Solutions of First Order Differential Equations in Iterated Strongly Normal Extensions

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Dedicated to

My Maa, Papa and Vai

Declaration

The work presented in this thesis has been carried out by me under the guidance of Dr. Varadharaj R. Srinivasan at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Ursashi Roy

In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Varadharaj R. Srinivasan

(Supervisor)

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Abstract

Let k be a differential field of characteristic zero with an algebraically closed field of constants C. This thesis concerns the problem of finding transcendental solutions of first order (nonlinear) differential equations in an iterated strongly normal extension of k. We deduce the structure of intermediate differential subfields of iterated strongly normal extensions of k that have transcendence degree one. We also produce a family of differential equations with no transcendental solutions in any iterated strongly normal extension of k. We show that if a first order differential equation has a transcendental solution in an iterated strongly normal extension of k, then there can only be a maximum of three $k-$algebraically independent solutions. We end the thesis with a conjecture regarding the algebraic dependence of solutions of a first order differential equation.

We give an independent proof of the fact that every intermediate subfield of a Picard-Vessiot extension is a solution field if and only if the differential Galois group has solvable identity component. This result is then used to give the structure of intermediate differential subfields of a Picard-Vessiot extension whose differential Galois group is connected and solvable.

We analyse transcendental liouvillian solutions of first order differential equations $y' = a_n y^n + \cdots + a_0$, where $a_i \in k$. In which case, the number of algebraic solutions is finite. We deduce a relation between the algebraic and the transcendental solutions. We also show that if a differential equation has a transcendental solution in an exponential extension then the differential equation can be written in terms of the algebraic solutions. When $k = C(x)$ with $x' = 1$, we provide a method of obtaining

transcendental solutions in an exponential extension of $C(x)$.

Contents

List of Notations

$\mathbb{C}$	The field of complex numbers with zero derivation
$\mathbb{Q}$	The field of rational numbers
$\mathbb{C}(x)$	Field of rational functions over $\mathbb{C}$ with derivation d/dx
C	An algebraically closed field with zero derivation
K	A differential field of characteristic zero
$\overline{K}$	Algebraic closure of K
C_K	Field of constants of K
k	A differential field of characteristic zero with a single derivation $'$ such that the field of constants C is algebraically closed
E	A differential field extension of k
$\text{tr.deg}(E/k)$	Transcendence degree of E over k
$\mathscr{G}(E\|k)$	Group of all differential automorphisms of E over k

Chapter 1

Introduction

Throughout this thesis, k stands for a differential field of characteristic zero equipped with a single derivation $'$ such that the field of constants C is algebraically closed. Let $f(Y,Z) \in k[Y,Z]$ be an irreducible polynomial involving the variable Z. By a *solution* of the differential equation $f(y,y') = 0$ we mean an element t in a differential field extension E of k such that the field of constants of $k(t,t')$ and k are the same. If t is transcendental (respectively, algebraic) over k, then it is called a transcendental (respectively, algebraic) solution of the differential equation. A differential field extension E of k is *strongly normal* if E is finitely differentially generated over k and every differential isomorphism σ of E over k satisfies the following conditions:

1. $\sigma|_{C_E} =$id and
2. $E\sigma E = EC(\sigma) = \sigma EC(\sigma)$, where $C(\sigma)$ is the field of constants of $E\sigma E$.

For example, a Picard-Vessiot extension of k is a strongly normal extension. In [17], Kolchin developed the notion of strongly normal extension as a differential analogue of normal extensions of polynomial Galois theory. Let E be a differential field extension of k and $k = E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_{n+1} = E$ be a tower of differential fields. If E_i is a strongly normal extension of E_{i-1} for all i, then E is called an *iterated strongly normal* extension of k. If for each i, $E_i = E_{i-1}(v_i)$, where v_i is either algebraic over E_{i-1} or $v_i' \in E_{i-1}$ or $v_i'/v_i \in E_{i-1}$, then E is called a liouvillian extension of k.

The study of first order differential equations dates back to the work of Fuchs and Poincaré in [12] and [28]. The simplest form of nonlinear differential equation is the Riccati equation. In [21], Kovacic gave an algorithm to determine the liouvillian solutions of a second order linear differential equation with rational function coefficients by providing a degree bound for the algebraic solutions of the corresponding Riccati equation. Thereafter, many algorithms have been given to find liouvillian solutions of linear homogeneous differential equations, as described in the book [40]. In the last few decades, many algorithms have been developed to compute the *rational* and *algebraic general solutions* of nonlinear differential equations using techniques from algebraic geometry [2, 10, 41].

In [37, Proposition 3.1], Srinivasan has provided a necessary and sufficient condition for the first order autonomous differential equation $y' = f(y)$, where $f(y) \in C(y)$, to have a transcendental solution in a liouvillian extension of C. The result of Srinivasan is an extension of [36, Corollary 2]. In [8], the authors give an algorithm to compute the rational liouvillian solutions of a first order autonomous differential equation. However, there is no systematic method to compute transcendental solutions of nonlinear (non-autonomous) differential equations. So it is natural to ask for a characterization of first order nonlinear, non-autonomous differential equations that have transcendental solutions in an iterated strongly normal extension of k. This thesis is focused on achieving that goal. Let E be an iterated strongly normal extension of k and $t \in E$ be a transcendental solution of a first order differential equation over k. Then $k \subseteq k(t, t') \subseteq E$ and $k(t, t')$ is a transcendence degree one extension of k. Therefore, our approach to determining the transcendental solutions of a first order differential equation is to find the structure of the transcendence degree one subfields of an iterated strongly normal extension of k. We also deduce the structure of intermediate subfields of liouvillian Picard-Vessiot extensions. As liouvillian extensions of k with C as its field of constants are well known examples of iterated strongly normal examples, we analyse first order differential equations that have transcendental liouvillian solutions and also provide an algorithm for finding such solutions.

In Chapter 2, we record known results from differential algebra for easy reference. We also prove a result (Lemma 2.3.3) that gives the relationship between the algebraic and transcendental solutions of the differential equation $y' = a_n y^n + a_{n-1} y^{n-1} + \cdots + a_0$, where $a_i \in k$, $a_n \neq 0$. The lemma is then used to prove that if the above differential equation has a transcendental liouvillian solution then the number of algebraic solutions is finite.

In [38, Theorem B], the author has classified transcendence degree one subfields of a liouvillian extension of a given field. In Chapters 3 and 4, we use the above result and polynomial computations to find transcendental liouvillian solutions of first order nonlinear differential equations over $C(x)$, where C is an algebraically closed field with zero derivation and $x' = 1$.

In Chapter 3, we analyse the transcendental liouvillian solutions of the variable separable differential equation

$$y' = r(x)\, F(y), \tag{1.1}$$

where $r(x)$ is a nonzero polynomial in $C[x]$ and $F(y)$ is a nonzero polynomial in $C[y]$. We show that if the above differential equation has a transcendental liouvillian solution, then all the roots of $F(y)$ are simple and these are the only algebraic solutions of the differential equation (Theorem 3.1.2). The solution lies in an exponential extension of $C(x)$ if and only if $\frac{1}{F(y)} = \sum_{i=1}^{n} \frac{m_i}{y-\alpha_i}$, where n is a positive integer, m_i's are nonzero integers and α_i's are pairwise distinct elements of C. We provide a class of differential equations that have transcendental liouvillian solutions but no algebraic solutions (Theorem 3.2.3).

Let y be a transcendental liouvillian solution of the differential equation

$$y' = a_n y^n + a_{n-1} y^{n-1} + \cdots + a_0, \tag{1.2}$$

where $a_i \in C(x)$ and $a_n \neq 0$. Using [37, Theorem 2.2] and [38, Theorem B], we conclude that only one of the following can occur: either there is an element $z \in C(x,y) \setminus C(x)$ such that $z' = az + b$, where a and b are nonzero elements of $C(x)$ or there is an element $z \in \overline{C(x)}(y) \setminus \overline{C(x)}$ satisfying $z' = az$, for some nonzero $a \in \overline{C(x)}$. In Chapter 4, we are mainly interested in the latter case. Using Lemma 2.3.3, we show that z can be written as $z = g\prod_{i=1}^{l}(y-\alpha_i)^{m_i}$, where g is a nonzero element of $\overline{C(x)}$, $\alpha_1, \ldots, \alpha_l$ are algebraic solutions of Equation (1.2) and $m_1, \ldots, m_l$ are nonzero integers. We give a necessary and sufficient condition (in terms of the algebraic solutions) for the existence of transcendental solutions of the differential equation in an exponential extension of $C(x)$ (see Theorem 4.1.3). We show that the number of algebraic solutions is at least n, where n is the degree of the polynomial $a_n y^n + \cdots + a_0 \,(= y')$. Many classes of differential equations satisfying the conditions of Theorem 4.1.3, such as autonomous differential equations, have precisely n distinct algebraic solutions. We show that if this phenomenon occurs, then differential equation (1.2) can be written as follow:

$$y' = a_n \prod_{i=1}^{n}(y-\alpha_i) + \sum_{i=1}^{n}\left(\prod_{j=1, j\neq i}^{n} \frac{y-\alpha_j}{\alpha_i-\alpha_j}\right)\alpha_i',$$

where $a_n \in C(x)$ and $\alpha_1, \ldots, \alpha_n$ are the algebraic solutions. So the differential equation can be expressed in terms of its algebraic solutions. This gives us a large class of differential equations that have transcendental liouvillian solutions. Using these results, we provide a method to determine whether a differential equation has a transcendental liouvillian solution if the minimal polynomials of the algebraic solutions are known. We apply our results to solve Abel's differential equation of the first kind.

The focus of this thesis is to classify first order differential equations that have transcendental solutions in iterated strongly normal extensions of k. To do so, we have to determine the structure of differential subfields of a liouvillian Picard-Vessiot extension of k. Chapter 5 is devoted to this purpose. We have used well known results of algebraic geometry, linear algebraic groups and the work of Kolchin in [17, 15] to obtain our results.

Let E be a Picard-Vessiot extension of k, K be a differential field intermediate to E and k. Let $T(K|k)$ be the set of all elements of K which are zeros of homogeneous linear differential equations over k. It is known that $T(E|k)$ is a finitely generated simple differential $k-$algebra whose field of fractions $\mathcal{Q}(T(E|k))$ equals the differential field E. We show that every intermediate differential field K is the field of fractions of $T(K|k)$ if and only if E is a liouvillian Picard-Vessiot extension of k (Theorem 5.2.1). We also show that if the differential Galois group $\mathscr{G}(E|k)$ is connected solvable linear algebraic group then any intermediate differential field K is given by $K = k(t_1, \cdots, t_n)$, where for each i, $t_i \in T(K|k)$, $t_i' = a_i t_i + b_i$ for $a_i \in k$ and $b_i \in k(t_1, \cdots, t_{i-1})$ (Corollary 5.2.2). It is possible to deduce Theorem 5.2.1 from [1]. We have shown this derivation as well.

In Chapter 6, we give a structure theorem for transcendence degree one intermediate differential subfields of an iterated strongly normal extension of k. In Theorem

6.2.5, we show that if E is an iterated strongly normal extension of k and K is an intermediate differential subfield of transcendence degree one, then there is a finite algebraic extension $\tilde{k}$ of k such that $\tilde{k}K = \tilde{k}(t, t', z)$, where z is algebraic over $\tilde{k}(t, t')$, and t is a transcendental solution of a Riccati or a Weierstrass differential equation over $\tilde{k}$. Using this we classify a first order differential equation $f(y, y') = 0$ into the following types:

- *Algebraic type*: All the solutions of $f(y, y') = 0$ are algebraic over k.
- *Riccati type*: The differential equation $f(y, y') = 0$ has a transcendental solution y such that there is a finite algebraic extension $\tilde{k}$ of k and an element $t \in \tilde{k}(y, y')$ such that $\tilde{k}(y, y')$ is a finite algebraic extension of $\tilde{k}(t)$ and t is a solution of a Riccati differential equation:

$$t' = a_2 t^2 + a_1 t + a_0, \text{ with } a_0, a_1, a_2 \in \tilde{k}, \text{ not all zero.}$$

- *Weierstrass type*: The differential equation $f(y, y') = 0$ has a transcendental solution y such that there is a finite algebraic extension $\tilde{k}$ of k and an element $t \in \tilde{k}(y, y')$ such that $\tilde{k}(y, y')$ is a finite algebraic extension of $\tilde{k}(t, t')$ and t is a solution of a Weierstrass differential equation:

$$(t')^2 = \alpha^2(4t^3 - g_2 t - g_3), \text{ with } g_2, g_3 \in C, \alpha \in \tilde{k} \text{ and } 27g_3^2 - g_2^3 \neq 0.$$

- *General type*: The differential equation is not of any of the above types.

In this thesis, we will not be concerned with differential equations of algebraic type. In Theorem 6.3.1, we describe a family of irreducible plane curves f such that the differential equation $f(y, y') = 0$ is of general type. A subfamily of examples includes Abel differential equations of the form

$$y' = a_n y^n + \cdots + a_2 y^2, \tag{1.3}$$

where $n \geq 3$, $a_i \in k$ and both a_2 and a_3 have no antiderivatives in k. The case when $n = 3$, $a_2 = -1$ and $a_3 = 1$ was extensively discussed in [32] and [39].

In [39], Top et al. classify first order autonomous differential equations. Our classification coincides with theirs when $k = C$. The authors show that an autonomous differential equation of general type has no transcendental solution in any iterated Picard-Vessiot extension of C. Thus, our results are a generalisation of their work. Our approach is more algebraic than geometric. The aforementioned paper discusses the algebraic independence of transcendental solutions of first order autonomous differential equations. It is shown that there is a subclass of the general type such that any number of distinct transcendental solutions are $C-$algebraically independent. We show that Equation (1.3) also satisfies this property (see Proposition 6.3.5) but differential equations (respectively, autonomous differential equations) of nongeneral type have at most three (respectively, at most one) $k-$algebraically independent (respectively, $C-$algebraically independent) solutions in any no new constant extension of k (respectively, C) (see Theorem 6.3.6).

In a recent article [11], it was shown that if any four (respectively, two) transcendental solutions of a first order differential equation (respectively, an autonomous differential equation) are $k-$algebraically (respectively, $C-$algebraically) independent, then any m distinct transcendental solutions are algebraically independent. Thus, in view of Theorem 6.3.6 we put forth the following:

Conjecture A first order differential equation (respectively, an autonomous differential equation) over k (respectively, over C) is not of general type if and only if it has at most three (respectively, one) $k-$algebraically independent (respectively, $C-$algebraically independent) solutions in any given no new constant extension of k (respectively, C).

We prove the above conjecture for rational autonomous differential equations, that is, $y' = f(y)$, where $f(y)$ is a nonzero rational function over C.

Chapter 2

Preliminaries

In this chapter, we will recall basic definitions and results from differential algebra. For details one may refer to [14], [23], [40] and [16].

2.1 Basic conventions

Definition 2.1.1. Let R be a ring. A mapping $\delta : r \in R \mapsto r' \in R$ is called a *derivation* if, for all $x, y \in R$, $(x+y)' = x' + y'$ and $(xy)' = x'y + xy'$. A ring together with a derivation map is called a *differential ring*.

In this section, R stands for a differential ring. The derivation on a ring that maps every element to zero is called the *zero or trivial derivation*. For example $R_1 = \mathbb{Q}[x], R_2 = \mathbb{Q}[z]$ with derivation $r' = 0$ for all $r \in \mathbb{Q}$ and $x' = 1, z' = z$ are differential rings.

Suppose R is a differential integral domain and Q_R is the associated quotient field.

Then any derivation $\delta : r \to r'$ on R extends to Q_R via the quotient rule

$$\left(\frac{r}{s}\right)' = \frac{r's - rs'}{s^2}$$

and this is the unique extension of δ to Q_R.

Definition 2.1.2. An ideal I of a differential ring R is called a *differential ideal* if it is closed under derivation, that is, $x' \in I$ for all $x \in I$. Furthermore, if I is a radical (respectively, prime) ideal of R, it is called a radical (respectively, prime) differential ideal.

Note that the differential ring R_1 has no proper differential ideals, whereas $I_n = \mathbb{Q}[z^n]$ are the differential ideals of R_2. R/I can be given a natural differential ring structure by defining $(r + I)' = r' + I$ for all $r \in R$.

Suppose R and S are differential rings with derivation δ_R and δ_S respectively. Suppose that $R \subseteq S$ such that $\delta_S|_R = \delta_R$. Then R is called a *differential subring* of S and $R \subseteq S$ is a *differential ring extension*, the derivation δ_R on R is said to extend to the derivation δ_S on S, and δ_S is said to be an extension of δ_R.

Definition 2.1.3. An element $c \in R$ is called a *constant* if $c' = 0$ and the set of all constants of R is denoted by C_R. A differential ring extension $R \subseteq S$ is called *no new constant extension* if $C_R = C_S$. If R is a differential field, then C_R is a differential subfield of R called the *field of constants.*

For example, the field of constants of $\mathbb{Q}(x)$ with$x' = 1$ is $\mathbb{Q}$.

Definition 2.1.4. Let (R, δ_R) and (S, δ_S) be two differential rings. A ring homomorphism $\phi : R \to S$ is called a *differential homomorphism* if ϕ commutes with the derivations, that is, $\phi \circ \delta_R = \delta_S \circ \phi$.

The kernel of any differential homomorphism $\phi : R \to S$ is a differential ideal of R and $\phi(R)$ is a differential subring of S. Let $K \subseteq E$ be differential fields. Let $\mathscr{G}(E|K)$ be the set of all differential automorphisms $\sigma : E \to E$ such that $\sigma|_K = \mathrm{id}$. Then $\mathscr{G}(E|K)$ is a group under usual composition. The following lemma describes the differential Galois group in two important cases.

Lemma 2.1.5 (cf. [14, Lemma 3.9 and Lemma 3.10])**.** *Let $K \subset M \subset E$ be differential fields with $C_E = C_K$. Suppose that $w, z \in E \setminus K$ such that w and z are transcendental over K.*

(i) If $M = K(w)$, where $w' \in K$, then $\mathscr{G}(M|K)$ is isomorphic to C_K. For each $c \in C_K$, $\sigma_c : M \to M$ defined by $\sigma_c(w) = w + c$ are the differential automorphisms of M over K.

(ii) If $M = K(z)$, where $z'/z \in K$, then $\mathscr{G}(M|K)$ is isomorphic to C_K^*. For each nonzero $c \in C_K$, $\sigma_c : M \to M$ defined by $\sigma_c(z) = cz$ are the differential automorphisms of M over K.

Consider the ring $R[y_0, y_1, y_2, \dots]$ of polynomials in infinite number of ordinary indeterminates. A unique derivation of $R[y_0, y_1, y_2, \dots]$ is determined by assigning $y_i' = y_{i+1}$. Change the notation so that

$$y_0 = y, \dots, y_n = y^{(n)}.$$

This procedure is called *adjunction of differential indeterminate* and is denoted by $R\{y\}$. The ring $R\{y\}$ is called the *ring of differential polynomials in the variable y* and its elements are called *differential polynomials.* If R is a differential integral domain with quotient field Q_R, then $R\{y\}$ is also a differential integral domain, whose fraction field is denoted by $Q_R\langle y\rangle$.

Note that the elements of $R\{y\}$ can also be regarded as differential operators on R as there is an obvious ring homomorphism $R\{y\} \to \mathrm{End}(R)$ which maps $y^{(i)}$ to δ^i.

Let $R \subseteq S$ be an extension of differential rings and X be a subset of S. Then $R\{X\}$ will denote the differential sub$-R-$algebra of S generated by X. If R and S are differential fields, then $R\langle X\rangle$ will denote the differential subfield of S generated by R and X.

2.2 Extending derivations

The following results provide important criteria for extending the derivation of a differential field to a no new constant extension.

Lemma 2.2.1 (cf. [4, Theorems 6.2.5, 6.2.6])**.** *Let F be a differential field of characteristic zero, E be a field extension of F and $w \in E$.*

(i) If w is algebraic over F, then the derivation on F extends uniquely to a derivation on $F(w)$. Moreover, if C_F is algebraically closed, then $C_{F(w)} = C_F$.

(ii) If w is transcendental over F, then $w' \in F(w)$ can be assigned arbitrarily and the derivation on F can be extended to $F(w)$. Moreover, if F is a field with trivial derivation, then for any $w' \in F(w) \setminus \{0\}$, $C_{F(w)} = C_F$.

Note that if w is transcendental over F, then $C_{F(w)}$ may not be equal to C_F even if C_F is algebraically closed.

Remark 2.2.2. Let C denote an algebraically closed field of characteristic zero with trivial derivation. By Lemma 2.2.1 (ii), the trivial derivation on C naturally extends to a derivation on the field $C(x)$ of rational functions by letting the derivative of x

be equal to 1. Also, the field of constants of $C(x)$ is C. Therefore by Lemma 2.2.1 (i), the field of constants of $\overline{C(x)}$ is also C.

The following result is well known. We prove the first part here. The other cases are similar. For proofs of the remaining parts one may refer to [4, Lemma 6.4.3 (b)].

Proposition 2.2.3. *Let $F \subseteq E$ be differential fields of characteristic zero, $w \in E$ be transcendental over F and $\alpha, \beta \in F \setminus \{0\}$. Then the following statements hold:*

(i) If there is no nonzero element $z \in F(w)$ such that $z' = n\alpha z$ for any nonzero integer n, then the differential field extension $F(w)$, where $w' = \beta w + \beta$ and $\beta \neq 0$, satisfies $C_{F(w)} = C_F$.

(ii) If there is no $z \in F$ such that $z' = \beta$, then the differential extension $F(w)$ of F satisfying $w' = \alpha$ also satisfies $C_F = C_{F(w)}$.

(iii) Suppose that for any positive integer l, there is no element $x \in F$ such that $x' = l\alpha x$. Then the derivation on F admits a unique extension to $F(w)$ satisfying both $w' = \alpha w$ and $C_{F(w)} = C_F$.

Proof. We will prove the first part. Suppose that $p \in F[w] \setminus F$ such that $p' = 0$. Let $p = a_n w^n + a_{n-1} w^{n-1} + \cdots + a_0$, where $a_i \in F$ and $a_n \neq 0$. Now,

$$0 = p' = \sum_{i=0}^{n} a_i' w^i + \sum_{i=1}^{n} i a_i w^{i-1} (\alpha w + \beta)$$

Comparing the coefficient of w^n we get that $a_n' = -n\alpha a_n$. This contradicts the hypothesis of the first part. Suppose that $(p/q)' = 0$, where p, q are nonzero relatively prime elements of $F[w]$. We may assume that q is a monic polynomial in $F[w] \setminus F$. Let $q = w^m + b_{m-1} w^{m-1} + \cdots + b_0$, where $b_i \in F$ and $m \geq 1$. Then $q' = m\alpha w^m + \left(b_{m-1}' + m\beta + (m-1) b_{m-1} \alpha\right) w^{n-1} + \cdots + (b_0' + b_1 \beta)$. Now,

$$0 = \left(\frac{p}{q}\right)' = \frac{qp' - pq'}{q^2} \implies q | q'.$$

From our previous observation $q' \neq 0$. Therefore $q' = m\alpha q$. Comparing the coefficient of w^{m-1} we get $b'_{m-1} + m\beta + (m-1)\alpha b_{m-1} = m\alpha b_{m-1}$. Now observe that $(mw + b_{m-1})' = \alpha\,(mw + b_{m-1})$. This again contradicts the hypothesis of the first part. Therefore $C_{F(w)} = C_F$. $\square$

Proposition 2.2.4 (cf. [23, Example 1.10, 1.11])**.** *Let $F \subseteq E = F\langle w\rangle$ be an extension of differential fields such that the characteristic of F is zero, C_F is algebraically closed and $C_E = C_F$.*

(i) If $w' \in F$, then either $E = F$ or $E = F(w)$ is a purely transcendental extension of F. In the latter case, there is no $z \in F$ such that $z' = w'$.

(ii) If $w'/w \in F$, then either $w^n \in F$ for some $n \in \mathbb{Z}$ or w is purely transcendental over F.

Let F be a differential field such that C_F is algebraically closed and w be an indeterminate over F. Then by Lemma 2.2.1, w' can be assigned arbitrarily so that $F(w)$ is a differential field extension of F. Suppose $b \in F$ such that b does not have an antiderivative in F and if we define $w' = b$, then by Proposition 2.2.3 (i), $F(w)$ is a no new constant extension of F. If b has an antiderivative x in F, then the adjunction of w gives new constants as

$$(w - x)' = b - b = 0.$$

Let $a \in F \setminus \{0\}$ and suppose for any $n \in \mathbb{N}$ there is no $f \in F$ such that $f' = naf$. If we define $w' = aw$, then by Proposition 2.2.3 (ii), $F(w)$ is a no new constant extension of F. If there exists $f \in F$ such that $f' = naf$, then

$$\left(\frac{w^n}{f}\right)' = 0.$$

These are two important types of building blocks for larger extensions. The first is called extension by *adjunction of integrals* and the second is called extension by adjunction of *exponential of an integral*. Next, we will discus differential fields constructed using these building blocks.

Definition 2.2.5. A differential field E is called a *liouvillian extension* (respectively, an *elementary extension*) of F if F is a differential subfield of E and $E = F(t_1, \cdots, t_n)$, where either

(i) t_i is algebraic over $F(t_1, \cdots, t_{i-1})$ or

(ii) $t_i' \in F(t_1, \cdots, t_{i-1})$ (respectively, $t_i' = s_i'/s_i$ for some $s_i \in F(t_1, \cdots, t_{i-1})$ or

(iii) $t_i'/t_i \in F(t_1, \cdots, t_{i-1})$ (respectively, $t_i'/t_i = s_i'$ for some $s_i \in F(t_1, \cdots, t_{i-1})$).

If (i) and (ii) (respectively, (i) and (iii)) hold, then E is called a *primitive* (respectively, *exponential*) extension of F. Given a liouvillian extension E of F, it is natural to ask for a criterion for algebraic independence of exponentials and primitive elements of E. The following result was first proved, using analytic techniques, by A. Ostrowski for a set of primitive elements over the field of meromorphic functions over complex numbers. Later, using the language of differential Galois theory, the theorem was reformulated and generalised by E. Kolchin.

Theorem 2.2.6 (Kolchin-Ostrowski). *[18, p. 1155] Let E be a no new constant extension of F. Let $a_1, \ldots, a_m \in E$ and $b_1, \ldots, b_n \in E \setminus \{0\}$ be such that $a_i' \in F$ for each i and $b_j'/b_j \in F$ for each j. Then either $a_1, \ldots, a_m, b_1, \ldots, b_n$ are algebraically independent over F or there exists $(c_1, \ldots, c_m) \in C^m \setminus \{(0, \ldots, 0)\}$ such that $\sum_{i=1}^m c_i a_i \in F$ or there exists $(r_1, \ldots, r_n) \in \mathbb{Z}^n \setminus \{(0, \ldots, 0)\}$ such that $\prod_{j=1}^n b_j^{r_j} \in F$.*

The following result is an extension of Corollary 2 of [36] and it gives an important criterion for checking whether a first order autonomous differential equation has a transcendental solution in a liouvillian extension of C or not.

Proposition 2.2.7 (cf. [37, Proposition 3.1])**.** *Let C be an algebraically closed field of characteristic zero with trivial derivation and let $H(Y)$ be a non-zero element of $C(Y)$. The equation $Y' = H(Y)$ has non-constant solution y which is liouvillian over C if and only if there exists an element z in $C(y) \setminus C$ such that $z' = 1$ or $z' = az$ for some nonzero constant a, that is,*

$$\frac{1}{H(y)} = \frac{\partial z}{\partial y} \quad \textit{or} \quad \frac{1}{H(y)} = \frac{1}{az}\frac{\partial z}{\partial y}.$$

2.3 First order differential equation

In this section, we will list some results about first order differential equations that will be needed in Chapters 3 and 4. But first, we define what is meant by solutions of a first order differential equation. As stated earlier, k is a differential field of characteristic zero with an algebraically closed field of constants C. Let $f(Y, Z) \in k[Y, Z]$ be an irreducible polynomial involving the variable Z. We can associate the following $k-$algebra to the first order differential equation $f(y, y') = 0$:

$$R_f = k[y, z, \frac{1}{d}] = k[Y, Z]/(f)\left[\frac{1}{d}\right].$$

Observe that R_f naturally becomes a differential integral domain by defining $y' = z$. The derivation uniquely extends to the field of fractions $k(f)$. Therefore $k(f) = k(y, y')$, where y is transcendental over k and $f(y, y') = 0$. Let X be a smooth projective curve such that $k(X) \cong k(f)$. The genus of X will also be called the *genus* of f. The following theorem on the genus will be used in Chapter 6.

Theorem 2.3.1. *[5, Theorem 5, p. 99] Let $F \subseteq M$ be an extension of fields. Let M be a function field of one variable over F and $\tilde{F}$ be the algebraic closure of F in M. Let L be a field extension of $\tilde{F}$. Then the genus of $M\langle L\rangle$ over L is at most* *equal to the genus of M over F and the equality holds whenever L is separable over F.*

By a *solution* of the differential equation $f(y, y') = 0$, we shall mean an element $t \in L$, where L is a differential field extension of k, such that $f(t, t') = 0$ and the field of constants of $k(t, t')$ is C (see [27, p. 47 - 48]). If t is transcendental (respectively, algebraic) over k, then it is called *a transcendental solution* (respectively, an *algebraic solution*). Therefore $f(y, y') = 0$ has a solution if and only if there is a no new constant extension L of k such that there is a differential homomorphism $\phi : R_f \to L$. In this case, $\phi(y)$ is the solution. If $\phi(y)$ is algebraic (respectively, transcendental) over k, then $\{0\} \subsetneq \ker(\phi)$ (respectively, $\{0\} = \ker(\phi)$). If the differential equation has a transcendental solution t, then the differential fields $k(f)$ and $k(t, t')$ are isomorphic. On the other hand, if all the solutions are algebraic, then there is an element $v \in k(f) \setminus k$ such that v is transcendental over k and $v' = 0$. Suppose we adjoin an indeterminate t to k and let t_1 be algebraic over $k(t)$ given by $f(t, t_1) = 0$. We extend the derivation of k to $k(t, t_1)$ by defining $t' = t_1$. If $k(t, t')$ is a no new constant extension of k, then the differential equation $f(y, y') = 0$ has a transcendental solution. Let t and s be two transcendental solutions of a first order differential equation. Then both $k(t, t')$ and $k(s, s')$ are isomorphic to the differential field $k(f)$ and hence $k(t, t')$ and $k(s, s')$ are isomorphic as differential fields.

If f is defined over the field of constants, then it is called an *autonomous differential equation*. Let $C(f) = C(y, y')$, where y is transcendental over C. We note that an autonomous differential equation always has a transcendental solution. This is easily seen by noting that if there is an element $v \in C(y, y') \setminus C$ such that $v' = 0$, then $x' = 0$ for all $x \in C(y, y')$. In particular, $y' = 0$, that is, $f(y, 0) = 0$. This contradicts the fact that y is a transcendental over C.

A solution t is called a *liouvillian solution* if L is a liouvillian extension of k and if

t is transcendental over k, then it is called a *transcendental liouvillian solution.*

Proposition 2.3.2. *Let t be a liouvillian solution of a first order differential equation over k. Then there is a liouvillian extension E of k containing a solution of the differential equation such that $C_E = C$.*

Proof. If t is an algebraic solution of a first order differential equation $f(y, y') = 0$, then since C is algebraically closed, $k(t)$ is a liouvillian extension of k such that $C_{k(t)} = C$. So let us assume that t is a transcendental liouvillian solution of f. Then by definition $k(t, t') \subseteq L$, where L is a liouvillian extension of k. Let $\tilde{k}$ be the algebraic closure of k in $k(t, t')$. By [38, Theorem B], one of the following occurs:

(i) $k(t, t') = \tilde{k}(z, \beta)$, where z satisfies a first order linear differential equation over $\tilde{k}$ and β is algebraic over $\tilde{k}(z)$.

(ii) $k(t, t') \subseteq K$, where K is a quadratic extension of $k(t, t')$ given by $K = k(\alpha, w, \beta)$, where $k(\alpha)$ is a quadratic extension of $\tilde{k}$, $w'/w \in k(\alpha) \setminus k$ and β is algebraic over $k(\alpha, w)$.

In the first case, $k(t, t')$ is a liouvillian extension of k with no new constants. In the second case, K is an algebraic extension of k. Therefore $C_K = C$. Also, note that K is a liouvillian extension of k. Thus in either case, the differential equation has a transcendental liouvillian solution. $\square$

The following lemma gives a relationship between the transcendental solutions and the algebraic solutions of a first order differential equation.

Lemma 2.3.3. *Let k be a differential field of characteristic zero with algebraically closed field of constants C. Let y be a transcendental solution of the differential* *equation*

$$y' = a_n y^n + a_{n-1} y^{n-1} + \cdots + a_0, \tag{2.1}$$

where $a_i \in k$, $a_n \neq 0$. Let $\gamma \in \overline{k}$ and $z \in \overline{k}(y) \setminus \overline{k}$ such that $z' = az + b$, where $a, b \in \overline{k}$.

- *(i) If $b \neq 0$ and there is no $w \in \overline{k}(y) \setminus \overline{k}$ such that $w'/w \in \overline{k}$. Then γ is an algebraic solution of differential equation (2.1) if and only if γ is a pole of z.*
- *(ii) If $a \neq 0$ and $b = 0$, then γ is an algebraic solution of the differential equation if and only if γ is a zero or a pole of z.*

Proof. First, we observe that for any $\alpha \in \overline{k}$, the differential equation (2.1) can be written as

$$(y' - \alpha') = (y - \alpha)\left(\sum_{i=1}^{n} \frac{1}{i!}\frac{\partial^i f}{\partial y^i}(\alpha)(y-\alpha)^{i-1}\right) + f(\alpha) - \alpha', \tag{2.2}$$

where $f(y) = a_n y^n + a_{n-1} y^{n-1} + \cdots + a_0$. Thus, it is easily seen that α is an algebraic solution of Equation (2.1) if and only if $y - \alpha$ divides $y' - \alpha'$ in the differential ring $\overline{k}[y]$. Let $\gamma \in \overline{k}$ be a pole of z of order l. Then we have the following series expansion for z about γ :

$$z = \frac{\beta_{-l}}{(y-\gamma)^l} + \cdots + \frac{\beta_{-1}}{(y-\gamma)} + \beta_0 + \beta_1(y-\gamma) + \cdots, \tag{2.3}$$

where $\beta_i \in \overline{k}$ and $\beta_{-l} \neq 0$. Now,

$$z' = \frac{-l\beta_{-l}(y' - \gamma')}{(y-\gamma)^{l+1}} + \frac{\beta'_{-l}}{(y-\gamma)^l} + \cdots$$

From Equation (2.2), z' can be written as follows:

$$z' = \frac{\alpha_{-l-1}}{(y-\gamma)^{l+1}} + \frac{\alpha_{-l}}{(y-\gamma)^l} + \cdots, \tag{2.4}$$

where $\alpha_{-l-1} = l\beta_{-l}\,(\gamma' - f(\gamma))$.

Note that $z' = az + b$, where $a, b \in \overline{k}$. Therefore, by comparing the coefficients of y^{-l-1} in Equations (2.3) and (2.4), we get $\alpha_{-l-1} = 0$. Since $l \neq 0$ and $\beta_{-l} \neq 0$, we obtain that $\gamma' = f(\gamma)$. Thus if γ is a pole of z, then it is an algebraic solution of differential equation (2.1).

Now we will prove the converse part of (i). Let $\gamma \in \overline{k}$ be an algebraic solution of Equation (2.1). Suppose that γ is not a pole of z, then we have

$$z = \beta_0 + \beta_1(y - \gamma) + \cdots, \text{ where } \beta_i \in \overline{k} \text{ and} \tag{2.5}$$

$$z' = \alpha_0 + \alpha_1(y - \gamma) + \cdots, \text{ where } \alpha_i \in \overline{k} \text{ and } \alpha_0 := \beta_0' + \beta_1 (f(\gamma) - \gamma'). \tag{2.6}$$

Note that $\alpha_0 = \beta_0'$ as $\gamma' - f(\gamma) = 0$. Also observe that $\beta_0 \neq 0$. Otherwise, $\alpha_0 = 0$ and $y - \gamma$ would divide both z' and z. This would imply that $y - \gamma$ divides $z' - az = b$, which is not possible as b lies in $\overline{k} \setminus \{0\}$. Therefore, $\beta_0 \neq 0$. Since $z' = az + b$, comparing the coefficient of $(y - \gamma)^0$ in Equations (2.5) and (2.6), we obtain $\alpha_0 = a\beta_0 + b$. Since $\alpha_0 = \beta_0'$, we get $\beta_0' = a\beta_0 + b$. Now consider the element $w := z - \beta_0$ and observe that $w \in \overline{k}(y) \setminus \overline{k}$ and $w' = aw$. This contradicts the hypotheses of (i). Therefore if γ is an algebraic solution of differential equation (2.1), then it must be a pole of z. This proves the first part.

To prove (ii), observe that we have $z' = az$ and that $(1/z)' = -a(1/z)$. Therefore, it follows from the above calculation that every zero and pole of z is an algebraic solution of the differential equation (2.1). If $\gamma \in \overline{k}$ is not a zero or a pole of z, then we have a power series expansion for z, as in Equation (2.5), with $\beta_0 \neq 0$. Now if $\gamma' = f(\gamma)$, then $\beta_0' = a\beta_0$ and we obtain that $(\beta_0^{-1} z)' = 0$. This is a contradiction as $C_{k(y)} = C$. □

Proposition 2.3.4. *For any $\alpha \in C(x)$, there exists $\gamma \in \overline{C(x)}$ such that $\gamma' = \alpha\gamma$ if and only if there are positive integers n, l, nonzero integers $m_1, \cdots, m_l$ and pairwise distinct elements $\beta_1, \cdots, \beta_l \in C$ such that $\alpha = \frac{1}{n} \sum_{i=1}^{l} \frac{m_i}{x - \beta_i}$.*

Proof. Suppose that there exists $\gamma \in \overline{C(x)}$ such that $\gamma' = \alpha\gamma$. Then by Proposition 2.2.4 (ii), $\gamma^n \in C(x)$ for some positive integer n. Therefore $\gamma^n = c\prod_{i=1}^{l}(x-\beta_i)^{m_i}$, where c is nonzero constant, l is a positive integer, $m_1, \cdots, m_l$ are nonzero integers and $\beta_1, \cdots, \beta_l \in C$ are pairwise distinct elements. Thus,

$$\alpha = \frac{\gamma'}{\gamma} = \frac{1}{n}\sum_{i=1}^{l}\frac{m_i}{x-\beta_i}.$$

To prove the converse, observe that if γ is an n^{th} root of $\prod_{i=1}^{l}(x-\beta_i)^{m_i}$, then $\gamma' = \alpha\gamma$. □

2.4 General solution

In this section, we will discuss the concept of general solutions of a differential equation. The reader may refer to [29, 2].

Let C be an algebraically closed field of characteristic zero with trivial derivation, $C(x)$ be the differential field with derivation $' := d/dx$, and $C(x)\{y\}$ be the ring of differential polynomials. Consider the differential equation $F(y, y', \cdots, y^{(n)}) = 0$, where $F \in C(x)\{y\} \setminus C(x)$. We may always assume that F is an irreducible polynomial in $C(x)[y, y', \ldots, y^{(n)}]$. The highest derivative of y in F is called the *order* of F denoted by $\mathrm{ord}(F)$. Let $o = \mathrm{ord}(F) > 0$: We may write F as follows:

$$F = \sum_{i=0}^{d} a_i y_o^i,$$

where a_i's are polynomials in $y, y_1, \cdots, y_{o-1}$ and $a_d \neq 0$; a_d is called the *initial* of F, $S_F := \partial F/\partial y_o$ is called the *separant* of F. The m^{th} derivative of F is denoted by $F^{(m)}$. Observe that

$$F^{(m)} = S_F y_{o+m} + R_m,$$

where R_m is a differential polynomial of order lower than $o+m$.

Lemma 2.4.1. *(see [29]) Let $F \in C(x)\{y\}$ such that F is an irreducible polynomial in $C(x)[y, y', \ldots, y^{(n)}]$. Then the ideal $\{F\}$ can be factored as:*

$$\{F\} = \Sigma_F \cap \{F, S_F\},$$

where $\Sigma_F := \{G \in C(x)\{y\} \mid GS_F \in \{F\}\}$ is a prime differential ideal.

The ideal Σ_F is a unique prime differential ideal that does not contain the separant S_F of F. On the other hand, the second component $\{F, S_F\}$ is the intersection of the other essential components of $\{F\}$.

Definition 2.4.2. Consider the differential equation $F(y, y', ..., y^{(n)}) = 0$.

(i) Let I be a nontrivial prime differential ideal of $C(x)\{y\}$. A zero η of I in a differential field extension of $C(x)$ is called a *generic zero* of I if for any differential polynomial P, $P(\eta) = 0$ implies that $P \in I$.

(ii) A generic zero of the differential ideal Σ_F is called a *general solution* of F. A zero of the ideal $\{F, S_F\}$ is called a *singular solution*.

(iii) An *algebraic general solution* of F is a general solution η of F which satisfies the following equation:

$$G(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n_j} a_{ij} x^i y^j,$$

where $a_{ij} \in C$ and $G(x, y)$ is irreducible in $C[x, y]$. If $n = 1$, then η is called a *rational general solution* of F.

A general solution of $F = 0$ is usually defined as a family of solutions with o independent parameters where $o = \text{ord}\,(F)$. The definition given by Ritt is more precise.

Remark 2.4.3. Let $F(y, y') = 0$ be a first order differential equation. Singular solutions are solutions of F and $S_F = \partial F/\partial y'$, therefore they are always algebraic. Since Σ_F is a prime differential ideal, whenever it has an algebraic generic zero, all of the other generic zeros are also algebraic. Therefore, if the differential equation has an algebraic general solution, then it has no transcendental liouvillian solution.

2.5 Picard-Vessiot theory

In this section, we will briefly discuss the theory of Picard-Vessiot extensions. One may refer to [40, Chapter 1] or [1].

A *Picard-Vessiot ring* R for a matrix differential equation $Y' = AY$, where $A \in M_n(k)$, is a simple differential ring such that there is a matrix $F \in \mathrm{GL}_n(R)$ satisfying $F' = AF$, called a *fundamental matrix* for $Y' = AY$ and R is minimal with respect to these properties, that is, R is generated as a ring by the entries of F and the inverse of the determinant $\det F$ of the matrix F.

By a differential $k-$module (M, ∂), we mean a finite dimensional $k-$module M together with an additive map $\partial : M \to M$ such that $\partial(\alpha m) = \alpha' m + \alpha\partial(m)$ for all $\alpha \in k$ and $m \in M$. Let M be a differential $k-$module. By fixing a $k-$basis $e_1, \ldots, e_n$ of M, we obtain a matrix $A = (a_{ij}) \in M_n(k)$ such that $\partial(e_i) = -\sum_j a_{ji}e_j$ and a corresponding matrix differential equation $Y' = AY$. Choosing any other basis will amount to obtaining a differential equation of the form $Y' = \tilde{A}Y$, where $\tilde{A} = B'B^{-1} + BAB^{-1}$ for some $B \in \mathrm{GL}_n(k)$. Furthermore, if R is a Picard-Vessiot ring for $Y' = AY$ with fundamental matrix F, then $(BF)' = \tilde{A}BF$ and thus R is also the Picard-Vessiot ring of $Y' = \tilde{A}Y$. This observation allows one to define a *Picard-Vessiot ring* for a differential module M to be a Picard-Vessiot ring of a corresponding matrix differential equation $Y' = AY$ of M.

Let M be a differential module with matrix differential equation $Y' = AY$ and $M^\vee$ be the dual of a differential module of M. Then $Y' = -A^tY$, where A^t is the transpose of A, is a matrix differential equation corresponding to $M^\vee$. Thus, if R is a Picard-Vessiot ring with fundamental matrix $F \in \mathrm{GL}_n(R)$, then $((F^t)^{-1})' = -A^t(F^t)^{-1}$ and thus M and $M^\vee$ have the same Picard-Vessiot ring R.

Let $k[\partial]$ be the ring of differential operators over k and $\mathscr{L} := \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0 \in k[\partial]$. Then there is a way of producing a matrix differential equation from $\mathscr{L}$. Let $M = k[\partial]/k[\partial]\mathscr{L}$, a matrix equation corresponding to the dual $M^\vee$ is $Y' = A_{\mathscr{L}}Y$. Thus if R is a Picard-Vessiot ring for $M^\vee$, then the fundamental matrix F for $Y' = A_{\mathscr{L}}Y$ is a Wronskian matrix, where

$$A_{\mathscr{L}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{pmatrix}, \quad F = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}.$$

Note that $y_1, \ldots, y_n$ are $C-$linearly independent and the C vector space V spanned by $y_1, \ldots, y_n$ is the set of all solutions of $\mathscr{L}(y) = 0$.

Picard-Vessiot rings are integral domains. A *Picard-Vessiot extension* for the equation $Y' = AY$ over k (or for a differential module M over k) is the field of fractions for the Picard-Vessiot ring for this equation. Let $Y' = AY$ be a differential equation of degree n, having Picard-Vessiot field E and differential Galois group $\mathscr{G}$. Then $\mathscr{G}$ considered as a subgroup of $\mathrm{GL}_n(C)$ is an algebraic group ([40, Theorem 1.27]).

2.6 Strongly normal extensions

In this section, we shall recall a few known results from the theory of strongly normal extensions, which was developed by Kolchin. One may refer to [17, 15, 19, 20].

Kolchin wanted to develop the concept of Galois extensions for differential fields. The main problem in developing such a theory is defining the concept of normal extensions of a differential field. He had two special cases of Galois theory for differential fields: finite Galois extension of a differential field and Picard Vessiot extension of a differential field. Naturally, he looked into these cases for hints.

In classical Galois theory, N is a normal extension of a field L if the fixed field of $\mathscr{G}(N|L)$ is L. In this case, if M is a field intermediate to N and L, then N is a normal extension of L. This is not the case for differential fields. Let E be a differential field extension of k. E is called a *weakly normal* extension of k if k is the fixed field of $\mathscr{G}(E|k)$. It is possible that there is an intermediate differential subfield K such that E is not weakly normal over K. To overcome this shortcoming, Kolchin defined a differential field E to be a *normal extension* of k if E is weakly normal over every intermediate differential subfield between E and k (see [16, Section 16]). In the aforementioned paper, it was shown that when E is normal over k, then there is a one-to-one Galois correspondence between the set of all differential fields intermediate to k and E and a certain set of subgroups of the group $\mathscr{G}(E|k)$. But there was no characterization of those " certain " subgroups that correspond to the intermediate differential fields. To overcome this problem, Kolchin looked into classical Galois theory and Picard-Vessiot extensions for possible hints.

In classical Galois theory, N is a normal extension of L if every field homomorphism $\phi: N \to \overline{L}$ has image N. This property is not shared by Picard-Vessiot extensions. The Picard-Vessiot theory, however, offers a suggestion for how to proceed. Let E

be a Picard-Vessiot extension of k. Then $C_E = C_k = C$, E is finitely generated and of finite transcendence degree over k. It is easy to varify that if σ is an isomorphism of E over k into an extension of E and if $C(\sigma)$ denotes the field of constants of the compositum $E\sigma E$, then $E\sigma E = EC(\sigma) = \sigma EC(\sigma)$. We recall that a *differential isomorphism* σ of E over k means a differential homomorphism (necessarily injective) of E into some differential field extension M of E with $\sigma|_k =$id. One would expect that there exists a "large enough" differential extension of E that contains an isomorphic copy of every M one could possibly encounter.

Definition 2.6.1. Let E^* be a differential field extension of E. We shall call E^* a *universal extension* of E if for every finitely generated differential field extension E_1 of E with $E_1 \subseteq E^*$ and every positive integer n and every prime differential ideal I of $E_1\{y_1, \ldots, y_n\}$, there exists a generic zero $(\eta_1, \ldots, \eta_n)$ of I with $\eta_i \in E^*$.

A necessary and sufficient condition for an extension E^* of E to be universal is that for every finitely generated extension E_1 of E with $E_1 \subseteq E^*$ and every finitely generated extension M of E_1 there exist an isomorphism of M over E_1 into E^*.

Theorem 2.6.2. *[17, p. 771] Every differential field has a universal field.*

With all this in our hands, we are ready to define a strongly normal extension of k.

Definition 2.6.3. A differential field extension E of k is *strongly normal* if E is finitely differentially generated over k and every differential isomorphism σ of E over k satisfies the following conditions:

1. $\sigma|_{C_E} =$id and

2. $E\sigma E = EC(\sigma) = \sigma EC(\sigma)$, where $C(\sigma)$ is the field of constants of $E\sigma E$.

A differential isomorphism σ of E over k is called *strong* if it satisfies the above properties. If E is a strongly normal extension of k then E is finitely generated over k (as fields) and that $C_E = C_k = C$ ([19, Propositions 12.2, 12.4]). Picard-Vessiot extensions are examples of strongly normal extensions.

Definition 2.6.4. Let E be a no new constant extension of k. A non constant element $t \in E$ is called *weierstrassian* over k if t satisfies the *Weierstrass* differential equation, that is, $t'^2 = \alpha^2(4t^3 - g_2 t - g_3)$ for some $\alpha \in k$ and $g_2, g_3 \in C$ with $27g_3^2 - g_2^3 \neq 0$. The extension $k\langle t\rangle$ is called an *elliptic extension* of k.

Elliptic extensions are examples of strongly normal extensions. For other examples, see [20, Example 14.2].

The group of all differential automorphisms of E over k is called the *Galois group* of E over k and is denoted by $\mathscr{G}(E|k)$. $\mathscr{G}(E|k)$ is an algebraic group (not necessarily affine) defined over C. Picard-Vessiot extensions are precisely those strongly normal extensions whose Galois groups are linear algebraic groups. Now we will discuss the existence of strongly normal extensions.

Theorem 2.6.5. *[15, Theorem 2, p. 880] Let X be a connected algebraic group over C. There exist differential fields $F \subseteq E$ such that E is a strongly normal extension of F whose Galois group is isomorphic to the group of C-rational points of X.*

In particular, the above theorem shows that every abelian variety over C can be realised as the differential Galois group of a strongly normal extension. We would like to point out that a linear homogeneous differential equation gives rise to a Picard-Vessiot extension, but a non-linear differential equation need not give rise to a strongly normal extension (as we will see in Section 6.3.1).

The fundamental theorem of strongly normal extension provides a bijective correspondence between differential subfields intermediate to E and k and the Zariski closed subgroups of $\mathscr{G}(E|k)$. If $\mathscr{H}$ is a closed subgroup of $\mathscr{G}(E|k)$ and K is an intermediate differential field, then the bijective correspondence is given by the maps

$$K \to \mathscr{G}(E|K) := \{\sigma \in \mathscr{G}(E|k) \mid \sigma(u) = u \ \forall \ u \in K\}$$
$$\mathscr{H} \to E^{\mathscr{H}} := \{u \ \in E \mid \sigma(u) = u \ \forall \ \sigma \in \mathscr{H}\}.$$

The field fixed by $\mathscr{G}(E|k)$ is k, that is $E^{\mathscr{G}(E|k)} = k$. Let K be a differential field intermediate to E and k. Then K is a strongly normal extension of k if and only if $\mathscr{G}(E|K)$ is a closed normal subgroup of $\mathscr{G}(E|k)$. In which case, the differential Galois group $\mathscr{G}(K|k)$ is isomorphic to the quotient group $\mathscr{G}(E|k)/\mathscr{G}(E|K)$. The algebraic closure of k in E is a finite Galois extension, which we denote by E^0. A strongly normal extension E over k is said to be *abelian* if $\mathscr{G}(E|k)$ is an abelian variety. Note that elliptic extensions are examples of abelian extensions.

The following result is called the Chevalley-Barsotti structure theorem.

Theorem 2.6.6. *[30, Theorem 16] Let $\mathscr{G}$ be a connected algebraic group over C. Then there exists a connected normal linear algebraic subgroup $\mathscr{H}$ of $\mathscr{G}$ such that $\mathscr{G}/\mathscr{H}$ is an abelian variety. $\mathscr{H}$ is unique and contains all the other linear algebraic subgroups of $\mathscr{G}$.*

Let E be a strongly normal extension of k and $\mathscr{G}$ be the group of differential automorphisms of E over k. From the fundamental theorem of strongly normal extensions, corresponding to the closed subgroups $\mathscr{G} \supseteq \mathscr{G}^0 \supseteq \mathscr{H} \supseteq 1$ we have a tower of fields:

$$k \subseteq E^0 \subseteq L \subseteq E,$$

where E^0 is a finite normal extension of k with Galois group $\mathscr{G}/\mathscr{G}^0$, L is a strongly normal extension of E^0 with Galois group $\mathscr{G}^0/\mathscr{H}$ isomorphic to an abelian variety and E is a Picard-Vessiot extension of L with a connected differential Galois group $\mathscr{H}$.

The following theorem classifies the transcendence degree one strongly normal extensions of k.

Theorem 2.6.7. *[15, Theorem 3] If E is a strongly normal extension of k is of transcendence degree one over k and if k is relatively algebraically closed in E, then there exists an element α such that either α is primitive over k and $E = k(\alpha)$, or α is exponential over k and $E = k(\alpha)$, or α is weierstrassian over k and E is an abelian algebraic extension of $k\langle\alpha\rangle$ of finite degree. In the latter case, if k is algebraically closed, then the weierstrassian element α may be chosen so that $E = k\langle\alpha\rangle$.*

Chapter 3

Liouvillian solutions of first order variable separable differential equations

In this chapter, we are mainly interested in finding the algebraic and transcendental liouvillian solutions of variable separable differential equations over $C(x)$, where $C(x)$ is endowed with usual derivation d/dx. We also want to understand the relationship between algebraic and transcendental solutions.

The liouvillian solutions of the following differential equation is analysed in the first section.

$$y' = r(x)\, F(y), \tag{3.1}$$

where $r(x)$ is a nonzero polynomial in $C[x]$ and $F(y)$ is a nonzero polynomial in $C[y]$. We use [37, Theorem 2.2] and [38, Theorem B] to provide a necessary and sufficient condition for the existence of a transcendental liouvillian solution of the above equation.

In [31], Rosenlicht has shown that if the differential equation $y^n = f(y, y', y'', \dots)$, where f is a polynomial in several variables with coefficients in $C(x)$ and of total degree less than n, has a liouvillian solution then it has an algebraic solution. We will provide a class of differential equations with transcendental liouvillian solutions but no algebraic solution.

3.1 Main results

In this section, first we will prove few results about the algebraic solutions of the differential equation (3.1) then we will prove our main results. If $F(y) = c$, where $c \in C \setminus \{0\}$ then there is an element $v \in C[x]$ such that $v' = cr(x) = r(x)F(y)$. In this case the differential equation does not have transcendental solutions. Since if t is a transcendental solution, then in the differential field $C(x, t)$, we have $(t - v)' = 0$, a contradiction. Thus, if $F(y) \in C \setminus \{0\}$ then the differential equation does not have any transcendental solution. Therefore from now on we shall assume that $F(y) \in C[y] \setminus C$.

Proposition 3.1.1. *The differential equation (3.1) does not have any nonconstant algebraic solutions if all the roots of F are simple.*

Proof. Before we proceed with the proof, we make the following observations. Since C is algebraically closed, all roots of F are in C. Let $\gamma \in C$ be a root of F. Then $F(\gamma) = 0 = \gamma'$ and thus, all the roots of $F(y)$ are solutions of the differential equation. We also have

$$F = \sum_{i=1}^{n} a_i \, (y - \gamma)^i,$$

where $a_i \in C$, $a_n = 1$ and as F has only simple roots, $a_1 \neq 0$.

First we show that the differential equation does not have solutions in $C(x) \setminus C$. Let $p \in C[x] \setminus C$. Then by comparing the degree of p' and $r(x)\, F(p)$, we conclude that differential equation (3.1) does not have any nonconstant solution in $C[x]$. Let $p,\, q \in C[x] \setminus \{0\}$ be relatively prime polynomials. We shall assume $q \notin C$ or equivalently, $q' \neq 0$. Suppose that p/q is a solution of the differential equation. Then, for $p_1 = p - \gamma q$, we have

$$\begin{aligned}(p_1/q)' &= (p/q)' = r(x)F(p/q)\\ &= r(x)\left((p_1/q)^n + a_{n-1}(p_1/q)^{n-1} + \cdots + a_1(p_1/q)\right).\end{aligned}$$

Therefore $q^n\,(qp_1' - p_1q') = r(x)\sum_{i=1}^{n} a_i\, p_1^i\, q^{n+2-i}$ and we obtain p_1 divides p_1'. Since deg $(p_1) >$ deg (p_1'), we must have $p_1' = 0$. Let m =deg(q). The degree of q^np_1q' is $m(n+1)-1$, whereas, since $a_1 \neq 0$, the degree of the polynomial $r(x)\sum_{i=1}^{n} a_i\, p_1^i\, q^{n+2-i}$ is at least $m(n+1)$. Thus, the differential equation (3.1) does not have any nonconstant rational solutions.

We will now show that there is no solution in $\overline{C(x)} \setminus C(x)$. Let $n =$ deg (F) and for $i = 1, \cdots, n$, let $\alpha_i \in C$ be the simple roots of $F(y)$. If $n = 1$, then the differential equation is of the form $y' = r(x)\,(y - \alpha_1)$ and the solutions of the differential equation are $c_1 \exp(\int r(x)) + \alpha_1$, where $c_1 \in C$. Thus the nonconstant solutions are transcendental over $C(x)$. Now assume that $n > 1$. Suppose that the differential equation has a solution α in $\overline{C(x)} \setminus C(x)$. Let Y be an indeterminate over $C(x)$ and $P(x, Y)$ be an irreducible polynomial in $C[x, Y]$ such that $P(x, \alpha) = 0$. Note that $\deg_Y(P) > 1$. Now consider the polynomial $P_1(x, Y) := \partial P/\partial x + (\partial P/\partial Y)\, r(x)F(Y)$. Note that $P_1 \neq 0$ as $\deg_Y((\partial P/\partial Y)\, r(x)F(Y)) > \deg_Y(\partial P/\partial x)$. Taking the derivative of $P(x, \alpha) = 0$, we obtain that $P_1(x, \alpha) = 0$. Therefore, P divides P_1 in $C(x)[Y]$ and there exists a nonzero polynomial $Q(x, Y) \in C(x)[Y]$ such that

$$\frac{\partial P}{\partial x} + \frac{\partial P}{\partial Y}\, r(x)\, F(Y) = Q(x, Y)P(x, Y), \tag{3.2}$$

Comparing the degrees of Y in (3.2), we get $\deg_Y(Q) = n-1$. Then Q may be written as $h(x)Q(x,Y) = \sum_{i=0}^{n-1} g_i(x)\,Y^i$, where $h(x) \in C[x]$ is a nonzero polynomial, $g_i(x) \in C[x]$, $g_{n-1}(x) \neq 0$ such that $\gcd(h(x), g_0, \cdots, g_{n-1}) = 1$. It follows from Equation (3.2) that $h(x)$ divides the irreducible polynomial $P(x,Y)$. Therefore $h(x) = 1$ and $Q \in C[x,Y]$. Since $P \in C[x,Y]$ is irreducible and $\deg_Y(P) > 1$, $P(x,\alpha_i) \neq 0$. On the other hand, since $\deg_Y(Q) = n-1$, there must be a j such that $Q(x,\alpha_j) \neq 0$. Substituting $Y = \alpha_j$ in Equation (3.2), we have

$$\frac{\partial P(x,\alpha_j)}{\partial x} = Q(x,\alpha_j)\,P(x,\alpha_j). \tag{3.3}$$

Since $Q(x,\alpha_j) \neq 0$ and $P(x,\alpha_j) \neq 0$, we have $\partial P(x,\alpha_j)/\partial x \neq 0$. But for a non zero polynomial $P(x,\alpha_j)$, $\deg_x(\partial P(x,\alpha_j)/\partial x) < \deg_x P(x,\alpha_j)$. Thus we have arrived at a contradiction. □

Now we shall prove our main result.

Theorem 3.1.2. *The differential equation (3.1) has a transcendental liouvillian solution y if and only if there exists $z \in C(x,y) \setminus C(x)$ satisfying only one of the following conditions:*

(i) $z' = cr(x)z$, *where c is a nonzero constant,*

(ii) $z' = (c_0 r(x) + h'/h)z + \beta$, *where c_0 is a nonzero constant, h and β are nonzero elements of $C(x)$.*

In this case, all the roots of $F(y)$ are simple roots and the roots of $F(y)$ are the algebraic solutions of the differential equation.

Proof. To prove the necessary part consider the differential field extension $C(x,y)$ of $C(x)$ where y is an indeterminate and $y' = r(x)F(y)$. Suppose that there is an element $z \in C(x,y) \setminus C(x)$ such that $z' = cr(x)z$ for some nonzero constant c. By Proposition 2.3.4, there is no element $t \in C(x)$ such that $t' = ncr(x)t$ for any nonzero integer n. Therefore by Proposition 2.2.3 (iii), the field of constants of $C(x,z)$ is C. Since $C(x,y)$ is an algebraic extension of $C(x,z)$, the field of constants of $C(x,y)$ is also C (by Lemma 2.2.1 (i)). Thus, in this case y is a transcendental liouvillian solution of the differential equation. Similarly, if case (ii) occurs, then by Proposition 2.2.3 (i) one can show that the field of constants of $C(x,y)$ is C.

Now we prove the sufficient part. Let y be a transcendental liouvillian solution of the differential equation (3.1) and E be a liouvillian extension of $C(x)$ such that $y \in E$. Then by definition the field of constants of $C(x,y)$ is C. If γ is a multiple root of F then $(\partial F/\partial y)(\gamma) = 0$ and from Equation (2.2), $(y-\gamma)' = (y-\gamma)R(y-\gamma)$, where $(y-\gamma)$ divides $R(y-\gamma)$. But, from [38, Proposition 4.1], it is known that the liouvillian solutions of such differential equations are algebraic over $C(x)$. Therefore all the roots of $F(y)$ must be simple roots. By Proposition 3.1.1, the algebraic solutions of the differential equation are the roots of $F(y)$.

Let γ be a simple root of F. Since $(y-\gamma)' = (y-\gamma)R(y-\gamma)$ and $y-\gamma$ is transcendental and liouvillian over $C(x)$ if and only if y is transcendental and liouvillian over $C(x)$, we shall replace $y - \gamma$ with y and obtain

$$y' = r(x)F(y), \text{ where } F(y) = y^n + a_{n-1}y^{n-1} + \cdots + a_1 y,\ a_i \in C \text{ and } a_1 \neq 0. \quad (3.4)$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the simple roots of $F(y)$. Note that $C(x,y)$ is purely transcendental over $C(x)$ and is an intermediate differential subfield of $C(x)$ and E. From Theorem 2.2 of [37], there is an element in $C(x,y) \setminus C(x)$ satisfying a linear homogeneous differential equation of order ≥ 1 over $C(x)$. Now $C(x,y)$ is finitely generated over $C(x)$ and therefore the hypotheses of [38, Theorem B] are satisfied. Thus,

1. there is an element $z \in C(x,y) \setminus C(x)$ which satisfies a first order linear differential equation over $C(x)$. Therefore either

   (a) $z' = \alpha z$, or

   (b) $z' = \beta$, or

   (c) cases (1a) and (1b) do not hold and $z' = \alpha z + \beta$, where $\alpha, \beta \in C(x) \setminus \{0\}$, or

2. there is an element $w \in \overline{C(x)}(y) \setminus \overline{C(x)}$ such that $w' = \overline{\alpha} w$ for some $\overline{\alpha} \in L \setminus C(x)$, where L is a quadratic extension of $C(x)$.

Note that the cases (1a), (1b), (1c) and (2) are pairwise disjoint. We will show that case (1b) and case (2) can not occur.

Suppose that there exists $w \in \overline{C(x)}(y) \setminus \overline{C(x)}$ satisfying the conditions of (2). Then by Proposition 3.1.1, all the algebraic solutions of equation (3.1) are constants. Then by Lemma 2.3.3 (ii), $w = g \prod_{i=1}^{n} (y - \alpha_i)^{m_i}$, where g is a nonzero element of $\overline{C(x)}$, m_i are nonzero integers. Taking the logarithmic derivative of w we get

$$\frac{w'}{w} = \overline{\alpha} = \frac{g'}{g} + \sum_{i=1}^{n} m_i \frac{y' - \alpha_i'}{y - \alpha_i}.$$

Now consider the element $w_1 \in C(x,y)$ given by $w_1 = \prod_{i=1}^{n} (y - \alpha_i)^{m_i}$. Note that $w_1 \in C(x,y) \setminus C(x)$ and $w_1'/w_1 = \overline{\alpha} - (g'/g)$. Also note that $w_1'/w_1 \in C(x,y)$. Therefore w_1'/w_1 lies in the intersection of $C(x,y)$ and $\overline{C(x)}$, which implies $w_1'/w_1 \in C(x)$. This contradicts the hypothesis of (2). Thus case (2) does not hold.

Suppose that case (1a) holds. Then by Lemma 2.3.3 (ii), $z = \prod_{i=1}^{n} (y - \alpha_i)^{m_i}$, where m_i are nonzero integers. Now,

$$\alpha = \frac{z'}{z} = \sum_{i=1}^{n} m_i \frac{(y' - \alpha_i')}{y - \alpha_i} = r(x)\, F(y) \sum_{i=1}^{n} \frac{m_i}{y - \alpha_i}.$$

Since the intersection of the subfields $C(x)$ and $C(y)$ is C, therefore

$$\frac{\alpha}{r(x)} = c = F(y)\left(\sum_{i=1}^{n}\frac{m_i}{y-\alpha_i}\right) \implies \frac{\alpha}{r(x)} = c \text{ and } \frac{1}{F(y)} = c^{-1}\sum_{i=1}^{n}\frac{m_i}{y-\alpha_i}, \quad (3.5)$$

where c is a nonzero constant.

Suppose that case (1a) does not hold and there is an element $z \in C(x,y) \setminus C(x)$ such that $z' = \alpha z + \beta$, for some nonzero $\beta \in C(x)$. By comparing the degrees of y in $\alpha z + \beta$ and z' one can easily show that $z \notin C(x)[y]$. By Lemma 2.3.3 (i), the algebraic solutions of the differential equation are the poles of z. Let l be the order of the pole at zero. Then z can be written as follows

$$z = \frac{\beta_{-l}}{y^l} + \cdots + \frac{\beta_{-1}}{y} + \beta_0 + \beta_1 y + \cdots . \quad (3.6)$$

Note that the denominator of z is of the form $\prod_{i=1}^{n}(y-\alpha_i)^{m_i}$, where α_i are the roots of $F(y)$ and m_i are nonzero integers. Therefore $\beta_i \in C(x)$ for all i and also note that $\beta_{-l} \neq 0$. Differentiating z, we obtain

$$\begin{aligned} z' = &- l\beta_{-l}r(x)y^{-l-1}\left(a_1 y + \cdots + y^n\right) + \cdots + (-\beta_{-1})r(x)y^{-2}\left(a_1 y + \cdots + y^n\right) + \cdots \\ &+ \frac{\beta'_{-l}}{y^l} + \cdots + \frac{\beta'_{-1}}{y} + \beta'_0 + \beta'_1 y + \cdots \end{aligned} \quad (3.7)$$

Since $z' = \alpha z + \beta$, comparing the coefficient of y^{-l} in Equations (3.6) and (3.7) we obtain $\beta'_{-l} = (\alpha + l\, a_1\, r(x))\, \beta_{-l}$. If $\alpha = 0$, then $la_1 r(x) = \beta'_{-l}/\beta_{-l}$, which contradicts Proposition 2.3.4 as $la_1 r(x)$ is a nonzero polynomial. So case (1b) is not possible. Thus if case (1a) does not hold then $z' = \alpha z + \beta$, where $\alpha \neq 0$ and $\beta \neq 0$ and $\alpha = -la_1 r(x) + \beta'_{-l}/\beta_{-l}$. □

Now we shall prove the following well known result.

Proposition 3.1.3. *Let $h(x)$ be a nonzero rational function in $C(x)$. Let $F(y)$ and $G(y)$ be nonzero relatively prime polynomials in $C[y]$ such that $G(y)/F(y) =$*

$\sum_{i=1}^{\lambda} \frac{n_i}{y-\alpha_i}$, *where* λ *is a positive integer,* n_i*'s are nonzero integers and* α_i *are pairwise distinct elements of* C. *Then the differential equation*

$$y' = h(x)\frac{F(y)}{G(y)} \tag{3.8}$$

has a transcendental liouvillian solution y *if and only if there is no nonzero* $\gamma \in \overline{C(x)}$ *such that* $\gamma' = h(x)\gamma$. *In this case,* $\alpha_1, \ldots, \alpha_\lambda$ *are the algebraic solutions and there is an element* z *in* $C(x,y) \setminus C(x)$ *such that* $z' = ch(x)z$, *where* c *is a nonzero constant.*

Proof. Suppose that there is a nonzero $\gamma \in \overline{C(x)}$ such that $\gamma' = h(x)\gamma$ then by Proposition 2.3.4, $h(x) = \frac{1}{l}\sum_{i=1}^{\delta} m_i/(x-\beta_i)$, where l, δ are positive integers, m_i are nonzero integers and β_i's are pairwise distinct elements of C. Let $m_i > 0$, for $i = 1, 2, \ldots \mu$ and $m_i < 0$, for $i = \mu+1, \ldots \delta$, where $0 \leqslant \mu \leqslant \delta$. Let $n_i > 0$, for $i = 1, 2, \ldots \tau$ and $n_i < 0$, for $i = \tau+1, \ldots \lambda$, where $0 \leqslant \tau \leqslant \lambda$. Consider the following polynomial

$$P_c(Y) = \left(\prod_{i=1}^{\tau}(Y-\alpha_i)^{ln_i}\right)\left(\prod_{i=\mu+1}^{\delta}(x-\beta_i)^{m_i}\right) - c\left(\prod_{i=\tau+1}^{\lambda}(Y-\alpha_i)^{ln_i}\right)\left(\prod_{i=1}^{\mu}(x-\beta_i)^{m_i}\right)$$

where c is a constant. It can be easily shown that $P_c(Y)$ is an algebraic general solution of Equation (3.8). Therefore the equation does not have transcendental liouvillian solutions.

Suppose that there is no nonzero element $\gamma \in \overline{C(x)}$ such that $\gamma'/\gamma = mh(x)$, for any nonzero integer m. Consider the differential extension of $C(x,z)$ of $C(x)$, where $z' = h(x)z$. Note that $C(x,z)$ is a no new constant extension of C (by Proposition 2.2.3 (iii)). Let $y_1 \in \overline{C(x,z)}$ be a root of the following polynomial

$$Q(Y) = \prod_{i=1}^{\tau}(Y-\alpha_i)^{n_i} - z\prod_{i=\tau+1}^{\lambda}(Y-\alpha_i)^{n_i}.$$

Then $C(x,z,y_1)$ is a differential field extension of $C(x,z)$. Observe that y_1 is transcendental over $C(x)$ and y_1 is a solution of the differential equation (3.8). Since

y_1 is algebraic over $C(x, z)$, the field of constants of $C(x, z, y_1)$ is C (by Lemma 2.2.1 (i)). Note that $C(x, z, y_1) = C(x, y_1)$. As in Lemma 2.3.3 (ii), one can show that $\alpha_1, \ldots, \alpha_\lambda$ are the algebraic solutions of Equation (3.8). $\square$

Corollary 3.1.4. *The differential equation (3.1) has a transcendental solution in an exponential extension of $C(x)$ if and only if $\frac{1}{F(y)} = \sum_{i=1}^{n} \frac{m_i}{y - \alpha_i}$, where n is a positive integer, m_i are nonzero integers and α_i are pairwise distinct elements of C. In this case $\alpha_1, \ldots, \alpha_n$ are the algebraic solutions.*

Proof. Observe that the necessary part of Corollary 3.1.4 follows from Proposition 3.1.3 and the sufficient part follows from Equation (3.5). $\square$

3.2 A few counterexamples

Here we provide few counterexamples where the conclusions of Theorem 3.1.2 do not hold.

1. If a variable separable differential equation: $y' = r(x)F(y)$, where $r(x) \in C[x], F(y) \in C[y]$ admits a transcendental liouvillian solution then by Theorem 3.1.2, the number of algebraic solutions is equal to the degree of F in Equation (3.1). This phenomenon need not hold in general. For example, consider the differential equation $y' = (1/x)\ y^3$. There is no $t \in C(x)$ such that $t' = -2/x$. Now consider the tower of differential fields $C(x) \subseteq C(x, t) \subseteq C(x, t, y)$, where t is transcendental and $t' := -2/x$ and y is algebraic over $C(x, t)$ satisfying the relation $y^2 - 1/t = 0$. Then $C(x, t, y)$ is a liouvillian extension of C. In particular $C(x, y)$ is a liouvillian extension of $C(x)$, y is transcendental over $C(x)$, $t = 1/y^2 \in C(x, y)$ and $y' = (1/x)y^3$. By Proposition 2.2.3 (ii) and

Lemma 2.2.1 (i), the field of constants of $C(x,y)$ is C. Now taking $z=t$ and $a=0$ in Lemma 2.3.3 (i), we obtain that the only algebraic solution is zero.

2. If the differential equation (3.1) has a transcendental liouvillian solution, then there are finitely many algebraic solutions. But the existence of finitely many algebraic solutions does not imply the existence of a transcendental liouvillian solution. For example, it is shown in [37, p. 421] that $y' = (1/x)y^2(y-1)$ does not have a transcendental liouvillian solution over $C(x)$. Using arguments similar to those given in Proposition 3.1.1, one can also show that the only algebraic solutions of the differential equation are 0 and 1.

3. By Proposition 3.1.3, there are variable separable differential equations with infinitely many algebraic solutions. For example, consider $y' = (1/x)y(y-1)(y-2)$. Now for any nonzero constant c, the roots of the polynomial $P_c(Y) = cx^2(Y-1)^2 - Y(Y-2)$ in $\overline{C(x)}$ are solutions of the above differential equation.

Until now we have discussed differential equations that have at least one algebraic solution. We shall now give necessary and sufficient conditions for the existence of a transcendental liouvillian solution of the following differential equation

$$f(y)\,y' = g(x), \text{ where } f(y) \in C[y] \setminus \{0\}\,,\ g(x) \in C(x) \setminus \{0\}\,. \tag{3.9}$$

We will also show that there are no algebraic solutions if such a solution exists. We will need the following propositions to prove our result.

Proposition 3.2.1. *Let q be a monic, irreducible polynomial in $C(x)[Y]$, where Y is an indeterminate. Then the roots of q are algebraic solutions of Equation (3.9) if and only if q divides the polynomial $H(Y) := f(Y)\,\frac{\partial q}{\partial x} + g(x)\,\frac{\partial q}{\partial Y}$.*

Proof. Suppose that q divides the polynomial $H(Y)$. Then there is a polynomial $Q \in C(x)[Y]$ such that

$$f(Y)\,\frac{\partial q}{\partial x} + g(x)\,\frac{\partial q}{\partial Y} = q\,Q. \tag{3.10}$$

Observe that $q \neq Y - c_1$, for any $c_1 \in C$, otherwise the above equation reduces to $g(x) = (Y - c_1)Q$, which is absurd. Let α be a root of q, then $\alpha \in \overline{C(x)} \setminus C$. Taking the derivative of $q(x, \alpha) = 0$, we get

$$\frac{\partial q}{\partial x}(x, \alpha) + \frac{\partial q}{\partial Y}(x, \alpha)\,\alpha' = 0. \tag{3.11}$$

Substituting $Y = \alpha$ in Equation (3.10) we obtain

$$f(\alpha)\,\frac{\partial q}{\partial x}(x, \alpha) + g(x)\,\frac{\partial q}{\partial Y}(x, \alpha) = 0. \tag{3.12}$$

Multiplying Equation (3.11) by $f(\alpha)$ and subtracting it from Equation (3.12), we get

$$\frac{\partial q}{\partial Y}(x, \alpha)\,(f(\alpha)\,\alpha' - g(x)) = 0. \tag{3.13}$$

Since q is the minimal polynomial of α, $\frac{\partial q}{\partial Y}(x, \alpha) \neq 0$. Note that $f(Y) \in C[Y]$ therefore $f(\alpha) \neq 0$. Thus $f(\alpha)\alpha' - g(x) = 0$, this proves the sufficient part.

Conversely, let the roots of $q(x, Y)$ be algebraic solutions of Equation (3.9). Let α be a root of q. Since the differential equation does not have any solutions in C, so $\alpha \in \overline{C(x)} \setminus C$. Suppose that $\alpha \in C(x)$, then since q is monic and irreducible, $q = (Y - \alpha)$. In this case, it can be easily shown that $H(\alpha) = 0$, thus q divides $H(Y)$. Let us assume that $\alpha \in \overline{C(x)} \setminus C(x)$. Then $C(x)[\alpha]$ is a finite algebraic extension of $C(x)$ and the derivation extends uniquely to $C(x)[\alpha]$. So there exists $h(x, Y) \in C(x)[Y]$ such that $\alpha' = h(x, \alpha)$. Note that by Proposition 2.2.4 (i), $\alpha' \notin C(x)$. Therefore $\deg_Y h(x, Y) \geq 1$. Differentiating $q(x, \alpha) = 0$, we have

$$\frac{\partial q}{\partial x}(x, \alpha) + \frac{\partial q}{\partial Y}(x, \alpha)\,\alpha' = 0. \tag{3.14}$$

Consider the polynomial $P \in C(x)[Y]$ given by $P(x,Y) = (\partial q/\partial x) + (\partial q/\partial Y)\, h(x,Y)$. Observe that $\deg_Y(q) > \deg_Y(\partial q/\partial x)$ as q is a nonzero monic polynomial in $C(x)[Y]$. Since $\deg_Y(h) \geq 1$, $\deg_Y((\partial q/\partial Y)\, h(x,Y)) \geq \deg_Y(q) > \deg_Y(\partial q/\partial x)$. Therefore P is a nonzero polynomial in $C(x)[Y]$ such that $P(x,\alpha) = 0$. Now consider the polynomial $Q(x,Y) = f(Y)h(x,Y) - g(x)$. Since $\deg_Y h(x,Y) \geq 1$, Q is a nonzero polynomial. Also, note that $Q(x,\alpha) = 0$. Therefore, there exists nonzero polynomials $P_1(x,Y), P_2(x,Y) \in C(x)[Y]$ such that

$$f(Y)\, h(x,Y) - g(x) = q(x,Y)\, P_1(x,Y) \tag{3.15}$$

$$\frac{\partial q}{\partial x} + \frac{\partial q}{\partial Y} h(x,Y) = q(x,Y)\, P_2(x,Y). \tag{3.16}$$

Multiplying Equations (3.15) and (3.16) by $(-\partial q/\partial Y)$ and $f(Y)$ respectively and adding them, we get

$$f(Y)\frac{\partial q}{\partial x} + \frac{\partial q}{\partial Y}\, g(x) = q(x,Y)\, P_3(x,Y),$$

where $P_3(x,Y) = f(Y)\, P_2(x,Y) - (\partial q/\partial Y) P_1(x,Y)$. Therefore q divides $H(Y)$. □

Proposition 3.2.2. *Suppose that the differential equation (3.9) has a transcendental liouvillian solution y. Then it has at most finitely many algebraic solutions.*

Proof. If the differential equation (3.9) has a transcendental liouvillian solution y, then by Theorem (B) of [38, p. 359] and Theorem 2.2 of [37, p. 414], there is an element $z \in C(x,y) \setminus C(x)$ that satisfies a first order linear differential equation over $C(x)$ or there is an element $w \in \overline{C(x)}(y) \setminus \overline{C(x)}$ satisfying $w' = \overline{a}\, w$, where $\overline{a} \in \overline{C(x)} \setminus C(x)$. We will prove the proposition assuming that there exists $z \in C(x,y) \setminus C(x)$ satisfying $z' = b$, where $b \in C(x) \setminus \{0\}$. The proofs of the other cases are similar. Suppose that $z = p/q$, where $p,\, q \in C(x)[y] \setminus \{0\}$ and $(p,q) = 1$. Now,

$$z' = b \implies qp' - q'p = bq^2 \implies f(y)\,(qp' - q'p) = bf(y)q^2. \tag{3.17}$$

Therefore q divides $f(y)q'$ and by Proposition 3.2.1, one can show that all the roots of q are solutions of differential equation (3.9).

Let α be an algebraic solution of the differential equation. Let $q_1 \in C(x)[y]$ be the minimal polynomial of α. Then since the differential equation (3.9) does not have any solutions in C, $\alpha \in \overline{C(x)} \setminus C$ and $q_1 \in C(x)[y] \setminus C[y]$. Suppose that q_1 divides p. Then p can be written as $p = q_1^n p_1$, where $p_1 \in C(x)[y] \setminus \{0\}$, n is a positive integer, and $(q_1, p_1) = 1$. Note that $q_1' \neq 0$ as $C(x, y)$ is a no new constant extension of C. Since the roots of q_1 are solutions of Equation (3.9), therefore by Proposition 3.2.1, $f(y)q_1' = q_1 Q$, where Q is a nonzero polynomial in $C(x)[y]$. Substituting $p = q_1^n p_1$ and $f(y)q_1' = q_1 Q$ in $f(y)\,(qp' - pq') = f(y)(bq^2)$, we get

$$q_1^n\,(nqQp_1 + f(y)qp_1' - f(y)p_1 q') = f(y)(bq^2).$$

This implies that q_1 divides q, which is a contradiction as $(p, q) = 1$. Therefore, q_1 does not divide p. Now we will show that q_1 is a factor of q. If it is not so, then $q(x, \alpha) \neq 0$. From the previous discussion, α is not a root of p; therefore $p(x, \alpha) \neq 0$. Note that $f(\alpha) \neq 0$ as $\alpha \in \overline{C(x)} \setminus C$. We may write

$$p = \sum_{i=0}^{n} c_i\,(y - \alpha)^i, \quad q = \sum_{i=0}^{m} d_i\,(y - \alpha)^i, \quad f(y) = \sum_{i=0}^{l} e_i (y - \alpha)^i$$

where $n, m, l,$ are non negative integers, $c_i, d_i, e_i \in \overline{C(x)}$ and $c_n, c_0, d_m, d_0, e_l, e_0$ are nonzero elements. Comparing the coefficient of $(y - \alpha)^0$ in $f(y)\,(qp' - pq') = f(y)(bq^2)$, we get

$$e_0\,(d_0\,c_0' - c_0\,d_0') = e_0\,(b\,d_0^2) \implies (c_0/d_0)' = b.$$

Now $z' = b$ which implies that $(z - c_0/d_0)' = 0$. This contradicts the fact that $\overline{C(x, y)}$ is a no new constant extension of C. Therefore $d_0 = 0$, which implies $q(\alpha) = 0$. Let $z_1 \in C(x, y) \setminus C(x)$ such that $z_1' \in C(x)$, then by Kolchin-Ostrowsky Theorem

$z_1 = c_1 z + h$, where c_1 is a nonzero constant and $h \in C(x)$. So the denominator of z is unique up to multiplication by nonzero scalars. Thus every root of q is an algebraic solution of the differential equation (3.9) and conversely. Since q has finitely many roots, the differential equation has finitely many algebraic solutions. □

Theorem 3.2.3. *The following differential equation*

$$f(y)y' = g(x), \text{ where } f(y) \in C[y] \setminus \{0\}, \; g(x) \in C(x) \setminus \{0\},$$

has a transcendental liouvillian solution y if and only if the antiderivative of $g(x)$ does not lie in $C(x)$. In this case, there exists an element $z \in C(x,y) \setminus C(x)$ such that $z' = g(x)$ and the differential equation does not have any algebraic solution.

Proof. Suppose the differential equation (3.9) has a transcendental liouvillian solution y. Let $f(y) = \sum_{i=0}^{n} a_i \, y^i$, where $a_i \in C$ and $a_n \neq 0$. If there exists $t \in C(x)$ such that $t' = g(x)$, then the differential equation has an algebraic general solution given by $Q_c(Y) = \sum_{i=0}^{n} (a_i/(i+1)) \, Y^{i+1} - t + c$, where c is a constant. In this case, the differential equation does not have a transcendental liouvillian solution. Thus $g(x)$ does not have any antiderivative in $C(x)$. This proves the sufficient part.

Consider the element $z \in C(x,y)$ given by $z = \sum_{i=0}^{n} (a_i/(i+1)) \, y^{i+1}$. Clearly, z is transcendental over $C(x)$ and $z' = g(x)$. Suppose that α is an algebraic solution of the differential equation. Note that the field of constants of $C(x,z)$ is C. Then by Proposition 3.2.2, the minimal polynomial of α divides the denominator of z. Since z is a polynomial in y, the differential equation does not have any algebraic solution.

Conversely, suppose that $g(x)$ has no antiderivative in $C(x)$. Then define a differential extension $C(x,z)$ of $C(x)$ by $z' = g(x)$. Note that $C(x,z)$ is a purely transcendental extension of $C(x)$ such that the field of constants of $C(x,z)$ is C (by Proposition 2.2.3 (ii)). Define an algebraic extension $C(x,z,y_1)$ of $C(x,z)$, where $y_1 \in \overline{C(x,z)}$ is a root of the polynomial $P(Y) = \sum_{i=0}^{n} (a_i/(i+1)) \, (Y)^{i+1} - z$. Observe that y_1 is transcendental over $C(x)$ and it can be easily shown that y_1 is a solution of the differential equation (3.9). Also note that the field of constants of $C(x, z, y_1)$ is C. Thus $C(x, z, y_1)$ is a liouvillian extension of $C(x)$ which contains a transcendental solution of the differential equation. $\square$

Chapter 4

Transcendental liouvillian solutions of first order nonlinear differential equations

The content of this chapter is based on the author's work in [33]. Consider the following first order nonlinear differential equation:

$$y' = a_n y^n + a_{n-1} y^{n-1} + \cdots + a_0, \tag{4.1}$$

where $a_i \in C(x)$ and $a_n \neq 0$. Let y_1 be a solution of differential equation (4.1). If $n = 2$ (respectively $n = 3$), then the above equation is called Riccati equation (respectively Abel's differential equation of the first kind).

In this chapter our focus is to develop methods to find transcendental solutions of the differential equation in an exponential extension of $C(x)$.

4.1 Relation between algebraic and transcendental solutions

In the following theorem we provide the relationship between the algebraic and transcendental liouvillian solutions of differential equation (4.1).

Theorem 4.1.1. *Suppose differential equation (4.1) has a transcendental liouvillian solution y. Then the differential equation has finitely many algebraic solutions.*

Proof. Suppose that y is a transcendental liouvillian solution of Equation (4.1). Then by definition, the field of constants of $C(x,y)$ is C. By Theorem 2.2 of [37], there is an element in $C(x,y)\setminus C(x)$ satisfying a linear homogeneous differential equation of order greater or equal to one over $C(x)$. Now $C(x,y)$ is finitely generated over $C(x)$, therefore the hypotheses of [38, Theorem B] are satisfied. Since y is transcendental over $C(x)$, the algebraic closure of $C(x)$ in $C(x,y)$ is $C(x)$ itself. Therefore there exists an element $z\in C(x,y)\setminus C(x)$ which satisfies a linear differential equation of order $\leqslant 2$. If there is no element in $C(x,y)\setminus C(x)$ that satisfies a first order linear differential equation, then z satisfies an irreducible linear homogeneous differential equation of order 2. In which case, there exists an element w in $\overline{C(x)}(y)\setminus\overline{C(x)}$ such that $w'=aw$, where a lies in a quadratic extension of $C(x)$. Therefore only one of the following two cases can occur:

1. there is an element $z\in\overline{C(x)}(y)\setminus\overline{C(x)}$ such that $z'=az$, for some nonzero $a\in\overline{C(x)}$.

2. there is an element $z\in\overline{C(x)}(y)\setminus\overline{C(x)}$ satisfying $z'=az+b$, where $a,b\in\overline{C(x)}$ and $b\neq 0$.

By Lemma 2.2.1 (i), the derivation of $C(x,y)$ extends uniquely to a derivation of $\overline{C(x)}(y)$ and the field of constants remains the same. Suppose that case (1) holds. Then by Lemma 2.3.3 (ii), algebraic solutions of Equation (4.1) are the zeros and poles of z. Since the number of zeros and poles of z is finite, the number of algebraic solutions of the differential equation is also finite. Similar arguments can be applied to case (2). Therefore in both cases, the number of algebraic solutions is finite. $\square$

The converse is not true. In Chapter 6 we will show that the differential equation $y' = (1/x)(y^3 - y^2)$ has transcendental solutions over $\mathbb{C}(x)$ but no transcendental liouvillian solution. We have already seen that it has only two algebraic solutions.

From this point on, we will discuss transcendental liouvillian solutions of Equation (4.1) where there exists $z \in \overline{C(x)}(y) \setminus \overline{C(x)}$ such that $z' = az$, for some nonzero $a \in \overline{C(x)}$, that is, the differential equation has a transcendental solution in an exponential extension of $C(x)$.

Proposition 4.1.2. *Let K be an algebraically closed differential field of characteristic zero. Let $K(y)$ be a no new constant extension of K. Suppose that there is an element $z \in K(y) \setminus K$ such that $z'/z \in K$. Then there is an element $w \in K(y) \setminus K$ with $w'/w \in K$ such that any $z_1 \in K(y) \setminus K$ satisfying $z_1'/z_1 \in K$ is of the form $z_1 = g_1 w^n$ for some nonzero element $g_1 \in K$ and nonzero integer n.*

Proof. Since K is algebraically closed, y is transcendental over K and C_K is algebraically closed. Consider the following set:

$$S = \{t \in K(y) \setminus K \mid t'/t \in K \text{ and } K(z) \subsetneq K(z,t)\}.$$

Since $K(y)$ is a finite algebraic extension of $K(z)$, the number of fields intermediate to $K(z)$ and $K(y)$ is finite. Therefore there is a finite set $\Omega = \{z, t_1, \ldots, t_m\}$ such that for any $t \in S$, $K(z,t) = K(z,t_i)$ for some $t_i \in \Omega$. By Kolchin-Ostrowski theorem $t_i = g_i z^{r_i/s_i}$, where $g_i \in K \setminus \{0\}$, $r_i, s_i \in \mathbb{Z} \setminus \{0\}$ and $(r_i, s_i) = 1$. We may assume that $|r_i| < |s_i|$. Therefore $K(z, t_i) = K(z^{1/s_i})$. Let $n = \max\{|s_i|\}$ and $w = z^{1/n}$.

We claim that any $z_1 \in k(y) \setminus K$ satisfying $z_1'/z_1 \in K$ lies in $K(w) = K(z^{1/n})$. If $z_1 \in K(z)$ then we are done. Suppose that $K(z) \subsetneq K(z, z_1)$. Then $K(z, z_1) = K(z, t_j) = K(z^{1/s_j})$, for some $j \in \{1, \ldots, m\}$. Note that $|s_j| \le n$. If $K(w) \subsetneq K(w, z_1)$. Then by Kolchin-Ostrowski theorem $z_1 = g w^{r/s}$, where $g \in K \setminus \{0\}$, $r, s \in \mathbb{Z} \setminus \{0\}$, $(r, s) = 1$ and $|r| < |s|$. Then $z_1 = g z^{r/(ns)}$. Let $r/(ns) = r_0/s_0$, where $r_0, s_0 \in \mathbb{Z} \setminus \{0\}$ and $(r_0, s_0) = 1$. Now we will show that $|s_0| > n$. Suppose that $d = (r, ns)$. Then since $(r, s) = 1$, we have $d = (r, n)$, $r_0 = r/d$ and $s_0 = (ns)/d$. Clearly, $d \le |r| < |s|$. This implies that $n = (n/d)d < (n/d)|s| = |s_0|$. Now, $K(z^{1/s_j}) = K(z, z_1) = K(z^{1/s_0})$. This implies $|s_0| = |s_j|$. But $n < |s_0|$ and $|s_j| \le n$. Thus $z_1 \in K(w)$. $\square$

Theorem 4.1.3. *Let y be a transcendental liouvillian solution of the differential equation*

$$y' = a_n y^n + a_{n-1} y^{n-1} + \cdots + a_0, \text{ where } a_i \in C(x) \text{ and } a_n \neq 0. \tag{4.2}$$

Suppose that there is an element $z \in \overline{C(x)}(y) \setminus \overline{C(x)}$ such that $z' = az$, where $a \in \overline{C(x)} \setminus \{0\}$. Then the following statements hold:

(i) There are at least n pairwise distinct algebraic solutions of differential equation (4.2).

(ii) $z = g \prod_{i=1}^{l} (y - \alpha_i)^{m_i}$ and $a = g'/g + a_n \sum_{i=1}^{l} m_i \alpha_i^{n-1}$, where g is a nonzero element of $\overline{C(x)}$, l is a positive integer, $\{\alpha_1, \ldots, \alpha_l\}$ is the set of algebraic solutions of the differential equation and $m_1, \ldots, m_l$ are nonzero integers.

(iii) α_i *and* m_i *satisfy the following equations:*

$$\sum_{i=1}^{l} m_i = 0,\ \sum_{i=1}^{l} m_i\alpha_i = 0, \dots,\ \sum_{i=1}^{l} m_i\alpha_i^{n-2} = 0,\ a_n\sum_{i=1}^{l} m_i\alpha_i^{n-1} \neq \frac{\gamma'}{\gamma}, \tag{4.3}$$

for any nonzero $\gamma \in \overline{C(x)}$.

Conversely, suppose that $\alpha_1, \alpha_2, \dots, \alpha_l$ *are algebraic solutions of differential equation (4.2) and* $m_1, m_2, \dots, m_l$ *are nonzero integers that satisfy Equation (4.3). Then the differential equation has a transcendental liouvillian solution* y *such that there exists* $z \in \overline{C(x)}(y) \setminus \overline{C(x)}$ *satisfying* $z' = \left(a_n \sum_{i=1}^{l} m_i\alpha_i^{n-1}\right) z$.

Proof. Let y be a transcendental liouvillian solution of differential equation (4.2) then $C(x, y)$ is a no new constant extension of $C(x)$. Suppose that there exist an element $z \in \overline{C(x)}(y) \setminus \overline{C(x)}$ such that $z' = az$, for some nonzero $a \in \overline{C(x)}$. Then by Theorem 4.1.1, the differential equation has finitely many solutions in $\overline{C(x)}$. Let $\{\alpha_1, \dots, \alpha_l\}$ be the set of all the algebraic solutions of Equation (4.2). Then by Lemma 2.3.3 (ii), $z = g\prod_{i=1}^{l}(y - \alpha_i)^{m_i}$, where g is a nonzero element of $\overline{C(x)}$ and m_i are nonzero integers. Note that z is transcendental over $\overline{C(x)}$. Therefore by Proposition 2.2.4 (ii), there is no nonzero $\gamma \in \overline{C(x)}$ such that $a = \gamma'/\gamma$. Now,

$$a = \frac{z'}{z} = \frac{g'}{g} + \sum_{i=1}^{l} m_i \frac{y' - \alpha_i'}{y - \alpha_i}. \tag{4.4}$$

Since each α_i is an algebraic solution of the differential equation, it follows from Equation (2.2) that $y - \alpha_i$ divides $(y' - \alpha_i')$ and we have

$$\frac{y' - \alpha_i'}{y - \alpha_i} = a_n y^{n-1} + (a_n\alpha_i + a_{n-1})y^{n-2} + \cdots + \left(a_n\alpha_i^{n-1} + a_{n-1}\alpha_i^{n-2} + \cdots + a_1\right). \tag{4.5}$$

From Equations (4.4) and (4.5), we get

$$\begin{aligned} a-\frac{g'}{g} =& \sum_{i=1}^{l} m_i \left(a_n y^{n-1} + (a_n\alpha_i + a_{n-1})y^{n-2} + \cdots + a_n\alpha_i^{n-1} + a_{n-1}\alpha_i^{n-2} + \cdots + a_1\right) \\ =& a_n \left(\sum_{i=1}^{l} m_i\right) y^{n-1} + \left(a_n \sum_{i=1}^{l} m_i\alpha_i + a_{n-1}\sum_{i=1}^{l} m_i\right) y^{n-2} + \cdots + \\ & \left(a_n \sum_{i=1}^{l} m_i\alpha_i^{n-1} + a_{n-1}\sum_{i=1}^{l} m_i\alpha_i^{n-2} + \cdots + a_1 \sum_{i=1}^{l} m_i\right). \end{aligned} \tag{4.6}$$

Note that $a - g'/g \neq \gamma'/\gamma$, for any nonzero $\gamma \in \overline{C(x)}$. Otherwise, $a = (g\gamma)'/(g\gamma)$, which is a contradiction. Since $a_n \neq 0$, we get Equations (4.3) by comparing the coefficients of y^i in the Equation (4.6) . If $l < n$, then since the α_i are pairwise distinct elements of $\overline{C(x)}$, the only solution of Equations (4.3) is $m_i = 0$ for all $i = 1, \ldots, l$. This contradicts the fact that m_i are nonzero integers. Therefore the differential equation has at least n distinct algebraic solutions.

Conversely, suppose that $\alpha_1, \ldots, \alpha_l$ are pairwise distinct algebraic solutions of Equation (4.2) and that there exists nonzero integers $m_1, \ldots, m_l$ satisfying Equations (4.3). Then consider the field extension $C(x) \subseteq C(x, \alpha_1, \ldots, \alpha_l, y_1)$, where y_1 is transcendental over $C(x, \alpha_1, \ldots, \alpha_l)$. Since α_i are algebraic over $C(x)$, therefore by Lemma 2.2.1 (i), the derivation of $C(x)$ extends uniquely to a derivation of $C(x, \alpha_1, \ldots, \alpha_l)$ and the field of constants remains the same. We define $y_1' = a_n y_1^n + \cdots + a_0$. Then by Lemma 2.2.1 (ii), $C(x, \alpha_1, \ldots, \alpha_l, y_1)$ is a differential field extension of $C(x)$. Now consider the element $z_1 \in C(x, \alpha_1, \ldots, \alpha_l, y_1)$ given by $z_1 = \prod_{i=1}^{l}(y_1 - \alpha_i)^{m_i}$. Note that α_i are solutions of Equation (4.2), therefore from

Equation (4.6) we obtain

$$\begin{aligned}\frac{z_1'}{z_1} =& \sum_{i=1}^{l} m_i \frac{y_1' - \alpha_i'}{y_1 - \alpha_i} \\ =& a_n \left(\sum_{i=1}^{l} m_i \right) y_1^{n-1} + \left(a_n \sum_{i=1}^{l} m_i \alpha_i + a_{n-1} \sum_{i=1}^{l} m_i \right) y_1^{n-2} + \cdots + \\ & \left(a_n \sum_{i=1}^{l} m_i \alpha_i^{n-1} + a_{n-1} \sum_{i=1}^{l} m_i \alpha_i^{n-2} + \cdots + a_1 \sum_{i=1}^{l} m_i \right). \end{aligned} \tag{4.7}$$

Since m_i and α_i satisfy Equations (4.3), we obtain $z_1' = \left(a_n \sum_{i=1}^{l} m_i \alpha_i^{n-1}\right) z_1$. Also, there is no nonzero element $\gamma \in \overline{C(x)}$ such that $\gamma' = \left(a_n \sum_{i=1}^{l} m_i \alpha_i^{n-1}\right) \gamma$, therefore by Proposition 2.2.3 (iii), $C(x, \alpha_1, \ldots, \alpha_l, z_1)$ is a purely transcendental differential field extension of $C(x, \alpha_1, \ldots, \alpha_l)$ such that the field of constants is C. Note that y_1 lies in an algebraic extension of $C(x, \alpha_1, \ldots, \alpha_l, z_1)$. So the field of constants of $C(x, \alpha_1, \ldots, \alpha_l, y_1)$ is C. Thus y_1 is a transcendental liouvillian solution of Equation (4.2). By Lemma 2.3.3 (ii), $\alpha_1, \ldots, \alpha_l$ are precisely the algebraic solutions of the differential equation. This proves the converse part. □

Remark 4.1.4. Let y and z be as defined in the above theorem. Now we will show that the nonzero integers m_i are not unique but their fractions are. By Proposition 4.1.2, there is an element $w \in \overline{C(x)}(y) \setminus \overline{C(x)}$ such that $w'/w \in \overline{C(x)}$ and any $z_1 \in \overline{C(x)}(y) \setminus \overline{C(x)}$ satisfying $z_1'/z_1 \in \overline{C(x)}$ is of the form $z_1 = g_1 w^r$ for some nonzero integer r. Then by Theorem 4.1.3 (ii), $w = \prod_{i=1}^{l} (y - \alpha_i)^{m_i}$. Therefore $z_1 = g \prod_{i=1}^{l} (y - \alpha_i)^{r m_i}$. Now

$$\frac{r m_i}{r m_1} = \frac{m_i}{m_1}, \ \forall \ i > 1.$$

Proposition 4.1.5. *Let y and z be as defined in Theorem 4.1.3. If $z \in C(x, y) \setminus C(x)$, then z can be written as $z = \prod_{i=1}^{r} G_i^{m_i}$, where $G_1, \ldots, G_r$ are the minimal polynomials of the algebraic solutions.*

Proof. Let $\{\alpha_1, \ldots, \alpha_l\}$ be the set of algebraic solutions of Equation (4.2). Let $\{G_1, \ldots, G_r\}$, where $1 \leqslant r \leqslant l$, be the set of minimal polynomials of the algebraic solutions of differential equation. Let $z = p/q$, where p, $q \in C(x)[y] \setminus \{0\}$ and $(p, q) = 1$. By Lemma 2.3.3 (ii), α_1 is a root of either p or q. Let α_1 be a root of p and G_1 be the minimal polynomial of α_1. Then all the roots of G_1 are algebraic solutions of the differential equation. Note that G_1 divides p since G_1 is an irreducible polynomial. Also, G_1 does not divide q as $(p, q) = 1$. Therefore p can be written as $p = G_1^{m_1} p_1$, where $p_1 \in C(x)[y]$ and $(G_1, p_1) = 1$. Therefore, if α_i and α_j are the roots of same minimal polynomial, then $m_i = m_j$ and z can be written as $z = \prod_{i=1}^{r} G_i^{m_i}$. □

Next we will prove two polynomial identities which will be used to prove Proposition 4.1.7 later.

Lemma 4.1.6. *Let K be a differential field of characteristic zero and $K(y)$ be the rational function field over K. Let $r_1, \ldots, r_n \in K$ be the roots of the polynomial $p(y) = \sum_{j=0}^{n} f_j y^{n-j}$, where $f_0 = 1$. Then*

$$\sum_{i=1}^{n} m_i \left(\prod_{j=1, j \neq i}^{n} (y - r_j) \right) = \sum_{j=0}^{n-1} \left(\sum_{i=0}^{j} b_i f_{j-i} \right) y^{n-1-j} \quad \text{and} \tag{4.8}$$

$$\sum_{i=1}^{n} m_i r_i' \left(\prod_{j=1, j \neq i}^{n} (y - r_j) \right) = \sum_{j=0}^{n-1} \left(\sum_{i=0}^{j} \frac{b_{i+1}'}{i+1} f_{j-i} \right) y^{n-1-j}, \tag{4.9}$$

where $m_1, \ldots, m_n$ are nonzero integers and $b_i = \sum_{j=1}^{n} m_j r_j^i$, for $i = 0, \ldots, n$.

Proof. Since $r_1, \ldots, r_n$ are the roots of $p(y)$, therefore using Vieta's formulas, we have

$$(-1)^j f_j = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant n} \left(\prod_{l=1}^{j} r_{i_l} \right), \text{ for } j = 1, \ldots, n. \tag{4.10}$$

Let $g_{i,0} = 1$ and $g_{i,j} = (-1)^j f_j - r_i g_{i,j-1}$, for all $1 \leqslant i, j \leqslant n$. Let $d_j = \sum_{i=1}^{n} m_i g_{i,j}$, for $0 \leqslant j \leqslant n-1$. Then $\prod_{j=1, j\neq i}^{n}(y - r_j) = \sum_{j=0}^{n-1}(-1)^j g_{i,j} y^{n-1-j}$ and

$$\sum_{i=1}^{n} m_i \left(\prod_{j=1, j\neq i}^{n} (y - r_j) \right) = \sum_{j=0}^{n-1} (-1)^j d_j y^{n-1-j}. \tag{4.11}$$

We will show that $(-1)^j d_j = \sum_{i=0}^{j} b_i f_{j-i}$, for all $0 \leqslant j \leqslant n-1$.

$$\begin{aligned} d_j =& \sum_{i=1}^{n} m_i g_{i,j} = \sum_{i=1}^{n} m_i \left((-1)^j f_j - r_i g_{i,j-1} \right) \\ =& (-1)^j \sum_{i=1}^{n} m_i f_j - \sum_{i=1}^{n} m_i r_i \left((-1)^{j-1} f_{j-1} - r_i g_{i,j-2} \right) \\ =& (-1)^j \sum_{i=1}^{n} m_i f_j + (-1)^j \sum_{i=1}^{n} m_i r_i f_{j-1} + (-1)^j \sum_{i=1}^{n} m_i r_i^2 f_{j-2} + \cdots + \\ & (-1)^j \sum_{i=1}^{n} m_i r_i^{j-1} f_1 + (-1)^j \sum_{i=1}^{n} m_i r_i^j f_0 \\ =& (-1)^j \left(b_0 f_j + b_1 f_{j-1} + \cdots + b_{j-1} f_1 + b_j f_0 \right). \end{aligned}$$

We obtain Equation (4.8) by substituting the value of $(-1)^j d_j$ in Equation (4.11). Similarly, one can show that Equation (4.9) also holds. □

In Theorem 4.1.3 (i), we have seen that if Equation (4.2) has a transcendental solution in an exponential extension of $k(x)$, then there are at least n algebraic solutions, where n is the degree of the polynomial $y' = a_n y^n + \cdots + a_0$ $(= y')$. In the following proposition we will discuss the structure of differential equation (4.2) when there are precisely n algebraic solutions.

Proposition 4.1.7. *Let y, z and a be as defined in Theorem 4.1.3. Suppose that differential equation (4.2) has precisely n pairwise distinct algebraic solutions $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then the differential equation is of the following form:*

$$y' = a_n \prod_{i=1}^{n} (y - \alpha_i) + \sum_{i=1}^{n} \left(\prod_{j=1, j\neq i}^{n} \frac{y - \alpha_j}{\alpha_i - \alpha_j} \right) \alpha_i'. \tag{4.12}$$

Proof. Given that $\{\alpha_1, \ldots, \alpha_n\}$ is the set of all the algebraic solutions of the differential equation. Note that by Theorem 4.1.3 (i) and (ii), Equation (4.2) has at least n distinct algebraic solutions and $z = g\prod_{i=1}^{n}(y-\alpha_i)^{m_i}$, where $g \in \overline{C(x)} \setminus \{0\}$ and $m_1, \ldots, m_n$ are nonzero integers. Let $z_1 = hz$, where h is a nonzero element of $\overline{C(x)}$. Then $z_1' = (h'/h + a)z_1$. Therefore we may take $g = 1$. Now,

$$
\begin{aligned}
a &= \frac{z'}{z} = \sum_{i=1}^{n} m_i \frac{y' - \alpha_i'}{y - \alpha_i} \\
\implies \left(\sum_{i=1}^{n} m_i \left(\prod_{j=1, j\neq i}^{n} (y-\alpha_j)\right)\right) y' &= a\prod_{i=1}^{n}(y-\alpha_i) + \sum_{i=1}^{n}\left(m_i\alpha_i' \prod_{j=1, j\neq i}^{n}(y-\alpha_j)\right).
\end{aligned} \tag{4.13}
$$

From Proposition 4.1.6, we obtain the following expression:

$$
\sum_{i=1}^{n} m_i \left(\prod_{j=1, j\neq i}^{n} (y-\alpha_j)\right) = \sum_{j=0}^{n-1}\left(\sum_{i=0}^{j} f_{j-i}b_i\right) y^{n-1-j}, \tag{4.14}
$$

where $b_i = \sum_{j=1}^{n} m_j\alpha_j^i$ and $f_i \in \overline{C(x)}$ such that $\prod_{i=1}^{n}(y-\alpha_i) = \sum_{i=0}^{n} f_i y^{n-i}$. Note that $b_i = 0$ for $i = 0, \ldots, n-2$ as the m_i and the α_i satisfy Equations (4.3). Therefore Equation (4.14) becomes

$$
\sum_{i=1}^{n} m_i \left(\prod_{j=1, j\neq i}^{n} (y-\alpha_j)\right) = \sum_{i=1}^{n} m_i\alpha_i^{n-1}. \tag{4.15}
$$

By Theorem 4.1.3 (ii), $\sum_{i=1}^{n} m_i\alpha_i^{n-1} = a/a_n$. Substituting the above values in Equation (4.13), we get

$$
y' = a_n\prod_{i=1}^{n}(y-\alpha_i) + \sum_{i=1}^{n}\left(\frac{a_n}{a} m_i\alpha_i' \prod_{j=1, j\neq i}^{n}(y-\alpha_j)\right). \tag{4.16}
$$

For any $\alpha_t \in \{\alpha_1, \ldots, \alpha_n\}$, if we substitute y by α_t in Equation (4.15), then we get

$$
\sum_{i=1}^{n} m_i\alpha_i^{n-1} = m_t \prod_{j=1, j\neq t}^{n} (\alpha_t - \alpha_j).
$$

Since $\sum_{i=1}^n m_i\alpha_i^{n-1} = a/a_n$, we obtain $a/(a_n m_t) = \prod_{j=1,j\neq t}^n(\alpha_t - \alpha_j)$, for all $t = 1,\ldots,n$. Thus Equation (4.16) can be written as

$$y' = a_n\prod_{i=1}^n(y-\alpha_i) + \sum_{i=1}^n\left(\prod_{j=1,j\neq i}^n \frac{y-\alpha_j}{\alpha_i-\alpha_j}\right)\alpha_i'.$$

□

Remark 4.1.8. Equation (4.12) is independent of the m_i and is only dependent on the algebraic solutions of the differential equation.

Let $r_1,\ldots,r_n \in C(x)$ be pairwise distinct rational functions. Suppose there exists nonzero integers $m_1,\ldots,m_n$ such that r_i and m_i satisfy Equation (4.3). Then by Theorem 4.1.3 and Proposition 4.1.7 we can choose an appropriate a_n and construct a differential equation that has a transcendental liouvillian solution, and r_i are the only algebraic solutions.

Corollary 4.1.9. *For any $f \in C(x)$, there are uncountably many first order nonlinear differential equations over $C(x)$ that have a transcendental liouvillian solution and f as an algebraic solution.*

Proof. Note that if $f = 0$ then by Corollary 3.1.4 the differential equation $y' = cy(y-1)$, where $c \in C\setminus\{0\}$, has transcendental solution in a exponential extension of $C(x)$. So we may assume that f is a nonzero rational function. Since C is an algebraically closed field of characteristic zero, there are uncountably many elements in C. Then by Proposition 2.3.4, there are uncountably many $c \in C\setminus\{0\}$ such that $cf \neq \gamma'/\gamma$, for any nonzero $\gamma \in \overline{C(x)}$. Consider the differential equation

$$y' = c(y)(y-f) + \frac{f'}{f}y. \tag{4.17}$$

Note that f and zero are algebraic solutions of the differential equation. Let $a_2 = c$, $\alpha_1 = 0$, $\alpha_2 = f$, $m_1 = 1$ and $m_2 = -1$. Now $m_1+m_2 = 0$ and $a_2(m_1\alpha_1+m_2\alpha_2) = cf$.

Therefore by the converse part of Theorem 4.1.3 and Proposition 4.1.7, differential equation (4.17) has a transcendental liouvillian solution such that zero and f are the algebraic solutions. The liouvillian tower is given by $C(x) \subseteq C(x,z)$, where $z' = (cf)z$ and y lies in $C(x,z)$ such that $y - (y-f)z = 0$. $\square$

Corollary 4.1.10. *Let α be a nonzero element of $\overline{C(x)}$ such that $\alpha^l = h \in C(x)$, for some positive integer l. Then there is a first order nonlinear differential equation over $C(x)$ that has a transcendental liouvillian solution such that α is an algebraic solution.*

Proof. Let $\alpha_1, \ldots, \alpha_l$ be the roots of the polynomial $Y^l - h = 0$, where $h \in C(x) \backslash \{0\}$. Let $\alpha_{l+1} = 0$, $m_1 = 1, m_2 = 1, \ldots, m_l = 1$ and $m_{l+1} = -l$. Observe that

$$\sum_{i=1}^{l+1} m_i = 0,\ \sum_{i=1}^{l+1} m_i\alpha_i = 0, \ldots,\ \sum_{i=1}^{l+1} m_i\alpha_i^{l-1} = 0,\ \sum_{i=1}^{l+1} m_i\alpha_i^{l} = lh.$$

By Proposition 2.3.4, we can choose a nonzero $a_{l+1} \in C(x)$ such that $a_{l+1}\sum_{i=1}^{l+1} m_i\alpha_i^l \neq \gamma'/\gamma$, for any nonzero $\gamma \in \overline{C(x)}$. In particular, we may choose $a_{l+1} = 1/h$. Now consider the following differential equation:

$$y' = a_{l+1}\prod_{i=1}^{l+1}(y-\alpha_i) + \sum_{i=1}^{l+1}\left(\prod_{j=1, j\neq i}^{l+1}\frac{y-\alpha_j}{\alpha_i-\alpha_j}\right)\alpha_i'. \tag{4.18}$$

Note that $b_j := \sum_{i=1}^{l+1} m_i\alpha_i^j \in C(x)$, for $j = 0, \ldots, l+1$. Therefore, it follows from Equations (4.8), (4.9) and (4.13) that the coefficients of y^i in Equation (4.18) are elements of $C(x)$. By Theorem 4.1.3 and Proposition 4.1.7, Equation (4.18) has a transcendental liouvillian solution and $\alpha_1, \ldots, \alpha_{l+1}$ are the algebraic solutions. $\square$

4.1.1 Method of finding transcendental liouvillian solution of differential equation (4.2)

Suppose that the minimal polynomial of all the algebraic solutions of differential equation (4.2) are known. We give a step-by-step procedure to determine whether differential equation (4.2) has a transcendental liouvillian solution y such that there is an element $z \in C(x,y) \setminus C(x)$ satisfying $z' = az$, for some nonzero $a \in C(x)$.

Suppose that differential equation (4.2) has a transcendental liouvillian solution satisfying the above conditions. Then by Theorem 4.1.3 (ii), z can be written as $z = \prod_{i=1}^{l}(y-\alpha_i)^{m_i}$, where $\alpha_1, \dots, \alpha_l$ are the algebraic solutions of the differential equation and $m_1, \dots, m_l$ are nonzero integers. Therefore if all the algebraic solutions of a differential equation are known, then by the converse part of Theorem 4.1.3 one has to find suitable $m_1, \dots, m_l$ to determine the existence of such transcendental liouvillian solution.

Let $\{G_i \in C(x)[y] | i = 1, \dots, r\}$ be the set of minimal polynomial of $\alpha_1, \dots, \alpha_l$, where $0 < r \leq l$. Let G_i be the minimal polynomial of $\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{il_i}$, after renaming the α_i, where l_i are positive integers such that $\sum_{i=1}^{r} l_i = l$. Let $G_i = \prod_{j=1}^{l_i}(y-\alpha_{ij}) = \sum_{j=0}^{l_i} f_{i,j} y^{l_i - j}$, where $f_{i,j} \in C(x)$ and $f_{i,0} = 1$. Then using Girard–Newton formula, we obtain

$$h_{i,t} := \sum_{j=1}^{l_i} \alpha_{ij}^t = \begin{cases} -t f_{i,t} - \sum_{j=1}^{t-1} f_{i,t-j}\, h_{i,j} & \text{if } 1 \leqslant t \leqslant l_i \\ -\sum_{j=1}^{l_i} f_{i,j}\, h_{i,t-j} & \text{otherwise.} \end{cases} \tag{4.19}$$

Note that $h_{i,t} \in C(x)$ for all i and t. By Proposition 4.1.5, $z = \prod_{i=1}^{r} G_i^{m_i} = \prod_{i=1}^{r}\left(\prod_{j=1}^{l_i}(y-\alpha_{ij})^{m_i}\right)$ and Equations (4.3) can be written as

$$\sum_{i=1}^{r} m_i l_i = 0,\ \sum_{i=1}^{r} m_i h_{i,1} = 0, \dots, \sum_{i=1}^{r} m_i\, h_{i,n-2} = 0,\ a_n \sum_{i=1}^{r} m_i\, h_{i,n-1} = a. \tag{4.20}$$

We solve the above equations for m_i. If suitable m_i exist, then we can find a. Following are the steps:

(i) Substitute $m_1 = \frac{1}{l_1}\sum_{i=2}^{r} -m_i l_i$ in $\sum_{i=1}^{r} m_i h_{i,1} = 0, \ldots, \sum_{i=1}^{r} m_i\, h_{i,n-2} = 0$ to obtain the following equations:

$$\sum_{i=2}^{r} m_i\,(l_1 h_{i,1} - l_i h_{1,1}) = 0, \ldots, \sum_{i=2}^{r} m_i\,(l_1 h_{i,n-2} - l_i h_{1,n-2}) = 0. \tag{4.21}$$

(ii) The coefficients of $m_2, m_3, \ldots, m_r$ in Equations (4.21) lie in $C(x)$. After multiplying by a suitable factor, we may assume that the coefficients lie in $C[x]$. Note that the coefficients of powers of x in Equations (4.21) are C linear combinations of $m_2, m_3, \ldots, m_r$. Then we can obtain $m_2, m_3, \ldots, m_r$ by equating the coefficients of powers of x to zero. Obtain m_1 from $m_1 = \frac{1}{l_1}\sum_{i=2}^{r} -m_i l_i$.

(iii) Let $m_1, m_2, \ldots m_r$ be a solution of Equations (4.21). Then each m_i must be nonzero and m_i/m_1 must be a nonzero rational number. Multiply the tuple $(m_1, \ldots, m_r)$ by a suitable nonzero constant, if required, so that each m_i is a nonzero integer. If $m_{11}, m_{12}, \ldots, m_{1r}$ be another solution of (4.21) satisfying the above conditions, then by Remark 4.1.4,

$$\frac{m_{1i}}{m_{11}} = \frac{m_i}{m_1},\ \forall\, i = 2, 3, \ldots, r.$$

If the above conditions are not met, then there is no solution. Else, define $a = a_n \sum_{i=1}^{r} m_i\, h_{i,n-1}$. Use Proposition 2.3.4 to check if there exists a nonzero $\gamma \in \overline{C(x)}$ such that $\gamma' = a\gamma$. If no such γ exists, then by Theorem 4.1.3, the required solution exists.

4.2 Application to Abel's differential equation of the first kind

The following differential equations are called Abel's differential equations of the first and second kind respectively:

$$y' = a_3y^3 + a_2y^2 + a_1y + a_0 \tag{4.22}$$

$$(g + y)y' = f_2y^2 + f_1y + f_0, \tag{4.23}$$

where $a_i, g, f_i \in C(x)$, $g \neq 0$ and $a_3 \neq 0$. The well known substitution $g + y = 1/u$ transforms Equation (4.23) into an Abel's differential equation of the first kind:

$$u' + (f_0 - f_1g + f_2g^2)u^3 + (f_1 - 2f_2g + g')u^2 + f_2u = 0. \tag{4.24}$$

We note that the coefficients of u^i lie in $C(x)$ for all i. If y_1 and u_1 are transcendental solutions of Equations (4.23) and (4.24), respectively, then the differential fields $k(x, y_1)$ and $k(x, u_1)$ are isomorphic and the number of algebraic solutions of the two equations is equal.

In this section, we will give a method of finding transcendental solutions of differential equation (4.22) and, by extension, of differential equation (4.23) in an exponential extension of $k(x)$ using the results from Section 4.1 and the following propositions.

Proposition 4.2.1. *Let K be any field of characteristic zero and α_1, α_2, α_3 be three pairwise distinct elements of K. Then there exist nonzero integers m_1, m_2, m_3 such that $\sum_{i=1}^{3} m_i = 0$ and $\sum_{i=1}^{3} m_i\alpha_i = 0$ if and only if $(\alpha_1 - \alpha_3)/(\alpha_2 - \alpha_1) \in \mathbb{Q} \setminus \{0\}$.*

Proof. Let $(\alpha_1 - \alpha_3)/(\alpha_2 - \alpha_1) = p/q \in \mathbb{Q} \setminus \{0\}$, then define $m_2 = p$, $m_3 = q$ and $m_1 = -(m_2 + m_3)$. Note that $m_1 \neq 0$ otherwise,

$$m_2 + m_3 = 0 \implies p + q = 0 \implies \alpha_1 - \alpha_3 = \alpha_1 - \alpha_2 \implies \alpha_3 = \alpha_2.$$

We have arrived at a contradiction as α_i are pairwise distinct elements of K. Therefore m_i are nonzero integers. It can be easily shown that $\sum_{i=1}^{3} m_i = 0$ and $\sum_{i=1}^{3} m_i\alpha_i = 0$. The converse part is obvious. $\square$

Proposition 4.2.2. *Let y be a transcendental liouvillian solution of Equation (4.22) such that there exists $z_1 \in \overline{C(x)}(y) \setminus \overline{C(x)}$ satisfying $z_1'/z_1 \in \overline{C(x)} \setminus \{0\}$. Suppose that the differential equation has exactly three distinct algebraic solutions β_1, β_2 and β_3. Then the following statements hold:*

(i) Either each $\beta_i \in C(x)$ or $\beta_1 \in C(x)$ and β_2, β_3 lie in a quadratic extension of $C(x)$. In both the cases, there exists $z \in C(x, y) \setminus C(x)$ such that $z'/z \in C(x) \setminus \{0\}$.

(ii) In the latter case, $\beta_1 = -a_2/(3a_3)$, $\beta_2 = \beta_1 + \beta$ and $\beta_3 = \beta_1 - \beta$, where $\beta^2 \in C(x)$ but $\beta \notin C(x)$. Moreover, $z = \left((y-\beta_1)^2 - \beta^2\right)/(y-\beta_1)^2$ and $z' = (2a_3\beta^2)z$.

Proof. If $\alpha \in \overline{C(x)}$ is an algebraic solution of Equation (4.22), then all the roots of the minimal polynomial of α are solutions of the differential equation. Since the differential equation has precisely three distinct algebraic solutions, the degree of the minimal polynomial of α is at most three. Now we show that $[C(x,\alpha) : C(x)] \leq 2$.

Suppose that β_1, β_2 and β_3 are the roots of an irreducible cubic polynomial over $C(x)$. Let $G(y) = b_3y^3 + b_2y^2 + b_1y + b_0$, where $b_i \in C(x)$, be the minimal polynomial of β_i. Then by Cardano's formula, $\beta_i = B_1 + D\xi^i + \frac{B_2}{D\xi^i}$, where ξ is a primitive third root of unity, B_1, $B_2 \in C(x)$ and $D \in \overline{C(x)}$ such that $D^2, D \notin C(x)$. By Theorem 4.1.3, there exist nonzero integers m_1, m_2, m_3 such that $\sum_{i=1}^{3} m_i = 0$, $\sum_{i=1}^{3} m_i\beta_i = 0$. Substituting the value of β_i and $\sum_{i=1}^{3} m_i = 0$ in $\sum_{i=1}^{3} m_i\beta_i = 0$, we get

$$D\sum_{i=1}^{3} m_i\xi^i + \frac{B_2}{D}\sum_{i=1}^{3} m_i\xi^{-i} = 0 \implies D^2\sum_{i=1}^{3} m_i\xi^i + B_2\sum_{i=1}^{3} m_i\xi^{-i} = 0.$$

Since $D^2 \notin C(x)$, we obtain $\sum_{i=1}^{3} m_i \xi^i = 0$. Note that $\sum_{i=1}^{3} m_i = 0$ and $\sum_{i=1}^{3} m_i \xi^i = 0$, therefore by Proposition 4.2.1 we obtain $\frac{\xi^2-1}{\xi-1} \in \mathbb{Q}$, which is not possible. Therefore either each $\beta_i \in C(x)$ or $\beta_1 \in C(x)$ and β_2, β_3 lie in a quadratic extension of $C(x)$.

Suppose that $\beta_1 \in C(x)$ and β_2, β_3 lie in a quadratic extension of $C(x)$. Then we may write $\beta_2 = r + \beta$ and $\beta_3 = r - \beta$, where $r \in C(x)$, $\beta \notin C(x)$ and $\beta^2 \in C(x)$. By Theorem 4.1.3, there exists nonzero integers m_1, m_2, m_3 such that $m_1 + m_2 + m_3 = 0$ and $m_1\beta_1 + m_2(r+\beta) + m_3(r-\beta) = 0$. This gives us $m_1(\beta_1 - r) + \beta(m_2 - m_3) = 0$. Now β lies in a quadratic extension of $C(x)$ and $\beta_1 - r$ lies in $C(x)$. Therefore $m_2 = m_3, m_1 = -2m_2, r = \beta_1$. We may assume that $m_2 = m_3 = 1$, and $m_1 = -2$. Then by Theorem 4.1.3 (ii),

$$z = \prod_{i=1}^{3} (y - \beta_i)^{m_i} = \frac{(y-\beta_1)^2 - \beta^2}{(y-\beta_1)^2} \tag{4.25}$$

$$a = a_3 \sum_{i=1}^{3} m_i \beta_i^2 = a_3 \left(-2r^2 + (r+\beta)^2 + (r-\beta)^2\right) = 2a_3\beta^2, \tag{4.26}$$

where $2a_3\beta^2 \neq \gamma'/\gamma$, for any nonzero $\gamma \in \overline{C(x)}$. Observe that $z \in C(x,y) \setminus C(x)$ and $a \in C(x)$. This proves the second part. By Proposition 4.1.7, differential equation (4.22) is of the form

$$y' = a_3 \prod_{i=1}^{3} (y - \beta_i) + \sum_{i=1}^{3} \left(\prod_{j=1, j\neq i}^{3} \frac{y - \beta_j}{\beta_i - \beta_j} \right) \beta_i'.$$

Substituting $\beta_1 = r$, $\beta_2 = r + \beta$ and $\beta_3 = r - \beta$ in the above equation, we get

$$y' = a_3 y^3 - 3a_3 r y^2 + \left(3a_3 r^2 + \frac{\beta'}{\beta} - a_3\beta^2\right) y + \left(r' - a_3 r^3 - r\,\frac{\beta'}{\beta} + a_3\, r\, \beta^2\right). \tag{4.27}$$

Note that in the above equation the coefficients of y^i lie in $C(x)$. Comparing Equations (4.22) and (4.27), we obtain $\beta_1 = r = -a_2/(3a_3)$. □

4.2.1 Method of solving Abel's differential equation of the first kind

Following are the steps to determine whether differential equation (4.22) has a transcendental liouvillian solution satisfying the hypotheses of Proposition 4.2.2. By Proposition 4.2.2, one of the following can occur:

(i) One rational and two algebraic solutions:

(a) If $r := -a_2/(3a_3)$ is not a solution of the differential equation, then go to step (ii). Else take the transformation $y \to y - r$. This would transform Equation (4.22) into an equation of the form

$$y' = a_3 y^3 + A_1 y, \text{ where } A_1 \in C(x). \tag{4.28}$$

(b) Taking the well known substitution $t = 1/y^2$ in Equation (4.28), we get

$$t' = (-2)\left(A_1\, t + a_3\right). \tag{4.29}$$

If Equation (4.29) has infinitely many algebraic solutions, then by Theorem 4.1.1, Equation (4.22) does not have any transcendental liouvillian solution y. If the above differential equation has exactly one rational solution r_1, then $\pm 1/(\sqrt{r_1})$ are algebraic solutions of Equation (4.28) as $t = 1/y^2$. Note that $r_1 \neq 0$ as $a_3 \neq 0$. Therefore $r, r \pm 1/(\sqrt{r_1})$ are algebraic solutions of Equation (4.22). From Equation (4.26) we obtain $a = (2a_3)/r_1$. If $(2a_3/r_1) = \gamma'/\gamma$, for some nonzero $\gamma \in \overline{C(x)}$, then there is no solution (by Theorem 4.1.3). Else we can obtain the transcendental liouvillian solution by solving Equation (4.29).

(ii) Three rational solutions: Find all the rational solutions of the differential equation using Algorithm 2 of [41]. If the differential equation has more than three rational solutions, then there is no solution of the desired type. Else, let r_1, r_2, r_3 be the only rational solutions. Then by Theorem 4.1.3 and Proposition 4.2.1, the differential equation has a transcendental liouvillian solution and exactly three distinct rational solutions if and only if $(r_1-r_3)/(r_2-r_1) \in \mathbb{Q}\setminus\{0\}$ and $a_3 \sum_{i=1}^{3} m_i r_i^2 \neq \gamma'/\gamma$, where m_i are obtained using Proposition 4.2.1 and γ is any nonzero element of $\overline{C(x)}$. In this case, the liouvillian tower is given by $C(x) \subseteq C(x,z) \subseteq C(x,z,y)$, where $z' = \left(a_3 \sum_{i=1}^{3} m_i r_i^2\right) z$ and y is an algebraic over $C(x,z)$ such that $\prod_{i=1}^{3}(y-r_i)^{m_i} - z = 0$.

Note that we can check whether there exists a nonzero $\gamma \in \overline{C(x)}$ such that $\gamma' = a\gamma$ by using Proposition 2.3.4. Next, we apply our method to solve differential equations.

Example 4.2.3. The following example appears in [25, p-1405].

$$y' = \frac{x^3}{9}y^3 - xy^2. \tag{4.30}$$

Here $a_3 = x^3/9,\ a_2 = -x$ and $a_1 = 0 = a_0$. Observe that $r := -a_2/(3a_3) = 3/x^2$ is a rational solution of the differential equation. We take the transformation $y \to y - r$ and the differential equation reduces to

$$y' = \frac{x^3}{9}y^3 + \frac{-3}{x}y. \tag{4.31}$$

One can show that roots of the polynomial $P_c(Y) := Y^2 - 9/(x^4(1+cx^2))$, where c is a constant, are algebraic solutions of Equation (4.31). Since c is an arbitrary constant, the differential equation has infinitely many algebraic solutions. Thus by Theorem 4.1.1, it does not have any transcendental liouvillian solution y.

Example 4.2.4. Consider the differential equation

$$y' = y^3 + \left(\frac{1}{2x} - x\right)y. \tag{4.32}$$

Here $a_3 = 1$, $a_1 = 1/(2x) - x$ and $a_2 = 0 = a_0$. Substituting $t = 1/y^2$, Equation (4.32) reduces to

$$t' + \left(\frac{1}{x} - 2x\right) t = -2 \tag{4.33}$$

Equation (4.33) is a first order linear differential equation and its solutions are given by $t = (1/x)\,(1 + c\,exp(-x^2))$, where c is a constant. The only rational solution of Equation (4.33) is $r_1 := 1/x$. Note that $(2a_3)/r_1 = 2a_3x = 2x \neq (\gamma'/\gamma)$, for any nonzero $\gamma \in \overline{C(x)}$ (using Proposition 2.3.4). Thus the given differential equation has a transcendental liouvillian solution. The liouvillian tower is given by $C(x) \subseteq C(x,z) \subseteq C(x,z,y)$ where $z = exp(x^2)$ and y is algebraic over $C(x,z)$ given by $(y^2 - x) = zy^2$. Note that $0, \pm\sqrt{x}$ are the only algebraic solutions.

Example 4.2.5. Consider the differential equation

$$y' = y^3 - 3y^2 + \frac{3 - 3x^2 - x^3}{x+1} y + x^2 + 2x - \frac{1}{x+1}. \tag{4.34}$$

We determine that $1, -x, x+2$ are the rational solutions of the differential equation. Let $r_1 = 1, r_2 = -x, r_3 = x+2$, then $(r_1 - r_3)/(r_2 - r_1) \in \mathbb{Q} \setminus \{0\}$. Using Proposition 4.2.1, we get $m_1 = -2, m_2 = 1, m_3 = 1$. Now $a_3 \sum_{i=1}^{3} m_i r_i^2 = 2x^2 + 4x + 2 \neq \gamma'/\gamma$, for any nonzero $\gamma \in \overline{C(x)}$. Thus by Theorem 4.1.3, the differential equation has a transcendental liouvillian solution. The liouvillian tower is given by $C(x) \subseteq C(x,z) \subseteq C(x,y)$, where $z' = (2x^2 + 4x + 2)z$ and y is algebraic over $C(x,z)$ given by $(y+x)(y-x-2) = z(y-1)^2$.

Few remarks:

Note that Theorems 4.1.1 and 4.1.3 hold if we replace $C(x)$ by any differential field K of characteristic zero such that C_K is algebraically closed.

As noted in the proof of Theorem 4.1.1, if Equation (4.1) has a transcendental liouvillian solution y, then the number of algebraic solutions is finite. It is an open problem to find the exact number of algebraic solutions.

In Proposition 4.1.7, we have given the structure of the differential equation in terms of its algebraic solutions. For any $\alpha \in \overline{C(x)}$, it is natural to ask whether there exists a first order nonlinear differential equation over $C(x)$ that has a transcendental liouvillian solution y such that and α is an algebraic solution.

Chapter 5

Differential subfields of liouvillian Picard-Vessiot extensions

This chapter is based on the author's work in [34]. First we recall the definition of Picard-Vessiot extension. Let $k[\partial]$ be the ring of differential operators over k and $\mathscr{L} \in k[\partial]$ be a monic operator of order n. A *Picard-Vessiot extension* E of k for $\mathscr{L}$ is a differential field extension of k having the same field of constants as k and satisfying the following conditions:

(a) The $C-$vector space V of all solutions of $\mathscr{L}(Y) = 0$ in E is of dimension n.

(b) $E = k\langle V \rangle$, that is, the smallest differential field containing k and V is E.

We had given another definition of Picard-Vessiot extensions in Chapter 2. One can show that both these definitions are equivalent ([40, Proposition 1.22]). Let E be a Picard-Vessiot extension of k, K be an intermediate differential subfield and $T(K|k)$ be the set of all elements of K which are zeros of homogeneous linear differential equations over k. It is known that $T(E|k)$ is a finitely generated simple differential $k-$algebra whose field of fractions $\mathcal{Q}(T(E|k))$ equals the differential field E. However, the quotient field of $T(K|k)$ need not be K. We show that $\mathcal{Q}(T(K|k)) = K$ for every intermediate differential field if and only if $\mathscr{G}(E|k)^0$ is solvable. Using this characterization we provide the structure of intermediate differential fields of a Picard-Vessiot extension E whose differential Galois group is connected solvable. The main result can also be proved using the article [1]. We also show this derivation.

5.1 The differential $k-$algebra $T(E|k)$

Let E be a Picard-Vessiot extension of k with differential Galois group $\mathscr{G}(E|k)$. In this section, we will discuss the differential $k-$algebra $T(E|k)$ where $T(E|k)$ is the differential subalgebra of E consisting of all solutions in E of linear homogeneous differential equations over k. From Corollary 1.38 of [40], $T(E|k)$ is the Picard-Vessiot ring. It plays a crucial role in Picard-Vessiot theory and is well understood. Let us list few known facts about $T(E|k)$ (see [23, 24]).

(i) $T(E|k)$ is a finitely generated simple differential $k-$algebra whose field of fractions $\mathcal{Q}(T(E|k))$ equals the differential field E.

(ii) $\mathscr{G}(E|k)$ stabilizes $T(E|k)$. The $\mathscr{G}(E|k)$ orbit set of an element $y \in E$ spans a finite dimensional C vector space if and only if $y \in T(E|k)$.

(iii) $T(E|k)$ can be described in terms of the coordinate ring of $\mathscr{G}(E|k)$ ([24], Theorem 5.12): If $\overline{k}$ is an algebraic closure of k then there is an $\overline{k}-$algebra isomorphism

$$\overline{k} \otimes_k T(E/k) \longrightarrow \overline{k} \otimes_C C[\mathscr{G}(E|k)].$$

The $\mathscr{G}(E|k)$ action is respected by the above isomorphism. Note that $\mathscr{G}(E|k)$ acts trivially on $\overline{k}$ and acts as right translations on the coordinate ring $C[\mathscr{G}(E|k)]$ of $\mathscr{G}(E|k)$.

(iv) When $\mathscr{G}(E|k)$ is a connected solvable group, it is also known that

$$T(E|k) \simeq k \otimes_C C[\mathscr{G}(E|k)]. \tag{5.1}$$

Again, the $\mathscr{G}(E|k)$ action is respected by the above isomorphism ([24, Corollary 5.29]).

Let $\mathscr{H}$ be a closed subgroup of $\mathscr{G}(E|k)$. Then by the fundamental theorem $\mathscr{H} = \mathscr{G}(E|K)$ for some intermediate differential field K. Note that $T(K|k) = T(E|k) \cap K = T(E|k)^{\mathscr{H}}$. The following proposition will be used to prove our main result.

Proposition 5.1.1. *([34, Proposition 2.1]) Let E be a Picard-Vessiot extension of k and $k(x)$ be the algebraic closure of k in E. Let K be a differential field intermediate to k and E. Then the following holds.*

(a) $T(K(x)|k(x)) = T(K(x)|k)$.

(b) $T(K(x)|k)$ is an integral extension of $T(K|k)$.

Proof. Every differential equation over k is also a differential equation $k(x)$. Therefore $T(K(x)|k) \subseteq T(K(x)|k(x))$. For any $y \in k(x)$, there is a nonnegative integer m such that $y, y', \dots, y^{(m)}$ are $k-$linearly dependent as $k(x)$ is finite dimensional k vector space. This implies that $k(x) \subseteq T(K(x)|k)$. Let $y \in T(K(x)|k(x)) \setminus k(x)$ and $\mathscr{L} = \partial^{(n)} + a_{n-1}\partial^{(n-1)} + \cdots + a_0 \in k(x)[\partial]$ be a monic operator of order $n \geq 1$ such that $\mathscr{L}(y) = 0$. Let V be the set of all solutions of $\mathscr{L}$ in E and for any $\sigma \in \mathscr{G}(E|k)$, let V_σ be the set of all solutions of $\mathscr{L}_\sigma = \partial^n + \sigma(a_{n-1})\partial^{(n-1)} + \cdots + \sigma(a_0)$ in E. Note that $\sigma(V) = V_\sigma$. Now $a_i \in k(x)$ for each i and $E^{\mathscr{G}(E|k)^0} = k(x)$. Therefore the set $S_i := \{\sigma(a_i)| \ \sigma \in \mathscr{G}(E|k)\}$ is finite set for each i. Thus, there are only finitely many $\mathscr{L}_\sigma$. Let $\sigma_0 \in \mathscr{G}(E|k)$ be the identity and $\mathscr{L} = \mathscr{L}_{\sigma_0}, \mathscr{L}_{\sigma_1}, \dots, \mathscr{L}_{\sigma_l}$ be the distinct operators. Let $W = V_{\sigma_0} + V_{\sigma_1} + \cdots + V_{\sigma_l}$. Then, W is a finite dimensional C vector space. For any $\sigma \in \mathscr{G}(E|k)$ and $y \in V_{\sigma_i}$, we have $\sigma(y) \in V_{\sigma\sigma_i} = V_{\sigma_j} \subseteq W$. This implies that W is a $\mathscr{G}(E|k)-$module. Thus, any element of W must be a solution of some operator in $k[\partial]$. In particular, $y \in T(K(x)|k)$. This proves the first part. Taking $K = E$, we have

$$T(E|k) = T(E|k(x)). \tag{5.2}$$

Let $s \in T(K(x)|k)$. Since $k(x)$ is a finite Galois extension of k, so is $K(x)$ over K. In this case, $\mathscr{G}(K(x)|K)$ is same as the ordinary Galois group $Aut(K(x)|K)$ ([40], Exercise 1.24). Let $\{s = s_1, s_2, \dots, s_m\} = \{\sigma(s) \mid \sigma \in \mathscr{G}(K(x)|K)\}$. Then for $\sigma \in \mathscr{G}(K(x)|K)$, we have $\sigma(s) \in T(K(x)|k)$ and thus $s_i \in T(K(x)|k)$ for all i. The coefficients of the minimal polynomial of s over K are symmetric polynomials in $s_1, \dots, s_m$. Therefore the coefficients of the minimal polynomial lie in $T(K(x)|k) \cap K = T(K|k)$. This shows that $T(K(x)|k)$ is an integral extension of $T(K|k)$. $\square$

5.2 Liouvillian Picard-Vessiot extensions

In this section, we will prove our main result which provides a characterization for liouvillian Picard-Vessiot extensions. First, we recall a few definitions. A Picard-Vessiot extension E of k is called a *liouvillian Picard-Vessiot* extension if E is a liouvillian extension as well as a Picard-Vessiot extension. In this case, the identity component of $\mathscr{G}(E|k)$ is solvable. Let G be a linear algebraic group defined over an algebraically closed field. A closed subgroup H of G is called *observable* if the quotient variety G/H is quasi affine.

Theorem 5.2.1. *([34, Theorem 3.1]) A differential field E is a liouvillian Picard-Vessiot extension of k if and only if $\mathcal{Q}(T(K|k)) = K$ for any differential field K intermediate to E and k. In this case, $T(K|k)$ is a finitely generated simple differential $k-$algebra.*

Proof. Let E be a liouvillian Picard-Vessiot extension of k, K be an intermediate differential subfield and $\mathscr{H} := \mathscr{G}(E|K)$. Let us first assume that $\mathscr{G}(E|k)$ is connected. Then since $\mathscr{G}(E|k)$ is solvable, we have $T(E|k) \simeq k \otimes_C C[\mathscr{G}(E|k)] = k[\mathscr{G}(E|k)]$. From [7, Theorem 4.3], we have that every closed subgroup of $\mathscr{G}(E|k)$ is observable. Now from [3, Theorem 3], we have $\mathcal{Q}(k[\mathscr{G}(E|k)]^{\mathscr{H}}) = \mathcal{Q}(k[\mathscr{G}(E|k)])^{\mathscr{H}}$. Thus

$$\mathcal{Q}(T(E|k)^{\mathscr{H}}) = \mathcal{Q}(T(E|k))^{\mathscr{H}} \implies \mathcal{Q}(T(K|k)) = K.$$

Now suppose that $\mathscr{G}(E|k)$ is not connected. Since $E^{\mathscr{G}(E|k)^0} = k(x) \subseteq K(x) \subseteq E$, we have $\mathcal{Q}(T(K(x)|k(x)) = K(x)$. From Proposition 5.1.1, $T(K(x)|k(x)) = T(K(x)|k)$ and thus $K(x) = \mathcal{Q}(T(K(x)|k)$. We have also shown that $T(K(x)|k)$ is an integral extension of $T(K|k)$. Let $S = T(K|k) \setminus \{0\}$. Then $S^{-1}T(K(x)|k)$ is also an integral extension of $S^{-1}T(K|k)$. Observe that the latter is a field; so is the former. Since $K(x) = \mathcal{Q}(T(K(x)|k))$ is the smallest field containing $T(K(x)|k)$, $S^{-1}T(K(x)|k) = K(x)$. Now for any $t \in K$, we have $t = f/g$, where $f \in T(K(x)|k)$ and $g \in S = T(K|k) \setminus \{0\}$. Therefore, $f = gt \in T(K(x)|k) \cap K = T(K|k)$ and this proves that $\mathcal{Q}(T(K|k)) = K$.

Now we will prove the converse part. Suppose that E is not a liouvillian extension of k. Then the identity component $\mathscr{G}(E|k)^0$ is not solvable and therefore it contains a nontrivial Borel subgroup $\mathscr{B}$. Let $K = E^{\mathscr{B}}$ and $k(x)$ be the algebraic closure of k in E. Let $s \in T(E|k(x))^{\mathscr{B}}$ and $\mathcal{O}_s = \{\sigma(s) \in \mathscr{G}(E|k)^0\}$. Clearly, $\mathcal{O}_s$ is contained in a $\mathscr{G}(E|k)^0$ stable finite dimensional C vector space V, say. Since $\mathscr{B}$ is a Borel subgroup of $\mathscr{G}(E|k)^0$, $\mathscr{G}(E|k)^0/\mathscr{B}$ is a projective variety. Therefore, the induced map $\phi : \mathscr{G}(E|k)^0/\mathscr{B} \to V$, given by $\phi(\bar{\sigma}) = \sigma(s)$ for $\sigma \in \mathscr{G}(E|k)^0$, is a morphism from a projective variety into some affine space containing $\mathcal{O}_s$. Thus ϕ must be a constant. That is, $s \in T(E|k(x))^{\mathscr{G}(E|k)^0} = k(x)$ and thus $k(x) = T(E|k(x))^{\mathscr{B}}$. Then

$$\mathcal{Q}(T(K|k)) = \mathcal{Q}(T(E|k)^{\mathscr{B}}) = \mathcal{Q}(T(E|k(x))^{\mathscr{B}}) = k(x) \neq K.$$

This proves the converse.

Next, we will show that $T(K|k)$ is a finitely generated differential $k-$algebra. First, assume that $\mathscr{G}(E|k)$ is a connected solvable group. Let $\mathscr{H}$ be a closed subgroup of $\mathscr{G}(E|k)$ and $K := E^{\mathscr{H}}$. We have $T(E|k) \simeq k \otimes_C C[\mathscr{G}(E|k)]$ and therefore

$$T(K|k) = T(E|k)^{\mathscr{H}} \simeq (k \otimes_C C[\mathscr{G}(E|k)])^{\mathscr{H}} = k \otimes_C C[\mathscr{G}(E|k)]^{\mathscr{H}}.$$

Since $\mathscr{G}(E|k)$ is solvable, the homogeneous space $\mathscr{G}(E|k)/\mathscr{H}$ is affine ([7], Theorem 4.3 and Corollary 4.6) and we obtain $C[\mathscr{G}(E|k)]^{\mathscr{H}} = C[\mathscr{G}(E|k)/\mathscr{H}]$ is a finitely generated $C-$algebra. This in turn implies $T(K|k) \simeq k \otimes_C C[\mathscr{G}(E|k)]^{\mathscr{H}}$ is a finitely generated $k-$algebra. Now assume that only $\mathscr{G}(E|k)^0$ is solvable. Let $k(x) = E^{\mathscr{G}(E|k)^0}$ and observe that $\mathscr{G}(E|k(x)) = \mathscr{G}(E|k)^0$ is connected. Then we know $T(K(x)|k(x))$ is a finitely generated $k(x)-$algebra and it follows that $T(K(x)|k(x))$ is a finitely generated $k-$algebra as well. Since $T(K(x)|k(x)) = T(K(x)|k)$ is an integral extension of $T(K|k)$, by Artin-Tate Theorem ([9], p.143) we obtain that $T(K|k)$ is a finitely generated $k-$algebra.

Now it only remains to show that $T(K|k)$ is a simple differential $k-$algebra. As done earlier, we will first prove simplicity when $\mathscr{G}(E|k)$ is connected. Let I be a nonzero differential ideal of $T(K|k)$ and choose $0 \neq y \in I$ so that $\mathscr{L}(y) = 0$ for some $\mathscr{L} \in k[\partial]$ of smallest positive order n. Since the Galois group is connected and solvable, $\mathscr{L} = \mathscr{L}_{n-1}\mathscr{L}_1$ for $\mathscr{L}_{n-1}, \mathscr{L}_1 \in k[\partial]$ of orders $n-1$ and 1 ([16], p.38). Let $\mathscr{L}_1 = \partial - a$ for $a \in k$ and observe that $\mathscr{L}_1(y) = y' - ay \in I$. Now since

$0 = \mathscr{L}(y) = \mathscr{L}_{n-1}(\mathscr{L}_1(y))$, from the choice of n, we obtain that $y' - ay = b \in k$. Thus $b = \mathscr{L}_1(y) \in I$. If $b \neq 0$ then $I = T(K|k)$. On the other hand if $b = 0$ then $y' = ay$ and therefore $(1/y)' = -a(1/y)$. Thus $1/y \in T(K|k)$ and we again obtain $I = T(K|k)$. This completes the proof when $\mathscr{G}(E|k)$ is connected. For an arbitrary liouvillian Picard-Vessiot extension E of k, we have $T(K(x)|k(x)) = T(K(x)|k)$ to be a finitely generated simple differential $k-$algebra, where $k(x)$ is the algebraic closure of k in E. Suppose that $T(K|k)$ is not simple and let $I \neq T(K|k)$ be a differential ideal that is maximal among all differential ideals not intersecting $\{1\}$. Then I is known to be a prime ideal. Let I^e be the extension ideal in $T(K(x)|k)$. It is easy to see that I^e is a differential ideal and therefore $I^e = T(K(x)|k)$. Since $T(K(x)|k)$ is integral over $T(K|k)$ and that I is prime, there must exist a prime ideal of $T(K(x)|k)$ that contracts to I. But any such prime ideal must contain $I^e = T(K(x)|k)$, a contradiction. $\square$

Let E be a liouvillian Picard-Vessiot extension E of k and $k(x)$ be the algebraic closure of k in E. Then $\mathscr{G}(E|k)^0$ is connected solvable group. Therefore $\mathscr{G}(E|k)^0 = \mathscr{U} \rtimes \mathscr{T}$, where $\mathscr{U}$ is the unipotent radical of $\mathscr{G}(E|k)$ and $\mathscr{T}$ is a maximal torus of $\mathscr{G}(E|k)^0$. Using this one can show that E has the following structure ([24], Proposition 6.7):

(a) $E = E^{\mathscr{U}}(\eta_1, \dots, \eta_n)$, where $\eta_1, \dots, \eta_n \in E$ are algebraically independent over $E^{\mathscr{U}}$ and $\eta_i' \in E^{\mathscr{U}}(\eta_1, \dots, \eta_{i-1})$,

(b) $E^{\mathscr{U}} = k(x)(\xi_1, \dots, \xi_m)$, where $\xi_1, \dots, \xi_m \in E^{\mathscr{U}}$ are algebraically independent over $k(x)$ such that $\xi_i'/\xi_i \in k(x)(\xi_1, \dots, \xi_{i-1})$. Moreover, $E^{\mathscr{U}}$ is a Picard-Vessiot extension of $k(x)$ with a differential Galois group isomorphic to a maximal torus of $\mathscr{G}(E|k)^0$.

From the inverse problem for tori ([24], p.99 or [40], Exercise 1.41), we can further assume that the $k(x)-$algebraically independent ξ_i are chosen so that $\xi_i'/\xi_i \in k(x)$ (as opposed to $\xi_i'/\xi_i \in k(x)(\xi_1,\ldots,\xi_{i-1})$). With this description of E, in the next corollary, we will decompose K into a tower of differential fields in the following manner: each differential field in the tower is obtained from its predecessor by adjoining the solution of a first order linear differential equation of a certain form.

Corollary 5.2.2. *([34, Corollary 3.3]) Let E be a liouvillian Picard-Vessiot extension of k and $\mathscr{G}(E|k)$ be connected. Let K be a differential field intermediate to E and k. Then $K = k(t_1,\ldots,t_n)$, where for each i, $t_i \in T(K|k)$, $t_i' = a_it_i + b_i$ for $a_i \in k$ and $b_i \in k(t_1,\ldots,t_{i-1})$. Furthermore,*

1. *if $\mathscr{G}(E|k)$ is unipotent then each a_i can be taken to be zero for each i and that $t_1,\ldots,t_n$ are algebraically independent.*

2. *if $\mathscr{G}(E|k)$ is a torus then each b_i can be taken to be zero.*

Proof. Let us assume that $k \subsetneq K \subsetneq E$. Let M be any differential field such that $k \subseteq M \subsetneq K$. We will show that there is a $y \in T(K|k) \setminus M$ such that $y' = ay + b$ for some $a \in k$ and $b \in M$ and that a can be taken to be zero if $\mathscr{G}(E|k)$ is unipotent and that b can be taken to be zero if $\mathscr{G}(E|k)$ is a torus.

Note that that $T(K|k) \setminus M \neq \emptyset$ (by Theorem 5.2.1). Choose $y \in T(K|k) \setminus M$ and $\mathscr{L} \in k[\partial]$ of smallest positive degree m such that $\mathscr{L}(y) = 0$. Since $\mathscr{G}(E|k)$ is connected and solvable, $\mathscr{L} = \mathscr{L}_{m-1}\mathscr{L}_1$, where $\mathscr{L}_{m-1}, \mathscr{L}_1 \in k[\partial]$ are of order $m-1$ and 1 respectively. Let $\mathscr{L}_1 = \partial - a$, $a \in k$. Observe that $\mathscr{L}_1(y) \in T(K|k)$ and $\mathscr{L}_{m-1}(\mathscr{L}_1(y)) = 0$. Observe that $\mathscr{L}_1(y) \in T(M|k) \subset M$ due to our choice of m. As a result, we have identified an element $y \in T(K|k) \setminus M$ such that $y' = ay + b$, where $a \in k$ and $b \in M$.

If $\mathscr{G}(E|k)$ is unipotent then $E = k(\eta_1, \ldots, \eta_s)$, where $\eta_i' \in k(\eta_1, \ldots, \eta_{i-1})$. By [38, Proposition 2.2], $\mathscr{L}$ has a nonzero solution $\alpha \in k$. Thus, in this case, we may choose $\mathscr{L}_1 = \partial - (\alpha'/\alpha)$ and obtain an element $y/\alpha \in T(K|k)$ such that $(y/\alpha)' = b/\alpha \in M$.

Finally, suppose that $\mathscr{G}(E|k)$ is a torus. Then $E = k(\xi_1, \ldots, \xi_s)$, where $\xi_i'/\xi_i \in k$ for each i. If $b \neq 0$ then we apply Proposition 2.2 of [38] to the extension $M(\xi_1, \ldots, \xi_s)$ of M with $\mathscr{L}_1(y) = y' - ay = b$. Therefore there is an element $\alpha \in M$ such that $\alpha' - a\alpha = b$. Now $y - \alpha \in T(K|k) \setminus M$ and $(y-\alpha)'/(y-\alpha) = a \in k$.

Taking $M = k$, first we obtain t_1 and taking $M = k(t_1, \ldots, t_{i-1})$, we obtain $t_i \in T(K|k) \setminus M$, with the desired properties. Since K is finitely generated over k, there is an n such that $K = k(t_1, \ldots, t_n)$. $\square$

In the above corollary, the hypothesis that $\mathscr{G}(E|k)$ is connected and solvable allowed us to factor the differential operator $\mathscr{L}$ over $k[\partial]$, which was a crucial step in the proof. One cannot drop the assumption that $\mathscr{G}(E|k)$ is connected. For example, consider the extension $E = \mathbb{C}(x)(\sqrt{x}, e^{\sqrt{x}})$. Then E is a liouvillian Picard-Vessiot extension of $\mathbb{C}(x)$ for the differential equation $\mathscr{L}(Y) = Y'' + (1/2x)Y' - (1/4x)Y = 0$. The set $V := \mathrm{span}_{\mathbb{C}}\{e^{\sqrt{x}}, e^{-\sqrt{x}}\}$ is the set of all solutions of $\mathscr{L}(Y) = 0$ in E. Since E contains the algebraic extension $\mathbb{C}(x)(\sqrt{x})$, $\mathscr{G}(E|k)$ is not connected. The differential Galois group $\mathscr{G}(E|\mathbb{C}(x))$ is isomorphic to $\mathrm{G}_m \rtimes \mathrm{Z}_2$. Being one dimensional it is solvable. One can show that the intermediate differential field $K := \mathbb{C}(x)(e^{\sqrt{x}} + e^{-\sqrt{x}})$ contains no elements satisfying a first order linear differential equation over $\mathbb{C}(x)$ other than the elements of $\mathbb{C}(x)$ itself ([38], p.376).

5.3 Intermediate differential subfields of Picard-Vessiot extensions

Let $(\mathbb{C}(x), d/dx)$ be the ordinary differential field of complex rational functions with derivation $' := d/dx$. Let E be a Picard-Vessiot extension of the Airy differential equation $\mathscr{L}(Y) := Y'' - xY = 0$. Then the differential Galois group is isomorphic to $\mathrm{SL}(2, \mathbb{C})$. Let K be the fixed field of upper triangular matrices in $\mathrm{SL}(2, \mathbb{C})$. Then $K = \mathbb{C}(x)(w)$, where w is transcendental over $\mathbb{C}(x)$ and w is a solution of the Ricatti equation $w' = x - w^2$. Also, $T(K|\mathbb{C}(x)) = T(E|\mathbb{C}(x)) \cap K = \mathbb{C}(x)$ ([24], pp. 86-87). Now we will show that the differential ring $\mathbb{C}(x)[w]$ is a finitely generated simple differential $k-$algebra whose field of fractions is K. It is enough to show that the differential ring $\mathbb{C}(x)[w]$ is simple. Let I be a maximal differential ideal of $\mathbb{C}(x)[w]$. Then I is a prime ideal. Since $\mathbb{C}(c)[w]$ is a P.I.D., $I = (v)$ for some monic irreducible polynomial $v \in \mathbb{C}(x)[w]$. Let $v = \prod_{i=1}^{m} w - \alpha_i$, where α_i are distinct elements of $\overline{\mathbb{C}(x)}$. Now,

$$v' = \sum_{j}^{m} (w' - \alpha_j') \prod_{i \neq j}^{m} (w - \alpha_i).$$

Since $v' \in I$, v divides v'. Then $w - \alpha_i$ must divide $w' - \alpha_i'$ and it follows that $\alpha_i' = x - \alpha_i^2$. This contradicts the fact that the Ricatti equation $w' = x - w^2$ has no solutions in $\overline{\mathbb{C}(x)}$. Thus $\mathbb{C}(x)[w]$ is a (finitely generated) simple differential $k-$algebra. This example prompts us to consider whether an intermediate differential field K of any arbitrary Picard-Vessiot extensions may be constructed as a field of fractions of a finitely generated simple differential $k-$subalgebras of K. Using the following propositions, we will provide a positive answer to this question in Theorem 5.3.3.

Proposition 5.3.1. *([34, Proposition 4.1]) Let K be a finitely generated differential field extension of k. Then K contains a finitely generated differential $k-$algebra whose field of fractions is K.*

Proof. Let $K = k(y_1, \ldots, y_{t-1})[y_t]$, where $y_1, \ldots, y_{t-1}$ is a transcendence base of K over k and y_t is algebraic over $k(y_1, \ldots, y_{t-1})$. For each y_i, we will construct a finitely generated differential $k-$algebra R_i whose field of fractions is $k\langle y_i \rangle = k(y_i, y_i', \ldots)$. Then the smallest $k-$algebra R containing $R_1, \ldots, R_t$ will be a finitely generated differential $k-$algebra whose field of fractions is K.

Let $y \in \{y_1, \ldots, y_t\}$. Consider the differential field $k\langle y \rangle$. Let n be the smallest positive integer such that $y, y', \ldots, y^{(n-1)}$ are algebraically independent over k and that $y^{(n)}$ be algebraic over the subalgebra $k[y, y', \ldots, y^{(n-1)}]$ of K. Let

$$P(X) := \sum_{i=0}^{m} a_i X^i \in k[y, y', \ldots, y^{(n-1)}][X]$$

be a minimal polynomial of $y^{(n)}$ with $a_m \neq 0$. Differentiating $P(y^{(n)}) = 0$ we get

$$\sum_{i=0}^{m} a_i'(y^{(n)})^i + \left(\sum_{i=0}^{m} i a_i (y^{(n)})^{i-1} \right) y^{(n+1)} = 0.$$

Note that $r := \sum_{i=0}^{m} i a_i (y^{(n)})^{i-1} \neq 0$ and therefore

$$y^{(n+1)} = \frac{-\sum_{i=0}^{m} a_i'(y^{(n)})^i}{r} \in k[y, y', \ldots, y^{(n)}][r^{-1}]. \tag{5.3}$$

Let $R_y := k[y, y', \ldots, y^{(n)}, r^{-1}]$. Then $(1/r)' = -r'/r^2 \in R_y$ and $y^{(i)} \in R_y$ for all i. Therefore R_y is a finitely generated differential $k-$algebra whose field of fractions is $k\langle y \rangle$. $\square$

Proposition 5.3.2. *([34, Proposition 4.2]) Let E be a Picard-Vessiot extension of k and S be a differential $k-$subalgebra of E such that $T(E|k) \subseteq S$. Then S is a simple differential $k-$algebra.*

Proof. Let I be a nonzero differential ideal of S and a be a nonzero element of I. Since E is the fraction field of $T(E|k)$, $a = f/g$ for $f, g \in T(E|k) \setminus \{0\}$. Therefore $(0 \neq)\ ga = f \in I^c = T(E|k) \cap I$. Since the contraction ideal I^c is a differential ideal and that $T(E|k)$ is a simple differential ring, we have $1 \in I^c \subseteq I$. This implies that S is a simple differential ring. $\square$

Theorem 5.3.3. *([34, Theorem 4.3]) Let E be a Picard-Vessiot extension of k and let $k \subseteq K \subseteq E$ be an intermediate differential field. Then K contains a finitely generated simple differential $k-$algebra whose field of fractions is K.*

Proof. Since Picard-Vessiot extensions are finitely generated field extensions, we apply Proposition 5.3.1 and obtain a finitely generated differential $k-$algebra R, whose field of fractions is K. Let $R := k[\frac{x_1}{y_1}, \frac{x_2}{y_2}, \ldots, \frac{x_n}{y_n}]$, where $x_i, y_i \in T(E/k)$. We will find an element $s \in R$ so that $R[1/s]$ is a simple differential $k-$algebra and this would complete the proof.

Let S be the subalgebra generated by the $T(E|k)$ and the set $\{\frac{1}{y_i} \mid 1 \leq i \leq n\}$. Then $S \supseteq T(E|k)$ is a finitely generated differential $k-$algebra. By Proposition 5.3.2, S is also simple.

Next, we will find a suitable candidate for s. Let $\overline{E}$ be the algebraic closure of E and $\overline{k} \subseteq \overline{E}$ be the algebraic closure of k in $\overline{E}$. Clearly, $\overline{k}$ is an algebraically closed field. Let $\overline{R}$ and $\overline{S}$ be the rings generated by R and S over $\overline{k}$, respectively. Note that $\overline{R} \subseteq \overline{S}$ and $\overline{R}$ and $\overline{S}$ are integral extensions of R and S respectively. The domains $\overline{R}$ and $\overline{S}$ are finitely generated $\overline{k}-$algebras and therefore they are coordinate rings of some irreducible affine varieties X and Y.

Let $\psi : Y \to X$ be the morphism induced by the inclusion $\overline{R} \subseteq \overline{S}$. Then ψ is dominant and therefore $\psi(Y)$ must contain an open set U of X. Choose $f \in \overline{R}$ so that $X_f := \{x \in X \mid f(x) \neq 0\} \subseteq U$. Since f must be integral over the domain R, there is a monic polynomial $P(X) = X^n + s_{n-1}X^{n-1} + \cdots + s \in R[X]$ such that $P(f) = 0$ and $s \neq 0$. Then $(f^{n-1} + s_{n-1}f^{n-2} + \cdots + s_1)f = -s$ and we have $X_s \subseteq X_f \subseteq U \subseteq \psi(Y)$. Thus ψ naturally restricts to a surjective morphism from

Y_s to X_s. Observe that $\mathscr{V}(I)$ is a non-empty subset of X_s for any proper ideal I of $\overline{R}[1/s]$. Since $X_s \subset \psi(Y)$, we obtain that $\psi^{-1}(\mathscr{V}(I))$ is a non-empty subset of Y_s. Then $\psi^{-1}(\mathscr{V}(I)) \subseteq \mathscr{V}(I^e)$, where I^e is the extension of I in $\overline{S}[1/s]$ and therefore I^e is also a proper ideal of $\overline{S}[1/s]$.

Now we will show that $R[1/s]$ is a simple differential ring. Suppose that $\mathfrak{a}$ is a nonzero proper differential ideal of $R[1/s]$. Since every differential ideal is contained in a maximal differential ideal and since maximal differential ideals are prime, we have a nonzero prime differential ideal $\mathfrak{p}$ containing $\mathfrak{a}$. But $\overline{R}[1/s]$ is an integral extension of $R[1/s]$ and therefore there is a prime ideal $\mathfrak{q}$ in $\overline{R}[1/s]$ such that $\mathfrak{q} \cap R[1/s] = \mathfrak{p}$. Let $\mathfrak{p}^e$ be the extension ideal of $\mathfrak{p}$ in $\overline{R}[1/s]$. Clearly $\mathfrak{p}^e \subseteq \mathfrak{q}$ and therefore $\mathfrak{p}^e$ is a proper ideal of $\overline{R}[1/s]$. Let $\mathfrak{b}$ be the extension ideal of $\mathfrak{p}^e$ in $\overline{S}[1/s]$. Then from our earlier observation, $\mathfrak{b}$ is a proper ideal. Since $\mathfrak{p}$ is a differential ideal, so is $\mathfrak{b}$. Now the contraction $\mathfrak{b}^c := S[1/s] \cap \mathfrak{b}$ must be a proper differential ideal of $S[1/s]$. Furthermore, $0 \neq \mathfrak{p} \subseteq R[1/s] \subseteq S[1/s]$ implies $\mathfrak{p} \subseteq \mathfrak{b}^c$ and thus $\mathfrak{b}^c$ must be a proper nonzero differential ideal of $S[1/s]$. This contradicts Proposition 5.3.2. $\square$

5.4 Solution algebras and solution fields

In this section, we will prove Theorem 5.2.1 using results from the article [1]. The contents of this section are part of the author's work in [22, Section 4]. First we will discuss the relevant definitions and results.

Let $k[\partial]$ be the usual ring of differential operators. Note that any differential module M over k is also a $k[\partial]-$module such that $\dim_k M$ is finite. Recall that finitely generated $k[\partial]$ modules are isomorphic to a finite direct sum $\bigoplus M_i$, where each M_i is isomorphic to either $k[\partial]$ or $k[\partial]/k[\partial]\mathscr{L}_i$ for some $\mathscr{L}_i \in k[\partial]$ with deg $\mathscr{L}_i > 0$. Let M be a differential $k-$module of dimension n. A no new constant extension E of k

is called a *solution field* for M if there is a morphism $\psi : M \to E$ of $k[\partial]-$modules such that $\psi(M)$ generates E; in which case, E is said to be generated by a solution ψ. Let S be a differential $k-$algebra and an integral domain such that the field of constants of $\mathcal{Q}(S)$ is C. Then S is called a *solution algebra* for M if there is a morphism $\psi : M \to S$ of $k[\partial]-$modules such that $\psi(M)$ generates S.

Proposition 5.4.1. *Let $S \supseteq k$ be a differential integral domain such that the field of constants of the field of fractions of S is C. Let E be a no new constant extension of k. Then S (respectively, E) is a solution algebra (respectively, solution field) if and only if S (respectively, E) is generated by a finite set of solutions of linear differential equations over k.*

Proof. Suppose that S is generated by solutions $\{y_1, \ldots, y_n\}$ of linear differential equations over k. Let $\mathscr{L}_i(y_i) = 0$ for $\mathscr{L}_i \in k[\partial]$. Consider the $k[\partial]-$morphism $\oplus_{i=1}^n \psi_i : \bigoplus_{i=1}^n k[\partial]/k[\partial]\mathscr{L}_i \to S$ defined by $\psi_i(1) = y_i$. Therefore S is a solution algebra. Conversely, let S be a solution algebra generated by a solution ψ. Then we have a map $\psi : M \to S$ of $k[\partial]-$modules such that $\psi(M)$ generates S as a $k-$algebra. Note that $\psi(M)$ is a (finite dimensional) differential $k-$module. Let $y_1, \ldots, y_n$ be a $k-$basis of $\psi(M)$. Then each y_i must satisfy a linear homogeneous differential equation over k.

Similarly, if E is a no new constant extension of k, then E is a solution field if and only if E is generated by solutions of linear differential equation over k. $\square$

Let $M^\vee$ be the dual of M. A differential field E is called a *Picard-Vessiot field* if it has the same field of constants as k and the $C-$modules $\mathrm{Sol}(M, E) := \mathrm{Hom}_{k[\partial]}(M, E)$ and $\mathrm{Sol}(M^\vee, E) := \mathrm{Hom}_{k[\partial]}(M^\vee, E)$ are of dimension n over C and E is minimal with respect to these properties. We have seen in Section 2.5 that M and $M^\vee$ have the same Picard-Vessiot extension. As a result the concept of Picard-Vessiot field for M and Picard-Vessiot extension for M are the same. Note that E is a solution field of M^n if it is a Picard-Vessiot field of a differential $k-$module M of dimension n.

Theorem 5.4.2. *([1, Lemma 4.2.2, Theorem 1.2.2]) Let M be a differential $k-$module.*

(i) The quotient field of a solution algebra S for M is a solution field for M.

(ii) Conversely, any solution field K for M is the quotient field of (non unique) solution algebra $\mathcal{S}$ for M.

(iii) Any solution field for M embeds as differential subfield of a Picard-Vessiot field for M.

(iv) Let E be the Picard-Vessiot field for M with differential Galois group G, then an intermediate differential field $k \subseteq K \subseteq E$ is a solution field if and only if the corresponding subgroup $H \subset G$ is observable.

Now we are ready to state and prove Theorem 5.2.1 in terms of solution fields.

Theorem 5.4.3. *([22, Proposition 4.3]) Let E be a Picard-Vessiot extension of k with differential Galois group $\mathscr{G}$. Then E is a liouvillian extension of k if and only if every differential field K intermediate to E and k is a solution field.*

Proof. Let E be a liouvillian extension of k, K be an intermediate differential subfield and $\mathscr{H} := \mathscr{G}(E|K)$. We know that $\mathscr{G}^0$, the connected component of $\mathscr{G}$, is a solvable linear algebraic group. From [13, Pages 6, 12], we have

(i) A closed subgroup $\mathscr{H}$ of an algebraic group $\mathscr{G}$ is observable if and only if $\mathscr{H} \cap \mathscr{G}^0$ is observable in $\mathscr{G}^0$.

(ii) If $\mathscr{G}$ is a solvable algebraic group then any closed subgroup $\mathscr{H}$ of $\mathscr{G}$ is observable.

Since $\mathscr{G}^0$ is solvable, every closed subgroup of $\mathscr{G}^0$ is observable. Thus in particular $\mathscr{H} \cap \mathscr{G}^0$ is observable in $\mathscr{G}^0$. This now implies our closed subgroup $\mathscr{H}$ is observable. Now by Theorem 5.4.2, K must be a solution field. To prove the converse, we suppose that E is not a liouvillian Picard-Vessiot extension of k. Then $\mathscr{G}^0$ is not solvable and therefore it contains a non-trivial Borel subgroup $\mathscr{B}$. Since $\mathscr{G}/\mathscr{B}$ is a projective variety, we obtain from Theorem 5.4.2 that the differential field K corresponding to $\mathscr{H}$ is not a solution field. $\square$

Chapter 6

Solutions of first order differential equations in iterated strongly normal extensions

This chapter is based on the author's work in [22]. In this chapter our aim is to classify the transcendental solutions of a first order differential equation in an iterated strongly normal extension of k. If t is a transcendental solution of a first order differential equation in an iterated strongly normal extension of k then $k(t, t')$ is a transcendence degree one differential subfield of an iterated strongly normal extension. Thus we find the structure of transcendence degree one subfields of a strongly normal extension. We also discuss the algebraic dependence of transcendental solutions of first order differential equations and give a large class of differential equations that do not have transcendental solutions in any iterated strongly normal extension of k.

6.1 Transcendence degree one subfields of strongly normal extensions

In this section, we will classify transcendence degree one subfields of strongly normal extensions. First, a strongly normal extension is decomposed into a well-known tower of differential subfields. Then we use the structure of liouvillian Picard-Vessiot extensions and Theorem 2.6.7 to derive our structure theorem.

Proposition 6.1.1. *Let E be a no new constant extension of k. Let L be a differential field intermediate to E and k. Suppose that there are two elements $y, t \in E$, each transcendental over L, such that $L(t) = L(y)$. Then t satisfies a Riccati equation over L if and only if y satisfies a Riccati equation over L.*

Proof. From Lüroth's theorem, we know that there are elements $a, b, c, d \in L$ such that $ad - bc \neq 0$ and that

$$y = \frac{at + b}{ct + d}.$$

Taking derivative of the above equation, we obtain

$$\begin{aligned} y'(ct+d)^2 &= ca't^2 + (a'd + b'c)t + db' - \big(c'at^2 + (bc' + ad')t + bd'\big) \\ &\quad + (ad - bc)t'. \end{aligned}$$

If $y' = f(y)$ is a polynomial of degree ≤ 2 then we see that $y'(ct+d)^2$ is a polynomial of degree ≤ 2. Since $0 \neq ad - bc \in L$, we will solve for t' and obtain $t' = g(t)$, where g is a polynomial in one variable over L of degree at most 2. $\square$

Let E be a strongly normal extension of k. Since $\mathscr{G} := \mathscr{G}(E|k)$ is an algebraic group, there is a chain of subgroups (see Theorem 2.6.6)

$$\mathscr{G} \supseteq \mathscr{G}^0 \supseteq \mathscr{H} \supseteq \{1\},$$

where $\mathscr{G}^0$ is the identity component of $\mathscr{G}$, $\mathscr{H}$ is a closed normal subgroup of $\mathscr{G}^0$ as well as a connected linear algebraic group such that $\mathscr{G}^0/\mathscr{H}$ is an abelian variety. The fundamental theorem of strongly normal extensions gives us the following tower of differential fields:

$$k \subseteq E^0 \subseteq F \subseteq E, \tag{6.1}$$

where E^0 is a finite Galois extension of k with Galois group $\mathscr{G}/\mathscr{G}^0$, F is an abelian extension of E^0, that is $\mathscr{G}(F|E^0) \cong \mathscr{G}^0/\mathscr{H}$ is an abelian variety and E is a Picard-Vessiot extension of F with $\mathscr{H} \cong \mathscr{G}(E|F)$.

Theorem 6.1.2. *([22, Theorem 5.2]) Let E be a strongly normal extension of k and K be a differential field intermediate to E and k. If tr.deg$(K|k) = 1$ then there is a finite algebraic extension $\tilde{k}$ of k and an element $t \in \tilde{k}K$ such that one of the following holds:*

(i) $\tilde{k}K = \tilde{k}(t)$ and t is a solution of a Riccati equation over $\tilde{k}$.

(ii) $\tilde{k}K = \tilde{k}(t, t')$ and t is a solution of a Weierstrass differential equation over $\tilde{k}$.

Proof. We decompose E into a tower of differential fields as in Equation (6.1). Then we have the following cases to consider.

Case (i): Suppose that $KF \supsetneq F$. We take a finite algebraic extension $\tilde{k}$ (inside the algebraic closure of E) so that there is a nonsingular projective curve Γ defined over $\tilde{k}$ satisfying the following properties: the function field $\tilde{k}(\Gamma) \cong \tilde{k}K$, $\tilde{k}$ is algebraically closed in $\tilde{k}K$ and that Γ has a $\tilde{k}-$point. We first claim that Γ is a rational curve, that is, $\tilde{k}K = \tilde{k}(t)$.

Since the compositum $\tilde{k}E$ remains a strongly normal extension of $\tilde{k}$ (see [17, Theorem 5]), for convenience of notation, we replace $\tilde{k}$ by k. Let $z \in KF \setminus F$. Since $\mathscr{G}(E|F)$

is a connected linear algebraic group and since a connected linear algebraic group is a union of its Borel subgroups, by the fundamental theorem, we obtain that

$$\bigcap_{\mathscr{B},\ \text{Borel subgroups}} E^{\mathscr{B}} = F.$$

Therefore z does not lie in every Borel subgroups of $\mathscr{G}(E|F)$. So there is a Borel subgroup $\mathscr{B}$ such that $z \in E \setminus E^{\mathscr{B}}$. Since $\mathscr{B}$ is a connected solvable group, $\mathscr{B} = \mathscr{U} \rtimes \mathscr{T}$, where $\mathscr{U}$ is the unipotent radical and $\mathscr{T}$ is a maximal torus. As a result, E can be further decomposed as follows:

$$E \supseteq E^{\mathscr{U}} \supseteq E^{\mathscr{B}} \supseteq F, \text{ where}$$

(i) E is a Picard-Vessiot extension of $E^{\mathscr{U}}$ with a unipotent differential Galois group $\mathscr{U}$.

(ii) $E^{\mathscr{U}}$ is a Picard-Vessiot extension of $E^{\mathscr{B}}$ with $\mathscr{G}(E^{\mathscr{U}}|E^{\mathscr{B}}) \cong \mathscr{B}/\mathscr{U} \cong \mathscr{T}$.

Therefore there are two possibilities, either $E^{\mathscr{U}} \subsetneq E^{\mathscr{U}}K$ or $E^{\mathscr{U}}K = E^{\mathscr{U}}$. We will discuss each of these cases in details.

Let us first consider the case where $E^{\mathscr{U}}K$ properly contains $E^{\mathscr{U}}$. Then since $\mathscr{U}$ is connected, $E^{\mathscr{U}}$ is algebraically closed in $E^{\mathscr{U}}K$ and we obtain that $\text{tr.deg}(E^{\mathscr{U}}K|E^{\mathscr{U}}) = 1$. Applying Corollary 5.2.2, we obtain $E^{\mathscr{U}}K = E^{\mathscr{U}}(y)$, where $y' \in E^{\mathscr{U}}$. Since $E^{\mathscr{U}}K = E^{\mathscr{U}}(\Gamma)$, the (geometric) genus of Γ over $E^{\mathscr{U}}$ is 0. By Theorem 2.3.1, the geometric genus remains invariant under base change by separable fields. Also, k is algebraically closed in K. Thus, the genus of K over k is zero. Since Γ has $k-$point, we obtain that $K = k(t)$.

Therefore we have $E^{\mathscr{U}}K = E^{\mathscr{U}}(t) = E^{\mathscr{U}}(y)$, where $y' \in E^{\mathscr{U}}$. It follows from Proposition 6.1.1 that t is a solution of a Riccati equation over $E^{\mathscr{U}}$. Say, $t' = g(t)$, where g is a polynomial over $E^{\mathscr{U}}$ of degree ≤ 2. Since $K = k(t)$ is a differential field,

$t' \in k(t) \setminus \{0\}$. Let $t' = h_1(t)/h_2(t)$, where h_1 and h_2 are nonzero relatively prime polynomials in $k[t]$. We may assume that h_2 is a monic polynomial. Therefore

$$g(t) = t' = \frac{h_1(t)}{h_2(t)} \implies g(t)h_2(t) = h_1(t).$$

Note that h_1 and h_2 remain relatively prime over any field extension of k. Thus $h_2 = 1$ and we obtain that the coefficients of g are in k.

Now we consider the case where $E^{\mathscr{U}}K = E^{\mathscr{U}}$, that is $K \subseteq E^{\mathscr{U}}$. Since $\mathscr{T}$ is a connected, commutative linear algebraic group, $KE^{\mathscr{B}}$ is a Picard-Vessiot extension of $E^{\mathscr{B}}$ such that $E^{\mathscr{B}}$ is algebraically closed in $E^{\mathscr{U}}$. Therefore $E^{\mathscr{B}}$ is also algebraically closed in $KE^{\mathscr{B}}$. Since $\mathrm{tr.deg}(KE^{\mathscr{B}}|E^{\mathscr{B}}) = 1$, $\mathscr{G}(KE^{\mathscr{B}}|E^{\mathscr{B}}) \cong \mathrm{G_m}$. From [23, Example 5.24] we have $E^{\mathscr{B}}K = E^{\mathscr{B}}(y)$, where $y'/y \in E^{\mathscr{B}}$. This shows that Γ is a genus zero curve in this case as well. Therefore $E^{\mathscr{B}}K = E^{\mathscr{B}}(t) = E^{\mathscr{B}}(y)$ with $y'/y \in E^{\mathscr{B}}$. We come to the conclusion that t satisfies a Riccati equation over k by making the same argument as in the preceding paragraph.

Case (ii). Suppose that $K \subseteq F$. Note that every intermediate differential subfield is strongly normal over E^0 as F is an abelian extension of E^0. In particular, KE^0 is strongly normal over E^0. Since E^0 is algebraically closed in F, $\mathrm{tr.deg}(KE^0|E^0) = 1$. It follows from Theorem 2.6.7 that either $KE^0 = E^0(y)$, where $y' \in E^0$ or $y'/y \in E^0 \setminus \{0\}$ or that $\bar{k}K = \bar{k}(t, t')$, where t is a solution of a Weierstrass differential equation over $\bar{k}$. If $KE^0 = E^0(y)$ with $y' \in E^0$ or $y'/y \in E^0 \setminus \{0\}$ then $\mathscr{G}(KE^0|E^0) \cong \mathrm{G_a}(C)$ or $\mathrm{G_m}(C)$ and we would get a surjective morphism from the abelian variety $\mathscr{G}(F|E^0)$ to the linear algebraic group $\mathscr{G}(KE^0|E^0)$, which is not possible. Therefore, $\bar{k}K = \bar{k}(t, t')$, where t is a transcendental solution of a Weierstrass differential equation over $\bar{k}$. Now K is a finitely generated differential field over k. Therefore there is a finite algebraic extension $\tilde{k}$ of k such that $\tilde{k}K = \tilde{k}(t, t')$. $\square$

Remark 6.1.3. If E is a Picard-Vessiot extension of k then only case (i) of the above theorem can occur. Moreover, one can show that KE^0 is a rational field generated by a solution of a Riccati differential equation. To see this, we consider the connected group $\mathscr{G}(E|E^0)$ and its codimension one closed subgroup $\mathscr{G}(E|KE^0)$. Then, the function field of the homogeneous space $\mathscr{G}(E|E^0)/\mathscr{G}(E|KE^0)$ is known to be rational; for example, see [6, Theorem 4.4]. Since KE^0 is isomorphic (as fields) to $E^0(\mathscr{G}(E|E^0)/\mathscr{G}(E|KE^0)) = E^0(x)$, we obtain that $KE^0 = E^0(y)$ [23, p. 87]. As in the proof of the previous theorem, there is a Borel subgroup of $\mathscr{G}(E|E^0)$ such that either $KE^{\mathscr{U}} = E^{\mathscr{U}}(t) = E^{\mathscr{U}}(y)$, where $t' \in E^{\mathscr{U}}$ or $KE^{\mathscr{B}} = E^{\mathscr{B}}(t) = E^{\mathscr{B}}(y)$, where $t'/t \in E^{\mathscr{B}}$. Again, arguing as above, one can show that y satisfies a Riccati differential equation over E^0.

6.2 Transcendence degree one subfields of iterated strongly normal extensions

In this section, our aim is to classify transcendence degree one subfields of an iterated strongly normal extension. Recall that E is called an iterated strongly normal extension of k if there is a tower of differential subfields $k =: E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_{n+1} = E$, where E_i is a strongly normal extension of E_{i-1}. Since each E_i is strongly normal over E_{i-1}, E_i is finitely generated over E_{i-1} such that $C_{E_i} = C_{E_{i-1}}$. Thus E is finitely generated over k such that $C_E = C$. Let K be a differential field intermediate to k and E. Then K is also finitely generated over k. Suppose that $E_{i-1} \subsetneq E_{i-1}K \subseteq E_{i-1}$. Then by Theorem 6.1.2, there is an element $t \in E_{i-1}K$ such that t is transcendental over E_{i-1} and satisfies a Riccati or a Weierstrass differential equation over an algebraic extension of E_{i-1}. In the next proposition and lemma we will devise techniques to find an element v satisfying a Riccati or a Weierstrass differential equation over an algebraic extension of k.

Proposition 6.2.1. *Let E be a strongly normal extension of k with differential Galois group $\mathscr{G}$ and $\tilde{E}$ be an algebraic extension of E. Suppose that K is a differential field intermediate to k and $\tilde{E}$ such that tr.deg$(K|k) = 1$. Then tr.deg$(K \cap E|k) = 1$ and there is an algebraic extension $\tilde{k}$ of k such that one of the following holds:*

(i) $\tilde{k}(K \cap E) = \tilde{k}(y)$, *where y satisfies a Riccati differential equation over $\tilde{k}$.*

(ii) $\tilde{k}(K \cap E) = \tilde{k}(y, y')$, *where y satisfies a Weierstrass differential equation over $\tilde{k}$.*

Proof. Since E is a strongly normal extension of k, KE is a strongly normal extension of K and the natural restriction map from the differential Galois group $\mathscr{G}(KE|K)$ to $\mathscr{G}$ is an isomorphism onto the subgroup $\mathscr{G}(E|K \cap E)$ of $\mathscr{G}$ (see [17, Theorem 5]). Let tr.deg$(E|k) = n$. Then $n \geq 1$ as tr.deg$(K|k) = 1$ and $k \subseteq K \subseteq \tilde{E}$, where $\tilde{E}$ is algebraic over E. Now tr.deg$(E|K \cap E) = \dim(\mathscr{G}(E|K \cap E)) = $ tr.deg$(KE|K) = n - 1$. Thus, tr.deg$((K \cap E)|k) = $ tr.deg$(E|k) -$ tr.deg$(E|(K \cap E)) = 1$. The remainder of the proof is derived from Theorem 6.1.2. $\square$

Lemma 6.2.2. *([22, Lemma 6.2]) Let L be a no new constant extension of $\bar{k}$. Suppose that K and E are differential subfields intermediate to L and $\bar{k}$ having the following properties: K is finitely generated over $\bar{k}$, tr.deg$(K|\bar{k}) = 1$, tr.deg$(EK|E) = 1$, E is algebraically closed, EK is weakly normal extension of K and that the group of differential automorphisms $\mathscr{G}(EK|K)$ stabilises E.*

(i) If there is a Riccati equation over E having a solution $t \in EK \setminus E$ then there is Riccati equation over $\bar{k}$ having a solution $v \in K \setminus \bar{k}$.

(ii) If there is a Weierstrass differential equation over E having a solution $t \in EK \backslash E$ then there is a Weierstrass differential equation over $\bar{k}$ having a solution $v \in K \setminus \bar{k}$.

Proof. Since K is finitely generated over $\bar{k}$, EK is also finitely generated over E such that EK is a finite algebraic extension of $E(t, t')$. Therefore there are finitely many fields intermediate to $E(t, t')$ and EK. Consider the set

$$S = \{\sigma \in \mathscr{G}(EK|K)|\ E(t, t') \subsetneq E(t, t', \sigma(t), \sigma(t)')\} \cup \{id\}\,.$$

Thus there are finitely many automorphisms $\sigma_1, \ldots, \sigma_n \in \mathscr{G}(EK|K)$, where σ_1 is the identity, such that $\mathscr{G}(EK|K)$ stabilizes the differential field $E^* := E(\sigma_1(t), \sigma_1(t)', \ldots,\, \sigma_n(t), \sigma_n(t)')$.

We claim that $(E^* \setminus E) \cap K \neq \emptyset$. Since $\text{tr.deg}(K|\bar{k}) = \text{tr.deg}(EK|E) = 1$, there is an element $u \in K \setminus \bar{k}$ such that u is transcendental over E. Let $X^m + \alpha_{m-1}X^{m-1} + \cdots + \alpha_0 \in E^*[X]$ be the minimal polynomial of u over E^*. Then for any $\sigma \in \mathscr{G}(EK|K)$, we have $\sigma(u) = u$, therefore $u^m + \sigma(\alpha_{m-1})u^{m-1} + \cdots + \sigma(\alpha_0) = 0$ and thus

$$(\sigma(\alpha_{m-1}) - \alpha_{m-1})\, u^{m-1} + \cdots + \sigma(\alpha_0) - \alpha_0 = 0.$$

Therefore $\sigma(\alpha_i) = \alpha_i$ for all $i = 0, \ldots, m-1$. This implies that $\alpha_i \in K$ as EK is assumed to be weakly normal over K. Thus $\alpha_i \in E^* \cap K$ for all i. Since u is transcendental over E, there is at least one i such that $s := \alpha_i \in E^* \setminus E$. This proves our claim.

$$
\begin{tikzcd}
EK \arrow[dd, dash, "\mathscr{G}(EK|K)"'] \arrow[dr, dash] & & & \mathcal{E} \arrow[ddd, dash, dashed] \\
 & E^* \arrow[urr, dash, dashed] \arrow[dr, dash] & & \\
K \arrow[ddrr, dash] & & E(s,s') \arrow[uur, dash, dashed] \arrow[dr, dash] \arrow[dd, dash] & \\
 & & & E \arrow[dd, dash] \\
 & & \bar{k}(s,s') \arrow[dr, dash] & \\
 & & & \bar{k}
\end{tikzcd}
$$

Now we are ready to prove the lemma. First, consider the case where $t \in EK \setminus E$ satisfies a Riccati equation over E. Let $t' = P(t)$. Note that for any $\sigma \in \mathscr{G}(EK|K)$, $\sigma(t)' = P_\sigma(\sigma(t))$ is also a Riccati equation, where $P_\sigma \in E[X]$ is the polynomial obtained by applying σ to the coefficients of P. Since each of $\sigma_1(t), \ldots, \sigma_n(t)$ is a solution of some Riccati equation over E, there is a Picard-Vessiot extension $\mathcal{E}$ of E containing $\sigma_1(t), \ldots, \sigma_n(t)$. Now $E(s,s')$ is a differential subfield of $\mathcal{E}$ with $\mathrm{tr.deg}(E(s,s')|E) = 1$. We apply Theorem 6.1.2 and we get that $E(s,s') = E(y)$, where y satisfies a Riccati differential equation over E. Using the fact that the genus of a field remains invariant under base change by separable extensions, we conclude that $\bar{k}(s,s')$ has genus 0 and thus $\bar{k}(s,s') = \bar{k}(v)$. By Proposition 6.1.1, we obtain that v satisfies a Riccati equation over $\bar{k}$. This proves (i).

Now we consider that case where t is a transcendental solution of a Weierstrass differential equation over E; $t'^2 = \alpha^2(4t^3 - g_2 t - g_3)$ with $g_2, g_3 \in C$. Note that $\sigma(t)'^2 = \sigma(\alpha)^2(4\sigma(t)^3 - g_2\sigma(t) - g_3)$ for $\sigma \in \mathscr{G}(EK|K)$. Since E^* is the compositum of finitely many strongly normal extensions $E(\sigma_i(t), \sigma_i(t)')$ of E, E^* is a strongly normal extension of E. Also, $E \subseteq E^* \subseteq EK$ and therefore $\mathrm{tr.deg}(E^*|E) = 1$. Now $\mathrm{genus}(E(t,t')|E) = 1$ and $E(t,t') \subseteq E^*$, therefore $\mathrm{genus}(E^*|E) \geq 1$. It follows from Theorem 2.6.7 that E^* is a genus one abelian extension of E. By case (ii) of Theorem 6.1.2, $E(s,s')$ is a genus one abelian extension of E. Thus $\bar{k}(s,s')$ is also a genus one field extension of $\bar{k}$.

Let $\mathscr{C}_1$ be a nonsingular projective model for $\bar{k}(s,s')$. Then $\mathscr{C}_1$ is also a model for $E(s,s')$. By Theorem 2.6.7, the group $\mathscr{G}(E(s,s')|E)$ is the $C-$points of a Weierstrass elliptic curve $\mathscr{C}_2$ defined over C :

$$\mathscr{C}_2 = X_2^2X_0 - 4X_1^3 + h_2X_0^2X_1 + h_3X_0^3,$$

where $h_2, h_3 \in C$. Furthermore, $E(s,s') = E(x,x')$, where $(1 : x : x'/\alpha)$ is a point of $\mathscr{C}_2$ for some $\alpha \in E$. Therefore $\mathscr{C}_2$ is also a nonsingular projective model for $E(s,s')$. Then, $\mathscr{C}_1$ and $\mathscr{C}_2$ are isomorphic over E. Since both the curves are defined over $\bar{k}$ and the $j-$ invariants $j(\mathscr{C}_1) = j(\mathscr{C}_2) \in C \subseteq \bar{k}$, the curves are isomorphic over $\bar{k}$ as well [35, Proposition 1.4]. Thus, there is a $\bar{k}(s,s')-$point $(1 : \omega : \rho)$ of $\mathscr{C}_2$ such that $\bar{k}(s,s') = \bar{k}(\omega,\rho)$. Then, as in the proof of [15, Theorem 3], we can show that $\bar{k}(s,s') = \bar{k}(\omega,\omega')$ and that $(1 : \omega : \omega'/\beta)$ is a point on $\mathscr{C}_2$ for some $\beta \in \bar{k}$. This implies $\omega'^2 = \beta^2(4\omega^3 - h_2\omega - h_3)$. □

Now we are ready to prove our main result.

Theorem 6.2.3. *([22, Theorem 6.3]) Let E be an iterated strongly normal extension of k and K be an intermediate differential field such that $tr.deg(K|k) = 1$. Then, there is a finite algebraic extension $\tilde{k}$ of k such that $\tilde{k}K = \tilde{k}(y,z)$, where z is algebraic over $\tilde{k}(y)$, y is transcendental over $\tilde{k}$ and either y is a solution of a Riccati differential equation or a Weierstrass differential equation over $\tilde{k}$.*

Proof. Let $k =: E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_{n+1} = E$ be a tower of strongly normal extensions, where we may assume that E_{n+1} is an algebraic extension of E_n and that E_n is a transcendental extension of E_{n-1}. Let $\overline{E}$ be the algebraic closure of E. Consider the following tower of algebraically closed differential fields

$$\overline{E}_0 \subseteq \overline{E}_1 \subseteq \overline{E}_2 \subseteq \cdots \subseteq \overline{E}_n = \overline{E}.$$

Since $\overline{E}_{n-1}E_n$ is strongly normal over $\overline{E}_{n-1}$ and $\overline{E}_{n-1}K$ is contained in $\overline{E}$, by Proposition 6.2.1, we obtain an element $y \in \overline{E}_{n-1}K$ transcendental over $\overline{E}_{n-1}$ such that y satisfies either a Riccati or a Weierstrass differential equation over $\overline{E}_{n-1}$.

Let us assume that the following statement holds.

(C1) For all $i = 1, \dots, n-1$, $\overline{E}_iK$ is a weakly normal extension of $\overline{E}_{i-1}K$ and that the group $\mathscr{G}(\overline{E}_iK|\overline{E}_{i-1}K)$ stabilises $\overline{E}_i$.

The differential field K is finitely generated over k as it is a subfield of an iterated strongly normal extension of k. Consequently, $\overline{E}_iK$ is also finitely generated over $\overline{E}_i$. We apply Lemma 6.2.2 repeatedly to obtain an element $y \in \bar{k}K \backslash \bar{k}$ having the desired properties. Since K is finitely generated over k, we only need a finite algebraic extension $\tilde{k}$ so that the differential field $\tilde{k}K$ contains both y and the coefficients of the differential equation of y. Therefore we have proved the theorem assuming that (C1) holds.

Now we will verify (C1). Since $\overline{E}_{i-1}E_i$ is strongly normal over $\overline{E}_{i-1}$, $L^* := \overline{E}_{i-1}KE_i$ is a strongly normal extension of $L := K\overline{E}_{i-1}$. Note that the field of constants of $\overline{E}$ and k are the same, therefore the group $\mathscr{G}(L^*|L)$ stabilizes the strongly normal extensions E_i of E_{i-1} and $\overline{E}_{i-1}E_i$ of $\overline{E}_{i-1}$. Let I be the set of all irreducible polynomials over E_i. Observe that the group $\mathscr{G}(L^*|L)$ acts on I via the map $P \mapsto P_\sigma$, where the automorphism σ is applied to the coefficients of P to obtain P_σ. Let $L^*(I) \subseteq \overline{E}$ be the splitting field of I. Observe that $L^*(I) = L^*\overline{E}_i = K\overline{E_i}$. Because the fixed field of the group $\mathscr{G}(K\overline{E}_i|L^*)$ is the field L^* and that the fixed field of $\mathscr{G}(L^*|L)$ is $L = \overline{E}_{i-1}K$, we deduce that the fixed field of $\mathscr{G}(\overline{E}_iK|\overline{E}_{i-1}K)$ is $\overline{E}_{i-1}K$. Therefore $\overline{E}_iK$ is weakly normal over $\overline{E}_{i-1}K$. Since $\mathscr{G}(\overline{E}_iK|\overline{E}_{i-1}K)$ stabilizes the strongly normal extension E_i of E_{i-1} as well as the set I of polynomials over E_i, $\mathscr{G}(\overline{E}_iK|\overline{E}_{i-1}K)$ stabilises $\overline{E}_i$ as well. This proves (C1). $\square$

Corollary 6.2.4. *If E is an iterated strongly normal extension of the field of constants C then every intermediate differential field K with tr.deg$(K|C) = 1$ is of the form $K = C(y, z)$, where z is algebraic over $C(y)$ and that either $y' = 1$ or $y' = cy$ for some nonzero $c \in C$ or y is a solution of a Weierstrass differential equation over C.*

Proof. We only need to take into account the case where there is an element $t \in K \backslash C$ such that $t' = at^2 + bt + c$ for $a, b, c \in C$. Since t satisfies a Riccati differential equation over C, the differential field $C(t)$ can be embedded in a Picard-Vessiot extension of C. We know that the Picard-Vessiot extensions of C have connected, commutative differential Galois group. Therefore $C(t)$ itself is a Picard-Vessiot extension of C with $\mathscr{G}(C(t)|C)$ isomorphic to either $\mathrm{G_a}(C)$ or $\mathrm{G_m}(C)$. Thus, $C(t) = C(y)$, where either $y' = 1$ or $y' = cy$ for some nonzero constant c. $\square$

To summarise, we have the following theorem.

Theorem 6.2.5. *Let E be a no new constant extension of k. Suppose that E contains a differential field K having transcendence degree, tr.deg$(K|k) = 1$.*

1. *If E is a strongly normal extension of k then there is a finite algebraic extension $\tilde{k}$ of k such that $\tilde{k}K = \tilde{k}(y, y')$, where y is transcendental over $\tilde{k}$ and one of the following holds.*

   (a) y is a solution of a Riccati differential equation over $\tilde{k}$; $y' = a_2y^2 + a_1y + a_0$ for some $a_0, a_1, a_2 \in \tilde{k}$

   (b) y is a solution of a Weierstrass differential equation over $\tilde{k}$; $(y')^2 = \alpha^2(4y^3 - g_2y - g_3)$ for constants g_2, g_3 such that $27g_3^2 - g_2^3 \neq 0$ and $\alpha \in \tilde{k}$.

2. *If E is an iterated strongly normal extension then there is a finite algebraic extension $\tilde{k}$ of k such that $\tilde{k}K$ is a finite algebraic extension of $\tilde{k}(y, y')$, where y is transcendental over $\tilde{k}$ and either (1a) or (1b) holds.*

If we restrict Theorem 6.2.5 to the case when $k = C$ then *(1a)* can be replaced with

*(1a)** $y' = 1$ *or* $y' = cy$ *for some nonzero constant* c *or*

As a consequence of the theorem, we obtain a classification of first order differential equations $f(y, y') = 0$ over k into the following types:

Algebraic type All solutions of $f(y, y') = 0$ are algebraic over k.

Riccati type The differential equation $f(y, y') = 0$ has a transcendental solution y such that there is a finite algebraic extension $\tilde{k}$ of k and an element $t \in \tilde{k}(y, y')$ such that $\tilde{k}(y, y')$ is a finite algebraic extension of $\tilde{k}(t)$ and that t is a solution of a Riccati differential equation:

$$t' = a_2 t^2 + a_1 t + a_0, \text{ with } \ a_0, a_1, a_2 \in \tilde{k}, \text{ not all zero.}$$

Weierstrass type The differential equation $f(y, y') = 0$ has a transcendental solution y such that there is a finite algebraic extension $\tilde{k}$ of k and an element $t \in \tilde{k}(y, y')$ such that $\tilde{k}(y, y')$ is a finite algebraic extension of $\tilde{k}(t, t')$ and that t is a solution of a Weierstrass differential equation:

$$(t')^2 = \alpha^2(4t^3 - g_2 t - g_3), \ g_2, g_3 \in C, \alpha \in \tilde{k} \text{ and } 27g_3^2 - g_2^3 \neq 0.$$

General type The differential equation $f(y, y') = 0$ is not of the above types.

Here we will not be concerned with differential equations of algebraic type. Such differential equations are discussed in [27]. In Proposition 3.1.3 we have listed a known class of differential equations of algebraic type. With this classification, Theorem 6.2.5 can be stated as follows:

Theorem 6.2.6. *([14, Theorem 1.1]) Let k be a differential field with an algebraically closed field of constants C and $f(y, y') = 0$ be a differential equation over k. Suppose that $f(y, y') = 0$ has a transcendental solution y in an iterated strongly normal extension E of k. Then the following statements hold.*

(i) The differential equation $f(y, y') = 0$ is of either Riccati or Weierstrass type.

(ii) If $k = C$ and $f(y, y') = 0$ is of Riccati type then there is an element $t \in C(y, y')$ such that either $t' = 1$ or $t' = ct$ for some nonzero $c \in C$.

(iii) If E is a strongly normal extension of k and $f(y, y') = 0$ is of Riccati type (respectively, Weierstrass type) then the finite algebraic extension $\tilde{k}$ and the element $t \in \tilde{k}(y, y')$, as in the definition of a Riccati type (respectively, Weierstrass type), can be chosen so that $\tilde{k}(y, y') = \tilde{k}(t)$ (respectively, $\tilde{k}(y, y') = \tilde{k}(t, t')$).

The papers [39], [26] and [27] are also pertinent to the aforementioned theorem. In the next subsections we shall briefly discuss how they are related to our theorem.

6.2.1 First order autonomous differential equations, [39].

Let $f \in C[X, Y]$ be irreducible and y be a transcendental solution of the autonomous differential equation $f(y, y') = 0$. Let Γ be the nonsingular projective model whose function field $C(\Gamma)$ is isomorphic to $C(y, y')$. The natural isomorphism between $C(y, y')$ and $C(\Gamma)$ makes the latter a differential field. Let $Der_C(C(\Gamma))$ be the $C(\Gamma)-$module of $C-$derivations on $C(\Gamma)$. Then, there is a natural $C(\Gamma)-$module isomorphism between $Der_C(C(\Gamma))$ and $Hom_{C(\Gamma)}\left(\Omega^1_{C(\Gamma)|C}, C(\Gamma)\right)$ given by $(a \mapsto a') \mapsto z$, where z is the differential such that $\Phi(z) = 1$. In this way, the tuple (Γ, z) can be associated to an autonomous differential equation. This identification allowed for the classification of autonomous equations into four categories.

1. $z = dg$ for some $g \in C(\Gamma)$ or equivalently, there is an element $t \in C(\Gamma)$ such that $t' = 1$; in this case the equations are of *exact* type.

2. $z = dg/cg$ for some $g \in C(\Gamma)$ and nonzero constant c or equivalently, there is a $t \in C(\Gamma)$ such that $t' = ct$ for some nonzero constant c; in this case the equations are of *exponential* type.

3. $z = dg/h$ for $h, g \in C(\Gamma)$ and $h^2 = g^3 + ag + b$ with $4a^3 + 27b^2 \neq 0$ or equivalently, there is an element $t \in C(\Gamma)$ such that $t'^2 = t^3 + at + b$ for $a, b \in C$ with $4a^3 + 27b^2 \neq 0$; in this case the equations are of *Weierstrass* type.

4. If the equation is not of the above three types then the it is of *general* type.

Furthermore, an autonomous equation (Γ, z) is called *new* if (Γ, z) is not a proper pull back. An autonomous equation of general and new type has the following interesting property [39, Theorem 2.1]: Any number of distinct transcendental solutions are $C-$algebraically independent. The authors used this property to demonstrate that no iterated Picard-Vessiot extension yields transcendental solutions to equations of general type ([39, Proposition 7.1]). In fact, by extending their reasoning, it is possible to demonstrate that autonomous differential equations of general type have no transcendental solutions in any iterated strongly normal extension. So we define a differential equation $f(y, y') \in k[y, y']$ to be of general type if it has no transcendental solution in any strongly normal extension of k.

Thus, when $k = C$, Theorem 6.2.5 can be recovered from their work. Since C is an algebraically closed field of constants, every nonconstant solution must be transcendental and every autonomous differential equation has a transcendental solution, the above classification of differential equations coincides with ours.

6.2.2 Painlevé property and transcendence degree 1 subfields of a strongly normal extension ([26, 27]):

Let k be a finite algebraic extension of the ordinary differential field $\mathbb{C}(x)$ of rational functions over complex numbers with $x' = 1$. A differential equation $f(y, y') = 0$ over k is said to have the Painlevé property if the set of all branch points and the set of all essential singularities of the solutions form a discrete set. Suppose that a differential equation $f(y, y') = 0$ has a transcendental solution $y \in K$, where K is a no new constant extension of k such that $\text{tr.deg}(K|k) = 1$. In [26, Theorem 4.5], the authors have shown that f has PP if and only if there exists a finite algebraic extension $\tilde{k}$ of k and an element $u \in \tilde{k}(y, y')$ such that $\tilde{k}(u, u') = \tilde{k}(y, y')$ and that u is a solution of either a Riccati or a Weierstrass differential equation over $\tilde{k}$. Thus, we can conclude the following from Theorem 6.2.5.

> Let k be the algebraic closure of $\mathbb{C}(x)$ with the derivation that restricts to the derivation d/dx on $\mathbb{C}(x)$. Let $K = k(y, y')$ be a no new constant extension of k such that $\text{tr.deg}(K|k) = 1$. Let $f \in k[X, Y]$ be an irreducible polynomial involving the variable Y such that $f(y, y') = 0$. Then K is a differential subfield of a strongly normal extension of k if and only if f has the Painlevé property.

6.3 Transcendental solutions of first order differential equations

In this section, we use Theorem 6.2.5 to deduce a few properties of differential equations of nongeneral type. We also give a large class of differential equations of the general type. Then we will discuss rational autonomous differential equations.

6.3.1 Differential equations of general type

In the previous chapters, we have given examples of differential equations that have transcendental solutions in a liouvillian extension of k. By Proposition 2.3.2, we may assume that the field of constants of the liouvillian extension is C. Such a liouvillian extension is an iterated Picard-Vessiot extension of K. Therefore we have ample examples of differential equations of the nongeneral type. Now, we will give a method to generate a large class of differential equations of general type.

Theorem 6.3.1. *([22, Theorem 7.6]) Let $f \in \bar{k}[Y, Z]$ be an irreducible polynomial having the following properties:*

(i) $p := (0, 0)$ is a simple point of f and Z is the tangent line at p.

(ii) With respect to the uniformizing parameter Y, both the coefficients λ_2 and λ_3 of the $Y-$adic expansion $Z = \lambda_2 Y^2 + \lambda_3 Y^3 + \cdots$ do not have any antiderivatives in $\bar{k}$.

Then the differential equation $f(y, y') = 0$ has a transcendental solution and the equation is of general type.

Proof. Let $\bar{k}(y,z)$ be the function field of f. We extend the derivation of $\bar{k}$ to $\bar{k}(y,z)$ dy defining $' : \bar{k}(y,z) \to \bar{k}(y,z)$, where $y' = z$. We need to demonstrate that $k(y,y')$ is a no new constant extension of k in order to show that the differential equation $f(y,y') = 0$ has a transcendental solution. Note that $\bar{k}(y,y')$ embeds as a differential field in $\bar{k}((y))$.

Therefore any element $v \in \bar{k}(y,y') \setminus \bar{k}$, can be written as $v = \sum_{i=r}^{\infty} a_i y^i$, where $a_i \in \bar{k}$ and $a_r \neq 0$. Taking derivative we get

$$\begin{aligned} v' &= a_r' y^r + a_{r+1}' y^{r+1} + \cdots + r a_r y^{r-1}(\lambda_2 y^2 + \cdots) + (r+1)a_{r+1} y^r (\lambda_2 y^2 + \cdots) + \cdots \\ &= a_r' y^r + (a_{r+1}' + r a_r \lambda_2) y^{r+1} + (a_{r+2}' + r a_r \lambda_3 + (r+1)a_{r+1}\lambda_2) y^{r+2} + \cdots . \end{aligned} \tag{6.2}$$

For a moment, let us assume the following statements.

(C2) There is no element $v \in \bar{k}(y,y') \setminus \bar{k}$ such that $v' = \alpha v + \beta$ for any $\alpha, \beta \in \bar{k}$.

(C3) There is no element in $v \in \bar{k}(y,y') \setminus \bar{k}$ satisfying a Riccati or a Weierstrass differential equation.

If we take $\alpha = 0$ and $\beta = 0$, then from (C1) we conclude that the differential equation has a transcendental solution. By Theorem 6.2.5, the differential equation is of general type. Now, we will prove (C1) and (C2).

Suppose (C2) does not hold, then substituting $v = \sum_{i=r}^{\infty} a_r y^r$ in $v' = av + b$ we have to consider the following cases.

Case (i). $\mathrm{ord}_p(v) = 0$. Here we have $a_0' = \alpha a_0 + \beta$ and if $m \geq 1$ is the least positive integer such that $a_m \neq 0$ then we have $a_m' = \alpha a_m$ and that $a_{m+1}' + m a_m \lambda_2 = \alpha a_{m+1}$. Note that such a m exists as $v \in \bar{k}(y,y') \setminus \bar{k}$. Thus, we obtain

$$\left(-\frac{a_{m+1}}{m a_m}\right)' = \lambda_2,$$

a contradiction to our assumption on λ_2.

Case (ii). $\mathrm{ord}_p(v) \geq 1$ or $\mathrm{ord}_p(v) \leq -2$. Then $a_r' = \alpha a_r$ and that $a_{r+1}' + r a_r \lambda_2 = \alpha a_{r+1}$. This implies
$$\left(-\frac{a_{r+1}}{r a_r}\right)' = \lambda_2$$
and as before we obtain a contradiction.

Case (iii). $\mathrm{ord}_p(v) = -1$. Then $a_{-1}' = \alpha a_{-1}$ and $a_1' = \alpha a_1 + \lambda_3 a_{-1}$. This implies
$$\left(\frac{a_1}{a_{-1}}\right)' = \lambda_3,$$
which contradicts our assumption on λ_3. This proves (C2).

Now we will show that there is no element in $\bar{k}(y, y') \setminus \bar{k}$ that satisfies a Riccati equation over $\bar{k}$. Suppose not. Let $v \in \bar{k}(y, y')$ such that $v' = b_2 v^2 + b_1 v + b_0$ for $b_2, b_1, b_0 \in \bar{k}$. Then from (C2), we must have $b_2 \neq 0$.

Case (i). If $\mathrm{ord}_p(v) \leq -1$ then from Equation (6.2) we obtain $\mathrm{ord}_p(v') \geq \mathrm{ord}_p(v)$. But $b_2 \neq 0$ and $\mathrm{ord}_p(v) \leq -1$ implies $\mathrm{ord}_p(v') = \mathrm{ord}_p(b_2 v^2 + b_1 v + b_0) = 2\,\mathrm{ord}_p(v)$. Thus we have obtained $\mathrm{ord}_p(v) \leq \mathrm{ord}_p(v') = 2\,\mathrm{ord}_p(v)$, a contradiction.

Case (ii). If $\mathrm{ord}_p(v) \geq 1$ then $b_0 = 0$ and we obtain that $(1/v)' = -b_1(1/v) - b_2$, which contradicts (C2).

Case (iii). If $\mathrm{ord}_p(v) = 0$ then $a_0 \in \bar{k}$ is a solution of the Riccati equation $a_0' = b_2 a_0^2 + b_1 a_0 + b_0$. One can easily show that
$$\left(\frac{1}{v - a_0}\right)' = \frac{-2b_2 a_0 - b_1}{v - a_0} - b_2$$
and this again contradicts (C2). This proves our claim.

Now we suppose that there is a transcendental element $v \in \bar{k}(y, y')$ satisfying a Weierstrass differential equation: $v'^2 = \alpha^2(4v^3 - g_2 v - g_3)$ for $g_2, g_3 \in C$ and $\alpha \in \bar{k}$.

Then from the $y-$adic expansion of v and from Equation (6.2), we have $v'^2 = a_r'^2 y^{2r} + \cdots$ and thus

$$2\ \mathrm{ord}_p(v) \leq \mathrm{ord}_p(v'^2) = \mathrm{ord}_p(\alpha^2(4v^3 - g_2 v - g_3)).$$

Case (i). If $\mathrm{ord}_p(v) < 0$ then $\mathrm{ord}_p(\alpha^2(4v^3 - g_2 v - g_3)) = 3\ \mathrm{ord}_p(v)$ and we obtain 2 $\mathrm{ord}_p(v) \leq 3\ \mathrm{ord}_p(v)$, a contradiction.

Case (ii). If $\mathrm{ord}_p(v) > 0$ then $\mathrm{ord}_p(\alpha^2(4v^3 - g_2 v - g_3)) \leq \mathrm{ord}_p(v)$, and we obtain 2 $\mathrm{ord}_p(v) \leq \mathrm{ord}_p(v)$, again a contradiction.

Case (iii). If $\mathrm{ord}_p(v) = 0$ and in the $y-$adic expansion of v, let m be the least positive integer such that $a_m \neq 0$. Then from Equation (6.2), we have the following equations

$$\begin{aligned} v'^2 &= \alpha^2(4v^3 - g_2 v - g_3) \\ a_0'^2 &= \alpha^2(4a_0^3 - g_2 a_0 - g_3) \\ 2a_0' a_m' &= \alpha^2\left(12a_0^2 a_m - g_2 a_m\right). \end{aligned} \tag{6.3}$$

We assume for the moment that a_0 is a constant. Then a_0 must be one of the distinct roots of the polynomial $4Y^3 - g_2 Y - g_3$ and in particular, a_0 is not a root of $12Y^2 - g_2$. Now Equation (6.3) becomes

$$\alpha^2(12a_0^2 - g_2)a_m = 0,$$

which is absurd. Thus there is no element in $\bar{k}(t,t') \setminus \bar{k}$ satisfying a Weierstrass differential equation over $\bar{k}$.

Now we will show that a_0 is indeed a constant. Assume otherwise and consider the nonsingular projective curve

$$X_2^2 X_0 - 4X_1^3 + g_2 X_0^2 X_1 + g_3 X_0^3. \tag{6.4}$$

We have two nonconstant points of this curve, namely, $(1 : v : v'/\alpha)$ and $(1 : a_0 : a_0'/\alpha)$. If $(1 : \eta : \xi) := (1 : v : v'/\alpha)(1 : a_0 : a_0'/\alpha)$ then

$$\eta = -v - a_0 + \frac{1}{4\alpha^2}\left(\frac{v' - a_0'}{v - a_0}\right)^2$$

and thus $\eta \in \bar{k}(y, y') \setminus \bar{k}$. Now we apply [17, Lemma 2] for the Weierstrass equations $v'^2 = \alpha^2(4v^3 - g_2 v - g_3)$ and $a_0'^2 = (-\alpha)^2(4a_0^3 - g_2 a_0 - g_3)$ and obtain that $\eta' = 0$. This contradicts the fact that field of constants of $\bar{k}(y, y')$ is the same as the field of constants of $\bar{k}$. Thus we have proved (C3). □

Remark 6.3.2. Given an irreducible affine curve of the form $Z - F_2 - F_3 - \cdots - F_n$, where F_i are forms of degree i, one can find the coefficients λ_2 and λ_3 of the $Y-$adic expansion of z as follows: Let $F_2 = x_{20}Y^2 + x_{11}ZY + x_{02}Z^2$ and $F_3 = x_{30}Y^3 + R_3$. Then, on the curve, the value of Z equals

$$x_{20}Y^2 + x_{11}ZY + x_{02}Z^2 + x_{30}Y^3 + R_3 + F_4 + \cdots + F_n.$$

In the above expression we substitute back for Z and obtain

$$x_{20}Y^2 + x_{11}\left(x_{20}Y^2 + x_{11}ZY + \cdots\right)Y + x_{02}\left(x_{20}Y^2 + \cdots\right)^2 + x_{30}Y^3 + \cdots .$$

Continuing this process one actually obtains the $Y-$adic expansion of Z;

$$Z = x_{20}Y^2 + \left(x_{11}x_{20} + x_{30}\right)Y^3 + \cdots .$$

Example 6.3.3. The following differential equation over $\mathbb{C}(x)$ is of the general type.

$$y' - \frac{1}{x}y^2 - xyy' - \frac{1}{x+1}y^3 + y(y')^2 = 0$$

as $1/x$ and $x(1/x) + 1/(x+1) = 1 + 1/(x+1)$ have no antiderivatives in $\mathbb{C}(x)$.

Example 6.3.4. The following differential equation

$$y' = a_n y^n + \cdots + a_3 y^3 + a_2 y^2, \tag{6.5}$$

where both a_2 and a_3 having no antiderivatives in k, is easily seen to be of general type. The autonomous differential equation $y' = y^3 - y^2$ is therefore of general type. Thus any autonomous equation of the form (6.5), with a_2 and a_3 nonzero, is of general type.

In the following proposition, we will demonstrate an interesting property of differential equation (6.5) that it can have infinitely many algebraically independent transcendental solutions. Therefore by [11], any m distinct transcendental solutions are algebraically dependent. We note that this property is also satisfied by first order autonomous differential equations of general and new type.

Proposition 6.3.5. *Let $k(t)$ be a differential field extension of k such that t is transcendental over k and $t' = a_n t^n + \cdots + a_2 t^2$, where $a_i \in k$, $a_n \neq 0$, $n \geq 3$ and a_2 does not have an antiderivative in k. Then the following holds:*

1. *$k(t)$ is a no new constant extension of k.*
2. *If both a_2 and a_3 have no antiderivatives in k then both a_2 and a_3 do not have any antiderivative in $k(t)$.*

Proof. Suppose that $u \in k(t) \setminus k$ such that $u' = 0$. Let the $t-$adic expansion of u be $u = \sum_{i=p}^{\infty} b_i t^i$, where $b_i \in \overline{k}$ and $b_p \neq 0$. Let us first assume that $p \neq 0$. Then

$$\begin{aligned} 0 = u' &= b_p' t^p + b_{p+1}' t^{p+1} + \cdots + p b_p (a_2 t^{p+1} + a_3 t^{p+2} + \cdots) + \cdots \\ &= b_p' t^p + (b_{p+1}' + p b_p a_2) t^{p+1} + \cdots . \end{aligned}$$

Note that $C_{\overline{k}} = C_k$ as C_k is algebraically closed. Therefore comparing the coefficients of t^p and t^{p+1} we get that $b_p \in C_k$ and $b_{p+1}' = -p b_p a_2$. Note that a_2 does not have an antiderivative in $\overline{k}$ as it does not have any antiderivative in k. Therefore we have a contradiction. So let us assume that $p = 0$. Then u can be written as follows:

$$u = b_0 + b_1 t + b_2 t^2 + \cdots$$

Taking the derivative, we have

$$\begin{aligned} 0 = u' = {} & b_0' + b_1' t + b_2' t^2 + \cdots \\ & + b_1(a_2 t^2 + \cdots + a_n t^n) + 2b_2(a_2 t^3 + \cdots + a_n t^{n+1}) + \cdots . \end{aligned}$$

Therefore comparing the coefficients of t^0 and t we get $b_0' = 0,\ b_1' = 0$ and for $i \geq 2$ we obtain the following:

$$\begin{aligned} & b_i' + (i-1)b_{i-1}a_2 + (i-2)b_{i-2}a_3 + \cdots + b_1 a_i = 0, \text{ if } i \leq n \text{ and} \\ & b_i' + (i-1)b_{i-1}a_2 + (i-2)b_{i-2}a_3 + \cdots + (i-n+1)b_{i-n+1}a_n = 0, \text{ if } i > n. \end{aligned}$$

Since $b_0, b_1 \in C_k$ and a_2 does not have an antiderivative in $\overline{k}$, we can successively show that $b_i = 0$ and $b_{i+1}' = 0$ for $i \geq 1$. Thus $u = b_0 \in C_k$. This proves the first part.

Now we will prove the second part. Suppose that there is an element $\eta \in k(t) \setminus k$ such that $\eta' = a_2$. Let the $t-$adic expansion of η be $\eta = \sum_{i=p}^{\infty} d_i t^i$, where $d_i \in \overline{k}$ and $d_p \neq 0$. Differentiating η we obtain

$$a_2 = \eta' = \sum_{i=p}^{\infty} d_i' t^i + \sum_{i=p}^{\infty} i d_i \left(a_2 t^{i+1} + a_3 t^{i+2} + \cdots + a_n t^{i+n-1}\right). \tag{6.6}$$

If $p = 0$, then $d_p' = a_2$, which contradicts our hypothesis. If $p > 0$ then comparing the coefficient of t^0 we obtain $a_2 = 0$, which again contradicts our hypothesis that a_2 does not have an antiderivative in k. So let us assume that $p < 0$. We will show that $p = -1$.

Suppose that $p < -1$. Then $p + 1 < 0$ and comparing the coefficient of t^p and t^{p+1} we get that $d_p' = 0$ and $d_{p+1}' + p d_p a_2 = 0$ respectively. Note that $d_p \in C_k \setminus \{0\}$.

This contradicts the fact that a_2 has no antiderivative in k. Therefore $p = -1$. Substituting $p = -1$ in Equation (6.6) we get that

$$a_2 = \eta' = d'_{-1}t^{-1} + d'_0 + d'_1 t + \cdots +$$
$$- d_{-1}(a_2 + a_3 t + \cdots + a_n t^{n-2}) + d_1(a_2 t^2 + a_3 t^2 + \cdots + a_n t^n) + \cdots .$$

Comparing the coefficient of t^{-1}, t^0 and t we get

$$d'_{-1} = 0, \quad d'_0 = (d_{-1} + 1)a_2, \quad d'_1 = d_{-1}a_3.$$

This contradicts our hypothesis that both a_2 and a_3 do not have antiderivatives in k. Similarly, if η' were a_3, we would get a contradiction. $\square$

Therefore if we adjoin indeterminates $t_1, t_2, \ldots$ to k and define $t'_i = a_n t_i^n + \cdots + a_2 t_i^2$, where both a_2 and a_3 have no antiderivatives in k, then $k(t_1, t_2, \ldots)$ is an no new constant extension of k. Thus the differential equation has infinitely many algebraically independent transcendental solutions.

6.3.2 Differential equations of nongeneral type

In this section we will study the algebraic dependence of first order differential equations of nongeneral type. We will show that a differential equation of nongeneral type can have only finitely many algebraically independent transcendental solutions, unlike differential equation (6.5). Let L be a no new constant extension of k.

Suppose that L contains a transcendental solution y of the equation $y' = by$, where $0 \neq b \in k$. Then the solution set of the differential equation is $V = \mathrm{span}_C \{y\}$. Thus all the solutions are $C-$linearly dependent.

Suppose that L contains transcendental solutions of the differential equation $y' = by + c$ for $b, c \in k$ and $c \neq 0$. Let E be a Picard-Vessiot extension of L for $M =$

$k[\partial]/k[\partial]\mathscr{L}$, where $\mathscr{L} = \partial^2 - (b + (c'/c))\partial + b(c'/c) - b'$ is obtained by homogenizing $y' = by + c$. Let $V \subset E$ be the set of all solutions of $\mathscr{L}(y) = 0$. Then $\dim_C V = 2$. The differential field $\mathcal{E} := k\langle V \rangle$ is a Picard-Vessiot extension of k (for M). For any $y \in L$ such that $y' = by + c$, we see that $\mathscr{L}(y) = 0$ and therefore $y \in V \subset \mathcal{E}$. Moreover, for any automorphism $\sigma \in \mathscr{G}(\mathcal{E}|k)$, we have $\mathscr{L}(\sigma(y)) = \sigma(\mathscr{L}(y)) = 0$ and that $(\sigma(y) - y)' = b(\sigma(y) - y)$. Let $\tau \in \mathscr{G}(\mathcal{E}|k)$ be an automorphism such that $\tau(y) \neq y$. Then since $\mathscr{L}(\tau(y) - y) = \mathscr{L}(\tau(y)) - \mathscr{L}(y) = 0$, $\{\tau(y) - y, y\}$ is a $C-$basis of V. Therefore $\mathcal{E} = k(\tau(y), y)$, where the fields $k(\tau(y) - y)$ and $k(y)$ are differential fields. This gives us $\mathrm{tr.deg}(\mathcal{E}|k) \leq 2$ and thus any three solutions in L of $y' = by + c$, where $b, c \in k$ and $c \neq 0$, must be $k-$algebraically dependent.

Suppose that the Riccati equation

$$y' = ay^2 + by + c \text{ for } a, b, c \in k \text{ with } a \neq 0 \tag{6.7}$$

has a transcendental solution in L. Let

$$\mathscr{L} = \partial^2 - \left(\frac{a'}{a} + b\right)\partial + ac$$

and E be a Picard-Vessiot extension of L for $M = k[\partial]/k[\partial]\mathscr{L}$. Let $V \subset E$ be the set of all solutions of $\mathscr{L}(Y) = 0$. Then $v \in V \setminus \{0\}$ if and only if $-v'/av$ is a solution of the Riccati equation (6.7). Note that $\mathcal{E} := k\langle V \rangle$ is a Picard-Vessiot extension of k for M. We will show that that every solution in L of (6.7) lies in $\mathcal{E}$.

Let $u \in L$ be a solution of the Riccati equation (6.7). Let L^* be a Picard-Vessiot extension of E for the differential equation $Y' = -auY$. Then $L^* = E(z)$ for some nonzero z such that $z' = -auz$. Note that $V \subset \mathcal{E} \subseteq L^*$ and that V is a two dimensional vector space over C. Since C is the field of constants of L^* and $\mathscr{L}(z) = 0$, we must have $z \in V \subset \mathcal{E}$. Consequently, $u = -z'/(az) \in \mathcal{E}$.

Let $\Gamma \subset \mathcal{E}$ be the set of all transcendental solutions of Equation (6.7). Now we will show that $\mathrm{tr.deg}(k(\Gamma)|k) \leq 3$. Since $\dim_C V = 2$, $\mathscr{G}(\mathcal{E}|k)$ is a closed subgroup of the algebraic group $\mathrm{GL}(V)$. Now dim $\mathrm{GL}(V) = 4$, therefore if $\mathscr{G}(\mathcal{E}|k)$ is a proper closed subgroup then dim $\mathscr{G}(\mathcal{E}|k) \leq 3$. In which case, $\mathrm{tr.deg}(k(\Gamma)|k) \leq \mathrm{tr.deg}(\mathcal{E}|k) = \dim \mathscr{G}(\mathcal{E}|k) \leq 3$. Now we consider the case where $\mathscr{G}(\mathcal{E}|k) = \mathrm{GL}(V)$. Let $\{y_1, y_2\}$ be a $C-$basis of V. We identify $\mathscr{G}(\mathcal{E}|k)$ with $\mathrm{GL}(2, C)$. Consider the differential field $K = \mathcal{E}^{\mathcal{Z}}$, where $\mathcal{Z}$ is the center of $\mathrm{GL}(2, C)$. Observe that for any nonzero solution $v = c_1y_1 + c_2y_2 \in V$ and any automorphism $\tau \in \mathcal{Z}$, we have $\tau(v) = c_\tau v$. Therefore $v'/v \in K$ and $\Gamma \subset K$. Since $\mathcal{Z}$ is normal, K is a Picard-Vessiot extension of k with Galois group

$$\mathscr{G}(K|k) \cong \mathscr{G}(\mathcal{E}|k)/\mathcal{Z} \cong \mathrm{PGL}(2, C).$$

Thus $\mathrm{tr.deg}(k(\Gamma)|k) \leq \mathrm{tr.deg}(K|k) = \dim \mathrm{PGL}(2, C) = 3$. This implies that any four distinct elements in Γ must be algebraically dependent over k. We would like to point out that the Riccati equation $y' = -y^2 + x$ over the differential field $\mathbb{C}(x)$ has exactly three $\mathbb{C}(x)-$algebraically independent solutions in any Picard-Vessiot extension of $\mathbb{C}(x)$ (see [23, Example 4.29]).

Suppose that L has a transcendental element y satisfying a Weierstrass differential equation; $y'^2 = \alpha^2(4y^3 - g_2y - g_3)$. Then since L is a no new constant extension of k, from [17, Lemma 2], any other transcendental element z such that $z'^2 = \alpha^2(4z^3 - g_2z - g_3)$ must belong to the field $k(y, y')$. We summarize the above discussions in the following theorem.

Theorem 6.3.6. *Let L be a no new constant extension of k.*

(i) If an autonomous differential equation over C is not of general type then there is at most one $C-$algebraically independent solution of the equation in L.

(ii) If a first order differential equation over k is not of general type then there are at most three $k-$algebraically independent solutions of the equation in L.

Proof. Let $f(y, y')$ be a differential equation of nongeneral type. If the differential equation is of algebraic type then all the solutions are algebraic and the theorem holds. Suppose that t is a transcendental solution in an iterated strongly normal extension of k. Let $s_1, s_2, s_3, s_4 \in L$ be distinct transcendental solutions. If necessary, we substitute k with a finite algebraic extension of k to ensure that $k(t, t')$ contains a transcendental solution of a Riccati or Weierstrass differential equation over k. Now for $i = 1, 2, 3, 4$, consider the natural differential embeddings

$$\psi_i : k(t, t') \to L, \text{ where } \psi_i|_k = \text{id and } \psi_i(t) = s_i.$$

Let us first consider the case where there is a transcendental solution $y \in k(t, t') \setminus k$ of a Riccati differential equation over k, say $y' = a_2 y^2 + a_1 y + a_0$. Then $\psi_i(y)' = a_2 \psi_i(y)^2 + a_1 \psi_i(y) + a_0$ for each $i = 1, 2, 3, 4$. Then as noted earlier, $\text{tr.deg}(k(\psi_1(y), \dots, \psi_4(y))|k) \leq 3$. Since $\psi_i(y) \in k(s_i)$, each s_i is algebraic over $k(\psi_i(y)) \subseteq k(\psi_1(y), \dots, \psi_4(y))$. Therefore $s_1, \dots, s_4$ are $k-$algebraically dependent.

Now consider the case where $y \in k(t, t') \setminus k$ is a transcendental solution of a Weierstrass differential equation, say $y'^2 = \alpha^2(4t^3 - g_2 t^2 - g_3)$, where $\alpha \in k$. For $i = 1, 2$, we have $(\psi_i(y))'^2 = \alpha^2(4(\psi_i(y))^3 - g_2\psi_i(y) - g_3)$. Again from [17, Lemma 2], we have

$$(1 : \psi_1(y) : \psi_1(y)'/\alpha) = (1 : \psi_2(y) : \psi_2(y)'/\alpha)(1 : c_1 : c_2), \text{ where } c_1, c_2 \in C.$$

This implies $\psi_1(k(y, y')) = \psi_2(k(y, y'))$ and we conclude that s_1 and s_2 are algebraically dependent over k. Consequently, in this case, any two transcendental solutions of $f(y, y') = 0$ are $k-$algebraically dependent.

Now we will verify (i). Only the case when there is an element in $y \in C(t, t')$ such that $y' = 1$ or $y' = cy$ for some nonzero constant c needs to be taken into account. Then, $\psi_i(y)' = 1$ or $\psi_i(y)' = c\psi_i(y)$. Thus, $\psi_1(y) = \psi_2(y) + e$ for some $e \in C$ or

$\psi_1(y) = e\psi_2(y)$ for some nonzero $e \in C$. Therefore $\psi_1(C(y)) = \psi_2(C(y))$ and we obtain that s_1 and s_2 are algebraically dependent over C. $\square$

6.3.3 Rational autonomous differential equations

In this section, we will discuss about rational autonomous differential equations:

$$y' = f(y), \text{ where } f(y) \in C(y) \setminus \{0\}\,.$$

First, we note that the differential field $C(y)$, where y is transcendental over C and $y' = f(y)$, is a no new constant extension of C (by Lemma 2.2.1 (ii)). Therefore the differential equation $y' = f(y)$ always has a transcendental solution over C. Since $C(y)$ is a genus zero extension of C, there is no weierstrassian element in $C(y) \setminus C$.

In the following proposition, we completely classify the type of a rational autonomous differential equation.

Proposition 6.3.7. *The following statements are equivalent.*

(i) An autonomous differential equation over C of the form $y' = f(y)$ has a transcendental solution in an iterated strongly normal extension of C, that is the equation is of nongeneral type.

(ii) There is a nonzero element $z \in C(y)$ such that either $z' = 1$ or $z' = cz$ for some $c \in C \setminus \{0\}$, that is,

$$\frac{1}{f(y)} = \frac{\partial z}{\partial y} \quad \text{or} \quad \frac{1}{f(y)} = \frac{1}{cz}\frac{\partial z}{\partial y}.$$

Proof. Follws from Theorem 6.2.5 and Proposition 2.2.7. $\square$

Therefore the rational autonomous equation $y' = f(y)$ is of nongeneral type if and only if either $1/f(y)$ has no residues at any element of C, that is, the partial fraction decomposition of $1/f(y)$ is of the form

$$h(y) + \sum_{i=1}^{n} \sum_{j=2}^{n_i} \frac{d_{ij}}{(y-c_i)^j}, \tag{6.8}$$

where $h(y) \in C[y]$, d_{ij} are constants and c_i are distinct constants, or $1/f(y)$ is of the form

$$c \sum_{i=1}^{n} \frac{m_i}{(y-c_i)}, \tag{6.9}$$

where m_i are nonzero integers and c is a nonzero constant.

Example 6.3.8. The equations $y' = y^3 - y^2$ and $y' = \dfrac{y}{y+1}$ were heavily discussed in [32] and [39], are now easily seen to be of general type.

Example 6.3.9. The differential equation $y' = y^n - 1$, for $n \geq 3$ is of general type. To see this, let $\xi \in C$ be a primitive $n - th$ root of unity and consider the partial fraction expansion of $1/(y^n - 1)$:

$$\sum_{i=0}^{n-1} \frac{\alpha_i}{y - \xi^i}, \quad \text{where } \alpha_i = \frac{1}{\prod_{j=1, j\neq i}^{n-1} (\xi^i - \xi^j)}.$$

If the differential equation admits a non-constant solution in a strongly normal extension, then α_i/α_j must be a nonzero rational number for all i, j. However,

$$\frac{\alpha_1}{\alpha_0} = \frac{\prod_{i=1}^{n-1}(1-\xi^i)}{\prod_{i=0, i\neq 1}^{n-1}(\xi - \xi^i)} = \frac{1-\xi^{n-1}}{(\xi-1)\xi^{n-2}} = \xi,$$

which is not a rational number.

The following conjecture is stated in the introduction. Now we will verify it for the class of rational autonomous differential equations. In the other direction, we have seen that a differential equation of general type may have infinitely many transcendental solutions (Proposition 6.3.5).

Conjecture A first order differential equation (respectively, An autonomous differential equation) over k (respectively, over C) is not of general type if and only if it has at most three (respectively, one) $k-$algebraically independent (respectively, $C-$algebraically independent) solutions in any given no new constant extension of k (respectively, C).

Let us first consider the case where f has no zero in C. Then $1/f$ is a polynomial over C and has the form (6.8). Thus if $C(y)$ is a transcendental extension of C with $y' = f(y)$, where $1/f(y) \in C[y]$, then by the above proposition, $C(y)$ has a nonzero element z such that $z' = 1$. Therefore rational autonomous equations; $y' = f(y)$ with f having no zeros in C, are of nongeneral type.

Suppose that $\alpha \in C$ is a zero of f. Consider the rational function $g \in C(y)$ defined by $f(y) := g(y - \alpha)$. Then, we have

$$(y - \alpha)' = y' = f(y) = g(y - \alpha).$$

Thus, given a nonzero rational autonomous differential equation $y' = f(y)$ with f having a zero in C, we may assume that f has a zero at $y = 0$. The conjecture is therefore proved true by Theorem 6.3.6 provided we establish the following: If an autonomous differential equation

$$y' = f(y), \quad f \neq 0 \quad \text{and} \quad f(0) = 0$$

has at most one transcendental solution in any given no new constant L of C then the equation is not of general type.

Consider the purely transcendental differential field extension $C(t, y)$ of C, where $t' = f(t)$ and $y' = f(y)$. From our hypothesis, there is an element $u \in C(t, y) \setminus C(t)$ such that $u' = 0$. Then we have following equations

$$y' = f(y) = \sum_{i=m}^{\infty} c_i y^i \quad \text{and} \quad u = \sum_{i=p}^{\infty} b_i y^i,$$

where $m \geq 1$, $c_i \in C$ for all $i \geq m$ and $c_m \neq 0$, p is an integer and $b_i \in \overline{C(t)}$ for all $i \geq p$ and $b_p \neq 0$. Taking derivatives, we obtain

$$0 = u' = \sum_{i=p}^{\infty} b_i' y^i + \left(\sum_{i=m}^{\infty} c_i y^i \right) \left(\sum_{i=p}^{\infty} i b_i y^{i-1} \right).$$

We observe that

(i) If $p = 0$ and $m \geq 2$ then letting l be the least positive integer such that $b_l \neq 0$, we get $b_l' = 0$ and that $b_{l+m-1}' = -l c_m b_l \in C \setminus \{0\}$.

(ii) If $p \neq 0$ and $m \geq 2$ then $b_p' = 0$ and $b_{p+m-1}' = -p b_p c_m \in C \setminus \{0\}$.

(iii) If $p = 0$ and $m = 1$ then letting l be the least positive integer such that $b_l \neq 0$, we get $b_l'/b_l = -l c_1 \in C \setminus \{0\}$.

(iv) If $p \neq 0$ and $m = 1$ then $b_p'/b_p = -p c_1 \in C \setminus \{0\}$.

Thus, in the event that (i) or (ii) holds, we obtain an element $z \in C(t)$ such that $z' = 1$ and in the event that (iii) or (iv) holds, we obtain an element $z \in C(t) \setminus \{0\}$ such that $z' = cz$ for some $c \in C \setminus \{0\}$. Now from Proposition 6.3.7, the equation $y' = f(y)$ is not of general type. Thus, we have verified the conjecture for the class of rational autonomous differential equations.

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