zeta b$ for $a,b \in V$, if a and b are connected by an edge in the super graph (G,Ψ), i.e., $(a,b) \in E(G)$.

A ***pre-heap*** E over (V,ζ) is a **finite** subset of $V \times \{0,1,2,\dots\}$ satisfying, if $(a,m),(b,n) \in E$ with $a\,\zeta\,b$, then $m \neq n$. Each element (a,m) of E is called a basic piece. For $(a,m) \in E$, the position and level of the piece (a,m) is denoted by

$$\pi(a,m) = a, \quad h(a,m) = m.$$

A basic piece will be simply denoted by a when we do not need to emphasize its level. The set $\pi(E)$ is defined to be the set of all positions occupied by the pieces of E.

A partial order $\leq_E$ is defined on a pre-heap E as follows:

$$(a,m) \leq_E (b,n) \quad \text{if} \quad a\zeta b \quad \text{and} \quad m < n.$$

Two pre-heaps E and F are said to be *isomorphic* if there exists a position preserving order isomorphism ϕ between $(E,\leq_E)$ and $(F,\leq_F)$.

A ***heap*** E over (V,ζ) is a pre-heap over (V,ζ) such that: if $(a,m) \in E$ with $m > 0$ then there exists $(b,m-1) \in E$ such that $a\zeta b$. Every isomorphism class of pre-heaps contains exactly one heap, and this is the unique pre-heap E in the class for which $\sum_{(a,m)\in E} h(a,m)$ is minimal. Observe that, for any $\alpha_i \in V$, $\{(\alpha_i, 1)\}$ is also a heap, which we shall denote by α_i.

Remark 5.8.1. The graph G can have a countably infinite number of vertices, but each heap E over the graph G has only a finite number of pieces by definition.

Let $\mathcal{H}(V,\zeta)$ denote the set of all heaps over (V,ζ). Let $|E|$ denote the number of pieces in E and for $a \in V$, $|E|_a$ denote the number of pieces of E in the position a. Define a map

$$\mathcal{H}(V,\zeta) \longrightarrow \mathbb{Z}_+^I$$

by $E \mapsto (k_i)_{i\in I}$ where k_i is the number of pieces of E at position α_i, i.e.,

$$k_i = |\{(a,m) \in E : \pi(a,m) = \alpha_i\}| = |E|_{\alpha_i}$$

This defines a **natural** $\mathbb{Z}_+^I$**-gradation** on the set $\mathcal{H}(V,\zeta)$ of all heaps. Let $\mathcal{H}_{\mathbf{k}}(V,\zeta)$ be the set of all heaps of grade $\mathbf{k}$ for $\mathbf{k} = (k_i)_{i\in I} \in \mathbb{Z}_+^I$.

Define the superposition $E \circ F$, of F over E, as heap F 'falls' over E. $\mathcal{H}(V,\zeta)$ is a monoid with product as **superposition** of heaps. Define a map

$$\psi : V^* \to \mathcal{H}(V,\zeta)$$

by, $\psi(p_1 p_2 \cdots p_k) = p_1 \circ p_2 \circ \cdots \circ p_k$. Observe that $\psi^{-1}(E)$ is the set of all linear orders compatible with $\leq_E$. Since $M(V,G,\Psi) = V^*/\sim$ where $a \sim b$ means $(a,b) \notin E(G)$, it is clear that ψ extends to grade and order-preserving isomorphism of the monoids $M(V,G,\Psi)$ and $\mathcal{H}(V,\zeta)$. This defines a total order on $\mathcal{H}(V,\zeta)$. It also defines a $\mathbb{Z}_2$-grading, $\mathcal{H}(V,\zeta) = \mathcal{H}_0(V,\zeta) \oplus \mathcal{H}_1(V,\zeta)$ where $\mathcal{H}_0(V,\zeta)$ (resp. $\mathcal{H}_1(V,\zeta)$) consists of all those heaps in which the number of pieces coming from V_1 is even (resp. odd). The **standard word** of a heap E is defined to be $\mathrm{st}(E) = \mathrm{st}(\psi^{-1}(E))$ [c.f. Equation (5.7.1)].

5.8.2 Pyramids and Lyndon heaps

For a heap E, $\min E$ is the heap composed of minimal pieces of E with respect to $\leq_E$, i.e., set of pieces of heap E of level 1. A heap E such that $\min E = \{a\}$ is said to be a ***pyramid*** with the basis a.

A heap E is said to be ***periodic*** if there exists a heap $F \neq 0$ (0 - empty heap) and an integer $k \geq 2$ such that $E = F^k$. Similarly, E is ***primitive*** if $E = U \circ V = V \circ U$ implies either $U = 0$ or $V = 0$. If the minimum piece of pyramid has the lowest position (with respect to the total order on I) then such a pyramid is known as ***admissible pyramid***.

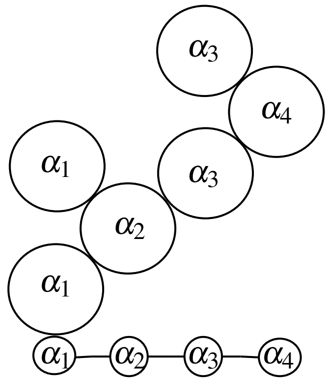
A pyramid E with the basis $\{p\}$ such that $|E|_p = 1$ is said to an **elementary pyramid**. An admissible pyramid that is also elementary is known as a ***super-letter***. The set of all super-letters in $\mathcal{H}(V,\zeta)$ is denoted by $\mathcal{A}(V,\zeta)$.

Let E be a heap. If $E = U \circ V$ for some heaps U and V, we say that $V \circ U$ is a ***transpose*** of E. The transitive closure of transposition is an equivalence relation on $\mathcal{H}(V,\zeta)$, which we call the conjugacy relation of heaps and is denoted by $\sim_c$.

A non-empty heap E is said to be ***Lyndon*** if E is primitive and minimal in its conjugacy class. Let $\mathcal{LH}(V,\zeta)$ denote the set of all Lyndon heaps over the super graph (G,Ψ).

The following diagram is an example of Lyndon heap. All the diagrams of heaps and Lyndon heaps in this chapter has the following assumptions: Let G be path graph on vertex set $V = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ and $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$.

Example 5.8.2. A Lyndon heap over the path graph on 4 vertices.



This is example of Lyndon heap $E = \alpha_1\alpha_2\alpha_3\alpha_1\alpha_4\alpha_3$.

Using Lyndon heaps, there is a Lyndon heaps basis defined for free partially commutative Lie algebras, we will generalize this notion to super Lyndon heaps and construct a basis of free partially commutative Lie superalgbera. Then we identify the free root spaces of a BKM superalgebra with the grade spaces of free partially commutative Lie superalgebra. This identification helps to construct one type of basis, known as the Lyndon heaps basis, for the free root spaces of a BKM superalgebra $\mathfrak{L}$. $\square$

Chapter 6

Basis of free root spaces

This chapter's objective is to introduce two types of free root space basis, namely Lyndon heaps basis and Lyndon Left Normed (LLN) basis. Throughout this chapter, we fix a tuple $\mathbf{k} = (k_i : i \in I) \in \mathbb{Z}_+^I$ such that $k_i \leq 1$ for $i \in I^{re} \sqcup \Psi_0$. Set, $\eta(\mathbf{k}) = \sum k_i \alpha_i$ and $\mathrm{wt}(\eta(\mathbf{k})) = \mathbf{k}$.

6.1 Main Result I: Lyndon heaps basis of free root spaces

The aim of this section is to identify the free root spaces of a BKM superalgebra with the grade spaces of free partially commutative Lie superalgebra. This identification helps to construct the Lyndon basis for the free root spaces of a BKM superalgebra $\mathfrak{L}$.

We begin by recalling the definition of standard factorization of Lyndon heaps and the Lyndon heaps basis of Lalonde from [43].

Definition 6.1.1. If E is a Lyndon heap then the standard factorization $\Sigma(E)$ of E is given by $\Sigma(E) = (F, N)$, where

1. $F \neq 0$ (empty heap)

2. $E = F \circ N$

3. N is Lyndon

4. N is minimal in the total order on $\mathcal{H}(V,\zeta)$.

We associate a Lie monomial $\Lambda(E)$ in $\mathcal{L}(G)$ corresponding to each Lyndon heap $E \in \mathcal{H}(V,\zeta)$ in the following way.

$$\Lambda(E) = \begin{cases} \alpha_i & \text{if } E = \alpha_i \in V \\ [\Lambda(F_1),\Lambda(F_2)] & \text{if } \Sigma(E) = (F_1,F_2) \end{cases}$$

The following theorem gives the Lyndon basis of the free partially commutative Lie algebra $\mathcal{L}(G)$.

Theorem 6.1.2. *[43] The set $\{\Lambda(E) : E \in \mathcal{H}(V,\zeta)$ is a Lyndon heap$\}$ forms a basis of $\mathcal{L}(G)$.*

6.1.1 The identification of the spaces $\mathfrak{L}_{\eta(\mathbf{k})}$ and $\mathcal{LS}_{\mathbf{k}}(G)$

Let $\mathcal{LS}(G,\Psi)$ be the free partially commutative Lie superalgebra associated with the supergraph (G,Ψ). The following lemma establishes a natural vector space isomorphism between the root space $\mathfrak{L}_{\eta(\mathbf{k})}$ of $\mathfrak{L}$ and the grade space $\mathcal{LS}_{\mathbf{k}}(G,\Psi)$ of $\mathcal{LS}(G,\Psi)$.

Lemma 6.1.3. *Let $\eta(\mathbf{k}) = \sum k_i\alpha_i \in \Delta_+$ with $\mathrm{wt}(\eta(\mathbf{k})) = \mathbf{k} = (k_i : i \in I) \in \mathbb{Z}_+^I$ such that $k_i \leq 1$ for $i \in I^{re} \sqcup \Psi_0$. Then*

(i) The root space $\mathfrak{L}_{\eta(\mathbf{k})}$ can be identified with the grade space $\mathcal{LS}_{\mathbf{k}}(G)$. In particular, $\dim \mathfrak{L}_{\eta(\mathbf{k})} = \dim \mathcal{LS}_{\mathbf{k}}(G,\Psi)$.

(ii) The root space $\mathfrak{L}_{\eta(\mathbf{k})}$ is independent of the Serre relations.

Proof. The positive part $\mathfrak{n}_+$ of $\mathfrak{L}$ can be written as $(\bigoplus_{\substack{\alpha\in\Delta_+ \\ \text{free}}} \mathfrak{L}_\alpha)\bigoplus(\bigoplus_{\substack{\alpha\in\Delta_+ \\ \text{non-free}}} \mathfrak{L}_\alpha)$. Using the defining relation (ix) of $\mathfrak{L}$, there exist a natural grade preserving surjective map

$$\Upsilon : \mathcal{LS}(G,\Psi) \twoheadrightarrow \mathfrak{n}_+$$

given by $\alpha_i \mapsto e_i$. Using the defining relations (vi)-(viii) of $\mathfrak{L}$, we see that the kernel of this map is generated by the elements

$$(\text{ad } \alpha_i)^{1-a_{ij}}\alpha_j \text{ if } i \in I^{re} \text{ and } i \neq j,$$

$$(\text{ad } \alpha_i)^{1-\frac{a_{ij}}{2}}\alpha_j \text{ if } i \in \Psi^{re} \text{ and } i \neq j, \text{ and}$$

$$(\text{ad}\alpha_i)^{1-\frac{a_{ij}}{2}}\alpha_j \text{ if } i \in \Psi_0 \text{ and } i = j$$

of $\mathcal{LS}(G,\Psi)$. Observe that in all these elements, some α_i's (corresponding to a real simple root or an odd simple root of norm zero) occur at least twice. Since Υ preserves the grading, the grade space $\mathcal{LS}_{\mathbf{k}}(G,\Psi)$ is injectively mapped onto the free root space $\mathfrak{L}_{\eta(\mathbf{k})}$ of $\mathfrak{L}$ by our assumption on $\mathbf{k}$. This completes the proof. □

6.1.2 Super Lyndon heaps and the standard factorization

We have seen in Theorem 6.1.2 that the Lyndon heaps in $\mathcal{H}(V,\zeta)$ parameterizes a basis for free partially commutative Lie algebra $\mathcal{L}(G)$. Now we generalize this result to free partially commutative Lie superalgebras by introducing the notion of super Lyndon heaps.

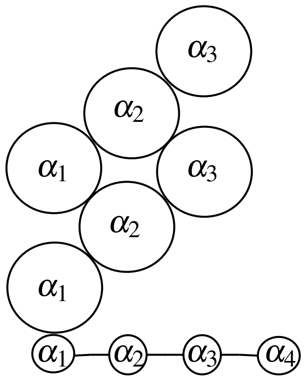
Definition 6.1.4. A heap $E \in \mathcal{H}(V,\zeta) = \mathcal{H}_0(V,\zeta)\oplus\mathcal{H}_1(V,\zeta)$ is said to be a **super Lyndon heap** if E satisfies one of the following conditions:

- E is a Lyndon heap.

- $E = F \circ F$ where $F \in \mathcal{H}_1(V,\zeta)$ is Lyndon.

Let $\mathcal{SLH}(V,\zeta)$ be the set of all super Lyndon heaps over the supergraph (G,Ψ).

Example 6.1.5. A super Lyndon heap over the path graph on 4 vertices with $I_1 = \{\alpha_1,\alpha_2,\alpha_3\}$ and $I_0 = \{\alpha_4\}$ is the following.



This is an example of a super Lyndon heap $E = \alpha_1\alpha_2\alpha_3\alpha_1\alpha_2\alpha_3$ with $F = \alpha_1\alpha_2\alpha_3$, a Lyndon heap in $\mathcal{H}_1(V,\zeta)$.

Definition 6.1.6. Let E be a super Lyndon heap in $\mathcal{H}(V,\zeta)$. If $E = F \circ F$ where F is a Lyndon heap in $\mathcal{H}_1(V,\zeta)$ then $\Sigma(E) = (F,F)$ is the standard factorization of E. We associate a Lie word $\Lambda(E)$ in $\mathcal{LS}(G)$ corresponding to each super Lyndon heap $E \in \mathcal{H}(V,\zeta)$ as follows

$$\Lambda(E) = \begin{cases} \alpha_i & \text{if } E = \alpha_i \in V \\ [\Lambda(F_1),\Lambda(F_2)] & \text{if } \Sigma(E) = (F_1,F_2) \end{cases}.$$

6.1.3 Basis of free partially commutative Lie superalgebras

The following theorem is the main result of this section. Here we construct the Lyndon heaps basis for free partially commutative Lie superalgebras.

Theorem 6.1.7. *The set* $\{\Lambda(E) : E \in \mathcal{H}(V,\zeta)$ *is super Lyndon*$\}$ *forms a basis of* $\mathcal{LS}(G,\Psi)$.

The rest of the section is dedicated to the proof of the above theorem. The proof of the following lemma is immediate.

Lemma 6.1.8. *Let $E \in \mathcal{SLH}_{\mathbf{k}}(V,\zeta)$. Then $\Lambda(E) = \sum_{F \in \mathcal{SLH}_{\mathbf{k}}(V,\zeta)} \alpha_F F$ where $\alpha_F \in \mathbb{Z}$. Since there are finite number of heaps of degree $\mathbf{k}$, the sum is a finite sum.*

Proposition 6.1.9. *The set $\mathcal{H}(V,\zeta)$ parameterizes a basis for the universal enveloping algebra of the free partially commutative Lie superalgebra $\mathcal{LS}(G,\Psi)$.*

Proof. Let $\mathfrak{U}$ be the $\mathbb{C}$-span of the heaps monoid $\mathcal{H}(V,\zeta)$ associated with the supergraph (G,Ψ). Then $\mathfrak{U}$ has an algebra structure induced from the multiplication in $\mathcal{H}(V,\zeta)$. This is the free partially commutative superalgebra associated with the supergraph (G,Ψ). Since it is the smallest associative superalgebra containing $\mathcal{LS}(G)$, $\mathfrak{U}$ is the universal enveloping algebra of the Lie superalgebra $\mathcal{LS}(G,\Psi)$. □

Proposition 6.1.10. *Let L be a super Lyndon heap of weight $\mathbf{k}$ over the supergraph (G,Ψ). Put $\Lambda(L) = \sum_{E \in \mathcal{SLH}_{\mathbf{k}}(V,\zeta)} \alpha_E E$. Then*

(i) $\alpha_L = 1$ if L is a Lyndon heap

(ii) $\alpha_L = 2$ if $L = L_1 \circ L_1$, L_1 is Lyndon heap in $\mathcal{H}_1(V,\zeta)$

(iii) If $\alpha_E \neq 0$ then $E \geq L$.

Proof. If E is a Lyndon heap, then part(i) and (iii) follow from [43, Theorem 4.2].

(ii) Let $L = L_1 \circ L_1$ where L_1 is Lyndon heap in $\mathcal{H}_1(V,\zeta)$. We have,

$$\begin{aligned}
\Lambda(L) &= [\Lambda(L_1), \Lambda(L_1)] \\
&= [\sum_{E \geq L_1} \alpha_E E, \sum_{E' \geq L_1} \alpha_{E'} E'] \text{ (Using part(i) and (iii)) for Lyndon heaps)} \\
&= [L_1, L_1] + \sum_{\substack{E > L_1 \\ E' > L_1}} \alpha_E \alpha_{E'} [E, E'] + \sum_{E > L_1} \alpha_E [E, L_1] + \sum_{E' > L_1} \alpha_{E'} [L_1, E'] \\
&= 2L + \textstyle\sum_{K > L} \alpha_K K.
\end{aligned}$$

This proves (ii).

(iii) Since, $st(E\circ E^{'})\geq st(E)\cdot st(E^{'})>st(L_1)\cdot st(L_1)=: st(L_1\circ L_1)$, this implies, $E\circ E^{'}>L$. Similarly, we have $E^{'}\circ E>L$. Also, $E\circ L_1>L,\ L_1\circ E>L$. Hence (iii) follows.

□

Corollary 6.1.11. *The set $\mathcal{B}=\{\Lambda(L): L$ is a super Lyndon heap$\}$ is linearly independent in $\mathcal{LS}(G,\Psi)$.*

Proof. Assume that

$$\sum_{L\in\mathcal{B}}\beta_L\Lambda(L)=0,\quad \beta_L\in\mathbb{C}$$

where all but finitely many β_L are zero. Then by the Proposition 6.1.10, we have

$$\sum_{L\in\mathcal{B}}\beta_L\Big(\sum_{\substack{E\geq L\\ \mathrm{wt}(E)=\mathrm{wt}(L)}}\alpha_E E\Big)=0$$

$$\sum_{L\in\mathcal{B}}\beta_L\left(\alpha_L L+\sum_{\substack{E>L\\ \mathrm{wt}(E)=\mathrm{wt}(L)}}\alpha_E E\right)=0 \text{ where } \alpha_L=\begin{cases}1, & \text{if } L\in\mathcal{LH}(V,\zeta)\\ 2, & \text{if } L=E\circ E,\ E\in\mathcal{H}_1(V,\zeta)\end{cases}$$

This implies,

$$\sum_{L\in\mathcal{LH}(V,\zeta)}\beta_L L+2\sum_{\substack{L=E\circ E\\ E\in\mathcal{LH}_1(V,\zeta)}}\beta_L L+\sum_{L\in\mathcal{B}}\sum_{\substack{E>L\\ \mathrm{wt}(E)=\mathrm{wt}(L)}}\beta_L\alpha_E E=0$$

Since $\mathcal{LS}$ is graded space, for each $\mathbf{k}$ in the grade space $\mathcal{LS}_{\mathbf{k}}(G,\Psi)$, we get,

$$\begin{cases}\displaystyle\sum_{L\in\mathcal{LH}_{\mathbf{k}}(V,\zeta)}\beta_L L+\sum_{\substack{E>L\\ \mathrm{wt}(E)=\mathrm{wt}(L)\\ E\in\mathcal{SLH}_{\mathbf{k}}(V,\zeta)}}\beta_L\alpha_E E=0, & \text{if } k_i \text{ is odd for some } i\in\mathrm{supp}(\mathbf{k})\\ \displaystyle\sum_{L\in\mathcal{LH}_{\mathbf{k}}(V,\zeta)}\beta_L L+2\sum_{\substack{L\in\mathcal{LH}_{\mathbf{k}}(V,\zeta)\\ L=E\circ E\\ E\in\mathcal{LH}_1(V,\zeta)}}\beta_L L+\sum_{\substack{E>L\\ \mathrm{wt}(E)=\mathrm{wt}(L)\\ L\in\mathcal{SLH}_{\mathbf{k}}(V,\zeta)}}\beta_L\alpha_E E=0, & \text{otherwise.}\end{cases}$$

Since heaps form a basis of $\mathfrak{U} = \mathbb{C}(\mathcal{H}(V,\zeta))$ it follows that $\beta_L = 0$ for all $L \in \mathcal{B}$ in the above equations. This completes the proof. $\square$

Proposition 6.1.12. *Let L and M be super Lyndon heaps such that $L < M$. Then*

$$[\Lambda(L),\Lambda(M)] = \sum_{\substack{N\in\mathcal{SLH}(V,\zeta)\\ N<M\\ deg(N)=deg(L\circ M)}} \alpha_N \Lambda(N).$$

Proof. We prove the result by considering the three different cases.

Case (i):- Suppose L, M are Lyndon heaps satisfying $L < M$, then result follows from [43, Theorem 4.4].

Case (ii):- Suppose exactly one of L, M is a super Lyndon heap. Without loss of generality, assume that $L = L_1 \circ L_1$ where L_1 is Lyndon heap in $\mathcal{H}_1(V,\zeta)$ and M is an arbitrary Lyndon heap. Now,

$$\begin{aligned}
[\Lambda(L),\Lambda(M)] &= [[\Lambda(L_1),\Lambda(L_1)],\Lambda(M)] \\
&= 2[\Lambda(L_1),[\Lambda(L_1),\Lambda(M)]] \\
&= 2[\Lambda(L_1), \sum_{\substack{N_1\in\mathcal{LH}(V,\zeta)\\ N_1<M\\ deg(N_1)=deg(L_1\circ M)}} \alpha_{N_1}\Lambda(N_1)] \quad (\because L_1 < L_1 \circ L_1 = L < M) \\
&= 2 \sum_{\substack{N_1\in\mathcal{LH}(V,\zeta)\\ N_1<M\\ deg(N_1)=deg(L_1\circ M)}} \alpha_{N_1}[\Lambda(L_1),\Lambda(N_1)] \\
&= 2\left(\sum_{\substack{N_1\in\mathcal{LH}(V,\zeta)\\ L_1<N_1<M\\ deg(N_1)=deg(L_1\circ M)}} \alpha_{N_1}[\Lambda(L_1),\Lambda(N_1)] + \sum_{\substack{N_1\in\mathcal{LH}(V,\zeta)\\ N_1<L_1\\ deg(N_1)=deg(L_1\circ M)}} \alpha_{N_1}[\Lambda(L_1),\Lambda(N_1)] \right)
\end{aligned}$$

Using case (i) in the first term of the above equation, we get,

$$\sum_{\substack{N_1\in\mathcal{LH}(V,\zeta)\\ L_1<N_1<M\\ deg(N_1)=deg(L_1\circ M)}} \alpha_{N_1}[\Lambda(L_1),\Lambda(N_1)] = \sum_{\substack{N_1\in\mathcal{LH}(V,\zeta)\\ N_1<M\\ deg(N_1)=deg(L_1\circ M)}} \alpha_{N_1}\left(\sum_{\substack{N_2\in\mathcal{LH}(V,\zeta)\\ N_2<N_1<M\\ deg(N_2)=deg(L_1\circ N_1)=deg(L\circ M)}} \alpha_{N_2}\Lambda(N_2)\right)$$

$$= \sum_{\substack{N_2\in\mathcal{LH}(V,\zeta)\\ N_2<M\\ deg(N_2)=deg(LoM)}} \underbrace{\left(\sum_{\substack{N'\in\mathcal{LH}(V,\zeta)\\ N_2<N'<M\\ deg(N')=deg(L_1\circ M)}} \alpha_{N'}\right)}_{\text{some constant } c_{N_2}} \alpha_{N_2}\Lambda(N_2)$$

$$= \sum_{\substack{N_2\in\mathcal{LH}(V,\zeta)\\ N_2<M\\ deg(N_2)=deg(L\circ M)}} (c_{N_2}\alpha_{N_2})\Lambda(N_2)$$

For the second summation, $[\Lambda(L_1),\Lambda(N_1)] = -(-1)^{a_{N_1}b_{L_1}}[\Lambda(N_1),\Lambda(L_1)]$ where $a_{N_1}, b_{L_1} \in \{0,1\}$ according to $N_1, L_1 \in \mathcal{H}_i(V,\zeta)$ for $i \in \{0,1\}$. Therefore,

$$[\Lambda(L_1),\Lambda(N_1)] = -(-1)^{a_{N_1}b_{L_1}} \sum_{\substack{K\in\mathcal{LH}(V,\zeta)\\ K<L_1<M\\ deg(K)=deg(N_1\circ L_1)=deg(L\circ M)}} \alpha_K\Lambda(K)$$

$$\sum_{\substack{N_1\in\mathcal{LH}(V,\zeta)\\ N_1<M\\ deg(N_1)=deg(L_1\circ M)\\ L_1>N_1}} \alpha_{N_1}[\Lambda(L_1),\Lambda(N_1)] = -\sum_{\substack{N_1\in\mathcal{LH}(V,\zeta)\\ N_1<M\\ deg(N_1)=deg(L_1\circ M)}} \alpha_{N_1}\left((-1)^{a_{N_1}b_{L_1}} \sum_{\substack{K\in\mathcal{LH}(V,\zeta)\\ K<L_1<M\\ deg(K)=deg(L\circ M)}} \alpha_K\Lambda(K)\right)$$

$$= -\sum_{\substack{K\in\mathcal{LH}(V,\zeta)\\ K<M\\ deg(K)=deg(L\circ M)}} \underbrace{\left(\sum_{\substack{N'\in\mathcal{LH}(V,\zeta)\\ K<N'<M\\ deg(N')=deg(L_1\circ M)}} (-1)^{a_{N'}b_{L_1}}\alpha_{N'}\right)}_{\text{constant } c_K} \alpha_K\Lambda(K)$$

$$= \sum_{\substack{K\in\mathcal{LH}(V,\zeta)\\ K<M\\ deg(K)=deg(L\circ M)}} \alpha'_K\Lambda(K) \text{ where } \alpha'_K = -c_K\alpha_K.$$

Thus,

$$[\Lambda(L),\Lambda(M)] = 2\Big(\sum_{\substack{N_2\in\mathcal{LH}(V,\zeta)\\ N_2<M\\ deg(N_2)=deg(L\circ M)}} (c_{N_2}\alpha_{N_2})\Lambda(N_2) + \sum_{\substack{K\in\mathcal{LH}(V,\zeta)\\ K<M\\ deg(K)=deg(L\circ M)}} \alpha'_K\Lambda(K)\Big).$$

Case (iii):- Suppose $L = L_1\circ L_1,\ M = M_1\circ M_1$ where L_1, M_1 are Lyndon heaps in $\mathcal{H}_1(V,\zeta)$ satisfying $L<M$. Then

$$\begin{aligned}
[\Lambda(L),\Lambda(M)] &= [[\Lambda(L_1),\Lambda(L_1)],\Lambda(M)]\\
&= 2[\Lambda(L_1),[\Lambda(L_1),\Lambda(M)]]\\
&= 2[\Lambda(L_1), \sum_{\substack{N\in\mathcal{LH}(V,\zeta)\\ N<M\\ deg(N)=deg(L_1\circ M)}} \alpha_N\Lambda(N)] \text{ (by the previous case)}\\
&= 2 \sum_{\substack{N\in\mathcal{LH}(V,\zeta)\\ N<M\\ deg(N)=deg(L_1\circ M)}} \alpha_N[\Lambda(L_1),\Lambda(N)]\\
&= 2\Big(\sum_{\substack{N\in\mathcal{LH}(V,\zeta)\\ L_1<N<M\\ deg(N)=deg(L_1\circ M)}} \alpha_N[\Lambda(L_1),\Lambda(N)] + \sum_{\substack{N\in\mathcal{LH}(V,\zeta)\\ N<M \text{ and } L_1>N\\ deg(N)=deg(L_1\circ M)}} \alpha_N[\Lambda(L_1),\Lambda(N)]\Big)
\end{aligned}$$

For those $N\in\mathcal{LH}(V,\zeta)$ such that, $L_1<N$ then by first case,

$$\begin{aligned}
[\Lambda(L_1),\Lambda(N)] &= \sum_{\substack{K\in\mathcal{LH}(V,\zeta)\\ K<N<M\\ deg(K)=deg(L_1\circ N)}} \beta_K\Lambda(K)\\
\Rightarrow \sum_{\substack{N\in\mathcal{LH}(V,\zeta)\\ N<M\\ deg(N)=deg(L_1\circ M)\\ L_1<N}} \alpha_N[\Lambda(L_1),\Lambda(N)] &= \sum_{\substack{N\in\mathcal{LH}(V,\zeta)\\ N<M\\ deg(N)=deg(L_1\circ M)}} \alpha_N \left(\sum_{\substack{K\in\mathcal{LH}(V,\zeta)\\ K<M\\ deg(K)=deg(L\circ M)}} \beta_K\Lambda(K)\right)\\
&= \sum_{\substack{K\in\mathcal{LH}(V,\zeta)\\ K<M\\ deg(K)=deg(L\circ M)}} \alpha'_K\Lambda(K)
\end{aligned}$$

For $N \in \mathcal{LH}(V,\zeta)$ such that $L_1 > N$ we have,

$$[\Lambda(L_1), \Lambda(N)] = -[\Lambda(N), \Lambda(L_1)] = -\sum_{\substack{N \in \mathcal{LH}(V,\zeta) \\ K_2 < L_1 \\ deg(K_2) = deg(N \circ L_1)}} \beta_{K_2} \Lambda(K_2)$$

$$\sum_{\substack{N \in \mathcal{LH}(V,\zeta) \\ N < M \\ deg(N) = deg(L_1 \circ M) \\ L_1 > N}} \alpha_N [\Lambda(L_1), \Lambda(N)] = \sum_{\substack{N \in \mathcal{LH}(V,\zeta) \\ N < M \\ deg(N) = deg(L_1 \circ M)}} \alpha_N \Bigg(\sum_{\substack{K_2 \in \mathcal{LH}(V,\zeta) \\ K < L_1 < M \\ deg(K_2) = deg(L \circ M)}} \beta_{K_2} \Lambda(K_2) \Bigg)$$

$$= \sum_{\substack{K_2 \in \mathcal{LH}(V,\zeta) \\ K_2 < M \\ deg(K_2) = deg(L \circ M)}} \alpha'_{K_2} \Lambda(K_2)$$

$$[\Lambda(L), \Lambda(M)] = 2\Bigg(\sum_{\substack{K \in \mathcal{LH}(V,\zeta) \\ K < M \\ deg(K) = deg(L \circ M)}} \alpha'_K \Lambda(K) + \sum_{\substack{K_2 \in \mathcal{LH}(V,\zeta) \\ K_2 < M \\ deg(K_2) = deg(L \circ M)}} \alpha'_{K_2} \Lambda(K_2) \Bigg)$$

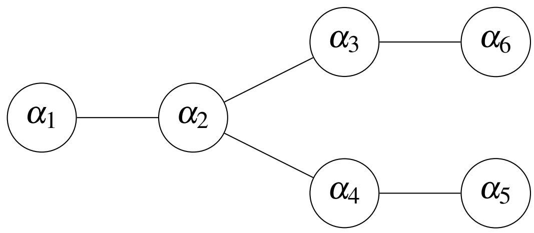
This completes the proof. □

By the above proposition, the Lie subsuperalgebra generated by $\mathcal{B} = \{\Lambda(L) : L$ is super Lyndon heap$\}$ in $\mathcal{LS}(G,\Psi)$ contains V. So this subalgebra is equal to $\mathcal{LS}(G,\Psi)$. This completes the proof of Theorem 6.1.7 and in turn, gives the Lyndon basis for the free roots spaces of BKM superalgebra $\mathfrak{L}$ whose associated supergraph is (G,Ψ) [c.f Lemma 6.1.3].

Example 6.1.13. Consider the BKM superalgebra $\mathfrak{L}$ associated with the BKM supermatrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -3 & -4 & -1 & 0 & 0 \\ 0 & -4 & -4 & 0 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 & -3 \end{bmatrix}.$$

The quasi-Dynkin diagram G of $\mathfrak{L}$ is as follows:



We have $I = \{1,2,3,4,5,6\}, \Psi = \{3,5\}$ and $I^{re} = \{1,4\}$. Assume the natural total order on I. Let $\mathbf{k} = (0,0,3,0,0,3) \in \mathbb{Z}_+[I]$. Then $\eta(\mathbf{k}) = 3\alpha_3 + 3\alpha_6 \in \Delta^1_+$. Fix $i = 3$ (minimal element in the support of $\mathbf{k}$), then the super Lyndon heaps of weight $\eta(\mathbf{k})$ are

$$\{\alpha_3\alpha_3\alpha_6\alpha_6\alpha_3\alpha_6,\ \ \alpha_3\alpha_3\alpha_3\alpha_6\alpha_6\alpha_6,\ \ \alpha_3\alpha_3\alpha_6\alpha_3\alpha_6\alpha_6\}$$

with standard factorization $\Sigma(\alpha_3\alpha_3\alpha_6\alpha_6\alpha_3\alpha_6) = (\alpha_3\alpha_3\alpha_6\alpha_6, \alpha_3\alpha_6),\ \Sigma(\alpha_3\alpha_3\alpha_3\alpha_6\alpha_6\alpha_6) = (\alpha_3, \alpha_3\alpha_3\alpha_6\alpha_6\alpha_6)$ and $\Sigma(\alpha_3\alpha_3\alpha_6\alpha_3\alpha_6\alpha_6) = (\alpha_3, \alpha_3\alpha_6\alpha_3\alpha_6\alpha_6)$. The associated Lie monomials

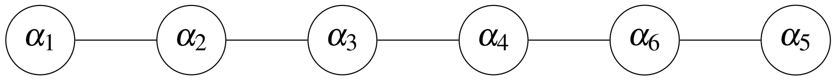
$$\Big\{[[e_3,[[e_3,e_6],e_6]],[e_3,e_6]],\quad [e_3,[e_3,[[[e_3,e_6],e_6],e_6]]],\quad [e_3,[[e_3,e_6],[[e_3,e_6],e_6]]]\Big\}$$

spans $\mathfrak{L}_{\eta(\mathbf{k})}$. We have $\dim \mathfrak{L}_{\eta(\mathbf{k})} = 3$ [c.f. Example 7.3.6]. So these Lie monomials form a basis for $\mathfrak{L}_{\eta(\mathbf{k})}$.

Example 6.1.14. Consider the BKM superalgebra $\mathfrak{L}$ associated with the BKM supermatrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -3 & -1 & 0 & 0 & 0 \\ 0 & -2 & -4 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & -1 & -3 \end{bmatrix}.$$

The quasi-Dynkin diagram G of $\mathfrak{L}$ is as follows:



We have $I = \{1,2,3,4,5,6\}$, $\Psi = \{3,5\}, I^{re} = \{1,4\}, \eta(\mathbf{k}) = 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$. Assume the natural total order on I. Fix $i = 3$, observe that the Lyndon heaps of weight $\eta(\mathbf{k})$ are $\{\alpha_3\alpha_3\alpha_4\alpha_5\alpha_5\alpha_6,\ \ \alpha_3\alpha_3\alpha_4\alpha_5\alpha_6\alpha_5\}$ with standard factorizations, $\Sigma(\alpha_3\alpha_3\alpha_4\alpha_5\alpha_6\alpha_5) = (\alpha_3, \alpha_3\alpha_4\alpha_5\alpha_6\alpha_5)$ and $\Sigma(\alpha_3\alpha_3\alpha_4\alpha_5\alpha_5\alpha_6) = (\alpha_3, \alpha_3\alpha_4\alpha_5\alpha_5\alpha_6)$. The associated Lie monomials are

$$\{[e_3,[e_3,[e_4,[[e_5,e_6],e_5]]]], \quad [e_3,[e_3,[e_4,[e_5,[e_5,e_6]]]]]\}$$

which form a spanning set of $\mathfrak{L}_{\eta(\mathbf{k})}$. Since $\dim \mathfrak{L}_{\eta(\mathbf{k})} = 2$ [c.f. Example 7.3.7]. These Lie monomials form a basis of $\mathfrak{L}_{\eta(\mathbf{k})}$.

6.2 Main result II: LLN basis of free root spaces

In this section, we extend the [3, Theorem 2] to the case of free root spaces of BKM Lie superalgebras $\mathfrak{L}$. In what follows in this section, we use the super-Jacobi identity (up to sign) to prove our results.

6.2.1 Initial alphabet and Left normed Lie word associated with a word

Given $[\tilde{\mathbf{w}}] \in M(V,G,\Psi)$ [c.f. Section 5.7] and $\mathbf{w} = b_1 \cdots b_r$ be an element in the class $[\tilde{\mathbf{w}}]$. We define,

$$\begin{aligned}
\text{length of } \mathbf{w} &= |\mathbf{w}| = r \\
|i(\mathbf{w})| &= |\{j : b_j = \alpha_i\}| \ \ \forall\, i \in I \\
\operatorname{supp}(\mathbf{w}) &= \{i \in I : |i(\mathbf{w})| \neq 0\} \\
\operatorname{wt}(\mathbf{w}) &= \sum_{i \in I} |i(\mathbf{w})|\alpha_i.
\end{aligned}$$

For $i \in I$, the initial multiplicity of α_i in $\mathbf{w}$ is defined to be the largest $k \geq 0$ for which there exists $\mathbf{u} \in M(V,G,\Psi)$ such that $\mathbf{w} = \alpha_i^k \mathbf{u}$. We define the ***initial alphabet*** $\mathrm{IA_m}(\mathbf{w})$ of $\mathbf{w}$ to be the multiset in which each $\alpha_i \in I$ occurs as many times as its initial multiplicity in $\mathbf{w}$. The underlying set is denoted by $\mathrm{IA}(\mathbf{w})$. The left normed Lie word associated with $\mathbf{w}$ is defined by

$$e(\mathbf{w}) = [[\cdots[[e_{b_1}, e_{b_2}], e_{b_3}]\cdots, e_{b_{r-1}}]e_{b_r}] \in \mathfrak{L}. \tag{6.2.1}$$

Using the Jacobi identity, it is easy to see that the association $\mathbf{w} \mapsto e(\mathbf{w})$ is well-defined and preserves the $\mathbb{Z}_2-$grading.

6.2.2 Lyndon words and their Standard factorization

For a fixed $i_0 \in I$ (which is assumed to be minimal in the total order on I), consider the set

$$\mathcal{X}_{i_0} = \{\mathbf{w} \in M(V,G,\Psi) : \mathrm{IA}_m(\mathbf{w}) = \{\alpha_{i_0}\} \text{ and } |i_0(\mathbf{w})| = 1\}.$$

Using (5.7.1) $\mathcal{X}_{i_0}$ (and hence $\mathcal{X}_{i_0}^*$) is totally ordered and $\mathbb{Z}_2-$graded. We denote by $FLS(\mathcal{X}_{i_0})$ the free Lie superalgebra generated by $\mathcal{X}_{i_0} = \mathcal{X}_{i_0,0} \sqcup \mathcal{X}_{i_0,1}$ where $\mathcal{X}_{i_0,0}$ (resp. $\mathcal{X}_{i_0,1}$) is the set of even (resp. odd) elements in $\mathcal{X}_{i_0}$.

Universal property of the free Lie superalgebra $FLS(\mathcal{X}_{i_0})$: Let $\mathfrak{l}$ be a Lie superalgebra and $\Phi : \mathcal{X}_{i_0} \to \mathfrak{l}$, a set map that preserves the $\mathbb{Z}_2$ grading. Then Φ can be extended to a Lie superalgebra homomorphism $\Phi : FLS(\mathcal{X}_{i_0}) \to \mathfrak{l}$.

Definition 6.2.1. A non–empty word $\mathbf{w} \in \mathcal{X}_{i_0}^*$ is called a **Lyndon word** if it satisfies one of the following equivalent definitions:

- $\mathbf{w}$ is strictly smaller than any of its proper cyclic rotations.
- $\mathbf{w} \in \mathcal{X}_{i_0}$ or $\mathbf{w} = \mathbf{uv}$ for Lyndon words $\mathbf{u}$ and $\mathbf{v}$ with $\mathbf{u} < \mathbf{v}$.

We say, $\mathbf{w} = \mathbf{uv}$ is a **standard factorization** of Lyndon word $\mathbf{w}$ when $\mathbf{u}, \mathbf{v}$ are Lyndon words such that $\mathbf{u} < \mathbf{v}$ and $\mathbf{v}$ is of maximal possible length satisfying this property. The standard factorization of $\mathbf{w}$ is denoted by $\sigma(\mathbf{w}) = (\mathbf{u}, \mathbf{v})$.

6.2.3 Super Lyndon words and their associated Lie word

Definition 6.2.2. A word $\mathbf{w} \in \mathcal{X}_{i_0}^*$ is said to be **super Lyndon** if $\mathbf{w}$ satisfies one of the following conditions [19]:

- $\mathbf{w}$ is a Lyndon word.
- $\mathbf{w} = \mathbf{uu}$ where $\mathbf{u} \in \mathcal{X}_{i_0,1}^*$ is Lyndon. In this case, we define $\mathbf{w} = \mathbf{uu}$ is the standard factorization of $\mathbf{w}$, i.e., $\sigma(\mathbf{w}) = (\mathbf{u}, \mathbf{u})$.

We will use Lyndon words (resp. super Lyndon words) to construct a basis for the Borcherds Lie algebras (resp. BKM Lie superalgebras). To each super Lyndon word $\mathbf{w} \in \mathcal{X}_{i_0}^*$, we associate a Lie word $L(\mathbf{w})$ in $FLS(\mathcal{X}_{i_0})$ as follows.

- If $\mathbf{w} \in \mathcal{X}_{i_0}$, then $L(\mathbf{w}) = \mathbf{w}$.
- If $\mathbf{w} = \mathbf{uv}$ is the standard factorization of $\mathbf{w}$, then $L(\mathbf{w}) = [L(\mathbf{u}), L(\mathbf{v})]$.

For more details about Lyndon words and super Lyndon words, see [20, 45, 52]. The following result can be seen in [20, 45] and the basis constructed there is known as the Lyndon basis for free Lie superalgebras.

Proposition 6.2.3. *The set $\{L(\mathbf{w}) : \mathbf{w} \in \mathcal{X}_{i_0}^*$ is super Lyndon$\}$ forms a basis of $FLS(\mathcal{X}_{i_0})$.*

Corollary 6.2.4. *If the set $\mathcal{X}_{i_0,1}$ is empty then $FLS(\mathcal{X}_{i_0})$ becomes the free Lie algebra $FL(\mathcal{X}_{i_0})$. In this case, $\{L(\mathbf{w}) : \mathbf{w} \in \mathcal{X}_{i_0}^*$ is Lyndon$\}$ forms a basis of $FL(\mathcal{X}_{i_0})$.*

6.2.4 LLN basis of BKM superalgebras

In this section, we construct another basis of free root spaces, which is known as Lyndon Left Normed (LLN) basis.

Define a map

$$\Phi : \mathcal{X}_{i_0} \to \mathfrak{L}, \quad \Phi(\mathbf{w}) = e(\mathbf{w})$$

where $e(\mathbf{w})$ is the left normed Lie word associated with $\mathbf{w}$. The map Φ preserves the $\mathbb{Z}_2$ grading. By the universal property, we have a Lie superalgebra homomorphism

$$\Phi : FLS(\mathcal{X}_{i_0}) \to \mathfrak{L}, \;\; \mathbf{w} \mapsto e(\mathbf{w}) \;\; \forall\, \mathbf{w} \in \mathcal{X}_{i_0}. \tag{6.2.2}$$

Since Φ preserves the Q_+-grading and $\mathfrak{L}$ can be infinite-dimensional, the map Φ need not be surjective. Let $\mathfrak{L}(i_0)$ be the image of the homomorphism Φ in $\mathfrak{L}$. Then $\mathfrak{L}(i_0)$ is Q_+-graded Lie sub-superalgebra of $\mathfrak{L}$ generated by $\{e(\mathbf{w}) : \mathbf{w} \in \mathcal{X}_{i_0}\}$. Observe that Φ maps any basis of the free Lie superalgebra $FLS(\mathcal{X}_{i_0})$ to $\mathfrak{L}(i_0)$ and the image spans $\mathfrak{L}(i_0)$. For any root $\eta(\mathbf{k}) = \sum k_i \alpha_i \in \mathbb{Q}_+$ satisfying $k_i \le 1$ for $i \in I^{re} \sqcup \Psi_0$, we construct a basis for the root space $\mathfrak{L}_{\eta(\mathbf{k})}$ from this spanning set. This is done by identifying the following combinatorial model from [3] with the set of super Lyndon heaps of weight $\mathbf{k}$.

$$C^{i_0}(\mathbf{k}, G) = \{\mathbf{w} \in \mathcal{X}_{i_0}^* : \mathbf{w} \text{ is a super Lyndon word, } \operatorname{wt}(\mathbf{w}) = \eta(\mathbf{k})\}, \;\; \iota(\mathbf{w}) := \Phi \circ L(\mathbf{w}). \tag{6.2.3}$$

This basis is known as **Lyndon Left Normed (LLN) basis**.

Theorem 6.2.5. *With the notations as defined above, the set* $\left\{\iota(\mathbf{w}) : \mathbf{w} \in C^{i_0}(\mathbf{k}, G)\right\}$ *is a basis of the root space* $\mathfrak{L}_{\eta(\mathbf{k})}$*. Moreover, if* $k_{i_0} = 1$*, the set* $\{e(\mathbf{w}) : \mathbf{w} \in \mathcal{X}_{i_0},\ \operatorname{wt}(\mathbf{w}) = \eta(\mathbf{k})\}$ *forms a left-normed basis of* $\mathfrak{L}_{\eta(\mathbf{k})}$*.*

We need the following lemmas in order to prove the theorem.

Lemma 6.2.6. *The root space* $\mathfrak{L}_{\eta(\mathbf{k})} = \mathfrak{L}(i_0)_{\eta(\mathbf{k})}$ *for* $\eta(\mathbf{k}) = \sum k_i\alpha_i \in \mathbb{Q}_+$ *satisfying* $k_i \leq 1$ *for* $i \in I^{re} \sqcup \Psi_0$.

Lemma 6.2.7. *With the notations as above, we have*

$$|C^{i_0}(\mathbf{k}, G)| = \dim FLS_{\mathbf{k}}(\mathcal{X}_{i_0}) = \dim \mathcal{LS}_{\mathbf{k}}(G, \Psi).$$

The proofs of these lemmas are postponed to the subsequent section. By assuming them, we prove the theorem.

Proof. By Lemma 6.2.6, $\mathfrak{L}_{\eta(\mathbf{k})} = \mathfrak{L}(i_0)_{\eta(\mathbf{k})}$. So, $\{\iota(\mathbf{w}) : \mathbf{w} \in C^{i_0}(\mathbf{k}, G)\}$ is a spanning set for $\mathfrak{L}_{\eta(\mathbf{k})}$ of cardinality equal to $|C^{i_0}(\mathbf{k}, G)|$. Now, Lemmas 6.2.7 and 6.1.3 show that $\{\iota(\mathbf{w}) : \mathbf{w} \in C^{i_0}(\mathbf{k}, G)\}$ is in fact a basis. □

We now show two examples to explain Theorem 6.2.5.

Example 6.2.8. From Example 6.1.13, consider the root space $\mathfrak{L}_{\eta(\mathbf{k})}$ where $\eta(\mathbf{k}) = 3\alpha_3 + 3\alpha_6$. Fix $i_0 = 3$. Super Lyndon words of weight $\eta(\mathbf{k})$ in $C^3(\mathbf{k}, G) = \{\mathbf{w} \in \mathcal{X}_3^* : \mathrm{wt}(\mathbf{w}) = \eta(\mathbf{k}), \mathbf{w} \text{ is super Lyndon}\}$ are $\{\alpha_3\alpha_3\alpha_6\alpha_6\alpha_3\alpha_6,\ \alpha_3\alpha_3\alpha_3\alpha_6\alpha_6\alpha_6,\ \alpha_3\alpha_3\alpha_6\alpha_3\alpha_6\alpha_6\}$ and corresponding Lie monomials are

$$\Big\{[[e_3,[[e_3,e_6],e_6]],[e_3,e_6]],[e_3,[e_3,[[[e_3,e_6],e_6],e_6]]],[e_3,[[e_3,e_6],[[e_3,e_6],e_6]]]\Big\}$$

which spans $\mathfrak{L}_{\eta(\mathbf{k})}$. But $\mathrm{mult}(\eta(\mathbf{k})) = 3$ [c.f. Example 7.3.6]. So, these Lie monomials form a basis for $\mathfrak{L}_{\eta(\mathbf{k})}$.

Example 6.2.9. From Example 6.1.14, consider the root space $\mathfrak{L}_{\eta(\mathbf{k})}$ where $\eta(\mathbf{k}) = 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$. Fix $i_0 = 3$, observe that

$$\mathcal{X}_3 = \{\alpha_3,\ \alpha_3\alpha_4,\ \alpha_3\alpha_4\alpha_5,\ \alpha_3\alpha_4\alpha_5\alpha_6,\ \alpha_3\alpha_4\alpha_4,\ \alpha_3\alpha_4\alpha_5\alpha_4\alpha_5,\ \cdots\}.$$

The only Lyndon words on $\mathcal{X}_3^*$ of weight $\eta(\mathbf{k})$ are $\{\alpha_3\alpha_3\alpha_4\alpha_5\alpha_5\alpha_6,\ \ \alpha_3\alpha_3\alpha_4\alpha_5\alpha_6\alpha_5\}$ with standard factorization, $\sigma(\alpha_3\alpha_3\alpha_4\alpha_5\alpha_5\alpha_6) = (\alpha_3, \alpha_3\alpha_4\alpha_5\alpha_5\alpha_6)$, and $\sigma(\alpha_3\alpha_3\alpha_4\alpha_5\alpha_6\alpha_5) = (\alpha_3, \alpha_3\alpha_4\alpha_5\alpha_6\alpha_5)$ respectively. Hence the corresponding Lie monomials

$$\Big\{[e_3,[[[[e_3,e_4],e_5],e_6],e_5]],[e_3,[[[[e_3,e_4],e_5],e_5],e_6]]\Big\}$$

spans $\mathfrak{L}_{\eta(\mathbf{k})}$. But $\text{mult}(\eta(\mathbf{k})) = 2$ [c.f. Example 7.3.7]. Hence, these Lie monomials form a basis for $\mathfrak{L}_{\eta(\mathbf{k})}$.

6.2.5 Proof of Lemma 6.2.6

We claim that $\mathfrak{L}_{\eta(\mathbf{k})} = \mathfrak{L}(i_0)_{\eta(\mathbf{k})}$. This is proved in multiple steps. First, we show that the left normed Lie words of weight $\mathbf{k}$ starting with a fixed $\alpha_{i_0} \in V$, for $i_0 \in I$, spans $\mathfrak{L}_{\eta(\mathbf{k})}$.

Lemma 6.2.10. *Fix an index $i \in I$. Then the root space $\mathfrak{L}_{\eta(\mathbf{k})}$ is spanned by the set of left normed Lie words* $\{e(\mathbf{w}) : \mathbf{w} \in \mathcal{X}_{i_0}^*, \text{wt}(\mathbf{w}) = \eta(\mathbf{k})\}$.

Proof. Observe, $\mathbf{w} \in \mathcal{X}_{i_0}^*$ if and only if $\text{IA}(\mathbf{w}) = \{\alpha_{i_0}\}$. Since, the set $\mathcal{B} = \{e(\mathbf{w}) : \mathbf{w} \in M_{\mathbf{k}}(V,G,\Psi)\}$ forms a spanning set for $\mathfrak{L}_{\eta(\mathbf{k})}$, to prove the required result, it is enough to show that each element of $\mathcal{B}$ can be written as a linear combination of left normed Lie words $e(\mathbf{w})$ satisfying $\text{wt}(\mathbf{w}) = \eta(\mathbf{k})$ and $\text{IA}(\mathbf{w}) = \{\alpha_{i_0}\}$. Let $\mathbf{w} = b_1 b_2 \cdots b_r \in M_{\mathbf{k}}(V,G,\Psi)$. Assume that $b_1 = \alpha_{i_0}$. If $|\text{IA}(\mathbf{w})| > 1$ then $e(\mathbf{w}) = 0$ and there is nothing to prove. If $|\text{IA}(\mathbf{w})| = 1$ then $\text{IA}(\mathbf{w}) = \{\alpha_{i_0}\}$ and the proof follows in this case. When $b_1 \neq \alpha_{i_0}$, consider the set $i(\mathbf{w}) = \{j : b_j = \alpha_{i_0}\}$. Assume $\min\{i(\mathbf{w})\} = p+1$ and set $\mathbf{w}' = b_1 b_2 \cdots b_p \alpha_{i_0}$. First, we claim,

$$
\begin{aligned}
& e(\mathbf{w}') \\
= \; & e(\alpha_{i_0} b_1 b_2 \cdots b_p) + \sum_{j_1=2}^{p} e(\alpha_{i_0} b_{j_1} b_1 b_2 \cdots \hat{b_{j_1}} \cdots b_p) \\
& + \sum_{1<j_2<j_1\le p} e(\alpha_{i_0} b_{j_1} b_{j_2} b_1 b_2 \cdots \hat{b_{j_2}} \cdots \hat{b_{j_1}} \cdots b_p) \\
& + \sum_{1<j_3<j_2<j_1\le p} e(\alpha_{i_0} b_{j_1} b_{j_2} b_{j_3} b_1 b_2 \cdots \hat{b_{j_3}} \cdots \hat{b_{j_2}} \cdots \hat{b_{j_1}} \cdots b_p) \\
& + \sum_{1<j_4<j_3<j_2<j_1\le p} e(\alpha_{i_0} b_{j_1} b_{j_2} b_{j_3} b_{j_4} b_1 b_2 \cdots \hat{b_{j_4}} \cdots \hat{b_{j_3}} \cdots \hat{b_{j_2}} \cdots \hat{b_{j_1}} \cdots b_p) + \cdots \\
& + \sum_{1<j_{p-2}<j_{p-3}<\cdots<j_2<j_1\le p} e(\alpha_{i_0} b_{j_1} b_{j_2} b_{j_3} \cdots b_{j_{p-3}} b_{j_{p-2}} b_1 b_2 \cdots \hat{b_{j_{p-2}}} \cdots \hat{b_{j_{p-3}}} \cdots \hat{b_{j_2}} \cdots \hat{b_{j_1}} \cdots b_p) \\
& + e(\alpha_{i_0} b_p b_{p-1} \cdots b_2 b_1)
\end{aligned}
$$

where $\hat{a}$ means the omission of the alphabet a in the expression. We prove the claim by apply induction on p. For $p = 1$,

$$\mathbf{w}' = b_1 \alpha_{i_0} \Rightarrow e(\mathbf{w}') = [e_{\alpha_{i_0}}, e_{b_1}] = e(\alpha_{i_0} b_1).$$

Assume that the result is true for $p = k$. Now, consider $p = k+1$

$$
\begin{aligned}
e(b_1 b_2 \ldots b_{k+1} \alpha_{i_0}) = \; & [[[[[[e_{b_1}, e_{b_2}], e_{b_3}]], e_{b_4}] \cdots, e_{b_k}], e_{b_{k+1}}], e_{\alpha_{i_0}}] \\
= \; & [[[[[e_{b_1'}, e_{b_3}] e_{b_4}], \cdots, e_{b_k}], e_{b_{k+1}}], e_{\alpha_{i_0}}]
\end{aligned}
\tag{6.2.4}
$$

by taking $[e_{b_1}, e_{b_2}] = e_{b_1'}, i.e., b_1' = [b_1, b_2]$. Using the induction hypothesis on the right-hand side of Equation 6.2.4, we get

$$
\begin{aligned}
& [[[[[[e_{b_1'}, e_{b_3}], e_{b_4}], \cdots, e_{b_k}], e_{b_{k+1}}], e_{\alpha_{i_0}}] = e(b_1' b_3 b_4 \ldots b_{k+1} \alpha_{i_0}) \\
= \; & e(\alpha_{i_0} b_1' b_3 \cdots b_{k+1}) + \sum_{3\le j\le k+1} e(\alpha_{i_0} b_j b_1' b_3 \cdots \hat{b_j} \cdots b_{k+1}) \\
& + \sum_{3\le j_2<j_1\le k+1} e(\alpha_{i_0} b_{j_1} b_{j_2} b_1' b_3 \cdots \hat{b_{j_2}} \cdots \hat{b_{j_1}} \cdots b_{k+1}) + \cdots \\
& + \sum_{3\le j_{k-1}<\cdots<j_2<j_1\le k+1} e(\alpha_{i_0} b_{j_1} b_{j_2} b_{j_3} \cdots b_{j_{k-1}} b_1' b_3 \cdots \hat{b_{j_{k-1}}} \cdots \hat{b_{j_2}} \cdots \hat{b_{j_1}} \cdots b_{k+1}) \\
& + e(\alpha_{i_0} b_{k+1} b_k \cdots b_1')
\end{aligned}
\tag{6.2.5}
$$

Now,

$$\begin{aligned}
&e(\alpha_{i_0}b_{j_1}b_{j_2}\cdots b_{j_t}b_1')\\
&=[\underbrace{[[[e_{\alpha_{i_0}},e_{b_{j_1}}],e_{b_{j_2}}],\cdots,e_{b_{j_t}}]}_{x},[\underbrace{e_{b_1}}_{y},\underbrace{e_{b_2}}_{z}]]\\
&=\underbrace{[[[[[e_{\alpha_{i_0}},e_{b_{j_1}}],e_{b_{j_2}}],\cdots,e_{b_{j_t}}],e_{b_1}],e_{b_2}]}_{[[x,y],z]}+\underbrace{[[[[[e_{\alpha_{i_0}},e_{b_{j_1}}],e_{b_{j_2}}],\cdots,e_{b_{j_t}}],e_{b_2}],e_{b_1}]}_{[[x,z],y]}\\
&=e(\alpha_{i_0}b_{j_1}b_{j_2}\cdots b_{j_t}b_1b_2)+e(\alpha_{i_0}b_{j_1}b_{j_2}\cdots b_{j_t}b_2b_1).
\end{aligned}$$

This implies,

$$\begin{aligned}
&e(\alpha_{i_0}b_{j_1}b_{j_2}\cdots b_{j_t}b_1'b_3\cdots b_{k+1})\\
&=[[\underbrace{[[[[e_{\alpha_{i_0}},e_{b_{j_1}}],e_{b_{j_2}}],\cdots e_{b_{j_t}}]e_{b_1'}]}_{e(\alpha_{i_0}b_{j_1}b_{j_2}\cdots b_{j_t}b_1')},e_{b_3}],\cdots e_{b_{k+1}}]=[[[e(\alpha_{i_0}b_{j_1}b_{j_2}\cdots b_{j_t}b_1'),e_{b_3}],\cdots,e_{b_{k+1}}]\\
&=[[[e(\alpha_{i_0}b_{j_1}b_{j_2}\cdots b_{j_t}b_1b_2)+e(\alpha_{i_0}b_{j_1}b_{j_2}\cdots b_{j_t}b_2b_1),e_{b_3}],\cdots,e_{b_{k+1}}]\\
&=[[e(\alpha_{i_0}b_{j_1}b_{j_2}..b_{j_t}b_1b_2),e_{b_3}],..,e_{b_{k+1}}]+[[e(ib_{j_1}b_{j_2}..b_{j_t}b_2b_1),e_{b_3}]..,e_{b_{k+1}}]\\
&=e(\alpha_{i_0}b_{j_1}b_{j_2}\cdots b_{j_t}b_1b_2b_3\cdots b_{k+1})+e(\alpha_{i_0}b_{j_1}b_{j_2}\cdots b_{j_t}b_2b_1b_3\cdots b_{k+1})
\end{aligned}\tag{6.2.6}$$

Using Equation 6.2.6 in Equation (6.2.5) we get,

$$\begin{aligned}
&[[[[[e_{b_1'},e_{b_3}]e_{b_4}],\cdots,e_{b_k}],e_{b_{k+1}}],e_{\alpha_{i_0}}]\\
=&e(\alpha_{i_0}b_1b_2b_3\cdots b_{k+1})+e(\alpha_{i_0}b_2b_1b_3\cdots b_{k+1})\\
&+\sum_{3\le j\le k+1}\big(e(\alpha_{i_0}b_jb_1b_2b_3\cdots\hat{b_j}\cdots b_{k+1})+e(\alpha_{i_0}b_jb_2b_1b_3\cdots\hat{b_j}\cdots b_{k+1})\big)\\
&+\sum_{3\le j_2<j_1\le k+1}\Big(e(\alpha_{i_0}b_{j_1}b_{j_2}b_1b_2b_3\cdots\hat{b_{j_2}}\cdots\hat{b_{j_1}}\cdots b_{k+1})+e(\alpha_{i_0}b_{j_1}b_{j_2}b_2b_1b_3\cdots\hat{b_{j_2}}\cdots\hat{b_{j_1}}\cdots b_{k+1})\Big)\\
&+\cdots+\sum_{3\le j_{k-1}<\cdots<j_2<j_1\le k+1}\Big(e(\alpha_{i_0}b_{j_1}b_{j_2}b_{j_3}\cdots b_{j_{k-1}}b_1b_2\cdots\hat{b_{j_{k-1}}}\cdots\hat{b_{j_2}}\cdots\hat{b_{j_1}}\cdots b_{k+1})\\
&+e(\alpha_{i_0}b_{j_1}b_{j_2}b_{j_3}\cdots b_{j_{k-1}}b_2b_1\cdots\hat{b_{j_{k-1}}}\cdots\hat{b_{j_2}}\cdots\hat{b_{j_1}}\cdots b_{k+1})\Big)\\
&+(e(\alpha_{i_0}b_{k+1}b_k\cdots b_1b_2)+e(\alpha_{i_0}b_{k+1}b_k\cdots b_2b_1))
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg(e(\alpha_{i_0} b_1 b_2 b_3 \cdots b_{k+1}) + e(\alpha_{i_0} b_2 b_1 b_3 \cdots b_{k+1}) + \sum_{3 \le j \le k+1} e(\alpha_{i_0} b_j b_1 b_2 b_3 \cdots \hat{b_j} \cdots b_{k+1}) \Bigg) \\
&\quad + \Bigg(\sum_{3 \le j \le k+1} e(\alpha_{i_0} b_j b_2 b_1 b_3 \cdots \hat{b_j} \cdots b_{k+1}) + \sum_{3 \le j_2 < j_1 \le k+1} e(\alpha_{i_0} b_{j_1} b_{j_2} b_1 b_2 b_3 \cdots \hat{b_{j_2}} \cdots \hat{b_{j_1}} \cdots b_{k+1}) \Bigg) \\
&\quad + \Big(\sum_{3 \le j_2 < j_1 \le k+1} e(\alpha_{i_0} b_{j_1} b_{j_2} b_2 b_1 b_3 \cdots \hat{b_{j_2}} \cdots \hat{b_{j_1}} \cdots b_{k+1}) \\
&\qquad\qquad + \sum_{3 \le j_3 < j_2 < j_1 \le k+1} e(\alpha_{i_0} b_{j_1} b_{j_2} b_{j_3} b_2 b_1 b_3 \cdots \hat{b_{j_3}} \cdots \hat{b_{j_2}} \cdots \hat{b_{j_1}} \cdots b_{k+1}) \Big) + \cdots + \\
&\quad + \Big(\sum_{3 \le j_{k-1} < \cdots < j_2 < j_1 \le k+1} e(\alpha_{i_0} b_{j_1} b_{j_2} b_{j_3} \cdots b_{j_{k-1}} b_2 b_1 \cdots \hat{b_{j_{k-1}}} \cdots \hat{b_{j_2}} \cdots \hat{b_{j_1}} \cdots b_{k+1}) \\
&\qquad\qquad + e(\alpha_{i_0} b_{k+1} b_k \cdots b_1 b_2) \Big) + e(\alpha_{i_0} b_{k+1} b_k \cdots b_2 b_1) \\
&= \sum_{j=1}^{k+1} e(\alpha_{i_0} b_j b_1 b_2 \cdots \hat{b_j} \cdots b_{k+1}) + \sum_{1 < j_2 < j_1 \le k+1} e(\alpha_{i_0} b_{j_1} b_{j_2} b_1 b_2 \cdots \hat{b_{j_2}} \cdots \hat{b_{j_1}} \cdots b_{k+1}) \\
&\quad + \sum_{1 < j_3 < j_2 < j_1 \le k+1} e(\alpha_{i_0} b_{j_1} b_{j_2} b_{j_3} b_1 b_2 \cdots \hat{b_{j_3}} \cdots \hat{b_{j_2}} \cdots \hat{b_{j_1}} \cdots b_{k+1}) + \cdots + \\
&\quad + \sum_{1 < j_{k-1} < \cdots < j_2 < j_1 \le k+1} e(\alpha_{i_0} b_{j_1} b_{j_2} b_{j_3} \cdots b_{j_{k-1}} b_1 b_2 \cdots \hat{b_{j_{k-1}}} \cdots \hat{b_{j_2}} \cdots \hat{b_{j_1}} \cdots b_{k+1}) \\
&\quad + e(\alpha_{i_0} b_{k+1} b_k \cdots b_2 b_1)
\end{aligned}
$$

Thus the result is true for p=k+1. This proves our claim. Hence we have,

$$
\begin{aligned}
e(\mathbf{w}') =& e(\alpha_{i_0} b_1 b_2 \cdots b_p) + \sum_{j_1=2}^{p} e(\mathbf{w}_{j_1}) + \sum_{1 < j_2 < j_1 \le p} e(\mathbf{w}_{j_1 j_2}) + \sum_{1 < j_3 < j_2 < j_1 \le p} e(\mathbf{w}_{j_1 j_2 j_3}) + \cdots + \\
&+ e(\mathbf{w}_{p(p-1)\cdots 2\cdot 1})
\end{aligned}
$$

where $\mathbf{w}_{j_1 j_2 \cdots j_\ell} = (\alpha_{i_0} b_{j_1} b_{j_2} \cdots b_{j_\ell} b_1 b_2 \cdots \hat{b_{j_\ell}} \cdots \hat{b_{j_{\ell-1}}} \cdots \hat{b_{j_1}} \cdots b_p)$. Observe that all the words $\mathbf{w}_{j_1,\cdots j_m}$ appearing in the summand of $e(\mathbf{w}')$ have the same weight. Further, they belong to $\mathcal{X}_{i_0}$, $e(\mathbf{w}_{j_1 j_2 \cdots j_\ell}) = 0$ whenever some b_{j_p} commutes with $\alpha_{i_0}, b_{j_1}, b_{j_2}, \cdots, b_{j_{p-1}}$. By the linearity of the brackets we have,

$$e(\mathbf{w}' \cdot b_{p+2}) = \sum_{j_1=1}^{p} e(\mathbf{w}_{j_1} \cdot b_{p+2}) + \sum_{1<j_2<j_1\leq p} e(\mathbf{w}_{j_1 j_2} \cdot b_{p+2}) + \sum_{1<j_3<j_2<j_1\leq p} e(\mathbf{w}_{j_1 j_2 j_3} \cdot b_{p+2})$$
$$+ \cdots + e(\mathbf{w}_{p(p-1)\cdots 2.1} \cdot b_{p+2})$$

Similarly, the remaining alphabets $b_{p+3}, \ldots, b_r$ can be added to the above expression. This gives,

$$e(\mathbf{w}) = \sum_{\substack{\mathbf{u}\in\mathcal{X}_{i_0}^* \\ \mathrm{wt}(\mathbf{u})=\eta(\mathbf{k})}} \alpha(\mathbf{u})e(\mathbf{u}) \text{ for some scalars } \alpha(\mathbf{u}).$$

Hence the lemma. □

Lemma 6.2.11. *If $\mathbf{u} \neq \mathbf{v} \in \mathcal{X}_{i_0}^*$ are Lyndon words then exactly one element of the set $\{\mathbf{uv}, \mathbf{vu}\}$ is Lyndon.*

Proof. We observe that if $\mathbf{u} < \mathbf{v}$ then $\mathbf{uv}$ is Lyndon, otherwise $\mathbf{vu}$ is Lyndon. □

Lemma 6.2.12. *Let $\mathbf{w}, \tilde{\mathbf{w}} \in \mathcal{X}_{i_0}^*$ be Lyndon words with standard factorization $\mathbf{w} = \mathbf{u_1 u_2}$, $\tilde{\mathbf{w}} = \mathbf{v_1 v_2}$. Assume that $\mathbf{w}\tilde{\mathbf{w}}$ is a Lyndon word. Then*

$$[L(\mathbf{w}), L(\tilde{\mathbf{w}})] \in \mathrm{span}\{\mathrm{L}(\mathrm{C}^{\mathrm{i}_0}(\mathrm{wt}(\mathbf{w}\tilde{\mathbf{w}})), \mathrm{G})\}.$$

Proof. We have two possibles situations: either $\mathbf{u_2} \geq \tilde{\mathbf{w}}$ or $\mathbf{u_2} < \tilde{\mathbf{w}}$.

If $\mathbf{u_2} \geq \tilde{\mathbf{w}}$ then $\sigma(\mathbf{w}\tilde{\mathbf{w}}) = (\mathbf{u_1 u_2}, \mathbf{v_1 v_2})$ is the standard factorization of $\mathbf{w}\tilde{\mathbf{w}}$. Indeed $\sigma(\mathbf{w}\tilde{\mathbf{w}}) \neq (\mathbf{u_1 u_{21}}, \mathbf{u_{22} v_1 v_2})$ for any standard factorization $\sigma(\mathbf{u_2}) = (\mathbf{u_{21}}, \mathbf{u_{22}})$, as $\mathbf{u_{22}} < \tilde{\mathbf{w}} \leq \mathbf{u_2} = \mathbf{u_{21} u_{22}}$ implies $\mathbf{u_{22}} < \mathbf{u_{21}}$ i.e. as $\mathbf{u_2} = \mathbf{u_{21} u_{22}}$ cannot be a Lyndon word. Thus, $\sigma(\mathbf{w}\tilde{\mathbf{w}}) = (\mathbf{u_1 u_2}, \mathbf{v_1 v_2})$ and $[L(\mathbf{w}), L(\tilde{\mathbf{w}})] = L(\mathbf{w}\tilde{\mathbf{w}}) \in \mathrm{span}\{\mathrm{L}(\mathrm{C}^{\mathrm{i}_0}(\mathrm{wt}(\mathbf{w}\tilde{\mathbf{w}})), \mathrm{G})\}$. Hence the lemma holds in this case.

If $\mathbf{u_2} < \tilde{\mathbf{w}}$ then $\sigma(\mathbf{w}\tilde{\mathbf{w}}) = (\mathbf{u_1}, \mathbf{u_2}\mathbf{v_1}\mathbf{v_2})$. Observe that $\sigma(\mathbf{w}\tilde{\mathbf{w}}) \neq (\mathbf{u_{11}}, \mathbf{u_{12}}\mathbf{u_2}\mathbf{v_1}\mathbf{v_2})$ for any standard factorization $\sigma(\mathbf{u_1}) = (\mathbf{u_{11}}, \mathbf{u_{12}})$ of $\mathbf{u_1}$as $\mathbf{u_{12}} < \mathbf{u_2}$ implies $\mathbf{u_{12}}\mathbf{u_2}$ is the longest right factor of $\mathbf{w} = \mathbf{u_{11}}\mathbf{u_{12}}\mathbf{u_2}$ which contradicts $\mathbf{w} = \mathbf{u_1}\mathbf{u_2}$ is the standard factorization.

If $\sigma(\mathbf{w}\tilde{\mathbf{w}}) = (\mathbf{u_1}, \mathbf{u_2}\mathbf{v_1}\mathbf{v_2})$ then

$$\begin{aligned}
[L(\mathbf{w}), L(\tilde{\mathbf{w}})] &= [L(\mathbf{u_1}\mathbf{u_2}), L(\tilde{\mathbf{w}})] \\
&= [[L(\mathbf{u_1}), L(\mathbf{u_2})], L(\tilde{\mathbf{w}})] \\
&= [L(\mathbf{u_1}), [L(\mathbf{u_2}), L(\tilde{\mathbf{w}})] + [[L(\mathbf{u_1}), L(\tilde{\mathbf{w}})], L(\mathbf{u_2})]
\end{aligned} \tag{6.2.7}$$

subcase(i):- If $\mathbf{u_2}\tilde{\mathbf{w}}$ is a Lyndon word with $\sigma(\mathbf{u_2}\tilde{\mathbf{w}}) = (\mathbf{u}_2, \tilde{\mathbf{w}})$ and $\mathbf{u_1}\tilde{\mathbf{w}}$ is a Lyndon word with $\sigma(\mathbf{u_1}\tilde{\mathbf{w}}) = (\mathbf{u_1}, \tilde{\mathbf{w}})$ then

$$\begin{aligned}
[L(\mathbf{w}), L(\tilde{\mathbf{w}})] &= [L(\mathbf{u_1}), L(\mathbf{u_2}\tilde{\mathbf{w}})] + [L(\mathbf{u_1}\tilde{\mathbf{w}}), L(\mathbf{u_2})] \\
&= [L(\mathbf{u_1}), L(\mathbf{u_2}\tilde{\mathbf{w}})] + L(\mathbf{u_1}\tilde{\mathbf{w}}\mathbf{u_2}) \\
&\text{as } \mathbf{u_2} < \tilde{\mathbf{w}} \text{ so } \sigma(\mathbf{u_1}\tilde{\mathbf{w}}\mathbf{u_2}) = (\mathbf{u_1}\tilde{\mathbf{w}}, \mathbf{u_2}).
\end{aligned}$$

We repeat the above procedure on $[L(\mathbf{u_1}), L(\mathbf{u_2}\tilde{\mathbf{w}})]$ and the subsequent terms till we get terms like $[L(\mathbf{v_1}), L(\mathbf{v_2})]$, where $\mathbf{v_1}, \mathbf{v_2}$ are Lyndon words with $\mathbf{v_1} \in \mathcal{X}_{i_0}$. This is possible since $\mathrm{wt}(\mathbf{u_1}) < \mathrm{wt}(\mathbf{w})$.

subcase(ii):- If $\mathbf{u_2}\tilde{\mathbf{w}}$ is a Lyndon word with $\sigma(\mathbf{u_2}\tilde{\mathbf{w}}) = (\mathbf{u_{21}}, \mathbf{u_{22}}\tilde{\mathbf{w}})$ where $\sigma(\mathbf{u_2}) = (\mathbf{u_{21}}, \mathbf{u_{22}})$ and $\sigma(\mathbf{u_1}\tilde{\mathbf{w}}) = (\mathbf{u_1}, \tilde{\mathbf{w}})$ then right-hand side of Equation 6.2.7 is equal to,

$$\begin{aligned}
&= [L(\mathbf{u_1}), [[L(\mathbf{u_{21}}), L(\mathbf{u_{22}})], L(\tilde{\mathbf{w}})]] + [L(\mathbf{u_1}\tilde{\mathbf{w}}), L(\mathbf{u_2})] \\
&= [L(\mathbf{u_1}), [[L(\mathbf{u_{21}}), L(\tilde{\mathbf{w}})], L(\mathbf{u_{22}})]] + [L(\mathbf{u_1}), [L(\mathbf{u_{21}}), [L(\mathbf{u_{22}}), L(\tilde{\mathbf{w}})]]] + L(\mathbf{u_1}\tilde{\mathbf{w}}\mathbf{u_2})
\end{aligned} \tag{6.2.8}$$

We repeat the above procedure firstly for $[L(\mathbf{u_{21}}), L(\tilde{\mathbf{w}})], [L(\mathbf{u_{22}}), L(\tilde{\mathbf{w}})]$, then using this in the Equation 6.2.8 and repeat the procedure for subsequent terms and stop when terms like $[L(\mathbf{v_1}), L(\mathbf{v_2})]$, are obtained where $\mathbf{v_1}, \mathbf{v_2}$ are Lyndon words with $\mathbf{v_1} \in \mathcal{X}_{i_0}$.

subcase(iii):- If $\mathbf{u_2}\tilde{\mathbf{w}}$ is Lyndon word with $\sigma(\mathbf{u_2}\tilde{\mathbf{w}}) = (\mathbf{u_{21}}, \mathbf{u_{22}}\tilde{\mathbf{w}})$ where $\sigma(\mathbf{u_2}) = (\mathbf{u_{21}}, \mathbf{u_{22}})$ and $\mathbf{u_1}\tilde{\mathbf{w}}$ is Lyndon word with $\sigma(\mathbf{u_1}\tilde{\mathbf{w}}) = (\mathbf{u_{11}}, \mathbf{u_{12}}\tilde{\mathbf{w}})$ where $\sigma(\mathbf{u_1}) = (\mathbf{u_{11}}, \mathbf{u_{12}})$ then then right-hand side of Equation 6.2.7 is equal to,

$$\begin{aligned}
&=[L(\mathbf{u_1}), [[L(\mathbf{u_{21}}), L(\mathbf{u_{22}})], L(\tilde{\mathbf{w}})]] + [[[L(\mathbf{u_{11}}), L(\mathbf{u_{12}})], L(\tilde{\mathbf{w}})], L(\mathbf{u_2})] \\
&=[L(\mathbf{u_1}), [[L(\mathbf{u_{21}}), L(\tilde{\mathbf{w}})], L(\mathbf{u_{22}})]] + [L(\mathbf{u_1}), [L(\mathbf{u_{21}}), [L(\mathbf{u_{22}}), L(\tilde{\mathbf{w}})]]] \\
&\quad + [[L(\mathbf{u_{11}}), [L(\mathbf{u_{12}}), L(\tilde{\mathbf{w}})]], L(\mathbf{u_2})] + [[[L(\mathbf{u_{11}}), L(\tilde{\mathbf{w}})], L(\mathbf{u_{12}})], L(\mathbf{u_1})]
\end{aligned} \tag{6.2.9}$$

We repeat the above procedure for $[L(\mathbf{u_{12}}), L(\tilde{\mathbf{w}})], [L(\mathbf{u_{11}}), L(\tilde{\mathbf{w}})]$, then using this in the Equation 6.2.9 and repeat the procedure for subsequent terms and stop when terms like $[L(\mathbf{v_1}), L(\mathbf{v_2})]$, are obtained where $\mathbf{v_1}, \mathbf{v_2}$ are super Lyndon words with $\mathbf{v_1} \in \mathcal{X}_{i_0}$. □

The following example explains the above lemma. Here we denote any word $\mathbf{w} = \alpha_{i_0}\alpha_{i+1}\dots\alpha_j$ by $\mathbf{w} = i(i+1)\dots j$ to avoid confusion.

Example 6.2.13. Consider the root space $\mathfrak{g}_{\eta(\mathbf{k})}$ where $\eta(\mathbf{k}) = 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$ from Example 6.1.14. Fix $i_0 = 3$. Let $\mathbf{w} = 334345, \mathbf{w}' = 34635364 \in \mathcal{X}_3^*$ then $\sigma(\mathbf{w}) = (3, 34345) = (\mathbf{u_1}, \mathbf{u_2}), \sigma(\mathbf{w}') = (346, 35364) = (\mathbf{v_1}, \mathbf{v_2})$. Thus $L(\mathbf{w}) = [L(\mathbf{u_1}), L(\mathbf{u_2})] = [3, [34, 345]]$ and $L(\mathbf{w}') = [L(\mathbf{v_1}), L(\mathbf{v_2})] = [346, [35, 364]]$. Since

$$\sigma(\mathbf{w}\mathbf{w}') = \sigma(\underbrace{334345}_{u_1u_2}\underbrace{34635364}_{\mathbf{w}'}) = (\underbrace{3}_{\mathbf{u_1}}, \underbrace{34345 34635364}_{\mathbf{u_2}\mathbf{w}'}),$$

we have,

$$\begin{aligned}
[L(\mathbf{w}), L(\mathbf{w}')] &= [\underbrace{[L(3), L(34345)]}_{[L(\mathbf{u_1}), L(\mathbf{u_2})]}, \underbrace{L(34635364)}_{L(\mathbf{w}')}] \\
&= \underbrace{[L(3), [L(34345), L(34635364)]]}_{[L(\mathbf{u_1}), [L(\mathbf{u_2}), L(\mathbf{w}')]]} + \underbrace{[[L(3), L(34635364)], L(34345)]}_{[[L(\mathbf{u_1}), L(\mathbf{w}')], L(\mathbf{u_2})]} \\
&= \underbrace{[L(3), [[L(34), L(345)], L(34635364)]]}_{[L(\mathbf{u_1}), [[L(\mathbf{u_{21}}), L(\mathbf{u_{22}})], L(\mathbf{w}')]]} + \underbrace{[L(334635364), L(34345)]}_{[L(\mathbf{u_1}\mathbf{w}'), L(\mathbf{u_2})]}
\end{aligned}$$

$$=[L(3),\underbrace{[[L(34),L(34635364)],L(345)]+[L(34),[L(345),L(34635364)]]}_{[[L(\mathbf{u_{21}}),L(\mathbf{w'})],L(\mathbf{u_{22}})]+[L(\mathbf{u_{21}}),[L(\mathbf{u_{22}}),L(\mathbf{w'})]]}]$$

$$+\underbrace{L(33463536434345)}_{L(\mathbf{u_1}\mathbf{w'}\mathbf{u_2})}$$

$$=[L(3),\underbrace{[L(3434635364),L(345)]+[L(34),L(34534635364)]}_{[L(\mathbf{u_{21}}\mathbf{w'}),L(\mathbf{u_{22}})]+[L(\mathbf{u_{21}}\mathbf{u_{22}}),L(\mathbf{w'})]}]+\underbrace{L(33463536434345)}_{L(\mathbf{u_1}\mathbf{w'}\mathbf{u_2})}$$

$$=[L(3),\underbrace{L(3434635364345)+L(3434534635364)}_{L(\mathbf{u_{21}}\mathbf{w'}\mathbf{u_{22}})+L(\mathbf{u_{21}}\mathbf{u_{22}})\mathbf{w'})}]+\underbrace{L(33463536434345)}_{L(\mathbf{u_1}\mathbf{w'}\mathbf{u_2})}$$

$$=\underbrace{L(33434635364345)}_{L(\mathbf{u_1}\mathbf{u_{21}}\mathbf{w'}\mathbf{u_{22}})}+\underbrace{L(33434534635364)}_{L(\mathbf{u_1}\mathbf{u_{21}}\mathbf{u_{22}}\mathbf{w'})}+\underbrace{L(33463536434345)}_{L(\mathbf{u_1}\mathbf{w'}\mathbf{u_2})}$$

Example 6.2.14. Let $I=\{1,2,3,4,5,6\}, \Psi=\{3,5\}, I_1=\{3,5\}, I_0=\{1,2,4,6\}, I^{re}=\{1,4\}$, $\eta(\mathbf{k})=2\alpha_3+\alpha_4+2\alpha_5+\alpha_6$. Fix $i_0=3$. Let $\mathbf{w}=334365, \mathbf{w'}=34635364\in\mathcal{X}_3^*$ then $\sigma(\mathbf{w})=(3,34365)=(\mathbf{u_1},\mathbf{u_2}), \sigma(\mathbf{w'})=(346,35364)=(\mathbf{v_1},\mathbf{v_2})$ be standard factorization of these words. Since $\sigma(\mathbf{ww'})=\sigma(\underbrace{334365}_{\mathbf{u_1u_2}}\underbrace{34635364}_{\mathbf{w'}})=(\underbrace{3}_{\mathbf{u_1}},\underbrace{3436534635364}_{\mathbf{u_2w'}})$,

$$[L(\mathbf{w}),L(\mathbf{w'})]=[\underbrace{[L(3),L(34365)]}_{[L(\mathbf{u_1}),L(\mathbf{u_2})]},\underbrace{L(34635364)}_{L(\mathbf{w'})}]$$

$$=\underbrace{[L(3),[L(34365),L(34635364)]]}_{[L(\mathbf{u_1}),[L(\mathbf{u_2}),L(\mathbf{w'})]]}+\underbrace{[[L(3),L(34635364)],L(34345)]}_{[[L(\mathbf{u_1}),L(\mathbf{w'})],L(\mathbf{u_2})]}$$

$$=\underbrace{[L(3),L(3436534635364)]}_{[L(\mathbf{u_1}),L(\mathbf{u_2w'})]}+\underbrace{[L(334635364),L(34345)]}_{[L(\mathbf{u_1w'}),L(\mathbf{u_2})]}$$

$$=\underbrace{L(33436534635364)}_{L(\mathbf{u_1u_2w'})}+\underbrace{L(33463536434345)}_{L(\mathbf{u_1w'u_2})}$$

Lemma 6.2.15. *If* $\mathbf{w_a}$ *and* $\mathbf{w_b}$ *are super Lyndon words then*

$$[L(\mathbf{w_a}),L(\mathbf{w_b})]\in\mathrm{span}\{\mathrm{L}(\mathrm{C}^{\mathrm{i_0}}(\mathrm{wt}(\mathbf{w_a}\mathbf{w_b})),\mathrm{G})\}.$$

Proof. Since $\mathbf{w_a}$ and $\mathbf{w_b}$ are super Lyndon words, we have the following cases:-

(i) $\sigma(\mathbf{w_a}) = (\mathbf{u},\mathbf{u})$, $\sigma(\mathbf{w_b}) = (\mathbf{v_1},\mathbf{v_2})$ where $\mathbf{v_1} \neq \mathbf{v_2}$.

(ii) $\sigma(\mathbf{w_a}) = (\mathbf{u_1},\mathbf{u_2})$, $\sigma(\mathbf{w_b}) = (\mathbf{v},\mathbf{v})$ where $\mathbf{u_1} \neq \mathbf{u_2}$.

(iii) $\sigma(\mathbf{w_a}) = (\mathbf{u},\mathbf{u})$, $\sigma(\mathbf{w_b}) = (\mathbf{v},\mathbf{v})$.

(iv) $\sigma(\mathbf{w_a}) = (\mathbf{u_1},\mathbf{u_2})$, $\sigma(\mathbf{w_b}) = (\mathbf{v_1},\mathbf{v_2})$ where $\mathbf{u_1} \neq \mathbf{u_2}$, $\mathbf{v_1} \neq \mathbf{v_2}$.

Case(i):- Let $\sigma(\mathbf{w_a}) = (\mathbf{u},\mathbf{u})$, $\sigma(\mathbf{w_b}) = (\mathbf{v_1},\mathbf{v_2})$. Then $\sigma(\mathbf{w_a}\mathbf{w_b}) = (\mathbf{u},\mathbf{u}\mathbf{v_1}\mathbf{v_2})$. Thus,

$$\begin{aligned}
[L(\mathbf{w_a}),L(\mathbf{w_b})] &= [[L(\mathbf{u}),L(\mathbf{u})],L(\mathbf{w_b})] \\
&= 2[L(\mathbf{u}),[L(\mathbf{u}),L(\mathbf{w_b})]] \\
&= 2[L(\mathbf{u}),L(\mathbf{u}\mathbf{w_b})] \\
&= L(\mathbf{u}\mathbf{u}\mathbf{w_b})
\end{aligned}$$

Case(ii):- Let $\sigma(\mathbf{w_a}) = (\mathbf{u_1},\mathbf{u_2})$, $\sigma(\mathbf{w_b}) = (\mathbf{v},\mathbf{v})$. If $\mathbf{u_2} < \mathbf{v}$ then $\sigma(\mathbf{w_a}\mathbf{w_b}) = (\mathbf{u_1},\mathbf{u_2}\mathbf{v}\mathbf{v})$. Thus,

$$\begin{aligned}
[L(\mathbf{w_a}),L(\mathbf{w_b})] &= [[L(\mathbf{u_1}),L(\mathbf{u_2})],L(\mathbf{w_b})] \\
&= [[L(\mathbf{u_1}),L(\mathbf{w_b})],L(\mathbf{u_2})] + [L(\mathbf{u_1}),[L(\mathbf{u_2}),L(\mathbf{w_b})]] \\
&= [L(\mathbf{u_1}\mathbf{w_b}),L(\mathbf{u_2})] + [L(\mathbf{u_1}),L(\mathbf{u_2}\mathbf{w_b})])]
\end{aligned}$$

Otherwise, $\sigma(\mathbf{w_a}\mathbf{w_b}) = (\mathbf{u_1}\mathbf{u_2},\mathbf{v}\mathbf{v})$. This implies,

$$[L(\mathbf{w_a}),L(\mathbf{w_b})] = L(\mathbf{w_a}\mathbf{w_b})$$

Case(iii):- Let $\sigma(\mathbf{w_a}) = (\mathbf{u},\mathbf{u})$, $\sigma(\mathbf{w_b}) = (\mathbf{v},\mathbf{v})$. Since $\mathbf{u} < \mathbf{v}$, $\sigma(\mathbf{w_a}\mathbf{w_b}) = (\mathbf{u},\mathbf{u}\mathbf{v}\mathbf{v})$. Thus,

$$\begin{aligned}
[L(\mathbf{w_a}),L(\mathbf{w_b})] &= [[L(\mathbf{u}),L(\mathbf{u})],L(\mathbf{w_b})] \\
&= [L(\mathbf{u}),L(\mathbf{u}\mathbf{w_b})].
\end{aligned}$$

Case(iv):- Let $\sigma(\mathbf{w_a}) = (\mathbf{u_1}, \mathbf{u_2})$, $\sigma(\mathbf{w_b}) = (\mathbf{v_1}, \mathbf{v_2})$ where $\mathbf{u_1} \neq \mathbf{u_2}$, $\mathbf{v_1} \neq \mathbf{v_2}$. This case follows from Lemma 6.2.12. □

Lemma 6.2.16. *The root space $\mathfrak{L}_{\eta(\mathbf{k})}$ is contained in the span $\{e(L(C^{i_0}(\mathbf{k}, G))\}$.*

Proof. Let $e(\mathbf{w}) \in \mathfrak{L}_{\eta(\mathbf{k})}$ for some $\mathbf{w} \in M_{\mathbf{k}}(V, \zeta)$. By Lemma 6.2.10, we can assume that $\mathrm{IA}(\mathbf{w}) = \{\alpha_{i_0}\}$. We prove the lemma by induction on $\mathrm{ht}(\eta(\mathbf{k}))$. If $\mathrm{ht}(\eta(\mathbf{k})) = 1$ then $\mathbf{w} = \alpha_{i_0}$ and there is nothing to prove. Assume that the result is true for any $\tilde{\mathbf{w}}$ such that $\mathrm{ht}(\mathrm{wt}(\tilde{\mathbf{w}})) < ht(\eta(k))$. Let $\mathbf{w} = \alpha_{i_0} b_1 b_2 \cdots b_r = \alpha_{i_0} \cdot \mathbf{u}$.

If $i_0(\mathbf{u}) = \phi$ then $\mathbf{w} \in \mathcal{X}_{i_0} \Rightarrow L(\mathbf{w}) = \mathbf{w} \Rightarrow e(\mathbf{w}) = e(L(\mathbf{w})) \in \mathrm{span}\{\mathrm{e(L(C^{i_0}(\mathbf{k},G))}\}$.

If $i_0(\mathbf{u}) \neq \phi$, suppose $min\{i_0(\mathbf{u})\} = p+1$. Then setting $\mathbf{w}' = \alpha_{i_0} b_1 b_2 \cdots b_p \alpha_{i_0}$. We have,

$$\begin{aligned} e(\mathbf{w}') &= [[[[e_{\alpha_{i_0}}, e_{b_1}], e_{b_2}] \cdots, e_{b_p}], e_{\alpha_{i_0}}] \\ &= -[e_{\alpha_{i_0}}, [[[e_{\alpha_{i_0}}, e_{b_1}], e_{b_2}] \cdots, e_{b_p}]] \\ &= -e(L(\alpha_{i_0} \alpha_{i_0} b_1 \cdots b_p)). \end{aligned}$$

This implies, $e(\mathbf{w}') \in \mathrm{span}\{e(L(C^{i_0}(wt(\mathbf{w}'), G))\}$ as $(\alpha_{i_0} \alpha_{i_0} b_1 \cdots b_p)$ is a super Lyndon word. Now,

$$\begin{aligned} e(\mathbf{w}' \cdot b_{p+2}) &= [[[[[e(\alpha_{i_0}), e_{b_1}], e_{b_2}] \cdots, e_{b_p}], e_{\alpha_{i_0}}], e_{b_{p+2}}] = [e(\mathbf{w}'), e_{b_{p+2}}] \\ &= [[e(\alpha_{i_0}), e(\alpha_{i_0} b_1 b_2 \cdots b_p)], e_{b_{p+2}}] \\ &= [e(\alpha_{i_0}), [e(\alpha_{i_0} b_1 b_2 \cdots b_p), e_{b_{p+2}}]] + [[e(\alpha_{i_0}), e_{b_{p+2}}], e(\alpha_{i_0} b_1 b_2 \cdots b_p)] \\ &= [e(\alpha_{i_0}), e(\alpha_{i_0} b_1 b_2 \cdots b_p b_{p+2})] + [e(\alpha_{i_0} b_{p+2}), e(\alpha_{i_0} b_1 b_2 \cdots b_p)] \end{aligned}$$

Similarly, using the Jacobi identity, we have

$$\begin{aligned}
e(\mathbf{w}) =& [e(\alpha_{i_0}), e(\alpha_{i_0} b_1 b_2 \cdots \widehat{b_{p+1}} \cdots b_r)] + \sum_{t=p+2}^{k} [e(\alpha_{i_0} b_1 b_2 \cdots \widehat{b_{p+1}} \cdots \hat{b_t} \cdots b_r), e(\alpha_{i_0} b_t)] + \\
&+ \sum_{p+2 \le t_1 < t_2 \le r}^{k} [e(\alpha_{i_0} b_1 b_2 \cdots \widehat{b_{p+1}} \cdots \widehat{b_{t_1}} \cdots \widehat{b_{t_2}} \cdots b_r), e(\alpha_{i_0} b_{t_1} b_{t_2})] \\
&+ \sum_{p+2 \le t_1 < t_2 < t_3 \le r}^{k} [e(\alpha_{i_0} b_1 b_2 \cdots \widehat{b_{p+1}} \cdots \widehat{b_{t_1}} \cdots \widehat{b_{t_2}} \cdots \widehat{b_{t_3}} \cdots b_r), e(\alpha_{i_0} b_{t_1} b_{t_2} b_{t_3})] + \cdots \\
&+ [e(\alpha_{i_0} b_1 \cdots b_p), e(\alpha_{i_0} b_{p+2} \cdots b_r)]
\end{aligned}$$

Using induction hypothesis, we see that each term on the right-hand side is of the form

$$[e(\alpha_{i_0} b_1 b_2 \cdots \hat{b_p} b_{p+1} \cdots \widehat{b_{t_1}} \cdots \widehat{b_{t_2}} \cdots \widehat{b_{t_j}} \cdots b_r), e(\alpha_{i_0} b_{t_1} b_{t_2} \cdots b_{t_j})] = \left[\sum_a e(L(\mathbf{w_a})), \sum_b e(L(\mathbf{w_b}))\right]$$

as $\mathrm{wt}(\alpha_{i_0} b_1 b_2 \cdots \hat{b_p} b_{p+1} \cdots \widehat{b_{t_1}} \cdots \widehat{b_{t_2}} \cdots \widehat{b_{t_j}} \cdots b_r) < \mathrm{wt}(\mathbf{w})$ and $\mathrm{wt}(\alpha_{i_0} b_{t_1} b_{t_2} \cdots b_{t_j}) < \mathrm{wt}(\mathbf{w})$. So,

$$\begin{aligned}
e(\alpha_{i_0} b_1 b_2 \cdots \hat{b_p} b_{p+1} \cdots \widehat{b_{t_1}} \cdots \widehat{b_{t_2}} \cdots \widehat{b_{t_j}} \cdots b_r) &= \sum_a e(L(\mathbf{w_a})) \\
e(\alpha_{i_0} b_{t_1} b_{t_2} \cdots b_{t_j}) &= \sum_b e(L(\mathbf{w_b}))
\end{aligned}$$

where $\mathbf{w_a}$ and $\mathbf{w_b}$ are super Lyndon words such that $\mathrm{wt}(\mathbf{w_b}) = \mathrm{wt}(\alpha_{i_0} b_{t_1} b_{t_2} \cdots b_{t_j})$ and $\mathrm{wt}(\mathbf{w_a}) = \mathrm{wt}(\alpha_{i_0} b_1 b_2 \cdots \hat{b_p} b_{p+1} \cdots \widehat{b_{t_1}} \cdots \widehat{b_{t_2}} \cdots \widehat{b_{t_j}} \cdots b_r)$. This implies,

$$\begin{aligned}
\left[\sum_a e(L(\mathbf{w_a})), \sum_b e(L(\mathbf{w_b}))\right] &= \sum_{a,b} [e(L(\mathbf{w_a})), e(L(\mathbf{w_b}))] \\
&= \sum_{a,b} e\left([L(\mathbf{w_a}), L(\mathbf{w_b})]\right).
\end{aligned}$$

By Lemma 6.2.15, $[L(\mathbf{w_a}), L(\mathbf{w_b})] \in \mathrm{span}\{\mathrm{L}(\mathrm{C}^{\mathrm{i}_0}(\mathbf{k}, \mathrm{G}))\}$. Thus,

$$\sum_{a,b} e\left([L(\mathbf{w_a}), L(\mathbf{w_b})]\right) \in \mathrm{span}\left\{\mathrm{e}(\mathrm{L}(\mathrm{C}^{\mathrm{i}_0}(\mathbf{k}, \mathrm{G}))\right\}$$

i.e.,

$$e(\mathbf{w}) \in \mathrm{span}\{\mathrm{e}(\mathrm{L}(\mathrm{C}^{\mathrm{i}_0}(\mathbf{k},\mathrm{G}))\}.$$

Hence $\mathfrak{L}_{\eta(\mathbf{k})}$ is contained in the span of $\{e(L(C^{i_0}(\mathbf{k},G))\} \subseteq \mathfrak{L}(i_0)$. □

6.2.6 Proof of Lemma 6.2.7 (Identification of $C^{i_0}(\mathbf{k},G)$ and super Lyndon heaps)

Fix $\mathbf{k} = (k_j : j \in I) \in \mathbb{Z}_+^I$ such that $k_j \leq 1$ for $j \in I^{re} \sqcup \Psi_0$. Fix $i_0 \in I$ and assume that i_0 is the minimum element in the total order of I such that $k_{i_0} \neq 0$. Consider,

$$\mathcal{X}_{i_0} = \{\mathbf{w} \in M(V,G,\Psi) : \mathrm{IA}_m(\mathbf{w}) = \{\alpha_{i_0}\} \text{ and } \alpha_{i_0} \text{ occurs only once in } \mathbf{w}\}.$$

Let $\mathbf{w} \in \mathcal{X}_{i_0}$ and $E = \psi(\mathbf{w})$ be the corresponding heap. Then

1. $\mathrm{IA}_m(\mathbf{w}) = \{\alpha_{i_0}\}$ implies that E is a pyramid.

2. α_{i_0} occurs exactly once in $\mathbf{w}$ implies that E is elementary.

3. α_{i_0} is the minimum element in the total order of V implies that E is an admissible pyramid.

Therefore,

$$\mathbf{w} \in \mathcal{X}_{i_0} \text{ if and only if } E = \psi(\mathbf{w}) \text{ is a super-letter.} \tag{6.2.10}$$

Let $\mathcal{A}_{i_0}(V,\zeta)$ be the set of all super-letters with basis $\{\alpha_{i_0}\}$ in $\mathcal{H}(V,\zeta)$. Let $\mathcal{A}^*_{i_0}(V,\zeta)$ be the monoid generated by $\mathcal{A}_{i_0}(V,\zeta)$ in $\mathcal{H}(V,\zeta)$. Then $\mathcal{A}^*_{i_0}(V,\zeta) = \mathcal{A}^*_{i_0,0}(V,\zeta) \oplus \mathcal{A}^*_{i_0,1}(V,\zeta)$ is also $\mathbb{Z}_2$-graded. From [44, Section 2.1, Proposition 1.3.5 and Proposition 2.1.5], it follows that this monoid is free. Since $\mathcal{H}(V,\zeta)$ is totally ordered, $\mathcal{A}_{i_0}(V,\zeta)$ is totally ordered. Hence

$\mathcal{A}_{i_0}^*(V,\zeta)$ is totally ordered by the lexicographic order induced from the order in $\mathcal{A}_{i_0}(V,\zeta)$ (call it $\leq^*$).

The following proposition from [44, Proposition 2.1.6] illustrates the relation between the total order $\leq$ on the heaps monoid $\mathcal{H}(V,\zeta)$ and the total order $\leq^*$ on the monoid $\mathcal{A}_{i_0}^*(V,\zeta)$.

Proposition 6.2.17. *Let $E,F \in \mathcal{A}_{i_0}^*(V,\zeta)$. Then $E \leq^* F$ if, and only if, $E \leq F$.*

With respect to this ordering, we can define Lyndon words over the alphabets $\mathcal{A}_{i_0}(V,\zeta)$. The following proposition from [44, Proposition 2.1.7] illustrates the relationship between the Lyndon words in $\mathcal{A}_{i_0}^*(V,\zeta)$ and the Lyndon heaps in $\mathcal{H}(V,\zeta)$.

Proposition 6.2.18. *For $E \in \mathcal{A}_{i_0}^*(V,\zeta)$, E is a Lyndon word in $\mathcal{A}_{i_0}^*(V,\zeta)$ if and only if E is a Lyndon heap as an element of $\mathcal{H}(V,\zeta)$.*

Next, we prove the following generalization of Proposition 6.2.18 for the case of super Lyndon words and super Lyndon heaps.

Proposition 6.2.19. *For $E \in \mathcal{A}_{i_0}^*(V,\zeta)$, E is a super Lyndon word in $\mathcal{A}_{i_0}^*(V,\zeta)$ if and only if E is a super Lyndon heap as an element of $\mathcal{H}(V,\zeta)$.*

Proof. Let $E \in \mathcal{A}_{i_0}^*(V,\zeta)$ be a super Lyndon word. Then two cases can occur.

Case(i). Suppose E is a Lyndon word $\mathcal{A}_{i_0}^*(V,\zeta)$ then, by Proposition 6.2.18, E is a Lyndon heap and hence is a super Lyndon heap in $\mathcal{H}(V,\zeta)$.

Case(ii). Suppose $E = F \circ F$ for some Lyndon word $F \in \mathcal{A}_{i_0,1}^*(V,\zeta)$ then by Proposition 6.2.18, F is a Lyndon heap in $\mathcal{H}(V,\zeta)$ and hence $E = F \circ F$ is a super Lyndon heap in $\mathcal{H}(V,\zeta)$.

Conversely, suppose E is a super Lyndon heap in $\mathcal{H}(V,\zeta)$.

Case(i). If E is a Lyndon heap in $\mathcal{H}(V,\zeta)$ then, by Proposition 6.2.18, E is a Lyndon word and hence is a super Lyndon word in $\mathcal{A}_{i_0}^*(V,\zeta)$.

Case(ii). If $E = F \circ F$ for some Lyndon heap $F \in \mathcal{H}(V, \zeta)$ then, by Proposition 6.2.18, F is a Lyndon word in $\mathcal{A}^*_{i_0,1}(V, \zeta)$ and hence $E = F \circ F$ is a super Lyndon word in $\mathcal{A}^*_{i_0}(V, \zeta)$. $\square$

Proof of Lemma 6.2.7: By Equation (6.2.10), we can identify $\mathcal{X}^*_{i_0}$ with $\mathcal{A}^*_{i_0}(V, \zeta)$. This implies,

$$\begin{aligned}
|C^{i_0}(\mathbf{k}, G)| &= |\{ \text{ super Lyndon words in } \mathcal{X}^*_{i_0} \text{ of weight } \mathbf{k}\}| \\
&= |\{ \text{ super Lyndon words of weight } \mathbf{k} \text{ in } \mathcal{A}^*_{i_0}(V, \zeta)\}| \\
&= |\{ \text{ super Lyndon heaps of weight } \mathbf{k} \text{ in } \mathcal{H}(V, \zeta)\}| \\
&= \dim \mathcal{LS}_{\mathbf{k}}(G) \\
&= \dim \mathfrak{L}_{\eta(\mathbf{k})} \text{ (By Theorem 6.1.7)}.
\end{aligned}$$

This shows that the elements of $C^{i_0}(\mathbf{k}, G)$ are precisely the Lyndon heaps of weight $\mathbf{k}$. Hence the lemma. $\square$

Chapter 7

Combinatorial properties of free roots of BKM superalgebras

In this chapter, we explore the combinatorial properties of free roots of BKM superalgebras. Let (G,Ψ) be a finite simple supergraph with vertex set $V = \{\alpha_i : i \in I\}$, edge set $E(G)$, and the set of odd vertices parameterized by $\Psi \subseteq I$ [c.f. Definition 5.2.1].

7.1 Free roots of BKM superalgebras

Let $(A = (b_{ij}), \Psi)$ be the adjacency matrix of a finite simple supergraph (G,Ψ). We construct a class of BKM supermatrices from (A,Ψ) as follows:

(i). Replace the diagonal zeros of A by arbitrary real numbers.

(ii). If one such number is positive, then replace all the non-zero entries in the corresponding row by arbitrary non-positive integers (resp. non-positive even integers) provided $i \notin \Psi$ (resp. $i \in \Psi$). Otherwise, replace the non-zero entries in the associated row of A with arbitrary non-positive real numbers.

Let $M_\Psi(G)$ be the set of BKM supermatrices associated with the supergraph (G,Ψ) constructed in this way. Let $M(G) = \bigcup_{\Psi\subseteq I} M_\Psi(G)$ and $\mathcal{C}(G)$ be the set of all BKM superalgebras whose quasi-Dynkin diagram is (G,Ψ) for some $\Psi \subseteq I$. We observe that the set $\mathcal{C}(G)$ consists of BKM superalgebras whose associated BKM supermatrices are in $M(G)$. In the following context, we recall the following lemma.

Lemma 7.1.1. *[56, Proposition 2.40] Let $i \in I^{im}$ and $\alpha \in \Delta_+\backslash\{\alpha_i\}$ such that $\alpha(h_i) < 0$. Then $\alpha + j\alpha_i \in \Delta_+$ for all $j \in \mathbb{Z}_+$.*

In the following proposition, we prove that all the BKM superalgebras belonging to $\mathcal{C}(G)$ share the same set of free roots and have equal respective multiplicities.

Proposition 7.1.2. *Let G be a graph. Let $\mathfrak{L}$ be a BKM superalgebra which is an element of $\mathcal{C}(G)$. Let Q_+ be the root lattice of $\mathfrak{L}$. We have,*

1. *If $\alpha \in Q_+$ is free, then α is a root in $\mathfrak{L}$ if and only if $\operatorname{supp}\alpha$ is connected in G. In particular, $\Delta^m(\mathfrak{L}_1) = \Delta^m(\mathfrak{L}_2)$ for $\mathfrak{L}_1, \mathfrak{L}_2 \in \mathcal{C}(G)$. Further, if $I^{im} = \emptyset$ then there exists a one-one correspondence between the connected subgraphs of G and the free roots of $\mathfrak{L}$.*

2. *For any $\mathfrak{L} \in \mathcal{C}(G)$, the multiplicity of a free root α depends only on the graph G and this multiplicity is equal to the number of super Lyndon heaps of weight $\mathbf{k} = (k_i : i \in I)$ where $\alpha = \sum_{i\in I} k_i\alpha_i$.*

Proof. The necessary part of (1) is straight forward, and we prove the sufficiency part. Assume $\alpha \in Q_+$ is free and $\operatorname{supp}\alpha$ is connected in G.

By applying induction on height of α, we show α is a root of $\mathfrak{L}$. Clearly, α is a root when $\operatorname{ht}(\alpha) = 1$. Suppose $\operatorname{ht}(\alpha) = 2$, i.e., $\alpha = \alpha_i + \alpha_j$ and $a_{ij} < 0$. If α_i is real, then $s_{\alpha_i}(\alpha_j) = \alpha_j - a_{ij}\alpha_i$ is a root of $\mathfrak{L}$ implying $\alpha_i + \alpha_j$ is a root as the root chain of α_j through

α_i contains $\alpha_j + m\alpha_i$ for all $0 \leq m \leq k$ for some $k \in \mathbb{N}$. If both α_i and α_j are imaginary, then Lemma 7.1.1 completes the proof.

Assume that the result is true for all connected free $\alpha \in Q_+$ of height $r-1$. Let β be a connected free element of height r in Q_+. Since $\operatorname{supp}\beta$ is connected, there exists a vertex $\alpha_i \in \operatorname{supp}\beta$ such that $\operatorname{supp}\beta \backslash \{\alpha_i\}$ is connected in G. Now, $\alpha = \sum_{\substack{\alpha_j \in \operatorname{supp}\beta \\ j \neq i}} \alpha_j$ is connected, free, and has height $r-1$. By the induction hypothesis α is a root in $\mathfrak{L}$. If α_i is real then $s_{\alpha_i}(\alpha)$ is a root, hence $\beta = \alpha + \alpha_i$ is also root. If α_i is imaginary then, by Lemma 7.1.1, β is a root. This completes the proof of (1). Now, the proof of (2) follows from Lemma 6.1.3 and Theorem 6.1.7. □

Example 7.1.3. Let $l_1 \geq 1, l_2 \geq 2$ and $l_3 \geq 3$ be positive integers satisfying $l_1 = l_2 = l_3$. Then the complex finite dimensional simple Lie algebras A_{l_1}, B_{l_2} and C_{l_3} have the same quasi-Dynkin diagram, the path graph on l_1 vertices with $\Psi = \emptyset$. These algebras have the same set of free roots by the Proposition 7.1.2.

Proposition 7.1.4. *[51, Corollary 2.1.23] A simple finite dimensional Lie superalgebra $\mathfrak{L}$ is a BKM superalgebra if and only if $\mathfrak{L}$ is contragredient of type $A(m,0) = \mathfrak{sl}(m+1,1), A(m,1) = \mathfrak{sl}(m+1,2), B(0,n) = \mathfrak{osp}(1,2n), B(m,1) = \mathfrak{osp}(2m+1,2), C(n) = \mathfrak{osp}(2,2n-2), D(m,1) = \mathfrak{osp}(2m,2), D(2,1,\alpha)$ for $\alpha = 0,-1, F(4)$, and $G(3)$.*

Using Proposition 7.1.4, we list, in Table 7.1, the BKM superalgebras for which the path graph on 4 vertices is the quasi-Dynkin diagram along with its free roots.

Remark 7.1.5. We observe that the number of free roots of a BKM superalgebra is equal to the number of connected subgraphs $C(G)$ of G. In particular, when G is a tree, this number is equal to the number of subtrees of G.

Table 7.1 BKM superalgebras with equal set of free roots

BKM superalgebras	Simple roots [39, Section 2.5.4]	Free roots
A_4	$\alpha_1=\varepsilon_1-\varepsilon_2, \alpha_2=\varepsilon_2-\varepsilon_3,$ $\alpha_3=\varepsilon_3-\varepsilon_4, \alpha_4=\varepsilon_4-\varepsilon_5.$	$\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_{12},\alpha_{13},$ $\alpha_{14},\alpha_{23},\alpha_{24},\alpha_{34}.$
B_4	$\alpha_1=\varepsilon_1-\varepsilon_2, \alpha_2=\varepsilon_2-\varepsilon_3,$ $\alpha_3=\varepsilon_3-\varepsilon_4, \alpha_4=\varepsilon_4.$	$\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_{12},\alpha_{13},$ $\alpha_{14},\alpha_{23},\alpha_{24},\alpha_{34}.$
C_4	$\alpha_1=\varepsilon_1-\varepsilon_2, \alpha_2=\varepsilon_2-\varepsilon_3,$ $\alpha_3=\varepsilon_3-\varepsilon_4, \alpha_4=2\varepsilon_4.$	$\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_{12},\alpha_{13},$ $\alpha_{14},\alpha_{23},\alpha_{24},\alpha_{34}.$
$A(3,0)$	$\alpha_1=\varepsilon_1-\varepsilon_2, \alpha_2=\varepsilon_2-\varepsilon_3,$ $\alpha_3=\varepsilon_3-\varepsilon_4, \alpha_4=\varepsilon_4-\delta_1.$	$\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_{12},\alpha_{13},$ $\alpha_{14},\alpha_{23},\alpha_{24},\alpha_{34}.$
$A(2,1)$	$\alpha_1=\varepsilon_1-\varepsilon_2, \alpha_2=\varepsilon_2-\varepsilon_3,$ $\alpha_3=\varepsilon_3-\delta_1, \alpha_4=\delta_1-\delta_2.$	$\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_{12},\alpha_{13},$ $\alpha_{14},\alpha_{23},\alpha_{24},\alpha_{34}.$
$B(0,4)$	$\alpha_1=\delta_1-\delta_2, \alpha_2=\delta_2-\delta_3,$ $\alpha_3=\delta_3-\delta_4, \alpha_4=\delta_4.$	$\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_{12},\alpha_{13},$ $\alpha_{14},\alpha_{23},\alpha_{24},\alpha_{34}.$
$B(3,1)$	$\alpha_1=\delta_1-\varepsilon_1, \alpha_2=\varepsilon_1-\varepsilon_2,$ $\alpha_3=\varepsilon_2-\varepsilon_3, \alpha_4=\varepsilon_3.$	$\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_{12},\alpha_{13},$ $\alpha_{14},\alpha_{23},\alpha_{24},\alpha_{34}.$
$C(4)$	$\alpha_1=\varepsilon_1-\delta_1, \alpha_2=\delta_1-\delta_2,$ $\alpha_3=\delta_2-\delta_3, \alpha_4=2\delta_3.$	$\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_{12},\alpha_{13},$ $\alpha_{14},\alpha_{23},\alpha_{24},\alpha_{34}.$
$F(4)$	$\alpha_1=\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta), \alpha_2=-\varepsilon_1,$ $\alpha_3=\varepsilon_1-\varepsilon_2, \alpha_4=\varepsilon_2-\varepsilon_3.$	$\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_{12},\alpha_{13},$ $\alpha_{14},\alpha_{23},\alpha_{24},\alpha_{34}.$

7.2 Multicoloring and the k-chromatic polynomial of G

For any finite set S, let $\mathcal{P}(S)$ be the power set of S. For a tuple of non–negative integers $\mathbf{k}=(k_i : i\in I)\in\mathbb{Z}_+^I$, set $\operatorname{supp}(\mathbf{k})=\{i\in I : k_i\neq 0\}$.

Definition 7.2.1. We call a map $\tau : V\to\mathcal{P}(\{1,\dots,q\})$ a proper vertex **k**–multicoloring of G if the following conditions are satisfied:

(i) For all $i\in I$, $|\tau(\alpha_i)|=k_i$,

(ii) For all $i,j\in I$ such that $(\alpha_i,\alpha_j)\in E(G)$ we have $\tau(\alpha_i)\cap\tau(\alpha_j)=\emptyset$.

For more details on multicoloring, we refer to [33]. The case $k_i=1$, for $i\in I$, corresponds to the classical graph coloring of graph G. The number of ways in which a graph G can be **k**–multicolored using q colors is a polynomial in q called the generalized **k**-chromatic polynomial (**k**-chromatic polynomial in short). It is denoted by $\pi^G_{\mathbf{k}}(q)$. A **k**-chromatic polynomial is defined as follows. Let $P_k(\mathbf{k},G)$ be the set of all ordered k–tuples $(P_1,\dots,P_k)$ such that:

1. each P_i is a non–empty independent subset of V, i.e. no two vertices have an edge between them;
2. For all $i \in I$, α_i occurs exactly k_i times in total in the disjoint union $P_1\dot{\cup}\cdots\dot{\cup}P_k$.

Then,

$$\pi^G_{\mathbf{k}}(q) = \sum_{k\geq 0} |P_k(\mathbf{k},G)| \binom{q}{k}. \tag{7.2.1}$$

Let $G(\mathbf{k})$ be the graph constructed as follows: For each $j \in \mathrm{supp}(\mathbf{k})$, take a clique (complete graph) of size k_j with vertex set $\{\alpha_j^1,\dots,\alpha_j^{k_j}\}$ and join all vertices of the r–th and s–th cliques if $(\alpha_r,\alpha_s)\in E(G)$. Let $\pi_{\mathbf{1}}^{G(\mathbf{k})}(q)$ be the chromatic polynomial of the graph $G(\mathbf{k})$. We have the following relation between the ordinary chromatic polynomials and the **k**-chromatic polynomials:

$$\pi^G_{\mathbf{k}}(q) = \frac{1}{\mathbf{k}!}\pi_{\mathbf{1}}^{G(\mathbf{k})}(q) \tag{7.2.2}$$

where $\mathbf{k}! = \prod_{i\in I} k_i!$.

7.2.1 Bond lattice and an isomorphism of lattices

For the rest of this paper, we fix a tuple of non-negative integers $\mathbf{k} = (k_i : i \in I)$ such that $k_i \leq 1$ for $i \in I^{re}\cup\Psi_0$ and set $\eta(\mathbf{k}) = \sum k_i\alpha_i$.

Definition 7.2.2. Let $L_G(\mathbf{k})$ be the weighted bond lattice of G, which is the set of $\mathbf{J} = \{J_1,\dots,J_k\}$ satisfying the following properties:

(i) $\mathbf{J}$ is a multiset, i.e. we allow $J_i = J_j$ for $i \neq j$;

(ii) each J_i is a multiset and the subgraph spanned by the underlying set of J_i is a connected subgraph of G for each $1 \leq i \leq k$;

(iii) For all $i \in I$, α_i occurs exactly k_i times in total in the disjoint union $J_1 \dot\cup \cdots \dot\cup J_k$.

For $\mathbf{J} \in L_G(\mathbf{k})$ we denote by $D(J_i, \mathbf{J})$ the multiplicity of J_i in $\mathbf{J}$ and set

$$\text{mult}(\beta(J_i)) = \dim \mathfrak{L}_{\beta(J_i)},$$

where, $\beta(J_i) = \sum_{\alpha \in J_i} \alpha$. Let $\mathbf{J}_0 = \{J_i \in \mathbf{J} : \beta(J_i) \in \Delta_+^0\}$ and $\mathbf{J}_1 = \mathbf{J} \backslash \mathbf{J}_0$.

In the following context, we recall the following lemma.

Lemma 7.2.3. *[3, Lemma 3.4] Let $\mathcal{P}$ be the collection of multisets $\gamma = \{\beta_1, \dots, \beta_r\}$ (we allow $\beta_i = \beta_j$ for $i \neq j$) such that each $\beta_i \in \Delta_+$ and $\beta_1 + \cdots + \beta_r = \eta(\mathbf{k})$. The map $\psi : L_G(\mathbf{k}) \to \mathcal{P}$ defined by $\{J_1, \dots, J_k\} \mapsto \{\beta(J_1), \dots, \beta(J_k)\}$ is a bijection.*

7.3 Main Result (Chromatic polynomial and root multiplicities)

Now we relate the **k**-chromatic polynomial with root multiplicities of BKM superalgebras.

Theorem 7.3.1. *Let G be the quasi Dynkin diagram of a BKM superalgebra $\mathfrak{L}$. Assume $\mathbf{k} = (k_i : i \in I) \in \mathbb{Z}_+^I$ is such that $k_i \leq 1$ for $i \in I^{re} \cup \Psi_0$. Then*

$$\pi_{\mathbf{k}}^G(q) = (-1)^{\text{ht}(\eta(\mathbf{k}))} \sum_{\mathbf{J} \in L_G(\mathbf{k})} (-1)^{|\mathbf{J}| + |\mathbf{J}_1|} \prod_{J \in \mathbf{J}_0} \binom{q \, mult(\beta(J))}{D(J, \mathbf{J})} \prod_{J \in \mathbf{J}_1} \binom{-q \, mult(\beta(J))}{D(J, \mathbf{J})}.$$

where $L_G(\mathbf{k})$ is the bond lattice of weight $\mathbf{k}$ of the graph G.

As a corollary, we obtained the following result, which gives a combinatorial formula for the multiplicities of free roots.

Corollary 7.3.2. *We have,*

$$\operatorname{mult}(\eta(\mathbf{k})) = \sum_{\ell|\mathbf{k}} \frac{\mu(\ell)}{\ell} \, |\pi^G_{\mathbf{k}/\ell}(q)[q]|, \quad \textit{if } \eta(\mathbf{k}) = \sum_{i\in I} k_i\alpha_i \in \Delta_0^+$$

and

$$\operatorname{mult}(\eta(\mathbf{k})) = \sum_{\ell|\mathbf{k}} \frac{(-1)^{\ell+1}\mu(\ell)}{\ell} \, |\pi^G_{\mathbf{k}/\ell}(q)[q]|, \quad \textit{if } \eta(\mathbf{k}) \in \Delta_1^+$$

where $|\pi^G_{\mathbf{k}}(q)[q]|$ denotes the absolute value of the coefficient of q in $\pi^G_{\mathbf{k}}(q)$ and μ is the Möbius function. If k_i's are relatively prime, in particular if for some $i \in I$, $k_i = 1$, we have,

$$\operatorname{mult}(\eta(\mathbf{k})) = |\pi^G_{\mathbf{k}}(q)[q]| \quad \textit{for any } \eta(\mathbf{k}) \in \Delta^+.$$

7.3.1 Proof of Theorem 7.3.1

For a Weyl group element $w \in W$, fix a reduced word $w = \mathbf{s}_{i_1}\cdots\mathbf{s}_{i_k}$ and let $I(w) = \{\alpha_{i_1},\dots,\alpha_{i_k}\}$. Observe that $I(w)$ is independent of the choice of the reduced expression of w. For $\gamma = \sum_{i\in I} m_i\alpha_i \in \Omega$, let $I_m(\gamma)$ be the multiset $\{\underbrace{\alpha_i,\dots,\alpha_i}_{m_i \text{ times}} : i \in I\}$ and $I(\gamma)$, the underlying set of $I_m(\gamma)$. Define

$$\Psi_0(\gamma) = I(\gamma) \cap \Psi_0,$$

$$\mathcal{J}(\gamma) = \{w \in W\backslash\{e\} : I(w) \cup I(\gamma) \text{ is an independent set}\}.$$

The following lemma is a generalization of [54, Lemma 2.3] (for Kac-Moody Lie algebras) and [3, Lemma 3.6] (for Borcherds algebras) in the setting of BKM superalgebras. Since the proof of this lemma is similar to the proof of the Borcherds algebras case, we omit the proof here.

Lemma 7.3.3. *Let $w \in W$, $\gamma = \sum\limits_{i \in I \setminus \Psi_0} \alpha_i + \sum\limits_{i \in \Psi_0} m_i \alpha_i \in \Omega$. For $w \in W$, set $\rho - w(\rho) + w(\gamma) = \sum_{\alpha \in \Pi} b_\alpha(w,\gamma)\alpha$. Then we have*

(i) $b_\alpha(w,\gamma) \in \mathbb{Z}_+$ for all $\alpha \in \Pi$ and $b_\alpha(w,\gamma) = 0$ if $\alpha \notin I(w) \cup I(\gamma)$.

(ii) $b_\alpha(w,\gamma) \geq 1$ for all $\alpha \in I(w)$.

(iii) $b_\alpha(w,\gamma) = 1$ if $\alpha \in I(\gamma) \setminus \Psi_0(\gamma)$ and $b_\alpha(w,\gamma) = m_\alpha$ if $\alpha \in \Psi_0(\gamma)$.

(iv) If $w \in \mathcal{J}(\gamma)$, then $b_\alpha(w,\gamma) = 1$ for all $\alpha \in I(w) \cup (I(\gamma) \setminus \Psi_0(\gamma))$, $b_\alpha(w,\gamma) = m_\alpha$ for all $\alpha \in \Psi_0(\gamma)$.

(v) If $w \notin \mathcal{J}(\gamma) \cup \{e\}$, then there exists $\alpha \in I(w) \subseteq \Pi^{\mathrm{re}}$ such that $b_\alpha(w,\gamma) > 1$.

The following proposition is an easy consequence of the above lemma and essential to prove Theorem 7.3.1. Let U be the sum-side of the denominator identity (Equation (5.5.1)), i.e.,

$$U := \sum_{w \in W} \sum_{\gamma \in \Omega} \varepsilon(w)\varepsilon(\gamma) e^{w(\rho-\gamma)-\rho}. \tag{7.3.1}$$

Proposition 7.3.4. *Let $q \in \mathbb{Z}$. Then*

$$U^q[e^{-\eta(\mathbf{k})}] = (-1)^{\mathrm{ht}(\eta(\mathbf{k}))}\, \pi^G_{\mathbf{k}}(q),$$

where $U^q[e^{-\eta(\mathbf{k})}]$ denotes the coefficient of $e^{-\eta(\mathbf{k})}$ in U^q.

Proof. Note, $U^q = (1 + (U-1))^q = \sum\limits_{k \geq 0} \binom{q}{k} (U-1)^k$. From Lemma 7.3.3, we get

$$w(\rho) - \rho - w(\gamma) = -\gamma - \sum_{\alpha \in I(w)} \alpha, \quad \text{for } w \in \mathcal{J}(\gamma) \cup \{e\}.$$

Since $k_i \leq 1$ for $i \in I^{re} \sqcup \Psi_0$, the coefficient of $e^{-\eta(\mathbf{k})}$ in $(U-1)^k$ is equal to

$$\Big(\sum_{\substack{\gamma\in\Omega\\ \gamma\neq 0}}\sum_{w\in\mathcal{J}(\gamma)} \varepsilon(\gamma)\varepsilon(w)e^{-\gamma-\sum_{\alpha\in I(w)}\alpha}\Big)^k[e^{-\eta(\mathbf{k})}]. \tag{7.3.2}$$

i.e.,

$$(U-1)^k[e^{-\eta(\mathbf{k})}] = \sum_{\substack{(\gamma_1,\dots,\gamma_k)\\ (w_1,\dots,w_k)}} \varepsilon(\gamma_1)\cdots\varepsilon(\gamma_k)\varepsilon(w_1)\cdots\varepsilon(w_k), \tag{7.3.3}$$

where the sum ranges over all k–tuples $(\gamma_1,\dots,\gamma_k)\in\Omega^k$ and $(w_1,\dots,w_k)\in W^k$ such that

- $w_i \in \mathcal{J}(\gamma_i)\cup\{e\},\ 1\leq i\leq k,$
- $I(w_1)\,\dot\cup\cdots\dot\cup\, I(w_k) = \{\alpha_i : i\in I^{\mathrm{re}}, k_i = 1\},$
- $I(w_i)\cup I(\gamma_i)\neq\emptyset$ for each $1\leq i\leq k$,
- $\gamma_1+\cdots+\gamma_k = \sum_{i\in I^{\mathrm{im}}} k_i\alpha_i.$

It follows that $(I(w_1)\cup I(\gamma_1),\dots,I(w_k)\cup I(\gamma_k))\in P_k(\mathbf{k},G)$, i.e., the sum ranges over all elements in $P_k(\mathbf{k},G)$. Hence,

$$(U-1)^k[e^{-\eta(\mathbf{k})}] = (-1)^{\mathrm{ht}(\eta(\mathbf{k}))}|P_k(\mathbf{k},G)|. \tag{7.3.4}$$

Therefore using Equation (7.2.1), we have

$$\begin{aligned} U^q[e^{-\eta(\mathbf{k})}] &= \sum_{k\geq 0}\binom{q}{k}(U-1)^k[e^{-\eta(\mathbf{k})}] = \sum_{k\geq 0}\binom{q}{k}(-1)^{\mathrm{ht}(\eta(\mathbf{k}))}|P_k(\mathbf{k},G)| \\ &= (-1)^{\mathrm{ht}(\eta(\mathbf{k}))}\,\pi^G_{\mathbf{k}}(q). \end{aligned}$$

□

Remark 7.3.5. In order to extend the [3, Theorem 1] to the case of BKM superalgebras, we need the extra assumption, $k_i \leq 1$ for $i\in\Psi_0$. If $k_i > 1$ for some $i\in\Psi_0$, then each γ_i

contributes $I_m(\gamma_i)$ to the required coefficient in Equation (7.3.2). Note that $I_m(\gamma_i)$ and hence $I(w_i)\cup I_m(\gamma_i)$ can be multisets. Since the independent sets considered in Equation (7.2.1) are not multisets, by assuming $k_i \leq 1$ for $i \in \Psi_0$, we dismiss the possibility of $I_m(\gamma_i)$ to be a multiset.

Proof of Theorem 7.3.1: By the denominator identity (5.5.1),

$$U = \frac{\prod_{\alpha\in\Delta_+^0}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}{\prod_{\alpha\in\Delta_+^1}(1+e^{-\alpha})^{\mathrm{mult}(\alpha)}}, \tag{7.3.5}$$

and hence,

$$\begin{aligned} U^q = \frac{\prod_{\alpha\in\Delta_+^0}(1-e^{-\alpha})^{q\,\mathrm{mult}(\alpha)}}{\prod_{\alpha\in\Delta_+^1}(1+e^{-\alpha})^{q\,\mathrm{mult}(\alpha)}} &= \prod_{\alpha\in\Delta_+}(1-\varepsilon(\alpha)e^{-\alpha})^{\varepsilon(\alpha)q\,\mathrm{mult}(\alpha)} \\ &= \prod_{\alpha\in\Delta_+}(\sum_{k\geq 0}(-\varepsilon(\alpha))^k\binom{\varepsilon(\alpha)q\,\mathrm{mult}(\alpha)}{k}e^{-k\alpha}). \end{aligned} \tag{7.3.6}$$

where $\varepsilon(\alpha)=1$ if $\alpha\in\Delta_+^0$ and -1 if $\alpha\in\Delta_+^1$.

On the other hand, by Proposition 7.3.4,

$$U^q[e^{-\eta(\mathbf{k})}] = (-1)^{\mathrm{ht}(\eta(\mathbf{k}))}\,\pi_{\mathbf{k}}^G(q).$$

Hence, from Equation 7.3.6, we get,

$$\pi_{\mathbf{k}}^G(q) = (-1)^{\mathrm{ht}(\eta(\mathbf{k}))}\sum_{\mathbf{J}\in L_G(\mathbf{k})}(-1)^{|\mathbf{J}|+|\mathbf{J}_1|}\prod_{J\in\mathbf{J}_0}\binom{q\ \mathrm{mult}(\beta(J))}{D(J,\mathbf{J})}\prod_{J\in\mathbf{J}_1}\binom{-q\ \mathrm{mult}(\beta(J))}{D(J,\mathbf{J})}. \quad \square$$

Formula for multiplicities of free roots

Let $\mathcal{A} := \mathbb{C}[[X_i : i\in I]]$ with $X_i = e^{-\alpha_i}$ be the algebra of formal power series. For $\zeta\in\mathcal{A}$ with constant term 1, $\log(\zeta) = -\sum_{k\geq 1}\frac{(1-\zeta)^k}{k}$ is well–defined.

Proof of Corollary 7.3.2: Considering U as an element of $\mathcal{A}$ [c.f. Equation 7.3.1], using Proposition 7.3.4,

$$\begin{aligned} -log(U) &= \sum_{k\geq 1}\frac{(1-U)^k}{k}, \\ i.e., -log(U)[e^{-\eta(\mathbf{k})}] &= \sum_{k\geq 1}\frac{(1-U)^k[e^{-\eta(\mathbf{k})}]}{k}, \\ &= \sum_{k\geq 1}\frac{(-1)^k}{k}(-1)^{\mathrm{ht}(\eta(\mathbf{k})}|P_k(\mathbf{k},G)| \quad (\text{ By Equation 7.3.4}) \\ &= (-1)^{\mathrm{ht}(\eta(\mathbf{k}))}\sum_{k\geq 1}\frac{(-1)^k}{k}|P_k(\mathbf{k},G)| \\ &= |\pi^G_{\mathbf{k}}(q)[q]| \quad (\text{ By Equation 7.2.1}). \end{aligned}$$

Now applying $-\log$ to Equation 7.3.5, we get,

$$\sum_{\substack{\ell\in\mathbb{N}\\ \ell|\mathbf{k}}}\frac{1}{\ell}\,\mathrm{mult}\,(\eta\,(\mathbf{k}/\ell)\,) = |\pi^G_{\mathbf{k}}(q)[q]| \quad \text{ if } \beta(\mathbf{k})\in\Delta^+_0 \tag{7.3.7}$$

and

$$\sum_{\substack{\ell\in\mathbb{N}\\ \ell|\mathbf{k}}}\frac{(-1)^{\ell+1}}{\ell}\,\mathrm{mult}\,(\eta\,(\mathbf{k}/\ell)\,) = |\pi^G_{\mathbf{k}}(q)[q]| \quad \text{ if } \beta(\mathbf{k})\in\Delta^+_1. \tag{7.3.8}$$

The statement of the corollary is now an easy consequence of the following Möbius inversion formula: $g(d)=\sum_{d|n}f(d) \iff f(n)=\sum_{d|n}\mu(\frac{n}{d})g(d)$ where μ is the möbius function. □

Example 7.3.6. Consider the BKM superalgebra $\mathfrak{L}$ and the root space $\eta(\mathbf{k})=3\alpha_3+3\alpha_6\in\Delta^1_+$ from Example 6.1.13. The $\mathbf{k}$-chromatic polynomial of the quasi Dynkin diagram G of $\mathfrak{L}$ is equal to

$$\pi^G_{\mathbf{k}}(q) = \binom{q}{3}\binom{q-3}{3} = \frac{1}{3!3!}q(q-1)(q-2)(q-3)(q-4)(q-5).$$

By Corollary 7.3.2, since $\eta(\mathbf{k})$ is odd,

$$\begin{aligned}\operatorname{mult}(\eta(\mathbf{k})) &= \sum_{\ell|\mathbf{k}} \frac{(-1)^{\ell+1}\mu(\ell)}{\ell}\, |\pi^G_{\mathbf{k}/\ell}(q)[q]| \\ &= |\pi^G_{\mathbf{k}}(q)[q]| + \frac{\mu(3)}{3} |\pi^G_{\mathbf{k}'}(q)[q]| \text{ where } \mathbf{k}' = (0,0,1,0,0,1) \\ &= \frac{10}{3} - \frac{1}{3} = 3\end{aligned}$$

Example 7.3.7. Consider the BKM superalgebra $\mathfrak{L}$ from the previous example. Let $\mathbf{k} = (2,1,0,1,2,0) \in \mathbb{Z}^I_+$. Then $\eta(\mathbf{k}) = 2\alpha_1 + \alpha_2 + \alpha_4 + 2\alpha_5 \in \Delta^0_+$. We have

$$\operatorname{mult}(\eta(\mathbf{k})) = \sum_{\ell|\mathbf{k}} \frac{\mu(\ell)}{\ell}\, |\pi^G_{\mathbf{k}/\ell}(q)[q]|$$

This implies that $\operatorname{mult}(\eta(\mathbf{k})) = |\pi^G_{\mathbf{k}}(q)[q]|$. We have $\pi^G_{\mathbf{k}}(q) = \frac{1}{4}q(q-1)^3(q-2)^2$. Therefore $\operatorname{mult}(\eta(\mathbf{k})) = 1$.

□

Bibliography

[1] E. Ardonne and R. Kedem. Fusion products of Kirillov-Reshetikhin modules and fermionic multiplicity formulas. *J. Algebra*, 308(1):270–294, 2007.

[2] G. Arunkumar. Chromatic symmetric function of graphs from Borcherds algebras. *J. Combin. Theory Ser. A*, 183:Paper No. 105492, 28, 2021.

[3] G. Arunkumar, D. Kus, and R. Venkatesh. Root multiplicities for Borcherds algebras and graph coloring. *J. Algebra*, 499:538–569, 2018.

[4] L. Barth and D. Kus. Graded decompositions of fusion products in rank 2. *Kyoto J. Math.*, 62(3):547–576, 2022.

[5] R. Biswal, V. Chari, L. Schneider, and S. Viswanath. Demazure flags, Chebyshev polynomials, partial and mock theta functions. *J. Combin. Theory Ser. A*, 140:38–75, 2016.

[6] R. Biswal, V. Chari, P. Shereen, and J. Wand. Macdonald polynomials and level two Demazure modules for affine $\mathfrak{sl}_{n+1}$. *J. Algebra*, 575:159–191, 2021.

[7] R. Biswal and D. Kus. A combinatorial formula for graded multiplicities in excellent filtrations. *Transform. Groups*, 26(1):81–114, 2021.

[8] R. Biswal and D. Kus. A combinatorial formula for graded multiplicities in excellent filtrations. *Transform. Groups*, 26(1):81–114, 2021.

[9] R. Borcherds. Generalized Kac-Moody algebras. *J. Algebra*, 115(2):501–512, 1988.

[10] M. Cederwall and J. Palmkvist. L_∞ algebras for extended geometry from Borcherds superalgebras. *Comm. Math. Phys.*, 369(2):721–760, 2019.

[11] V. Chari. Integrable representations of affine Lie-algebras. *Invent. Math.*, 85(2):317–335, 1986.

[12] V. Chari and B. Ion. BGG reciprocity for current algebras. *Compos. Math.*, 151(7):1265–1287, 2015.

[13] V. Chari and M. Kayla. Graded representations of current algebras., 2018.

[14] V. Chari and S. Loktev. Weyl, Demazure and fusion modules for the current algebra of $\mathfrak{sl}_{r+1}$. *Adv. Math.*, 207(2):928–960, 2006.

[15] V. Chari and A. Pressley. Weyl modules for classical and quantum affine algebras. *Represent. Theory*, 5:191–223, 2001.

[16] V. Chari and R. Venkatesh. Demazure modules, fusion products and Q-systems. *Comm. Math. Phys.*, 333(2):799–830, 2015.

[17] M. C. N. Cheng and A. Dabholkar. Borcherds-Kac-Moody symmetry of $\mathcal{N} = 4$ dyons. *Commun. Number Theory Phys.*, 3(1):59–110, 2009.

[18] S.-J. Cheng, J.-H. Kwon, and W. Wang. Irreducible characters of Kac-Moody Lie superalgebras. *Proc. Lond. Math. Soc. (3)*, 110(1):108–132, 2015.

[19] E. S. Chibrikov. A right normed basis for free Lie algebras and Lyndon-Shirshov words. *J. Algebra*, 302(2):593–612, 2006.

[20] E. S. Chibrikov. The right-normed basis of a free Lie superalgebra and Lyndon-Shirshov words. *Algebra Logika*, 45(4):458–483, 504, 2006.

[21] M.-K. Chuah and R. Fioresi. Equal rank real forms of affine non-twisted Kac-Moody Lie superalgebras. *J. Pure Appl. Algebra*, 224(7):106278, 18, 2020.

[22] A. Darehgazani, F. Shirnejad, and M. Yousofzadeh. Affine Lie superalgebras. *J. Algebra*, 570:636–677, 2021.

[23] B. Feigin and S. Loktev. On generalized Kostka polynomials and the quantum Verlinde rule. In *Differential topology, infinite-dimensional Lie algebras, and applications*, volume 194 of *Amer. Math. Soc. Transl. Ser. 2*, pages 61–79. Amer. Math. Soc., Providence, RI, 1999.

[24] G. Fourier. New homogeneous ideals for current algebras: filtrations, fusion products and Pieri rules. *Mosc. Math. J.*, 15(1):49–72, 181, 2015.

[25] G. Fourier and D. Hernandez. Schur positivity and Kirillov-Reshetikhin modules. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 10:Paper 058, 9, 2014.

[26] G. Fourier and P. Littelmann. Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions. *Adv. Math.*, 211(2):566–593, 2007.

[27] G. Fourier, V. Martins, and A. Moura. On truncated Weyl modules. *Comm. Algebra*, 47(3):1125–1146, 2019.

[28] P. D. Francesco and R. Kedem. Proof of the combinatorial Kirillov-Reshetikhin conjecture. *Int. Math. Res. Not. IMRN*, (7):Art. ID rnn006, 57, 2008.

[29] I. B. Frenkel and N. Y. Reshetikhin. Quantum affine algebras and holonomic difference equations. *Comm. Math. Phys.*, 146(1):1–60, 1992.

[30] M. R. Gaberdiel, S. Hohenegger, and D. Persson. Borcherds algebras and $\mathcal{N} = 4$ topological amplitudes. *J. High Energy Phys.*, (6):125, 34, 2011.

[31] A. Gadde, P. Liendo, L. Rastelli, and W. Yan. On the integrability of planar $\mathcal{N} = 2$ superconformal gauge theories. *J. High Energy Phys.*, (8):015, front matter+17, 2013.

[32] H. Garland. The arithmetic theory of loop algebras. *J. Algebra*, 53(2):480–551, 1978.

[33] M. M. Halldórsson and G. Kortsarz. Multicoloring: problems and techniques. In *Mathematical foundations of computer science 2004*, volume 3153 of *Lecture Notes in Comput. Sci.*, pages 25–41. Springer, Berlin, 2004.

[34] M. Henneaux, B. L. Julia, and J. Levie. E_{11}, Borcherds algebras and maximal supergravity. *J. High Energy Phys.*, (4):078, front matter+30, 2012.

[35] J. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Graduate texts in mathematics. Springer, 1972.

[36] B. Ion. Nonsymmetric Macdonald polynomials and Demazure characters. *Duke Math. J.*, 116(2):299–318, 2003.

[37] E. Jurisich. An exposition of generalized Kac-Moody algebras. In *Lie algebras and their representations (Seoul, 1995)*, volume 194 of *Contemp. Math.*, pages 121–159. Amer. Math. Soc., Providence, RI, 1996.

[38] E. Jurisich. Generalized Kac-Moody Lie algebras, free Lie algebras and the structure of the Monster Lie algebra. *J. Pure Appl. Algebra*, 126(1-3):233–266, 1998.

[39] V. G. Kac. Lie superalgebras. *Advances in Math.*, 26(1):8–96, 1977.

[40] T. Khandai and S. Rani. Fusion Product Modules for current algebra of type A_2. *arXiv e-prints*, page arXiv:2303.03337, Mar. 2023.

[41] J.-A. Kim and D.-U. Shin. Construction of Kac-Moody superalgebras as minimal graded Lie superalgebras and weight multiplicities for Kac-Moody superalgebras. *J. Math. Phys.*, 41(7):4981–5001, 2000.

[42] N. Kwon. Weyl-Kac type character formula for admissible representations of Borcherds-Kac-Moody Lie superalgebras. *Math. Z.*, 295(1-2):711–725, 2020.

[43] P. Lalonde. Bases de Lyndon des algèbres de Lie libres partiellement commutatives. volume 117, pages 217–226. 1993. Conference on Formal Power Series and Algebraic Combinatorics (Bordeaux, 1991).

[44] P. Lalonde. Lyndon heaps: an analogue of Lyndon words in free partially commutative monoids. *Discrete Math.*, 145(1-3):171–189, 1995.

[45] A. A. Mikhalëv. Subalgebras of free colored Lie superalgebras. *Mat. Zametki*, 37(5):653–661, 779, 1985.

[46] K. Naoi. Weyl modules, Demazure modules and finite crystals for non-simply laced type. *Adv. Math.*, 229(2):875–934, 2012.

[47] K. Naoi. Tensor products of Kirillov-Reshetikhin modules and fusion products. *Int. Math. Res. Not. IMRN*, (18):5667–5709, 2017.

[48] J. Palmkvist. Borcherds and Kac-Moody extensions of simple finite-dimensional Lie algebras. *J. High Energy Phys.*, (6):003, front matter+9, 2012.

[49] S. Rani and G. Arunkumar. A study on free roots of Borcherds-Kac-Moody Lie Superalgebras. *arXiv e-prints*, page arXiv:2103.12332, Mar. 2021.

[50] U. Ray. A character formula for generalized Kac-Moody superalgebras. *J. Algebra*, 177(1):154–163, 1995.

[51] U. Ray. *Automorphic forms and Lie superalgebras*, volume 5 of *Algebra and Applications*. Springer, Dordrecht, 2006.

[52] C. Reutenauer. *Free Lie algebras*, volume 7 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications.

[53] R. Venkatesh and S. Viswanath. Unique factorization of tensor products for Kac-Moody algebras. *Adv. Math.*, 231(6):3162–3171, 2012.

[54] R. Venkatesh and S. Viswanath. Chromatic polynomials of graphs from Kac-Moody algebras. *J. Algebraic Combin.*, 41(4):1133–1142, 2015.

[55] G. X. Viennot. Heaps of pieces. I. Basic definitions and combinatorial lemmas. In *Combinatoire énumérative (Montreal, Que., 1985/Quebec, Que., 1985)*, volume 1234 of *Lecture Notes in Math.*, pages 321–350. Springer, Berlin, 1986.

[56] M. Wakimoto. *Infinite-dimensional Lie algebras*, volume 195 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2001. Translated from the 1999 Japanese original by Kenji Iohara, Iwanami Series in Modern Mathematics.