Naively $\mathbb{A}^1$-Connected Components of Varieties

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To my parents and my bungdi

Declaration

The work presented in this thesis has been carried out by me under the guidance of Dr. Chetan Tukaram Balwe at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Bandna Rani

In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Chetan Tukaram Balwe

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Abstract

$\mathbb{A}^1$-homotopy theory is a homotopy theory for schemes in which the affine line $\mathbb{A}^1$ plays the role of the unit interval. The main objects of study are simplicial sheaves on the Nisnevich site of smooth schemes of finite type over a field. For these objects, one constructs analogues of various devices from the classical homotopy theory of topological spaces. One such device is the sheaf of $\mathbb{A}^1$-connected components of a simplicial sheaves.

For a general simplicial sheaf $\mathcal{X}$, the sheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ of $\mathbb{A}^1$-connected components of $\mathcal{X}$ is generally hard to compute. However, one can attempt to study it by means of the sheaf of naively $\mathbb{A}^1$-connected components, denoted by $\mathcal{S}(\mathcal{X})$. The sheaf $\mathcal{S}(\mathcal{X})$ may be viewed as a crude approximation to $\pi_0^{\mathbb{A}^1}(\mathcal{X})$, but it is easier to define and compute, at least when $\mathcal{X}$ is a sheaf of sets. The functor $\mathcal{S}$ is the main object of study in this thesis.

When $\mathcal{X}$ is a sheaf of sets, the direct limit of the sheaves $\mathcal{S}^n(\mathcal{X})$, which we denote by $\mathcal{L}(\mathcal{X})$ can be proved to be $\mathbb{A}^1$-invariant. In fact, this is the universal $\mathbb{A}^1$-homotopic quotient of $\mathcal{X}$. When $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-invariant, it can be proved to be isomorphic to $\mathcal{L}(\mathcal{X})$. A recent example of Ayoub has shown that $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is not always $\mathbb{A}^1$-invariant. However, we show that there is a natural bijection between field valued points of the sheaves $\mathcal{L}(\mathcal{X})$ and $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ for any sheaf of sets $\mathcal{X}$.

The sheaf $\mathcal{L}(\mathcal{X})$ is obtained by iterating $\mathcal{S}$ on a the sheaf $\mathcal{X}$ infinitely many times. Our second main result is to show that the infinitely many iterations are indeed necessary. We achieve this by constructing a family of sheaves $\{\mathcal{X}_n\}_n$, indexed by the positive integers, such that $\mathcal{S}^i(\mathcal{X}_n) \neq \mathcal{S}^{i+1}(\mathcal{X}_n)$ for any $i < n$.

The third main result of this thesis is regarding retract rational varieties over an infinite field k. A result of Kahn and Sujatha shows that for a retract rational variety X, the sheaf $\pi_0^{\mathbb{A}^1}(X)$ is the point sheaf. We strengthen this result by showing that $\mathcal{S}(X)$ is the point sheaf.

Notations

$Mor_{\mathcal{C}}(X, Y)$	Morphisms from an object X to an object Y of a category $\mathcal{C}$.
$\kappa(x)$	Residue field at a point x of a scheme
$\mathcal{O}_{X,x}$	Local ring at a point x of a scheme X
R^h	Henselization of a ring with respect a specified ideal.
$\mathcal{O}^h_{X,x}$	Henselization of the ring $\mathcal{O}_{X,x}$ with respect to its maximal ideal
X_x	$\operatorname{Spec} \mathcal{O}^h_{X,x}$ where x is a point on a scheme X
$\mathcal{Z}(I)$	Closed subscheme associated to an ideal (or ideal sheaf) I

Contents

Chapter 1

Introduction

1.1 $\mathbb{A}^1$-homotopy theory

Much of the development of algebraic geometry has involved adopting techniques from algebraic topology. In the context of topological spaces, certain invariants of interest, such as the fundamental group, are observed to be preserved under homotopy equivalences. This has led to the development of homotopy theory. The homotopy category of topological spaces provides us with the natural framework to study functors that are homotopy invariant, i.e. functors $\mathcal{F}$ on the category of topological spaces such that the morphism $\mathcal{F}(X) \to \mathcal{F}(X \times [0,1])$, induced by the projection $X \times [0,1] \to X$, is an isomorphism. In the context of algebraic geometry, the analogous notion is that of $\mathbb{A}^1$-invariance. We say that a contravariant functor $\mathcal{G}$, defined on some suitable category of schemes, is $\mathbb{A}^1$-*invariant* if for any object X of the category, the morphism $\mathcal{G}(X) \to \mathcal{G}(X \times \mathbb{A}^1)$, induced by the projection morphism $X \times \mathbb{A}^1 \to X$, is an isomorphism. Many interesting functors, such as Chow groups, motivic cohomology, étale cohomology, K-theory, etc. exhibit the property of $\mathbb{A}^1$-invariance. This has led to the development of $\mathbb{A}^1$-homotopy theory, which is a homotopy theory for algebraic varieties where the affine line $\mathbb{A}^1$ plays the role of the unit interval.

The foundations of this theory were laid by Morel and Voevodsky in [24].

For a finite-dimensional noetherian scheme S, they constructed the $\mathbb{A}^1$-homotopy category $\mathcal{H}(S)$ which provides an appropriate framework for homotopy-theoretic constructions for schemes over S.

Let Sm/S denote the Nisnevich site of smooth schemes of finite type over S. Let $\Delta^{op}Shv(Sm/S)$ denote the category of simplicial sheaves over this site. A simplicial version of the Yoneda lemma allows us to embed Sm/S into $\Delta^{op}Shv(Sm/S)$. The category $\Delta^{op}Shv(Sm/S)$ has a model structure, called the *locally injective model structure*. The $\mathbb{A}^1$-*model structure* on this category is obtained by performing an appropriate process of localization of this model structure with respect to the set of all projection morphisms of the form $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$ where $\mathcal{X}$ is a simplicial sheaf.

One can construct analogues of many concepts from classical homotopy theory in this context. For example, given any simplicial sheaf $\mathcal{X}$, we can associate to it the sheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$, called the sheaf of $\mathbb{A}^1$-connected components of $\mathcal{X}$. This is the $\mathbb{A}^1$-homotopic analogue of the set of connected components of a topological space. Similarly, given a pointed sheaf $(\mathcal{X}, x)$ and any integer $i \geq 1$, we can associate to it the group sheaf $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x)$.

1.2 Sheaf of naively $\mathbb{A}^1$-connected components

In [23], Morel establishes analogues of several results from classical homotopy theory under the assumption that S is the spectrum of a perfect field.

For instance, if X is a topological space, we may interpret its set of connected components and its homotopy groups as *discrete* topological spaces. A CW-complex X is discrete if and only if any morphism from a space of the form $U \times [0,1]$ factors through the projection $U \times [0,1] \to U$. Thus, we see that an $\mathbb{A}^1$-invariant space may be seen as the $\mathbb{A}^1$-homotopic analogue of the notion of a discrete space. Thus, the following result of Morel can be seen as the $\mathbb{A}^1$-homotopic analogue of the statement that the homotopy groups of a pointed topological space are discrete topological spaces.

Fact 1.1 ([23, Theorem 1.9])**.** *Let k be a perfect field and let $(\mathcal{X},\ x)$ be a pointed simplicial sheaf over Sm/k. Then the sheaves $\pi_i^{\mathbb{A}^1}(\mathcal{X},\ x)$ are $\mathbb{A}^1$-invariant for any $i > 0$.*

Morel conjectured that for any simplicial sheaf $\mathcal{X}$, the sheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ should also be $\mathbb{A}^1$-invariant. Some evidence for this conjecture was provided by verifying the $\mathbb{A}^1$-invariance in special cases. For instance, it was proved in [10] that $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-invariant if $\mathcal{X}$ is an H-group. In [3] and[7], it is proved that if X is a smooth projective curve over an algebraically closed field of characteristic zero, then $\pi_0^{\mathbb{A}^1}(X)$ is $\mathbb{A}^1$-invariant. However, a counter-example to Morel's conjecture was found by Ayoub (see [2]).

The sheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is hard to compute for a general simplicial sheaf $\mathcal{X}$. However, one can attempt to understand it through the related notion of the sheaf of naively $\mathbb{A}^1$-connected components, which we denote by $\mathcal{S}(\mathcal{X})$. When $\mathcal{X}$ is a sheaf of sets, $\mathcal{S}(\mathcal{X})$ has a rather simple, geometric interpretation. It is simply the Nisnevich sheafification of the presheaf $U \mapsto \mathcal{X}(U)/\sim$, where $\sim$ denotes the equivalence relation on $\mathcal{X}(U)$ generated by $\mathbb{A}^1$-homotopy. (See Section 2.3 for a detailed definition.) There is a canonical sequence of epimorphisms morphism

$$\mathcal{X} \to \mathcal{S}(\mathcal{X}) \to \pi_0^{\mathbb{A}^1}(\mathcal{X}).$$

This sheaf $\mathcal{S}(\mathcal{X})$ was first defined in [1, Definition 2.2.4] where it was denoted by $\pi_0^{ch}(\mathcal{X})$). Asok and Morel refer to this as the sheaf of $\mathbb{A}^1$-*chain connected components of* $\mathcal{X}$. However, they define (see [1, 2.2.2]) a variety X to be $\mathbb{A}^1$-chain connected if $\pi_0^{\mathbb{A}^1}(X)(L) = *$ for any finitely generated, separable field extension L/k. This is weaker than the notion of naive $\mathbb{A}^1$-connectedness that we wish to study. Hence, in order to avoid confusion, we will refer to $\mathcal{S}(\mathcal{X})$ as the sheaf of naively $\mathbb{A}^1$-connected components of $\mathcal{X}$. We will say that $\mathcal{X}$ is *naively* $\mathbb{A}^1$*-connected* if $\mathcal{S}(\mathcal{X}) = *$.

The following result of Asok and Morel is the first example of how the sheaf $\mathcal{S}(\mathcal{X})$ may be used to study the sheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$.

Fact 1.2 (see [1, 2.4.3]). *Let X be a proper variety over a field k. Then, for any finitely generated separable field extension L/k, the map $\mathcal{S}(X)(L) \to \pi_0^{\mathbb{A}^1}(X)(L)$ is a bijection.*

Asok and Morel conjectured that $\mathcal{S}(X) \to \pi_0^{\mathbb{A}^1}(X)$ should be an isomorphism of sheaves for any proper variety X over k. A counter-example in [3] shows that this need not be so. However, one still has the following relationship between the functors $\mathcal{S}$ and $\pi_0^{\mathbb{A}^1}$ in general.

Fact 1.3 (see [3, Theorem 1]). *Let $\mathcal{X}$ be a sheaf of sets over Sm/k. Let $\mathcal{L}(\mathcal{X})$ denote the direct limit $\varinjlim \mathcal{S}^n(\mathcal{X})$. Then, $\mathcal{L}(\mathcal{X})$ is an $\mathbb{A}^1$-invariant sheaf and we have a canonical factorization*

$$\mathcal{X} \to \pi_0^{\mathbb{A}^1}(\mathcal{X}) \to \mathcal{L}(\mathcal{X}).$$

The canonical morphism $\pi_0^{\mathbb{A}^1}(\mathcal{X}) \to \mathcal{L}(\mathcal{X})$ is an isomorphism if and only if $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-invariant.

The sheaf $\mathcal{L}(\mathcal{X})$ is the *universal $\mathbb{A}^1$-homotopic quotient of $\mathcal{X}$* in the sense that the morphism $\mathcal{X} \to \mathcal{L}(\mathcal{X})$ is the initial object in the category of all morphisms of the form $\mathcal{X} \to \mathcal{Z}$ where $\mathcal{Z}$ is an $\mathbb{A}^1$-invariant sheaf of sets.

As we noted above, Ayoub's counter-example shows that $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ need not always be $\mathbb{A}^1$-invariat and thus $\pi_0^{\mathbb{A}^1}(\mathcal{X}) \to \mathcal{L}(\mathcal{X})$ need not always be an isomorphism. However, we will prove the following result:

Theorem 1.4 (see 3.3). *Let $\mathcal{X}$ be a sheaf of sets on Sm/k. For any finitely generated, separable field extension L/k, the natural map $\pi_0^{\mathbb{A}^1}(\mathcal{X})(L) \to \mathcal{L}(\mathcal{X})(L)$ is a bijection.*

Theorem 1.4 may be seen as a generalization of Fact 1.2 to arbitrary sheaves of sets, but where the functor $\mathcal{S}$ has to be replaced by $\varinjlim \mathcal{S}^n$. (If X is a proper variety over a field k, we have a bijection $\mathcal{S}(X)(L) \to \mathcal{L}(X)(L)$ for any separable, finitely generated field extension L/k – see [3, Theorem 3.9].) We show, by constructing a sequence of examples that the iterations of $\mathcal{S}$ in Theorem 1.4 is indeed necessary.

Theorem 1.5. *Let k be a field. There exists a sequence $\mathcal{X}_n$ of $\mathbb{A}^1$-connected sheaves of sets over Sm/k such that $\mathcal{S}^{n+1}(\mathcal{X}_n) = \pi_0^{\mathbb{A}^1}(\mathcal{X}_n)$ is the trivial one-point sheaf, but $\mathcal{S}^i(\mathcal{X}_n) \neq \mathcal{S}^{i+1}(\mathcal{X}_n)$, for every $i < n+1$.*

1.3 Near-rationality properties

Let X be a variety over a field k. We say that two points $x, y \in X(k)$ are *elementarily R-equivalent* if there exists a rational map $\phi : \mathbb{A}^1_k \dashrightarrow X$ such that ϕ is defined at 0 and 1, and satisfies $\phi(0) = x$ and $\phi(1) = y$.

The relation of being elementarily R-equivalent generates an equivalence relation on $X(k)$ which we call as *R-equivalence*. The set of R-equivalence classes of $X(k)$ will be denoted by $X(k)/R$. We say that a variety X is R-trivial over k if $X(k)/R = *$. We say that X is *universally R-trivial* if for any finitely generated, separable field extension L/k, the variety $X_L := X \times_{\operatorname{Spec} k} \operatorname{Spec} L$ is R-trivial over L.

If X is a proper variety over k, a rational map $\mathbb{A}^1_k \dashrightarrow X$ actually extends to a morphism $\mathbb{A}^1_k \to X$. Thus, in this case, to say that X is R-trivial over k is equivalent to the condition $\mathcal{S}(X)(k) = *$. Thus, X is universally R-trivial if and only if $\mathcal{S}(X)(L) = *$ for any finitely generated, separable field extension L/k. By Fact 1.2, this condition is equivalent to saying that $\pi_0^{\mathbb{A}^1}(X)(L) = *$ for any finitely generated, separable field extension L/k. A result of Morel (see [21, Lemma 3.3.6] and [22, Lemma 6.1.3]) shows that this is equivalent to saying that $\pi_0^{\mathbb{A}^1}(X) = *$. Thus, we see that a proper variety over a field k is universally R-trivial if and only if it is $\mathbb{A}^1$-connected.

Thus, for proper varieties, $\mathbb{A}^1$-connectedness has a very simple algebro-geometric characterization. Since smooth, proper rational varieties are easily seen to be $\mathbb{A}^1$-

connected, it is natural to compare the notion of $\mathbb{A}^1$-connectedness to the notion of rationality as well as its weaker variants (such as stable rationality, retract rationality, unirationality, etc.). Note that, for such a comparison, it is important to assume that the variety is smooth, and not just proper over k. Indeed a singular rational variety need not be $\mathbb{A}^1$-connected in general. (Consider, for instance, the curve cut out by the homogeneous polynomial $Y^2Z - X^3 + X^2Z$ in $\mathbb{P}^2_{\mathbb{R}}$, which is rational but not $\mathbb{A}^1$-connected over $\mathbb{R}$ (see [4, Remark 2.3]).

It was proved by Asok and Morel (see [1, Theorem 2.3.6]) that if k is a field of characteristic 0, then retract rational varieties are $\mathbb{A}^1$-connected. This result was proved for arbitrary fields by Kahn and Sujatha (see [17, Theorems 8.5.1 and 8.6.2]). In the case when k is an infinite field, we have the following improvement of this result:

Theorem 1.6. *Let k be an infinite field. Let X be a smooth, proper, retract rational variety over k. Then $\mathcal{S}(X) = *$.*

Thus, if X is a smooth, proper variety over an infinite field k, we have the following implications:

$$\text{retract rational} \implies \text{naively } \mathbb{A}^1\text{-connected} \implies \mathbb{A}^1\text{-connected}$$

It is not known whether either of the above implications is strict.

The following result of Sawant provides a little more context for the second implication in the above diagram:

Fact 1.7 (see [28, Theorem 3.2])**.** *Let k be an infinite field and let $\mathcal{X}$ be a simplicial sheaf over Sm/k such that $\mathcal{S}(\mathcal{X})(L) = *$ for any finitely generated, separable field extension L/k. Then $\mathcal{S}^2(\mathcal{X}) = *$.*

Thus, if X is any $\mathbb{A}^1$-connected, smooth proper variety over an infinite field k, we see that $\mathcal{S}^2(X) = *$. However, it is not clear whether such a variety is naively $\mathbb{A}^1$-connected. It is easy to construct an example of a singular, proper variety X

which is $\mathbb{A}^1$-connected, but not naively $\mathbb{A}^1$-connected. However, no example of a smooth, proper variety with this property is known.

For a variety X, one might ask for necessary and sufficient conditions for the morphism $\mathcal{S}(X) \to \mathcal{L}(X)$ (or the morphism $\mathcal{S}(X) \to \pi_0^{\mathbb{A}^1}(X)$) to be an isomorphism. Theorem 1.6 says that if X is a smooth, proper, retract rational variety over an infinite field, then $\mathcal{S}(X) \to \mathcal{L}(X)$ (which can be identified with $\mathcal{S}(X) \to \pi_0^{\mathbb{A}^1}(X)$ in this case) is an isomorphism. This should be contrasted with the results in [7] and [8] which says that if X is a smooth, proper surface that is birationally ruled over a curve of genus > 0, then the morphism $\mathcal{S}(X) \to \mathcal{L}(X)$ is an isomorphism if and only if X is minimal.

Chapter 2

Preliminaries on $\mathbb{A}^1$-homotopy theory

In this chapter, we will review the construction of the $\mathbb{A}^1$-homotopy category. Our treatment of this subject is far from comprehensive since we focus only on the notion of $\mathbb{A}^1$-connectedness and the related notion of naive $\mathbb{A}^1$-connectedness.

2.1 The $\mathbb{A}^1$-homotopy category

Let k be a field. Let Sm/k denote the category of smooth schemes of finite type over k, equipped with the Nisnevich topology (see Appendix B). We consider the category $\mathbf{\Delta}^{op}Shv(Sm/k)$, of (Nisnevich) sheaves of simplicial sets on Sm/k. (We refer to the appendix, Section A.2 for some details regarding the category of simplicial sets.) We will refer to the objects of this category as *spaces*.

The category $\mathbf{\Delta}^{op}Shv(Sm/k)$ has a model structure, called the *injective local model structure* (see the Appendix, Section A.4). The weak equivalences for this model structure are the morphism $\mathcal{X} \to \mathcal{Y}$ of spaces which induce weak equivalences on stalks. We will refer to these as *simplicial weak equivalences*. The cofibrations for this model structure are the monomorphisms. In particular, every space is cofibrant. Fibrations are defined to be those having the right lifting property with respect to cofibrations which are also weak equivalences. We will denote the homotopy category associated with this model structure as the *simplicial homotopy category* and denote it by $\mathcal{H}_s(k)$. Let $Ex : \mathbf{\Delta}^{op}Shv(Sm/k) \to \mathbf{\Delta}^{op}Shv(Sm/k)$ be a fibrant approximation functor for this model structure.

We note that the category $\mathbf{\Delta}^{op}Shv(Sm/k)$ is actually a simplicial model category. This means that given any objects $\mathcal{X}$ and $\mathcal{Y}$, we can associate to it a simplicial set, denoted by $Map(\mathcal{X}, \mathcal{Y})$, called the simplicial mapping space from $\mathcal{X}$ to $\mathcal{Y}$ such that the set $Map(\mathcal{X}, \mathcal{Y})_0$ of 0-simplices in this simplicial set is equal to $Mor_{\mathbf{\Delta}^{op}Shv(Sm/k)}(\mathcal{X}, \mathcal{Y})$. Also, the association $(\mathcal{X}, \mathcal{Y}) \mapsto Map(\mathcal{X}, \mathcal{Y})$ satisfies certain properties that make it compatible with the model structure. We refer to the Appendix, Sections A.3 and A.4 for details.

The $\mathbb{A}^1$-*model structure* on $\mathbf{\Delta}^{op}Shv(Sm/k)$ is defined to be the left Bousfield localization with respect the set $\mathcal{A}$, of all the projection morphisms of the form $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$. (We refer to the Appendix, Section A.5 for further details regarding left Bousfield localizations.) We say that an object $\mathcal{X} \in \mathbf{\Delta}^{op}Shv(Sm/k)$, or its image in $\mathcal{H}_s(k)$, is $\mathbb{A}^1$-*local*, if it is $\mathcal{A}$-local in the sense of Section A.5. In other words, a space $\mathcal{X}$ is $\mathbb{A}^1$-local if and only if the map

$$Mor_{\mathcal{H}_s(k)}(\mathcal{X}, \mathcal{Y}) \to Mor_{\mathcal{H}_s(k)}(\mathcal{X}, \mathcal{Y} \times \mathbb{A}^1)$$

is a bijection for any space $\mathcal{Y}$. The weak equivalences in this model structure are called as $\mathbb{A}^1$-*weak equivalences*. The homotopy category corresponding to this model structure is called the $\mathbb{A}^1$-*homotopy category over* k and is denoted by $\mathcal{H}(k)$. We note the cofibrations in this model structure are the same as the ones in the locally injective model structure, i.e. the are precisely all the cofibrations. In particular, every space $\mathcal{X}$ is a cofibrant object in the $\mathbb{A}^1$-model structure as well.

Let $L_{\mathbb{A}^1}$ denote a fibrant replacement functor for the $\mathbb{A}^1$-model structure. Recall that it comes equipped with a natural transformation $\eta : Id \to L_{\mathbb{A}^1}$ such that $\eta_{\mathcal{X}} : \mathcal{X} \to L_{\mathbb{A}^1}(\mathcal{X})$ is an $\mathbb{A}^1$-equivalence for any space $\mathcal{X}$. We define the following:

1. For any space $\mathcal{X}$, let $\pi_0^{\mathbb{A}^1}(\mathcal{X}) := \pi_0(L_{\mathbb{A}^1}(\mathcal{X}))$.

2. For any integer $i > 0$ and any pointed space $(\mathcal{X}, x)$ (i.e. a space $\mathcal{X}$, along with a morphism $x : \mathrm{Spec}\,(k) \to \mathcal{X}$), we define $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x) := \pi_i(L_{\mathbb{A}^1}(\mathcal{X}), \eta_{\mathcal{X}}(x))$.

In general, it is quite difficult to compute the sheaves $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ and $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x)$. This is because it is not easy to do explicit computations involving an fibrant replacement functor for the $\mathbb{A}^1$-model structure. Morel and Voevodsky provide an explicit description of an $\mathbb{A}^1$-fibrant replacement functor, which we recall briefly in the next section.

2.2 $\mathbb{A}^1$-fibrant replacement functor

We begin by recalling the construction of the Morel-Voevodsky $Sing_*$ construction. This is the $\mathbb{A}^1$-homotopic analogue of the singular complex in classical algebraic topology.

In Appendix A.2, we construct the cosimplicial object Δ in the category of simplicial set, which mapped the object $[n]$ in the cosimplicial indexing category $\boldsymbol{\Delta}$ to the standard n-simplex Δ^n. Similarly, we constructed the cosimplicial object $|\Delta|$ in **Top** which mapped $[n]$ to the standard topological n-simplex. We perform an analogous construction in the category of smooth schemes.

In the affine $\mathbb{A}^{n+1}$, we use the coordinate functions $t_0, \ldots, t_n$, so that

$$\mathbb{A}^{n+1} = \mathrm{Spec}\, k[t_0, \ldots, t_n].$$

(Thus, we are labelling the coordinates from 0 to n instead of 1 to $n + 1$.) For $0 \le i \le n$, let e_i^n denote the point $(0, \ldots, 1, \ldots, 0)$ (having 1 in the i-th coordinate and 0 elsewhere). Then, we define $\Delta^n_{\mathbb{A}^1}$ to be the n-dimensional linear variety passing through $e_0^n, \ldots, e_n^n$. Thus,

$$\Delta^n_{\mathbb{A}^1} := \mathrm{Spec}\left(\frac{k[t_0, \ldots, t_n]}{\sum_{i=1}^n t_i = 1}\right).$$

If $f : [m] \to [n]$ is a morphism in the cosimplicial indexing category $\boldsymbol{\Delta}$, the corresponding morphism $\Delta_{\mathbb{A}^1}(f) : \Delta^m_{\mathbb{A}^1} \to \Delta^n_{\mathbb{A}^1}$ maps $\Delta^n_{\mathbb{A}^1}$ to the variety spanned by the points $e^n_{f(0)}, \ldots, e^n_{f(m)}$ so that e_i^m is mapped to the point $e^n_{f(i)}$.

Definition 2.1. Let $\mathcal{X}$ be a simplicial sheaf on Sm/k. Then, we define $Sing_*(\mathcal{X})$ to be the diagonal of the bisimplicial sheaf $\underline{Hom}(\Delta^n_{\mathbb{A}^1}, \mathcal{X})$ (where $\underline{\text{Hom}}$ denotes the internal Hom in the category of sheaves). Thus, for a smooth scheme U over k, we have

$$(Sing(\mathcal{X}))_n(U) = Mor_{Shv(Sm/k)}(U \times \Delta^n_{\mathbb{A}^1}, \mathcal{X}_n).$$

For every non-negative integer n and smooth scheme U, the projection morphism $U \times \Delta^n_{\mathbb{A}^1} \to U$ induces a map $\mathcal{X}_n(U \times \Delta^n_{\mathbb{A}^1}) \to \mathcal{X}_n(U)$. These maps give us a morphism $\mathcal{X} \to Sing_*(\mathcal{X})$, which is an $\mathbb{A}^1$-weak equivalence (see [24, page 89, Corollary 3.8]).

Let Ex denote a fibrant replacement functor for the locally injective model structure on $\boldsymbol{\Delta}^{op}Shv(Sm/k)$. An explicit example of such a functor is provided by Morel and Voevodsky in [24, page 70, Theorem 1.66]. Then Morel and Voevodsky have the following result.

Lemma 2.2 (see [24, page 107, Lemma 2.6])**.** *Let $L_{\mathbb{A}^1} : \boldsymbol{\Delta}^{op}Shv(Sm/k) \to \boldsymbol{\Delta}^{op}Shv(Sm/k)$ denote the functor*

$$L_{\mathbb{A}^1} = Ex \circ (Ex \circ Sing_*)^{\mathbb{N}} \circ Ex.$$

Then, for any simplicial sheaf $\mathcal{X}$, the simplicial sheaf $L_{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-fibrant. The canonical morphism $\mathcal{X} \to L_{\mathbb{A}^1}(\mathcal{X})$ is a trivial cofibration for the $\mathbb{A}^1$-model structure.

2.3 Naively $\mathbb{A}^1$-connected components

Definition 2.3. Let $\mathcal{X}$ be a simplicial sheaf on Sm/k. Then, we define $\mathcal{S}(\mathcal{X})$ to be the sheaf $\pi_0(Sing_*(\mathcal{X}))$.

We will only be interested in this functor in the case when $\mathcal{X}$ is a sheaf of sets. In this situation, $\mathcal{S}(\mathcal{X})$ has a somewhat simpler description. For any scheme U, we have two morphisms $\sigma_0, \sigma_1 : U \to U \times \mathbb{A}^1$ which map U isomorphically onto the closed subschemes $U \times \{0\}$ and $U \times \{1\}$ of $U \times \mathbb{A}^1$ respectively. Correspondingly, we have the two restriction morphisms $\sigma_0^*, \sigma_1^* : \mathcal{X}(U \times \mathbb{A}^1) \to \mathcal{X}(U)$. We say that two elements f and g of $\mathcal{X}(U)$ are $\mathbb{A}^1$-*homotopic* if there exists an element $h \in \mathcal{X}(U \times \mathbb{A}^1)$ such that $\sigma_0^*(h) = f$ and $\sigma_1^*(h) = g$. This is a symmetric and reflexive relation on $\mathcal{X}(U)$, but fails to be transitive in general. The equivalence relation on $\mathcal{X}(U)$ generated by this relation is called $\mathbb{A}^1$-*chain homotopy*. Let us denote this equivalence relation by $\sim_U$. It is easy to check that $\mathcal{S}(\mathcal{X})$ is the Nisnevich sheaf associated to the presheaf $U \mapsto \mathcal{X}(U)/\sim_U$.

Using the description of the $\mathbb{A}^1$-fibrant approximation functor in Lemma 2.2, we see that there exists a natural transformation $Sing_* \to L_{\mathbb{A}^1}$. For any object $\mathcal{X}$, the morphism $Sing_*(\mathcal{X}) \to L_{\mathbb{A}^1}(\mathcal{X})$ is an $\mathbb{A}^1$-weak equivalence. Applying the functor $\pi_0(-)$, we obtain a natural transformation $\mathcal{S}(-) \to \pi_0^{\mathbb{A}^1}(-)$ which is generally not an isomorphism. However, if $Sing_*(\mathcal{X})$ is $\mathbb{A}^1$-local then the morphism $Sing_*(\mathcal{X}) \to L_{\mathbb{A}^1}(\mathcal{X})$ is a simplicial weak equivalence. (If $\mathcal{M}$ is a model category and $\mathcal{A}$ is a set of morphisms in $\mathcal{M}$ such that the left Bousfield localization with respect to $\mathcal{A}$ exists, it is easy to prove that a morphism between $\mathcal{A}$-local objects is an $\mathcal{A}$-local equivalence if and only if it is a weak equivalence.) It follows that the canonical morphism $\mathcal{S}(\mathcal{X}) \to \pi_0^{\mathbb{A}^1}(\mathcal{X})$ is an isomorphism if $Sing_*(\mathcal{X})$ is $\mathbb{A}^1$-local. (This argument was used in [3], [6] and [7] to show that $Sing_*$ of certain varieties is not $\mathbb{A}^1$-local.)

Chapter 3

Universal $\mathbb{A}^1$-homotopic quotient

In this chapter, we prove that for any sheaf of sets $\mathcal{F}$, the morphism $\pi_0^{\mathbb{A}^1}(\mathcal{F}) \to \mathcal{L}(\mathcal{F})$ induces a bijection on L-valued points where L/k is any finitely generated, separable field extension. In Section 3.1, we prove that any Nisnevich cover of a smooth curve can be refined by an elementary Nisnevich cover. In Section 3.2, we explain how sections of a simplicial sheaf over an elementary Nisnevich cover of a smooth variety X may be glued to obtain a section over X. In Section 3.3, we present the proof of Theorem 1.4.

3.1 Nisnevich covers of curves

As discussed in Appendix B, it is easier to construct sections of a Nisnevich sheaf over an elementary Nisnevich cover. In this section, we will see that if we want to construct sections over a curve, we can always reduce the situation to that of an elementary Nisnevich cover.

Lemma 3.1. *Let C be a smooth curve and let $p : V \to C$ be a Nisnevich cover. Then, there exists an elementary Nisnevich cover $(p_1 : V_1 \to C, p_2 : V_2 \to C)$ refining the cover p, in the sense that there exists an open immersion $V_1 \coprod V_2 \to V$ such that the composition $V_1 \coprod V_2 \to V \xrightarrow{p} C$ is equal to $p_1 \coprod p_2$.*

Proof. Let $\eta : \operatorname{Spec} L \to C$ be the generic point of C. Then, η lifts to V and thus there exists an open immersion $U \hookrightarrow C$ which lifts to V. In fact, the lifting morphism maps U into one of the components of V, which we denote by V_1. Then $p|_{V_1} : V_1 \to C$ is a birational étale map, and so it is an open immersion. We write $V = V_1 \coprod V_2'$ and denote the morphism $p|_{V_1}$ by p_1.

The set $Z := C \backslash p_1(V_1)$ consists of a finite number of closed points of C. Each of those points can be lifted to V_2'. Suppose $Z = \{z_1, \ldots, z_m\}$. For $1 \leq i \leq m$, we pick a point $y_i \in V_2$ lifting z_i. We define

$$V_2 = \left[V_2' \backslash \left(\bigcup_{i=1}^{m} p^{-1}(z_i) \right) \right] \cup \{y_1, \ldots, y_m\}.$$

Denote the morphism $p|_{V_2}$ by p_2. Then, we see that the pair $(p_1 : V_1 \to C, p_2 : V_2 \to C)$ is an elementary Nisnevich cover refining the given cover $p : V \to C$. □

Lemma 3.2. *Let C be a smooth curve and let $(p_1 : V_1 \to C, p_2 : V_2 \to C)$ be an elementary Nisnevich cover of C. Let W be a dense open subscheme of $V_1 \times_C V_2$. Then, there exists an open subscheme V_2' of V_2 such that $(p_1 : V_1 \to C, p_2|_{V_2'} : V_2' \to C)$ is an elementary Nisnevich cover of C and the morphism $V_1 \times_C V_2' \to V_1 \times_C V_2$ induced by the inclusion $V_2' \hookrightarrow V_2$ maps $V_1 \times_C V_2'$ isomorphically onto W.*

Proof. We may assume that C is irreducible since the result can be proved for each component of C.

Let $\pi_2 : V_1 \times_C V_2 \to V_2$ be the projection on the second factor. Since p_1 is an open immersion, π_2 is an open immersion. We have

$$\pi_2(V_1 \times_C V_2) = \{v \in V_2 \mid p_2(v) \in p_1(V_1)\}.$$

Since $p_1(V_1)$ is dense in C and $p_2 : V_2 \to C$ is etale, it is easy to check that $\pi_2(V_1 \times_C V_2)$ is dense in V_2. Let Z be the complement of $\pi_2(V_1 \times_C V_2)$ in V_2. Thus, we have

$$Z = \{v \in V_2 | p_2(v) \notin p_1(V_1)\}.$$

As V_2 is 1-dimensional, Z is a finite set, each point of which is closed.

We define $V_2' = \pi_2(W) \cup Z$. From the above description of Z, it is easy to see that $(p_1 : V_1 \to C, p_2|_{V_2'} : V_2' \to C)$ is an elementary Nisnevich cover of C with the required property. □

3.2 Extending homotopies

A key idea in the proof of Theorem 1.4 is to use the homotopy extension property to glue morphisms. In this section, we will briefly explain this idea.

Let $\mathcal{F}$ be a simplicial sheaf over Sm/k. Let X be a smooth scheme over k and let $(p_1 : U \to X, p_2 : V \to X)$ be an elementary Nisnevich cover of X (see B.2). Thus, p_1 is an open immersion and p_2 is an étale morphism such that if $Z = X \backslash p_1(U)$, the morphism $V \times_X Z \to Z$ induced by p_2 is an isomorphism. Let W denote the scheme $U \times_X V$. Observe that the projection morphism $W = U \times_X V \to V$ is an open immersion.

Suppose we want to construct a morphism of X into $\mathcal{F}$ using this cover. So, we have a morphism $f : U \to \mathcal{F}$ and a morphism $g : V \to \mathcal{F}$. By the discussion following the statement of Fact B.3, we see that if the restrictions of f and g to W are equal, then f and g can be glued to give a morphism $h : X \to \mathcal{F}$.

However, suppose that the morphisms f and g which we have chosen do not satisfy this property. Then, we would like to modify them so that they can be glued together. Suppose that the morphisms $f|_W$ and $g|_W$ are simplicially homotopic, i.e. there exists a morphism $h : W \times \Delta^1 \to \mathcal{F}$ such that if $i_0, i_1 : \Delta_0 \to \Delta_1$ are the endpoint morphisms (see the appendix, Section A.2), then we have $h \circ (id_W \times i_0) = f|_W$ and $h \circ (id_W \times i_1) = g|_W$. Also assume that we can extend h to a morphism $H : V \times \Delta^1 \to \mathcal{F}$ such that $H \circ (id_W \circ i_0) = f$. Then, if $f' = H \circ (id_W \circ i_1)$, we see that f' and g can be glued together to obtain a morphism from X to $\mathcal{F}$.

This is a "homotopy extension" problem, which one sees often in the classical homotopy theory of topological spaces. In this situation, the existence of the homotopy extension H can be guaranteed by assuming that $\mathcal{F}$ is a fibrant sheaf.

Indeed, the morphism $i_0 : \Delta^0 \to \Delta^1$ is a trivial cofibration and the morphism $\mathcal{F} \to *$ is a fibration. Thus by Lemma A.14, the induced morphism $i_0^* : \mathcal{F}^{\Delta^1} \to \mathcal{F}^{\Delta^0} \cong \mathcal{F}$ induced by i_0 is a trivial fibration. The morphism $h : W \times \Delta^1 \to \mathcal{F}$ induces (by adjointness relations) a morphism $\widetilde{h} : W \to \mathcal{F}^{\Delta^1}$ such that the following square commutes

$$\begin{array}{ccc} W & \longrightarrow & \mathcal{F}^{\Delta^1} \\ \downarrow & & \downarrow \\ V & \xrightarrow[f]{} & \mathcal{F}. \end{array}$$

Since the left vertical map is a monomorphism, it is a cofibration. Thus, there exists a morphism $\widetilde{H} : V \to \mathcal{F}^{\Delta^1}$ making the diagram commutative. This map corresponds (via adjointness relations) to a morphism $H : V \times \Delta^1 \to \mathcal{F}$, which is the homotopy extension morphism we require.

3.3 Field valued points of $\mathcal{L}(\mathcal{X})$

Theorem 3.3. *Let $\mathcal{F}$ be a sheaf of sets. For any finitely generated, separable field extension K/k, the natural map $\pi_0^{\mathbb{A}^1}(\mathcal{F})(K) \to \mathcal{L}(\mathcal{F})(K)$ is a bijection.*

Proof. If $x_1, x_2 \in \mathcal{F}(K)$ map to the same element in $\mathcal{L}(\mathcal{F})(K)$, we want to prove that they map to the same element in $\pi_0^{\mathbb{A}^1}(\mathcal{F})(K)$. The hypothesis implies that there exists some non-negative integer n such that x_1 and x_2 map to the same element in $\mathcal{S}^{n+1}(\mathcal{F})(K)$. We will prove by induction on n that this implies that they map to the same element in $\pi_0^{\mathbb{A}^1}(\mathcal{F})(K)$. The case $n = 0$ is obvious. So we now assume that the claim is known to be true for $n < r$ for some positive integer r and now suppose that x_1, x_2 map to the same element in $\mathcal{S}^{r+1}(\mathcal{F})(K)$.

The images of x_1 and x_2 in $\mathcal{S}^r(\mathcal{F})(K)$ are connected by a chain of $\mathbb{A}^1$-homotopies. In other words, there is a sequence of elements $y_0 = x_1, y_1, \dots, y_m = x_2$ in $\mathcal{S}^r(\mathcal{F})(K)$ such that for $0 \leq i \leq m-1$, y_i is connected to y_{i+1} by an $\mathbb{A}^1$-homotopy $\mathbb{A}^1_K \to \mathcal{F}$. We claim that all the y_i's map to the same element of $\pi_0^{\mathbb{A}^1}(\mathcal{F})(K)$. It will suffice to prove the result in the case when $m = 1$, i.e. we may assume that there is a single $\mathbb{A}^1$-homotopy $\widetilde{h} : \mathbb{A}^1_K \to \mathcal{S}^r(\mathcal{F})$ connecting x_1 and x_2.

Since the morphism $\mathcal{F} \to \mathcal{S}^r(\mathcal{F})$ is an epimorphism, there exists a Nisnevich cover $p : V \to \mathbb{A}^1_K$ and a morphism $h : V \to \mathcal{F}$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{h} & \mathcal{F} \\ {\scriptstyle p}\downarrow & & \downarrow \\ \mathbb{A}^1_K & \xrightarrow{\widetilde{h}} & \mathcal{S}^r(\mathcal{F}) \end{array}$$

commutes. Using Lemma 3.1, we may assume that V is of the form $V = V_1 \coprod V_2$ such that if $p_i = p|_{V_i}$ for $i = 1, 2$, then $(p_1 : V_1 \to \mathbb{A}^1_K, p_2 : V_2 \to \mathbb{A}^1_K)$ is an elementary Nisnevich cover. Let $h_i = h|_{V_i}$.

For $i = 1, 2$, let $\pi_i : V_1 \times V_2 \to V_i$ be the projection morphism. Then, the two compositions

$$V_1 \times_{\mathbb{A}^1_K} V_2 \xrightarrow{\pi_i} V_i \xrightarrow{h_i} \mathcal{F} \to \mathcal{S}^r(\mathcal{F})$$

for $i = 1, 2$ are equal. Thus, there exists a Nisnevich cover $U \to V_1 \times_{\mathbb{A}^1_K} V_2$ such that the two compositions

$$U \to V_1 \times_{\mathbb{A}^1_K} V_2 \xrightarrow{\pi_i} V_i \xrightarrow{h_i} \mathcal{F} \to \mathcal{S}^{r-1}(\mathcal{F})$$

for $i = 1, 2$ are connected by a chain of $\mathbb{A}^1$-homotopies. There exists a scheme W, which is a union of some components of U such that the composition $W \hookrightarrow U \to V_1 \times_{\mathbb{A}^1_K} V_2$ is a dense open immersion. Using Lemma 3.2, we can shrink V_2 to reduce to the situation where $W \to V_1 \times_{\mathbb{A}^1_K} V_2$ is an isomorphism. So, now we will actually denote the scheme $V_1 \times_{\mathbb{A}^1_K} V_2$ by W (for typographical reasons). Thus, the two compositions

$$W = V_1 \times_{\mathbb{A}^1_K} V_2 \xrightarrow{\pi_i} V_i \xrightarrow{h_i} \mathcal{F} \to \mathcal{S}^{r-1}(\mathcal{F})$$

for $i = 1, 2$ are connected by a chain of $\mathbb{A}^1$-homotopies, which we denote by H. Thus, H is an ordered sequence $(h_1, \ldots, h_m)$ where each h_i is a a morphism $W \times \mathbb{A}^1_K \to \mathcal{S}^{r-1}(\mathcal{F})$. Also, for each $1 \leq i < m$, the morphisms $h_i|_{W \times \{1\}}$ and $h_{i+1}|_{W \times \{0\}}$ are the same. (Here we identify both $W \times \{1\}$ and $W \times \{0\}$ with W.)

Let $W = \coprod_{j=1}^{p} W_j$ be the decomposition of W into irreducible components. Let $\eta_j : \operatorname{Spec} L_j \to W_j$ be the generic point of W_j. Then the restriction of H to $\operatorname{Spec} L_j \times \mathbb{A}^1_K$ is a chain of $\mathbb{A}^1$-homotopies of $\operatorname{Spec} L_j$ in $\mathcal{S}^{r-1}(\mathcal{F})$. By the induction hypothesis, we can conclude that for every j, the two compositions

$$\operatorname{Spec} L_j \to W \xrightarrow{\pi_i} V_i \xrightarrow{h_i} \mathcal{F} \to \pi_0^{\mathbb{A}^1}(\mathcal{F})$$

for $i = 1, 2$ are equal.

Let $\mathcal{X} = L_{\mathbb{A}^1}(\mathcal{F})$. Thus, $\mathcal{X}$ is simplicially fibrant and also $\mathbb{A}^1$-local. Also, we have by definition $\pi_0(\mathcal{X}) = \pi_0^{\mathbb{A}^1}(\mathcal{F})$. Thus, the two compositions

$$\operatorname{Spec} L_j \to W \xrightarrow{\pi_i} V_i \xrightarrow{h_i} \mathcal{F} \to \pi_0^{\mathbb{A}^1}(\mathcal{X})$$

for $i = 1, 2$ are connected by a chain of simplicial homotopies of the form $\operatorname{Spec} L_j \times \Delta^1 \to \mathcal{X}$. (Actually, they can be connected by a single simplicial homotopy using [14, Proposition 9.5.24(2)], but we do not really need this detail.) These simplicial homotopies extend to an open subset W_j' of W_j for each j. Thus, if $W' = \cup_{j=1}^{p} W_j'$, then we see that the compositions

$$W' \hookrightarrow W \xrightarrow{\pi_i} V_i \xrightarrow{h_i} \mathcal{F} \to \pi_0^{\mathbb{A}^1}(\mathcal{X})$$

for $i = 1, 2$ are connected by a chain of simplicial homotopies of the form $W' \times \Delta^1 \to \mathcal{X}$.

Now, by using Lemma 3.2 to further shrink V_2, we may assume that $W' = W$. Thus, we have reduced to the situation where the two compositions

$$W \xrightarrow{\pi_i} V_i \xrightarrow{h_i} \mathcal{F} \to \pi_0^{\mathbb{A}^1}(\mathcal{F})$$

for $i = 1, 2$ are connected by a chain of simplicial homotopies of the form $W \times \Delta^1 \to \mathcal{X}$.

Now, we may use the discussion in Section 3.2 to extend these homotopies to V_2. Thus, we obtain a morphism $h_2' : V_2 \to \mathcal{X}$ such that the composition

$V_2 \xrightarrow{h_2} \mathcal{F} \to \mathcal{X}$ is connected to h_2' by a chain of simplicial homotopies, and the compositions

$$W \xrightarrow{\pi_1} V_1 \xrightarrow{h_1} \mathcal{F} \to \mathcal{X}$$

and

$$W \xrightarrow{\pi_2} V_2 \xrightarrow{h_2'} \mathcal{X}$$

are equal. These two homomorphisms may be glued together to give an $\mathbb{A}^1$-homotopy $\mathbb{A}^1_K \to \mathcal{X}$ connecting the image of the point x_1 in $\mathcal{X}(K)$ to a point x_2' where x_2' is simplicially homotopic to the image of x_2 in $\mathcal{X}(K)$. Since $\mathcal{X}$ is $\mathbb{A}^1$-local, the images of the points x_1 and x_2' in $\mathcal{X}(K)$ are actually simplicially homotopic. Thus, we see that the images of x_1 and x_2 in $\mathcal{X}(K)$ are simplicially homotopic. Hence, the images of x_1 and x_2 in $\pi_0^{\mathbb{A}^1}(\mathcal{X})(K)$ are equal. This completes the proof by induction. $\square$

Chapter 4

Iterations of the $\mathcal{S}$ functor

In this chapter, we construct a family of examples to show that the term $\mathcal{L}(\mathcal{X})$ in the statement of Theorem 1.4 cannot be replaced by $\mathcal{S}^n(\mathcal{X})$ for any positive integer. In other words, the infinitely many iterations of $\mathcal{S}$ are indeed necessary in general.

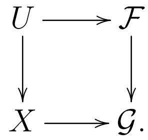
4.1 Closed embeddings of sheaves

Definition 4.1. Let $\mathcal{F}$ and $\mathcal{G}$ be Nisnevich sheaves of sets on Sm_k and let $i : \mathcal{F} \to \mathcal{G}$ be a monomorphism. We say that i is a *closed embedding of sheaves* if it has the right lifting property with respect to any dense open immersion $U \hookrightarrow X$, where X is a smooth variety over k.

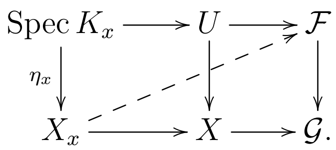
Observe that if X is a smooth variety over k and $\eta : \operatorname{Spec} K \to X$ is the inclusion of the generic point of X, then any closed embedding of schemes has the right lifting property with respect to η. The analogue of this fact for Nisnevich sheaves is as follows.

Lemma 4.2. *A monomorphism $i : \mathcal{F} \to \mathcal{G}$ is a closed embedding of sheaves if and only if for any smooth henselian local scheme X with generic point $\eta : \operatorname{Spec} K \to X$, the morphism i has the right lifting property with respect to η.*

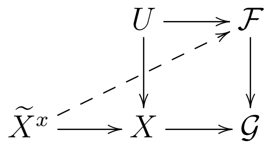
Proof. Let X be a smooth variety over k and let U be an open subset of X. Suppose that we have a diagram



For any point x of X, let X_x denote the scheme $\operatorname{Spec} \mathcal{O}^h_{X,x}$ and let $\eta_x : \operatorname{Spec} K_x \to X_x$ denote the generic point of X_x. Then, in the diagram

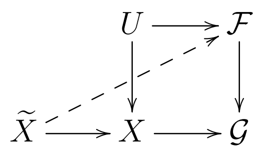


we obtain a morphism $X_x \to \mathcal{F}$ (indicated by the dashed arrow) making the diagram commute. This means that there exists a smooth variety $X'(x)$ with an étale morphism $\pi : \widetilde{X}^x \to X$, which is an isomorphism on $\pi^{-1}(x)$ and such that we have a morphism $\widetilde{X}^x \to \mathcal{F}$ making the diagram



commute. (Note: The superscript x in $\widetilde{X}^x$ is only intended to indicate the dependence on x and has no other meaning.)

Thus, we see that there exists a Nisnevich cover $\widetilde{X} \to X$ and a morphism $\widetilde{X} \to \mathcal{F}$ such that the diagram



commutes. The morphism $\widetilde{X} \to \mathcal{F} \to \mathcal{G}$ descends to a morphism $X \to \mathcal{G}$. Since $\mathcal{F} \to \mathcal{G}$ is a monomorphism, we see that the morphism $\widetilde{X} \to \mathcal{F}$ also descends to a morphism $X \to \mathcal{F}$ making the lower triangle in the diagram

$$\begin{array}{ccc} U & \longrightarrow & \mathcal{F} \\ \downarrow & \nearrow & \downarrow \\ X & \longrightarrow & \mathcal{G}. \end{array}$$

commute, which in turn, makes the upper triangle commute. □

Lemma 4.3. *Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be Nisnevich sheaves of sets on Sm_k.*

(*a*) *If $\mathcal{F} \to \mathcal{G}$ is a closed embedding of sheaves, then for any morphism $\mathcal{H} \to \mathcal{G}$, the morphism $\mathcal{F} \times_{\mathcal{G}} \mathcal{H} \to \mathcal{G}$ is a closed embedding of sheaves.*

(*b*) *Let $p : \mathcal{F} \to \mathcal{G}$ be an epimorphism and let $i : \mathcal{H} \to \mathcal{G}$ be a monomorphism of sheaves. If $i' : \mathcal{F} \times_{\mathcal{G}} \mathcal{H} \to \mathcal{F}$ is a closed embedding, then so is i.*

Proof. Part (*a*) is obvious; we prove part (*b*). Let $p' : \mathcal{F} \times_{\mathcal{G}} \mathcal{H} \to \mathcal{H}$ be the projection on the second factor. Let X be a smooth henselian local scheme with generic point $\eta : \operatorname{Spec} K \to X$. Let $\alpha : X \to \mathcal{G}$ such that $\alpha \circ \eta$ factors through i. As p is an epimorphism, there exists $\beta : X \to \mathcal{F}$ such that $p \circ \beta = \alpha$. Since $p \circ \beta \circ \eta = \alpha \circ \eta$ factors through i, the morphism $\beta \circ \eta$ factors through i'. As i' is a closed embedding, β factors through i'. Thus, $\beta = i' \circ \beta'$ and we have

$$\alpha = p \circ \beta = p \circ i' \circ \beta' = i \circ p' \circ \beta'.$$

Hence, α factors through i. This proves that i is a closed embedding. □

4.2 Construction of examples

Consider the Zariski cover of $\mathbb{A}^1_k$ given by $V_1 = \mathbb{A}^1_k \setminus \{1\}$ and $V_2 = \mathbb{A}^1_k \setminus \{0\}$. Let $p_1 : V_1 \to \mathbb{A}^1_k$ and $p_2 : V_2 \to \mathbb{A}^1_k$ be the inclusion morphisms. Let $W := V_1 \times_{\mathbb{A}^1_k} V_2 = \mathbb{A}^1 \setminus \{0, 1\}$. For $i = 1, 2$, let $\pi_i : V_1 \times_{\mathbb{A}^1_k} V_2 \to V_i$ be the projection (which is an open immersion).

We will now inductively construct a sequence of sheaves $\{\mathcal{X}_n\}_{n\in\mathbb{Z}_{\geq -1}}$ on Sm_k and morphisms $\alpha_n, \beta_n : \operatorname{Spec} k \to \mathcal{X}_n$. Set $\mathcal{X}_{-1} := \operatorname{Spec} k$ and let $\alpha_{-1}, \beta_{-1} : \operatorname{Spec} k \to \operatorname{Spec} k$ be the identity maps.

If $\mathcal{X}_{n-1}, \alpha_{n-1}, \beta_{n-1}$ are defined, we define $\mathcal{X}_n$ to be the pushout of the diagram

$$\begin{array}{ccc} W \coprod W & \xrightarrow{\phi_n} & V_1 \coprod V_2 \\ {\scriptstyle \psi_n}\downarrow & & \downarrow{\scriptstyle \psi'_n} \\ W \times \mathcal{X}_{n-1} & \xrightarrow[\phi'_n]{} & \mathcal{X}_n \end{array}$$

where $\phi_n = \pi_1 \coprod \pi_2$ and ψ_n is the composition

$$W \coprod W \xrightarrow{\sim} W \times (\operatorname{Spec} k \coprod \operatorname{Spec} k) \xrightarrow{id_W \times (\alpha_{n-1} \coprod \beta_{n-1})} W \times \mathcal{X}_{n-1}.$$

We define $\alpha_n : \operatorname{Spec} k \to \mathcal{X}_n$ to be the composition of $\operatorname{Spec} k \xrightarrow{0} V_1 \to \mathcal{X}_n$ and $\beta_n : \operatorname{Spec} k \to \mathcal{X}_n$ to be the composition of $\operatorname{Spec} k \xrightarrow{1} V_2 \to \mathcal{X}_n$. Clearly, $\mathcal{X}_0 = \mathbb{A}^1$ and α_0, β_0 are the morphisms $\operatorname{Spec} k \to \mathbb{A}^1$ corresponding to the points 0 and 1.

Lemma 4.4. *The morphism $\alpha_n \coprod \beta_n : \operatorname{Spec} k \coprod \operatorname{Spec} k \to \mathcal{X}_n$ is a closed embedding.*

Proof. For $n = 0$, the conclusion of lemma is clear. Now, assume that $n > 0$. Let $\mathcal{P} = \operatorname{Spec} k \coprod \operatorname{Spec} k$ and let $\gamma_n : \mathcal{P} \to \mathcal{X}_n$ denote the morphism $\alpha_n \coprod \beta_n$. Let $\mathcal{Q} := \mathcal{P} \times_{\gamma_n, \mathcal{X}_n, \psi'_n} (V_1 \coprod V_2)$, and let pr_1 and pr_2 denote the projections of this fiber product to the first and second factors respectively. As ψ'_n is a monomorphism, the projection pr_1 is also a monomorphism.

The composition $\mathcal{P} \xrightarrow{0 \coprod 1} V_1 \coprod V_2$ and the identity morphism $id_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}$ induce a morphism $\mathcal{P} \to \mathcal{Q}$ such that the composition $\mathcal{P} \to \mathcal{Q} \xrightarrow{pr_1} \mathcal{P}$ is equal to $id_{\mathcal{P}}$. Thus, we see that pr_1 is an epimorphism. Thus, pr_1 is an isomorphism.

If we identify $\mathcal{Q}$ with $\mathcal{P}$ using pr_1, then pr_2 may be identified with the morphism $0 \coprod 1$, which is a closed embedding of $\mathcal{P}$ into $V_1 \coprod V_2$. Thus, by apply Lemma 4.3(b) to disjoint union of ϕ'_n and ψ'_n, we see that γ_n is a closed embedding. □

In what follows, the following simple observation will be useful.

Lemma 4.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a monomorphism of Nisnevich sheaves. Let τ be a Grothendieck topology on Sm_k which is finer than the Nisnevich topology. Suppose that $\mathcal{X}$ is a sheaf for τ. Then, f has the right lifting property with respect to any τ-cover $V \to U$ where U is an essentially smooth scheme over k.*

Proof. Suppose we have a diagram

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & \mathcal{X} \\ \downarrow & & \downarrow f \\ U & \xrightarrow[\beta]{} & \mathcal{Y}. \end{array}$$

Since $\mathcal{X}$ is a sheaf for the topology τ, it suffices to prove that the two morphisms

$$V \times_U V \rightrightarrows V \xrightarrow{\alpha} \mathcal{X}$$

are equal. However, we see that the compositions of these morphisms with f are equal since $f \circ \alpha$ factors through β. Thus, the result follows since f is a monomorphism. □

Define $\mathcal{Y}_n := (W \times \mathcal{X}_{n-1}) \coprod V_1 \coprod V_2$ and let $p_n : \mathcal{Y}_n \to \mathcal{X}_n$ be the morphism induced by $\phi'_n : W \times \mathcal{X}_{n-1} \to \mathcal{X}_n$ and $\psi'_n : V_1 \coprod V_2 \to \mathcal{X}_n$. Let $\phi''_n : W \times \mathcal{X}_{n-1} \to \mathcal{Y}_n$ and $\psi''_n : V_1 \coprod V_2 \to \mathcal{Y}_n$ be the obvious (inclusion) maps. Thus, $\phi'_n = p_n \circ \phi''_n$ and $\psi'_n = p_n \circ \psi''_n$.

Lemma 4.6. *Let $n \geq 1$ be an integer. For any essentially smooth irreducible scheme Z, any morphism $\alpha : Z \to \mathcal{X}_n$ factors through p_n.*

Proof. Clearly, p_n is an epiomorphism of sheaves. Thus, for any scheme Z and any morphism $\alpha : Z \to \mathcal{X}_n$, there exists a Nisnevich cover $\{\gamma_i : Z_i \to Z\}_{i \in I}$ such that for each i, the morphism $\alpha|_{Z_i} := \alpha \circ \gamma_i$ is equal to $p_n \circ \beta_i$ for some $\beta_i : Z_i \to \mathcal{Y}_n$. Let us assume that each Z_i is irreducible and let $\eta_i : \operatorname{Spec} K_i \to Z_i$ be the generic points. Note that I may be taken to be a finite set.

Each β_i factors through ϕ''_n or ψ''_n. Suppose that all the β_i factor through ϕ''_n. Then, as ϕ'_n is a monomorphism, by Lemma 4.5 we see that there exists a morphism $\beta : Z \to W \times \mathcal{X}_{n-1}$ such that $\alpha = \phi'_n \circ \beta$. Similarly, if all the β_i factor through ψ''_n, then there exists a morphism $\beta : Z \to V_1 \coprod V_2$ such that $\alpha = \psi'_n \circ \beta$. We claim that neither of these conditions hold, then we can change some of the β_i's to reduce to the situation where they all factor through ψ''_n.

Thus, now let us assume that we can find two indices $i, j \in I$ such that β_i factors through ψ''_n and β_j factors through ϕ''_n. Thus, there exists a morphism $\beta'_i : Z_i \to V_1 \coprod V_2$ such that $\beta_i = \psi''_n \circ \beta'_i$ and a morphism $\beta'_j : Z_j \to W \times \mathcal{X}_{n-1}$ such that $\beta_j = \phi''_n \circ \beta'_j$.

Let P be a component of $Z_i \times_Z Z_j$ and let $\operatorname{Spec} L \to P$ be the generic point. Let ρ_i denote the composition

$$\operatorname{Spec} L \to \operatorname{Spec} K_i \xrightarrow{\eta_i} Z_i \xrightarrow{\beta'_i} V_1 \coprod V_2 \xrightarrow{\psi''_n} \mathcal{Y}_n$$

and let ρ_j denote the composition

$$\operatorname{Spec} L \to \operatorname{Spec} K_j \xrightarrow{\eta_j} Z_j \xrightarrow{\beta'_j} W \times \mathcal{X}_{n-1} \xrightarrow{\phi''_n} \mathcal{Y}_n.$$

Then, $p_n \circ \rho_i = p_n \circ \rho_j$. Thus, ρ_i factors through $\phi_n : W \coprod W \to V_1 \coprod V_2$ and ρ_j factors through $\psi_n : W \coprod W \to W \times \mathcal{X}_{n-1}$.

Since L/K_j is a separable field extension and since $W \coprod W$ is a scheme (and hence, an étale sheaf), by Lemma 4.5, the morphism $\beta'_j \circ \eta_j$ factors through ψ_n. By Lemma 4.3(a) and Lemma 4.5, ψ_n is a closed embedding. Thus, we see that β'_j factors through ψ_n. Let $\beta''_j : Z_j \to W \coprod W$ be such that $\beta'_j = \psi_n \circ \beta''_j$. Let $\widetilde{\beta}_j : Z_i \to \mathcal{Y}_n$ be the composition

$$Z_j \xrightarrow{\beta''_j} W \coprod W \xrightarrow{\phi_n} V_1 \coprod V_2 \to \mathcal{Y}_n.$$

Observe that $p_n \circ \widetilde{\beta}_j = p_n \circ \beta_j$. Thus, we may now replace β_j by $\widetilde{\beta}_j$.

We now repeat this process until we come to a situation where all the β_i's factor through ψ''_n. This completes the proof. $\square$

Lemma 4.7. *Let $n \geq 1$ be an integer. The square*

$$\begin{array}{ccc} W \coprod W & \longrightarrow & V_1 \coprod V_2 \\ \downarrow & & \downarrow \\ W \times \mathrm{Sing}_*^{\mathbb{A}^1} \mathcal{X}_{n-1} & \longrightarrow & \mathrm{Sing}_*^{\mathbb{A}^1} \mathcal{X}_n \end{array}$$

is a pushout square.

Proof. We will prove that for any essentially smooth irreducible scheme Z over k, the square

$$\begin{array}{ccc} (W \coprod W)(Z) & \xrightarrow{(\phi_n)_Z} & (V_1 \coprod V_2)(Z) \\ {\scriptstyle (\psi_n)_Z}\downarrow & & \downarrow{\scriptstyle (\psi'_n)_Z} \\ (W \times \mathcal{X}_{n-1})(Z) & \xrightarrow[(\phi'_n)_Z]{} & \mathcal{X}_n(Z) \end{array}$$

is a pushout square. Once this is proved, we take $Z = U \times \mathbb{A}^m$ for $m \geq 0$ where U is a smooth henselian local scheme over k. Now, using the fact that V_1, V_2 and W are $\mathbb{A}^1$-rigid, the result follows.

By Lemma 4.6, the function

$$(W \times \mathcal{X}_{n-1})(Z) \coprod (V_1 \coprod V_2)(Z) \to \mathcal{X}_n(Z)$$

is a surjection. Also, the pushouts of cofibrations are cofibrations, so the functions $(\phi_n)_Z$, $(\psi_n)_Z$, $(\phi'_n)_Z$ and $(\psi'_n)_Z$ are injective. Thus, it suffices to show that if $\alpha \in (V_1 \coprod V_2)(Z)$ and $\beta \in (W \times \mathcal{X}_{n-1})(Z)$ are such that $(\psi'_n)_Z(\alpha) = (\phi'_n)_Z(\beta)$, then there exists $\gamma \in (W \coprod W)(Z)$ such that $(\phi_n)_Z(\gamma) = \alpha$ and $(\psi_n)_Z(\gamma) = \beta$.

If $(\psi'_n)_Z(\alpha) = (\phi'_n)_Z(\beta)$, there exists a Nisnevich cover $Z' \to Z$ such that there exists $\gamma' \in (W \coprod W)(Z')$ such that $(\phi_n)_{Z'}(\gamma') = \alpha|_{Z'}$ and $(\psi_n)_{Z'}(\gamma') = \beta|_{Z'}$. By Lemma 4.5, γ' factors through a morphism $\gamma : Z \to W \coprod W$. This completes the proof. □

Theorem 4.8. *Let $n \geq 0$ be an integer.*

(1) $\mathrm{Sing}_*^{\mathbb{A}^1} \mathcal{X}_n$ *is simplicially equivalent to* $\mathcal{X}_{n-1}$.

(2) $\mathcal{S}(\mathcal{X}_n) \cong \mathcal{X}_{n-1}$.

Proof. We prove this theorem by induction on n. Since $\mathcal{X}_0 = \mathbb{A}^1$, the result is easily seen to be true for $n = 0$.

Suppose the result is known to be true for $n \leq m$ where $m \geq 0$. In the diagram

$$\begin{array}{ccccc}
W \times \mathrm{Sing}_*^{\mathbb{A}^1} \mathcal{X}_m & \longleftarrow & W \coprod W & \longrightarrow & V_1 \coprod V_2 \\
\downarrow & & \downarrow & & \downarrow \\
W \times \mathcal{X}_{m-1} & \longleftarrow & W \coprod W & \longrightarrow & V_1 \coprod V_2
\end{array}$$

the vertical arrows are simplicial equivalences and the horizontal arrows are cofibrations. Thus, for $n = m + 1$ we see that the pushouts of these diagrams are simplicially equivalent. Thus, $\mathrm{Sing}_*^{\mathbb{A}^1} \mathcal{X}_n$ is simplicial equivalent to $\mathcal{X}_{n-1}$. This proves (1). Since $\mathcal{S}(\mathcal{X}_n) = \pi_0(\mathrm{Sing}_*^{\mathbb{A}^1} \mathcal{X}_n) = \pi_0(\mathcal{X}_{n-1}) = \mathcal{X}_{n-1}$ by definition, this proves (2). □

Finally, we are able to prove Theorem 1.5, which we restate here:

Theorem 4.9. *Let k be a field. There exists a sequence $\mathcal{X}_n$ of $\mathbb{A}^1$-connected sheaves of sets over Sm/k such that $\mathcal{S}^{n+1}(\mathcal{X}_n) = \pi_0^{\mathbb{A}^1}(\mathcal{X}_n)$ is the trivial one-point sheaf, but $\mathcal{S}^i(\mathcal{X}_n) \neq \mathcal{S}^{i+1}(\mathcal{X}_n)$, for every $i < n + 1$.*

Proof. A repeated application of Theorem 4.8 shows that $\mathcal{S}^{n+1}(\mathcal{X}_n)$ is the trivial one point sheaf. It also shows that $\mathcal{X}_n$ is $\mathbb{A}^1$-connected. It is clear from the construction of $\mathcal{X}_n$ that $\mathcal{S}^i(\mathcal{X}_n) \neq \mathcal{S}^{i+1}(\mathcal{X}_n)$, for every $i < n + 1$. □

Chapter 5

Naive $\mathbb{A}^1$-connectedness of retract rational varieties

In this chapter, we will prove that smooth, proper, retract rational varieties over an infinite field k are naively $\mathbb{A}^1$-connected. Section 5.1 shows how one can construct the germ of a homotopy, which we call as an infinitesimal homotopy, on any smooth variety. In a rational variety, one can use an infinitesimal homotopy to construct $\mathbb{A}^1$-homotopy by the simple process of truncating power series. The main idea behind our proof is to execute a more sophisticated version of this idea by using the Weierstrass preparation theorem for henselian rings.

5.1 Infinitesimal homotopies

Notation 5.1. If X is a scheme and x is a point on X, the local scheme $\operatorname{Spec} \mathcal{O}^h_{X,x}$ will be denoted by X_x. (We have already used this notation in the proof of Lemma 4.2.) The canonical morphism $X_x \to X$ will be denoted by ω_x.

For any ring R, an *infinitesimal homotopy* of $\operatorname{Spec} R$ in a scheme X is a morphism $h : \operatorname{Spec} R\{t\} \to X$ (Recall (Appendix C.2) the meaning of notation $R\{t\}$). Let $\widehat{\sigma}_0 : \operatorname{Spec} R \to \operatorname{Spec} R\{t\}$ be the morphism induced by the quotient homomorphism $R\{t\} \to R\{t\}/tR\{t\} \cong R$ (see Appendix C.2). We say that this infinitesimal homotopy *starts* from the morphism $h \circ \widehat{\sigma}_0 : \operatorname{Spec} R \to X$.

The following lemma shows that for a smooth variety X and a point $x \in X$, one can easily construct the germ of a homotopy of X_x in X starting from the canonical morphism $\omega_x : X_x \to X$.

Lemma 5.2. *Let X be a smooth d-dimensional variety over k. Let x be a point of X. Let U be an open subset of X. Let $R = \mathcal{O}^h_{X,x}$ and let $\omega_x : \operatorname{Spec}(R) =: X_x \to X$ be the canonical morphism. Then, there exists a morphism $h : \operatorname{Spec} R\{t\} \to X$ starting from ω_x such that*

$$\operatorname{Spec} \kappa(x)\{t\} \to \operatorname{Spec} R\{t\} \xrightarrow{h} X$$

maps the generic point of $\operatorname{Spec} \kappa(x)\{t\}$ *into U.*

Proof. If $x \in U$, we may take h to be the composition

$$\operatorname{Spec} R\{t\} \to \operatorname{Spec} R \xrightarrow{\omega_x} X$$

where the first morphism is induced by the inclusion $R \hookrightarrow R\{t\}$. Thus, we will now assume that $x \notin U$. Let $Z = X \backslash U$.

For any non-negative integer n, consider the functor

$$U \mapsto Mor_{Sch/k}(U \times \operatorname{Spec} k[t]/\langle t^{n+1} \rangle, X)$$

on the category of k-schemes. This functor is known to be representable by a k-scheme of finite type (see [13]), which we denote by $J_n(X)$. Let $J(X) = \varprojlim J_n(X)$, where the inverse limit is computed in the category of k-schemes. The quotient homomorphism $k[[t]] \to k[t]/\langle t^{n+1} \rangle$ induces the morphism $\pi^X_n : J(X) \to J_n(X)$. For $n \geq m$, let $\pi^X_{n,m} : J_n(X) \to J_m(X)$ denotes the morphism induced by the quotient homomorphism $k[t]/\langle t^{n+1} \rangle \to k[t]/\langle t^{m+1} \rangle$.

Choose a morphism $\gamma : \operatorname{Spec} \kappa(x)[[t]] \to X$ which maps the closed point of $\operatorname{Spec} \kappa(x)[[t]]$ to x and the generic point into U. For any $n \geq 0$, let γ_n denote the composition

$$\operatorname{Spec} \kappa(x)[t]/\langle t^{n+1} \rangle \to \operatorname{Spec} \kappa(x)[[t]] \xrightarrow{\gamma} X.$$

We identify γ_n with a $\kappa(x)$-valued point of $J_n(X)$, which we denote by $\widetilde{\gamma}_n$. We define $\widetilde{g}_0 : \operatorname{Spec} R \to J_0(X) = X$ to be the morphism ω_x. For $n \geq 1$, we will inductively construct a morphism $\widetilde{g}_n : \operatorname{Spec} R \to J_n(X)$ such that:

(i) the composition $\operatorname{Spec} \kappa(x) \to \operatorname{Spec} R \xrightarrow{\widetilde{g}_n} J_n(X)$ is equal to $\widetilde{\gamma}_n$, and

(ii) the composition $\pi^X_{n+1,n} \circ \widetilde{g}_{n+1}$ equals $\widetilde{g}_n$.

Suppose $\widetilde{g}_n$ has been chosen for some non-negative integer n. Since X is smooth, $J_{n+1}(X) \to J_n(X)$ is smooth. (In fact, it is an affine bundle for all n — see [20, Lemma 9.1]). So we can choose a morphism $\widetilde{g}_{n+1} : \operatorname{Spec} R \to J_{n+1}(X)$ satisfying the conditions (i) and (ii) (see [12, Corollary 17.16.3, (ii)]).

The collection $\{\widetilde{g}_n\}_{n\geq 0}$ defines a morphism $\widetilde{g} : \operatorname{Spec} R \to J(X)$, which corresponds to a morphism $g : \operatorname{Spec} R[[t]] \to X$. The restriction of g to $\kappa(x)[[t]]$ is equal to γ.

There exists an integer n such that if $\gamma' : \operatorname{Spec} \kappa(x)[[t]] \to X$ satisfies

$$\pi_{n,\operatorname{Spec} \kappa(x)}(\gamma) = \pi_{n,\operatorname{Spec} \kappa(x)}(\gamma'),$$

then γ' maps the generic point of $\operatorname{Spec} \kappa(x)[[t]]$ into U. (Indeed, if there is no such n, then since $J(Z) = \varprojlim J_n(X)$, it will follow that $\gamma \in J(Z)$, which is not true.) By Facts C.6 and C.7, there exists a morphism $h : \operatorname{Spec} R\{t\} \to X$ such that $\pi^X_n(h) = \pi^X_n(g)$. This proves the lemma. □

Remark 5.3. As we see in the above proof, it is very easy to construct a morphism $\operatorname{Spec} R[[t]] \to X$. We could have used this as our notion of "infinitesimal homotopy", if we had an analogue of Fact C.4 for the ring $R[[t]]$, at least when R is regular. (Such a result was proved in characteristic 0 by Lafon in [19].) The proof of the preparation theorem for $R\{t\}$ in [9] crucially uses the fact that the functor $R\{t\}$ is a colimit of finite type R-algebras and we do not know if it can be adapted to give an analogous result for $R[[t]]$. So we choose to work with the ring $R\{t\}$ instead.

5.2 Rational curves in projective space

Let us fix a base field k. Let L be any field containing k. We will use T_0 and T_1 as homogeneous coordinates on $\mathbb{P}^1_L$. In other words, we will write $\mathbb{P}^1_L = \operatorname{Proj} L[T_0, T_1]$. We will identify $\mathbb{A}^1_L = \operatorname{Spec} L[t]$ with the open subscheme $\mathbb{P}^1_L \backslash \mathcal{Z}(T_1)$ by identifying t with T_0/T_1. We will denote the point $(0 : 1)$ of $\mathbb{P}^1_L$ by $\mathbf{0}_L$ and the point $(1 : 0)$ by ∞_L. Thus, t is a parameter at $\mathbf{0}_L$ and $1/t$ is a parameter at ∞_L. To avoid making the notation cumbersome, we will write $\mathbf{0}$ and ∞ instead of $\mathbf{0}_L$ and ∞_L in the following discussion.

A morphism $\phi : \mathbb{P}^1_L \to \mathbb{P}^N_k$ can be represented by an $(N+1)$-tuple $(P_0, \ldots, P_N)$ of homogeneous polynomials of a fixed degree d, such that $P_i \neq 0$ for some i. (Of course, some of the P_i's may be equal to 0. The zero polynomial can be assigned any degree.) Such a representation is not unique, but if we require the polynomials to be coprime, it is unique up to multiplication by a unit. Dehomogenizing this $(N+1)$-tuple with respect to T_1 gives an $(N+1)$-tuple of polynomials in t, which describes the restriction of ϕ to the open subscheme $D(T_1)$.

Recall that given a morphism from $\mathbb{P}^1_L \backslash \mathcal{Z}(T_1)$ to $\mathbb{P}^N_k$, it can be uniquely extended to a morphism $\mathbb{P}^1_L \to \mathbb{P}^N_k$. A morphism from $\mathbb{P}^1_L \backslash \mathcal{Z}(T_1)$ to $\mathbb{P}^N_k$ is given by an $(N+1)$-tuple of polynomials in $L[t]$, such that at least one of the polynomials is non-zero. Thus, we see that a morphism $\mathbb{P}^1_k \to \mathbb{P}^N_k$ can be represented in three ways — using $(N+1)$-tuples of homogeneous polynomials of a same degree in (T_0, T_1) or by using $(N+1)$-tuples of polynomials in either t or $1/t$. (Again, note that these representations are unique up to multiplication by a unit if we require the polynomials to be coprime.)

Given a morphism $\phi : \mathbb{P}^1_k \to \mathbb{P}^N_k$, we choose a representation of ϕ by an $(N+1)$-tuple of polynomials $(P_0, \ldots, P_N)$ in $k[t]$ which are coprime. Let $m \geq 0$ be any integer. For $0 \leq i \leq N$, let P'_i be the polynomial obtained by truncating P_i to degree m. Then, the $(N+1)$-tuple $(P'_0, \ldots, P'_N)$ is called the m-jet of ϕ at $\mathbf{0}$. Similarly, we can define the m-jet of ϕ at ∞. Note that these are well-defined up to multiplication by a unit.

The following lemma shows that given a closed subscheme W of $\mathbb{P}^N_k$ of codimension ≥ 2, and non-negative integers m_1 and m_2, there exists a morphism $\phi : \mathbb{P}^1_k \to \mathbb{P}^N_k$ such that it maps $\mathbb{P}^1_k \backslash \{\mathbf{0}, \infty\}$ into $\mathbb{P}^N_k \backslash W$, has a prescribed m_1-jet at $\mathbf{0}$ and a prescribed m_2-jet at ∞.

Lemma 5.4. *Let k be an infinite field and let L be a field containing k. Let N be a positive integer. Let W be a closed subscheme of $\mathbb{P}^N_k$ of codimension ≥ 2. Let $P = (P_0, \ldots, P_N)$ and $(Q_0, \ldots, Q_N)$ be $(N+1)$-tuples of polynomials in $L[t]$. Assume that $P_i \neq 0$ and $Q_j \neq 0$ for some indices i, j. Let m_1 be an integer such that $m_1 \geq \max_i degP_i$. Assume that $P_i(0) \neq 0$ for some i. Then, there exists an $(N+1)$-tuple $(c_0, \ldots, c_N) \in k^{N+1}$ such that the following conditions hold:*

(a) For $0 \leq i \leq N$, let $R_i(t) = P_i(t) + t^{m_1+1}c_i + t^{m_1+2}Q_i(t)$. Then, the polynomials $R_0, \ldots, R_N$ are coprime.

(c) Let $\phi : \mathbb{P}^1_L \to \mathbb{P}^N_k$ be the morphism represented by the $(N+1)$-tuple $(R_0, \ldots, R_N)$. Then $\phi(\mathbb{P}^1_L \backslash \{\mathbf{0}, \infty\}) \subset \mathbb{P}^N_k \backslash W$.

Note that if ϕ is as described in the lemma, the m_1-jet of ϕ at $\mathbf{0}$ is $(P_0(t), \ldots, P_N(t))$ and if $m_2 = \max_i \deg Q_j$, then the m_2-jet of ϕ at ∞ is $(Q_0(1/t), \ldots Q_N(1/t))$.

Proof. For any point $x = (x_0, \ldots, x_N)$ of $\mathbb{A}^{N+1}_k$ and $0 \leq i \leq N$, we define

$$R^x_i(t) = P_i(t) + x_i t^{m_1+1} + Q_i(t)t^{m_1+2}.$$

Let $\phi_x : \mathbb{A}^1_{\kappa(x)} \to \mathbb{A}^{N+1}_k$ be the morphism defined by

$$\phi_x(s) = (R^x_0(s), \ldots, R^x_N(s))$$

for any $s \in \mathbb{A}^1_{\kappa(x)}$. Let $C(W) \subset \mathbb{A}^{N+1}_k$ be the cone over W.

We define

$$B := \{(x, s, z) | \phi_x(s) = z\} \subset \mathbb{A}^{N+1}_k \times (\mathbb{A}^1_k \backslash \{0\}) \times C(W).$$

This is a closed subset of $\mathbb{A}_k^{N+1} \times (\mathbb{A}_k^1 \backslash \{0\}) \times C(W)$. Let $pr_1 : B \to \mathbb{A}_k^{N+1}$ be the projection map onto the first factor. We need to show that the complement of the image of pr_1 contains some k-rational point.

Let $pr_{23} : \mathbb{A}_k^{N+1} \times (\mathbb{A}_k^1 \backslash \{0\}) \times C(W) \to (\mathbb{A}_k^1 \backslash \{0\}) \times C(W)$ by the projection map onto the product of the second and third factors. We would like to estimate the dimension of the fibre $pr_{23}^{-1}(\gamma)$ where $\gamma = (s, z) \in \mathbb{A}_k^1 \backslash \{0\} \times C(W)$. The equation $\phi_x(s) = z$ imposes $N+1$ linear conditions on $\mathbb{A}_k^{N+1}$. Thus, the fibre has dimension 0. Thus, it follows that $\dim(B) \leq 0 + 1 + \dim(C(W)) \leq N$.

It follows that the closure of $pr_1(B)$ is of dimension $\leq N$. The result follows since k is an infinite field. □

Remark 5.5. This lemma is one of the main reasons for requiring the field k to be infinite in Theorem 1.6. The lemma need not hold if k is a finite field since all the k-rational points of $\mathbb{P}_k^N$ may be contained in W.

5.3 Retract rational varieties

We first set up some notation.

Notation 5.6. Given a scheme X (resp. an affine scheme $X = \operatorname{Spec} R$), and an ideal sheaf $\mathcal{I}$ (resp. an ideal $I \subset R$), we will denote by $\mathcal{Z}(\mathcal{I})$ (resp. $\mathcal{Z}(I)$) the closed subscheme of X associated to the ideal sheaf $\mathcal{I}$ (resp. the ideal I). If $\mathcal{L}$ is a line bundle on X and S is a set of sections of $\mathcal{L}$, we may also write $\mathcal{Z}(S)$ for the closed subschemes defined by the vanishing of the elements of S.

Theorem 5.7. *Let k be an infinite field. Let X be a smooth, proper, retract rational variety over k. Then $\mathcal{S}(X) = *$.*

Proof. Since X is retract rational, there exists a positive integer $N \geq 1$, and rational maps $\phi : X \dashrightarrow \mathbb{P}_k^N$ and $\psi : \mathbb{P}_k^N \dashrightarrow X$ such that $\psi \circ \phi$ is the identity map on X. Since k is infinite, this implies that $X(k)$ is non-empty. Since X is a smooth, proper, retract rational variety, we have $\pi_0^{\mathbb{A}^1}(X) = *$, and so $\mathcal{S}(X)(k) = *$. Thus, to prove that X is naively $\mathbb{A}^1$-connected, it suffices to prove that for any point x, there exists a chain of $\mathbb{A}^1$-homotopies of X_x (see Notation 5.1) in X connecting the canonical morphism $\omega_x : X_x \to X$ to a morphism that factors through some morphism $\operatorname{Spec} k \to X$.

Let us fix a point $x \in X$. We will denote the ring $\mathcal{O}^h_{X,x}$ by R. We now set up some notation for working with the scheme $\mathbb{P}^1_R$.

We use the notation in Section 5.2, so that $\mathbf{0}$ and ∞ denote the points $(0:1)$ and $(1:0)$ of $\mathbb{P}^1_k = \operatorname{Proj} k[T_0, T_1]$ respectively. Let σ_0 and σ_∞ be the sections of the projection morphism $\mathbb{P}^1_R \cong \mathbb{P}^1_k \times \operatorname{Spec} R \to \operatorname{Spec} R$, mapping $\operatorname{Spec} R$ isomorphically onto the closed subschemes $\mathcal{Z}(T_0) = \{\mathbf{0}\} \times \operatorname{Spec} R$ and $\mathcal{Z}(T_1) = \{\infty\} \times \operatorname{Spec} R$, respectively. We will denote the rational function T_0/T_1 by t and thus identify the open subscheme $\mathbb{P}^1_R \backslash \mathcal{Z}(T_1)$ with $\operatorname{Spec} R[t]$.

We will construct a morphism $H : \mathbb{P}^1_R \to X$ such that $H \circ \sigma_0 = \omega_x$ and $H \circ \sigma_\infty$ factors through some morphism $\operatorname{Spec} k \to X$. Clearly, this will prove the result.

There exists an ideal sheaf $\mathcal{K}$ on $\mathbb{P}^N_k$ such that if $\pi : Y \to \mathbb{P}^N_k$ is the blowup of $\mathbb{P}^N_k$ at $\mathcal{K}$, the map $\chi := \psi \circ \pi$ is a morphism from Y to X. The sheaf $\mathcal{K}$ can be chosen so that $W := \mathcal{Z}(\mathcal{K})$ is a variety (possibly reducible) of codimension ≥ 2. Let $V = \mathbb{P}^N_k \backslash W$. Let $U \subset X$ be an open subset on which ϕ is defined and such that $\phi(U) \subset V$. The ideal sheaf $\mathcal{K}$ corresponds to a homogeneous ideal of $k[X_0, \ldots, X_N]$ generated by homogeneous polynomials $p_1, \ldots, p_r$. We may assume, without loss of generality, that the polynomials $p_1, \ldots, p_r$ are all of the same degree l. Note that $r \geq 2$.

The polynomials $p_1, \ldots, p_r$ define global sections of $\mathcal{K}(l)$, which generate $\mathcal{K}(l)$. Thus, we obtain a surjective morphism $\mathcal{O}^r_{\mathbb{P}^N} \to \mathcal{K}(l)$. Thus, we have the following sequence of homomorphisms of sheaves of graded $\mathcal{O}_{\mathbb{P}^N}$-rings

$$Sym(\mathcal{O}^r_{\mathbb{P}^N}) \to Sym(\mathcal{K}(l)) \to \bigoplus_{j=0}^{\infty} \mathcal{K}^j(l).$$

This gives us a closed embedding of Y into $\mathbb{P}^N_k \times \mathbb{P}^{r-1}_k$.

Using Lemma 5.2, we choose an infinitesimal homotopy $h : \operatorname{Spec} R\{t\} \to X$

starting at ω_x such that h maps the point η_0 of $\operatorname{Spec} R\{t\}$, corresponding to the ideal $\mathfrak{m}R\{t\}$, into U. This gives us a rational map $\phi \circ h : \operatorname{Spec} R\{t\} \dashrightarrow \mathbb{P}^N_k$, which can be represented by an $(N+1)$-tuple $\mathbf{f} := (f_0, \dots, f_N)$ where $f_i \in R\{t\}$. We choose the f_i to be coprime in the unique factorization domain $R\{t\}$. Let I denote the ideal $\langle f_0, \dots, f_N \rangle$. The rational map $\phi \circ h$ is a morphism if and only if this ideal is principal. Let J denote the ideal $\langle p_1(\mathbf{f}), \dots, p_r(\mathbf{f}) \rangle$.

Recall that we have chosen h in such a way that the point η_0 of $\operatorname{Spec} R\{t\}$, corresponding to the ideal $\mathfrak{m}R\{t\}$, is mapped into U. Thus, the rational map $\phi \circ h$ is well-defined on η_0. Since we have chosen the f_i to be coprime elements of the unique factorization domain $R\{t\}$, it follows that at least one of the f_i does not vanish on η_0. By performing a change of coordinates on $\mathbb{P}^N_k$, we may reduce to the situation where none of the f_i vanishes on η_0. (Such a change of coordinates exists since k is an infinite field.) Thus, for each i, we have $f_i = u_i \widetilde{f_i}$ where u_i is a unit in $R\{t\}$ and $\widetilde{f_i}$ is a Weierstrass polynomial.

Recall that $\phi \circ h$ maps the point η_0 into V. The zero set of the ideal $\langle p_1, \dots, p_r \rangle$ is contained in the complement of V. Thus, at least one of the polynomials p_i does not vanish on η_0. We may assume that all the p_i's are of the same degree. Suppose p_1 does not vanish on η_0. Then, for each $i \neq 1$, we can replace p_i by $p_i + \epsilon_i p_1$ where ϵ_i is 0 if p_i does not vanish at η_0 and is equal to 1 otherwise. Thus, we may assume that none of the polynomials p_i vanishes at η_0.

Thus, $p_i(\mathbf{f}) = v_i \cdot P_i$ where v_i is a unit in $R\{t\}$ and P_i is a Weierstrass polynomial. Let $p := t \cdot \left(\prod_i \widetilde{f_i}\right)^2 \cdot \left(\prod_j P_j\right)^2$. Then, p is an element of the ideal IJ and it is a Weierstrass polynomial with $\deg_t(p) \geq 1$.

For $0 \leq i \leq N$, we can express f_i in the form

$$f_i = \alpha_i + p\beta_i$$

where $\alpha_i \in R[t]$ with $\deg_t(\alpha_i) < \deg_t(p)$ and $\beta_i \in R\{t\}$.

For $0 \leq i \leq N$, let $\overline{\alpha}_i(t) \in \kappa(x)[t]$ be the image of $\alpha_i(t)$ under the quotient homomorphism $R[t] \to R[t]/\mathfrak{m}R[t] = \kappa(x)[t]$. Let d be the largest non-negative integer such that t^d divides $\overline{\alpha}_i(t)$ for all i. Let $(\lambda_0, \ldots, \lambda_N) \in k^{N+1}$ such that $(\lambda_0 : \ldots : \lambda_N) \in V(k)$. (The existence of such an $(N+1)$-tuple $(\lambda_0, \ldots, \lambda_N)$ follows from the assumption that k is infinite.) We apply Lemma 5.4 to the two $(N+1)$-tuples

$$(\overline{\alpha}_0(t)/t^d, \overline{\alpha}_1(t)/t^d, \ldots, \overline{\alpha}_N(t)/t^d) \qquad \text{and} \qquad (\lambda_0, \ldots, \lambda_N)$$

of polynomials in $\kappa(x)[t]$. We see that there exists an $(N+1)$-tuple $(\mu_0, \ldots, \mu_N) \in k^{N+1}$ such that the $(N+1)$ polynomials $R_0(t), \ldots, R_N(t)$ in $\kappa(x)[t]$ the polynomials defined by

$$R_i(t) = \overline{\alpha}_i(t)/t^d + (\mu_i + \lambda_i t) \cdot t^{\deg_t(p)-d},$$

are coprime and define a morphism $u : \mathbb{P}^1_{\kappa(x)} \to \mathbb{P}^N_k$ mapping $\mathbb{P}^1_{\kappa(x)} \backslash \{\mathbf{0}, \infty\}$ into $\mathbb{P}^N_k \backslash W$.

Let $\mathbf{g} = (g_0, \ldots, g_N)$ where $g_i(t) = \alpha_i(t) + (\mu_i + \lambda_i t) \cdot p(t)$ for $0 \leq i \leq N$. Then, if $\overline{g}_i(t) \in \kappa(x)[t]$ is the image of $g_i(t)$ under the quotient homomorphism $R[t] \to \kappa(x)[t]$, we see that $\overline{g}_i(t) = t^d R_i(t)$ for all i. Thus, the following conditions hold:

(A) The polynomials $\overline{g}_0(t), \ldots, \overline{g}_N(t)$ have no common zeros in $\mathbb{A}^1_{\kappa(x)} \backslash \{0\}$.

(B) If $\overline{\mathbf{g}}$ denotes the $(N+1)$-tuple $(\overline{g}_0, \ldots, \overline{g}_N)$, then the collection of polynomials $p_1(\overline{\mathbf{g}}), \ldots, p_r(\overline{\mathbf{g}})$ in $\kappa(x)[t]$ has no common zero in $\mathbb{A}^1_{\kappa(x)} \backslash \{0\}$.

Let $\widetilde{I}$ and $\widetilde{J}$ be the ideals $\langle g_0, \ldots, g_N \rangle$ and $\langle p_1(\mathbf{g}), \ldots, p_r(\mathbf{g}) \rangle$ of $R[t]$ respectively.

For $0 \leq i \leq N$,

$$g_i = f_i + p \cdot (\mu_i + \lambda_i t - \beta_i) = f_i[1 + (p/f_i) \cdot (\mu_i + \lambda_i t - \beta_i)]. \tag{5.3.1}$$

As t divides p/f_i, it is a non-unit in $R\{t\}$, and so g_i is a unit multiple of f_i in $R\{t\}$, for each i . In particular, we have $\widetilde{I} R\{t\} = I$.

Similarly, for $1 \leq i \leq r$,

$$p_i(\mathbf{g}) - p_i(\mathbf{f}) = \sum_j (g_j - f_j) \cdot Q_{ij}$$

where Q_{ij} is some element of $R\{t\}$. Thus,

$$p_i(\mathbf{g}) = p_i(\mathbf{f})[1 + (p/p_i(\mathbf{f})) \cdot Q'_i] \tag{5.3.2}$$

for some $Q'_i \in R\{t\}$. As t divides $p/p_i(\mathbf{f})$, it is a non-unit in $R\{t\}$, and so $p_i(\mathbf{g})$ is a unit multiple of $p_i(\mathbf{f})$ in $R\{t\}$. This proves that $\widetilde{J}R\{t\} = J$.

We would now like to show that the R-algebra homomorphism $R[t]/(\widetilde{I}\widetilde{J}) \to R\{t\}/(IJ)$ is an isomorphism. This statement is trivially true if $\widetilde{I}\widetilde{J}$ is the unit ideal. Thus, let us assume, for now, that $\widetilde{I}\widetilde{J}$ is not the unit ideal. We would like to apply Lemma C.5, and so we verify that $\widetilde{I}\widetilde{J}$ satisfies the hypothesis of that lemma.

Condition (A), which was imposed on the $(N+1)$-tuple $\mathbf{g}$, implies that if $\widetilde{I}$ is not the unit ideal, then the only prime ideal containing $\widetilde{I}$ and $\mathfrak{m}R[t]$ is $\langle m, t\rangle$. Condition (B) implies the same for $\widetilde{J}$. Since at least one of the ideals $\widetilde{I}$ and $\widetilde{J}$ is not the unit ideal, it follows that $\widetilde{I}\widetilde{J}$ satisfies condition (2) of Lemma C.5.

Now, we verify that the homomorphism $R \to R[t]/(\widetilde{I}\widetilde{J})$ is a finite extension. For this, it suffices to find an element of $\widetilde{I}\widetilde{J}$ such that its leading coefficient (as a polynomial in t) is an invertible element of R.

First, we note that there exists an index i_0, $0 \le i_0 \le N$ such that $\lambda_{i_0} \neq 0$. Thus, g_{i_0} is a polynomial in t with a leading coefficient that is a unit in R. Secondly, we observe that for $1 \le j \le r$, $p_j(\mathbf{g})$ is a polynomial in t with degree $\le \deg(p_j) \cdot (\deg_t(p) + 1)$. The coefficient of $t^{\deg(p_j)\cdot(\deg_t(p)+1)}$ is equal to $p_j(\lambda_0, \dots, \lambda_N)$. Since $(\lambda_0 : \dots, \lambda_N)$ is in V, there exists an index j_0 such that $p_{j_0}(\lambda_0, \dots, \lambda_N) \neq 0$. Thus, $p_{j_0}(\mathbf{g})$ is a polynomial in t with a leading coefficient that is an invertible element in R. Thus, $g_{i_0} \cdot p_{j_0}(\mathbf{g}) \in \widetilde{I}\widetilde{J}$ is a polynomial in t with a leading coefficient that is an invertible element of R. Thus, we see that the homomorphism $R \to R[t]/(\widetilde{I}\widetilde{J})$ is a finite extension.

Thus, if $\widetilde{I}\widetilde{J}$ is not the unit ideal, we may now apply Lemma C.5 to conclude that the morphism

$$\theta : \operatorname{Spec} R\{t\} \to \operatorname{Spec} R[t]$$

induces an isomorphism of the closed subscheme $\mathcal{Z}(IJ) \subset \operatorname{Spec} R\{t\}$ with the closed subscheme $\mathcal{Z}(\widetilde{I}\widetilde{J}) \subset \operatorname{Spec} R[t]$. Of course, as we noted above, if $\widetilde{I}\widetilde{J}$ happens to be the unit ideal, $\mathcal{Z}(IJ) \to \mathcal{Z}(\widetilde{I}\widetilde{J})$ is trivially an isomorphism.

Let $\tau : B \to \operatorname{Spec} R\{t\}$ denote the blowup of $\operatorname{Spec} R\{t\}$ at the ideal IJ and let $\widetilde{\tau} : \widetilde{B} \to \operatorname{Spec} R[t]$ denote the blowup of $\operatorname{Spec} R[t]$ at the ideal $\widetilde{I}\widetilde{J}$. Since $\widetilde{I}\widetilde{J}R\{t\} = IJ$, we see that

$$B \cong \widetilde{B} \times_{\operatorname{Spec} R[t]} \operatorname{Spec} R\{t\}.$$

Let us denote the projection morphism $B \to \widetilde{B}$ by θ'. Since θ maps $\mathcal{Z}(IJ)$ isomorphically onto the closed subscheme $\mathcal{Z}(\widetilde{I}\widetilde{J})$ of $\operatorname{Spec} R[t]$, it follows that θ' maps $\tau^{-1}(\mathcal{Z}(IJ))$ isomorphically onto $\widetilde{\tau}^{-1}(\mathcal{Z}(\widetilde{I}\widetilde{J}))$.

Since the ideal sheaf $\tau^{-1}(I)\cdot\mathcal{O}_B$ is invertible, the rational map $h' : \operatorname{Spec} R\{t\} \to \mathbb{P}^N_k$ defined by the $(N+1)$-tuple $(f_0, \ldots, f_N)$ lifts to a morphism $B \to \mathbb{P}^N$. Since the ideal sheaf $\tau^{-1}(J)\cdot\mathcal{O}_B$ is also invertible, it further lifts to a morphism $h'' : B \to Y$. Thus, the diagram

$$\begin{array}{ccc} B & \xrightarrow{h''} & Y \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\chi} \\ \operatorname{Spec} R\{t\} & \xrightarrow{h} & X \end{array}$$

commutes.

Similarly, the rational map $\widetilde{h}' : \operatorname{Spec} R[t] \dashrightarrow \mathbb{P}^N_k$, defined by the $(N+1)$-tuple $(g_0, \ldots, g_N)$ lifts to a morphism $\widetilde{h}'' : \widetilde{B} \to Y$. Thus, the diagram

$$\begin{array}{ccccc} B & \xrightarrow{\theta'} & \widetilde{B} & \xrightarrow{\widetilde{h}''} & Y \\ {\scriptstyle\tau}\downarrow & & {\scriptstyle\widetilde{\tau}}\downarrow & & \downarrow{\scriptstyle\chi} \\ \operatorname{Spec} R\{t\} & \xrightarrow{\theta} & \operatorname{Spec} R[t] & & X \end{array}$$

commutes. Notice that, in the above diagram, we do not yet have a morphism from $\operatorname{Spec} R[t]$ to X making the diagram commute. We will prove that such a morphism exists.

We have the two morphisms h'' and $\widetilde{h}'' \circ \theta'$ from B to Y. These need not be equal. However we will show that they agree on $\tau^{-1}(\mathcal{Z}(IJ))$. This claim is trivial if IJ is the unit ideal. Thus, we now assume that IJ is not the unit ideal.

Let z be any point of $\tau^{-1}(\mathcal{Z}(IJ))$. We want to prove that that $h''(z) = \widetilde{h}'' \circ \theta'(z)$. Recall that we have fixed an embedding of Y into $\mathbb{P}^N_k \times \mathbb{P}^{r-1}_k$. Let $pr_1 : \mathbb{P}^N_k \times \mathbb{P}^{r-1}_k \to \mathbb{P}^N_k$ and $pr_2 : \mathbb{P}^N_k \times \mathbb{P}^{r-1}_k \to \mathbb{P}^{r-1}_k$ be the projection morphisms. It will suffice to prove that $pr_i \circ h''(z) = pr_i \circ \widetilde{h}'' \circ \theta'(z)$ for $i = 1, 2$.

For any element $r \in R\{t\}$, we will denote its image in $\mathcal{O}_{B,z}$ by r as well. Let $\mathfrak{n}_z$ denote the maximal ideal of the local ring $\mathcal{O}_{B,z}$. Since $\tau(z) \in \mathcal{Z}(IJ)$, there exists at least one element in the set $\{f_0, \ldots, f_N, p_1(\mathbf{f}), \ldots, p_r(\mathbf{f})\}$ which is a non-unit in $\mathcal{O}_{B,z}$. Let us pick one such element and denote it by q_z. Recall that we had chosen p to be equal to $t \cdot \left(\prod_i \widetilde{f}_i\right)^2 \cdot \left(\prod_j P_j\right)^2$. Thus, it follows that, in the ring $\mathcal{O}_{B,z}$, the non-unit element q_z divides p/f_i for every i and $p/p_j(\mathbf{f})$ for every j. We will use this observation in the following discussion.

The restriction of $pr_1 \circ h''$ to $\operatorname{Spec} \mathcal{O}_{B,z}$ is given by the $(N+1)$-tuple $(f_0, \ldots, f_N)$. We know that the ideal $I \cdot \mathcal{O}_{B,z}$ is principal. Thus, there exists an index i_z such that $f_i/f_{i_z} \in \mathcal{O}_{B,z}$ for all i. Let $f'_i = f_i/f_{i_z}$ for $0 \leq i \leq N$. If $\overline{f'_i}$ is the image of f'_i in $\mathcal{O}_{B,z}/\mathfrak{n}_z =: \kappa(z)$, the composition

$$\operatorname{Spec} \kappa(z) \xrightarrow{z} B \xrightarrow{pr_1 \circ h''} \mathbb{P}^N_k$$

is given by the $(N+1)$-tuple $(\overline{f'_0}, \overline{f'_1}, \cdots, \overline{f'_N})$. of elements in $\kappa(z)$. Note that $\overline{f'_{i_z}} = 1$.

Recall (see equation (5.3.1)) that $g_i = f_i(1 + (p/f_i)(\mu_i + \lambda_i t - \beta_i))$. Let $g'_i = f'_i(1 + (p/f_i)(\mu_i + \lambda_i t - \beta_i))$ for every i. Thus, we have $g_i = f_{i_z} g'_i$ for every i.

Let $\overline{g'_i}$ denote the image of g'_i in $\kappa(z)$. As we observed above, p/f_i is in $\mathfrak{n}_z$. Thus, we see that $\overline{f'_i} = \overline{g'_i}$ for every i.

The composition

$$\operatorname{Spec} \kappa(z) \xrightarrow{z} B \xrightarrow{pr_1 \circ \widetilde{h}'' \circ \theta'} \mathbb{P}^N_k$$

is given by the $(N+1)$-tuple $(\overline{g'_0}, \overline{g'_1}, \cdots, \overline{g'_N})$. Since $\overline{g'_i} = \overline{f'_i}$ for every i, we see that $pr_1 \circ h''(z) = pr_1 \circ \widetilde{h}'' \circ \theta'(z)$.

Similarly, using equation (5.3.2), we can show that $pr_2 \circ h''(z) = pr_2 \circ \widetilde{h}'' \circ \theta'(z)$ for any point $z \in \tau^{-1}(\mathcal{Z}(IJ))$. Thus, we conclude that $h''(z) = \widetilde{h}'' \circ \theta'(z)$ for any

$z \in \tau^{-1}(\mathcal{Z}(IJ))$.

Now, suppose z_1 and z_2 are two distinct points of $\widetilde{B}$ such that $\widetilde{\tau}(z_1) = \widetilde{\tau}(z_2) = z$. Then, as $\widetilde{\tau}$ is an isomorphism on the complement of $\mathcal{Z}(\widetilde{I}\widetilde{J})$, we see that $z \in \mathcal{Z}(\widetilde{I}\widetilde{J})$. Recall that θ maps $\theta^{-1}(\mathcal{Z}(\widetilde{I}\widetilde{J})) = \mathcal{Z}(IJ)$ isomorphically onto $\mathcal{Z}(\widetilde{I}\widetilde{J})$ and that θ' maps $\tau^{-1}(\mathcal{Z}(IJ))$ maps isomorphically onto $\widetilde{\tau}^{-1}(\mathcal{Z}(\widetilde{I}\widetilde{J}))$. Thus, there exist unique points y_1 and y_2 in $\psi^{-1}(\mathcal{Z}(IJ))$ such that $\theta'(y_i) = z_i$ for $i = 1, 2$. Also, $\widetilde{\tau} \circ \theta' = \theta \circ \tau$, and since θ is an injective on $\mathcal{Z}(IJ)$, we see that $\tau(y_1) = \tau(y_2)$. Since $\chi \circ h'' = h \circ \tau$, we see that $\chi \circ h''(y_1) = \chi \circ h''(y_2)$. Thus,

$$\begin{aligned} \chi \circ \widetilde{h}''(z_1) &= \chi \circ \widetilde{h}'' \circ \theta'(y_1) \\ &= \chi \circ h''(y_1) \\ &= \chi \circ h''(y_2) \\ &= \chi \circ \widetilde{h}'' \circ \theta'(y_2) = \chi \circ \widetilde{h}''(z_2). \end{aligned}$$

Note that $\widetilde{\tau}$ is a proper, birational morphism. Also, as $\operatorname{Spec} R[t]$ is normal, we have $\tau_*(\mathcal{O}_{\widetilde{B}}) \cong \mathcal{O}_{\operatorname{Spec} R[t]}$. Thus, we may apply [30, Lemma 8.11.1] and [11, chapter III Cor. 11.4] to conclude that there exists a morphism $\widetilde{h} : \operatorname{Spec} R[t] \to X$ such that $\chi \circ \widetilde{h}'' = \widetilde{h} \circ \widetilde{\tau}$.

Now, we will show that $\widetilde{h}$ can be extended to a morphism $H : \mathbb{P}^1_R \to X$. (Recall that $\mathbb{P}^1_R = \operatorname{Proj} R[T_0, T_1]$ contains $\operatorname{Spec} R[t]$ as the open subscheme $\mathbb{P}^1_R \backslash (\operatorname{Spec} R \times \{\infty\})$, via the identification $t = T_0/T_1$.)

For $0 \le i \le N$, let $G_i(T_0, T_1) \in R[T_0, T_1]$ be defined by

$$G_i(T_0, T_1) = T_1^{\deg_t(p)+1} \cdot g_i(T_0/T_1).$$

Thus, each G_i is a homogeneous polynomial of degree $\deg_t(p) + 1$ in $R[T_0, T_1]$. The coefficient of $T_0^{\deg_t(p)+1}$ in $G_i(T_0, T_1)$ is λ_i. Recall that there exists an index i_0 such that λ_{i_0} is a non-zero element of k. Thus, G_{i_0} has no zero in the closed subscheme $\mathcal{Z}(T_1)$ of $\mathbb{P}^1_R$. Thus, the rational map $H' : \mathbb{P}^1_R \dashrightarrow \mathbb{P}^N_k$ defined by the $(N+1)$-tuple $(G_0, \dots, G_N)$ is defined on an open subscheme of $\mathbb{P}^1_R$ containing the closed subscheme $\mathcal{Z}(T_1)$. The restriction of H' to the open subscheme $\mathbb{P}^1_R \backslash \mathcal{Z}(T_1) =$

$\operatorname{Spec} R[t]$ is given by the $(N+1)$-tuple $(g_0(t), \ldots, g_N(t))$. Thus, it is the same as the rational map $\widetilde{h}'$. Thus, H' is defined on the open subscheme $\mathbb{P}^1_R \backslash \mathcal{Z}(\widetilde{I})$. (Note that the morphism $\mathcal{Z}(\widetilde{I}) \hookrightarrow \operatorname{Spec} R[t] \hookrightarrow \mathbb{P}^1_R$ is closed since $\mathcal{Z}(\widetilde{I})$ is finite over $\operatorname{Spec} R$. The same is true for the closed subschemes $\mathcal{Z}(\widetilde{J})$ and $\mathcal{Z}(\widetilde{I}\widetilde{J})$ of $\operatorname{Spec} R[t]$. So, we view $\mathcal{Z}(\widetilde{I})$, $\mathcal{Z}(\widetilde{J})$ and $\mathcal{Z}(\widetilde{I}\widetilde{J})$ as closed subschemes of $\mathbb{P}^1_R$.)

On the open subscheme $\operatorname{Spec} R[t] \backslash \mathcal{Z}(\widetilde{I}\widetilde{J})$ of $\operatorname{Spec} R[t]$, where $\psi \circ \widetilde{h}'$ is well-defined, it agrees with the restriction of the morphism $\widetilde{h}$. Thus, we see that $\psi \circ H'$ agrees with $\widetilde{h}$ on $\mathbb{P}^1_R \backslash (\mathcal{Z}(\widetilde{I}\widetilde{J}) \cup \mathcal{Z}(T_1))$. Since $\mathbb{P}^1_R \backslash \mathcal{Z}(T_1)$ (where $\widetilde{h}$ is defined) and $\mathbb{P}^1_R \backslash \mathcal{Z}(\widetilde{I}\widetilde{J})$ (where $\psi \circ H'$ is defined) form a Zariski open cover of $\mathbb{P}^1_R$, we see that there exists a morphism $H : \mathbb{P}^1_R \to X$ extending $\widetilde{h}$. Now we compute the morphisms $H \circ \sigma_0$ and $H \circ \sigma_\infty$ from $\operatorname{Spec} R$ to X.

The morphism $H \circ \sigma_0$ is the same as $\widetilde{h} \circ \sigma_0$. So we will now compute $\widetilde{h} \circ \sigma_0$. We will prove that it is the same as the canonical morphism ω_x. Note that $h \circ \widehat{\sigma}_0 = \omega_x$. (Recall from Section 5.1 that $\widehat{\sigma}_0$ is the morphism $\operatorname{Spec} R \to \operatorname{Spec} R\{t\}$ corresponding to the quotient homomorphism $R\{t\} \to R\{t\}/tR\{t\} \cong R$.) Thus, it is enough to prove that $\widetilde{h} \circ \sigma_0 = h \circ \widehat{\sigma}_0$. For this, it will suffice to show that if η denotes the generic point of $\operatorname{Spec} R$, then the two compositions

$$\operatorname{Spec} \kappa(\eta) \longrightarrow \operatorname{Spec} R \underset{\widetilde{h} \circ \sigma_0}{\overset{h \circ \widehat{\sigma}_0}{\rightrightarrows}} X$$

are equal.

The rational map $h' : \operatorname{Spec} R\{t\} \dashrightarrow \mathbb{P}^N_k$ is defined on the open subscheme $\operatorname{Spec} R\{t\} \backslash \mathcal{V}(\widetilde{I})$ of $\operatorname{Spec} R\{t\}$ and it is represented by the $(N+1)$-tuple $(f_0, \ldots, f_N)$. Since t divides p, the image of f_i under the quotient homomorphism $R\{t\} \to R\{t\}/tR\{t\} \cong R$ is $\alpha_i(0)$, i.e. the constant term in the polynomial $\alpha_i(t) \in R[t]$. Since the f_i were chosen to be coprime, we see that at least one of the elements $\alpha_0(0), \ldots, \alpha_N(0)$ is non-zero. Thus, the morphism $h' \circ \widehat{\sigma} \circ \eta : \operatorname{Spec} \kappa(\eta) \to \mathbb{P}^N$ is represented by the $(N+1)$-tuple $(\alpha_0(0), \ldots, \alpha_N(0))$. Similarly, the morphism $\widetilde{h}' \circ \sigma_0 \circ \eta : \operatorname{Spec} \kappa(\eta) \to \mathbb{P}^N$ is also represented by the same $(N+1)$-tuple. Composing with ψ, we obtain the desired conclusion that the morphism $h \circ \widehat{\sigma}_0 \circ \eta$ is equal to $\widetilde{h} \circ \sigma_0 \circ \eta$. Thus, it follows that $\widetilde{h} \circ \sigma_0$ is equal to ω_x.

The restriction of H' to the closed subscheme $\mathcal{Z}(T_1)$ of $\mathbb{P}^1_R$ maps $\mathcal{Z}(T_1)$ to the point $(\lambda_0 : \ldots : \lambda_N)$ of $\mathbb{P}^N_k$. The morphism H agrees with H' on an open subscheme of $\mathbb{P}^1_R$ containing $\mathcal{Z}(T_1)$. Thus $H \circ \sigma_\infty$ maps $\operatorname{Spec} R$ to the k-valued point $\psi((\lambda_0 : \ldots : \lambda_N))$ of X. This completes the proof. □

Appendix A

Model categories

The theory of model categories provides a framework for doing computations in the localization of a category with respect to a class of morphisms. The prototypical example is that of the category of topological space, which one localizes with respect to the class of weak equivalences to obtain the homotopy category. The theory of model categories axiomatizes the machinery of classical homotopy theory. Model categories were introduced by Quillen in [25]. We will use [14] and [15] as our main references for this material.

A.1 Basic definitions

Before we define a model category, we fix two pieces of terminology. Firstly, recall that if $\mathcal{C}$ is a category and X and Y are objects of $\mathcal{C}$, we say that X is a *retract* of Y if there exist morphisms $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = 1_X$. This notion is easily extended to morphisms as well. Recall that the morphisms in a category $\mathcal{C}$ also form a category, which we denote by $\mathbf{Mor}_{\mathcal{C}}$. Given two morphisms $f : X \to Y$ and $g : U \to V$ in $\mathcal{C}$, a morphism from f to g in the category $\mathbf{Mor}_{\mathcal{C}}$ is a commutative square of the form

$$\begin{CD} X @>{p}>> U \\ @V{f}VV @VV{g}V \\ Y @>{q}>> V. \end{CD}$$

Thus, we say that f is a retract of g if we have a commutative diagram of the form

$$\begin{CD} X @>{p}>> U @>{p'}>> X \\ @V{f}VV @VV{g}V @VV{f}V \\ Y @>{q}>> V @>{q'}>> Y \end{CD}$$

such that $p' \circ p = 1_X$ and $q' \circ q = 1_Y$.

The second piece of terminology we recall is that of right/left lifting properties of morphisms. Given morphisms $i : A \to B$ and $p : X \to Y$, we say that (i, p) is a *lifting pair* if given any commutative square of the form

$$\begin{CD} A @>>> X \\ @V{i}VV @VV{p}V \\ B @>>> Y, \end{CD}$$

there exists a morphism $B \to X$ making the resulting diagram commute. In this case, we will also say that i has the *left-lifting property* with respect to p, and that p has the *right-lifting property* with respect to i.

Definition A.1. A *model category* is a category $\mathcal{M}$ equipped with three distinguished classes of morphisms, called *weak equivalences*, *cofibrations* and *fibrations*, satisfying the axioms (1)-(5) given below. A cofibration (resp. fibration) that is also a weak equivalence is called a *trivial cofibration* (resp. *trivial fibration*).

1. $\mathcal{M}$ is closed under small limits and colimits.

2. Weak equivalences, cofibrations and fibrations are closed under retracts.

3. Let f and g be two composable morphisms. Then if any two out of f, g and $g \circ f$ is a weak equivalence, so is the third.

4. Cofibrations have the left lifting property with respect to all trivial fibrations and trivial cofibrations have the left lifting property with respect to all fibrations.

5. Every morphism $f : X \to Y$ has two functorial factorizations as follows:

   (i) $f = q \circ i$ where i is a cofibration and q is a trivial fibration, and

   (ii) $f = p \circ j$ where j is a trivial cofibration and p is a fibration.

The three distinguished classes of morphisms are said to define a *model structure* on the category $\mathcal{M}$.

The object of interest in such a situation is the localization of $\mathcal{M}$ with respect to the class of weak equivalences. Fibrations and cofibrations constitute the machinery that makes it possible to compute morphisms in the localization.

Definition A.2. Let $\mathcal{M}$ be a model category. The localization of $\mathcal{M}$ with respect to the class of weak equivalences is called the *homotopy category* of $\mathcal{M}$. We denote it by $Ho(\mathcal{M})$.

Remark A.3. It can be proved that a morphism is a trivial cofibration if and only if it has the left-lifting property with respect to all fibrations. Analogous characterizations of cofibrations, fibrations and trivial fibrations in terms of lifting properties also hold. Thus, we see that any two of the three distinguished classes mentioned in Definition A.1 determine the third.

Remark A.4. Observe that the axioms for a model category are self-dual. In other words, a given model structure on $\mathcal{M}$ defines a model structure on the opposite category $\mathcal{M}^{op}$ where the weak equivalences are the opposites of weak equivalences in $\mathcal{M}$, the cofibrations are the opposites of fibrations in $\mathcal{M}$ and the fibrations are the opposites of cofibrations in $\mathcal{M}$.

Thus, given a statement about model categories, one can write the *dual* of the statement by reversing the directions of all morphisms and interchanging the role of fibrations and cofibrations, and of limits and colimits. The proof of any statement can also be a dualized to obtain a proof of the dual. This is the *principle of duality* in model categories.

Since a model category $\mathcal{M}$ admits all small colimits and limits, it has an initial object, which we denote by $\emptyset$, and a final object, which we denote by $*$. (It is possible that $\emptyset$ and $*$ are isomorphic, in which case the model category is said to be *pointed*.) An object X is said to be *cofibrant* (resp. *fibrant*) if the canonical morphism $\emptyset \to X$ (resp. $X \to *$) is a cofibration (resp. fibration). An object is said to be *cofibrant-fibrant* if it is both, cofibrant as well as fibrant.

A *fibrant approximation functor* on the model category $\mathcal{M}$ is a functor $F : \mathcal{M} \to \mathcal{M}$ equipped with a natural transformation $Id \to F$ such that for any object X of $\mathcal{M}$, the object $F(X)$ is fibrant and the morphism $X \to F(X)$ is a weak equivalence. We can similarly define the notion of cofibrant approximation functors. The existence of both, fibrant and cofibrant approximation functors, is guaranteed by condition (5) in Definition A.1.

We mention a few classical examples.

Example A.5. Let **Top** denote the category of topological spaces. We now describe the classical model structure on this category. See [15, Section 2.4] for the proof.

A continuous map $f : X \to Y$ is said to be a weak equivalence if it induces a bijection $\pi_0(X) \xrightarrow{\sim} \pi_0(Y)$ and group isomorphisms $\pi_i(X, x) \xrightarrow{\sim} \pi_i(Y, f(x))$ for any $x \in X$.

For $n \geq 0$, let D^n denote the closed unit ball in $\mathbb{R}^n$. A continuous map is said to be a fibration if it has the right lifting property with respect to morphisms of the form $D^n \hookrightarrow D^n \times [0, 1]$, $x \mapsto (x, 0)$ for all non-negative integers n. (Such maps are also called *Serre fibrations*.) It can be proved that every topological space is a fibrant object.

The cofibrations are the maps which have the left lifting property with respect to maps which are fibrations as well as weak equivalences.

Example A.6. The category of simplicial sets is an important example which we will discuss in some detail in Section A.2.

Example A.7. Let R be a ring and let $Ch(R)$ denote the category of chain complexes of (left) R-modules. A morphism $f : X_\bullet \to Y_\bullet$ between chain complexes is said to be a weak equivalence if the induced homomorphism $H_n(f) : H_n(X) \to H_n(Y)$ between homology modules is an isomorphism for all n. We define a morphism to be a cofibration if it is an injection. The fibrations are defined to be the morphisms having the right lifting property with respect to cofibrations which are also weak equivalences. (See [15, Theorem 2.3.13] for the proof.)

Now we briefly explain how cofibrations and fibrations provide the machinery to compute the morphisms in the homotopy category.

Let $\mathcal{M}_{cf}$ denote the full subcategory of cofibrant-fibrant objects of $\mathcal{M}$. Let $Ho(\mathcal{M}_{cf})$ denote the localization of $\mathcal{M}_{cf}$ with respect to the class of weak equivalences in $\mathcal{M}_{cf}$. Then, we have an obvious functor $Ho(\mathcal{M}_{cf}) \to Ho(\mathcal{M})$. Since every object of $\mathcal{M}$ is weakly equivalent to an object of $\mathcal{M}_{cf}$ (which can actually be functorially chosen), it is easy to prove that $Ho(\mathcal{M}_{cf}) \to Ho(\mathcal{M})$ is an equivalence of categories. (See [15, Proposition 1.2.3].) However, it is possible to get a much more elegant description of morphisms in $Ho(\mathcal{M}_{cf})$.

Definition A.8. Let $\mathcal{M}$ be a model category.

1. Let X be an object of $\mathcal{M}$. A *cylinder object* for $\mathcal{M}$ is a factorization

$$X \coprod X \xrightarrow{i} Cyl(X) \xrightarrow{p} X$$

of the morphism $1_X \coprod 1_X : X \coprod X \to X$, where i is a cofibration and p is a weak equivalence. (Such a factorization exists due to condition (5) in Definition A.1.) The morphism i can be written in the form $i = i_0 \coprod i_1$ where i_0 and i_1 are morphisms from X to $Cyl(X)$. (Cylinder objects for a given object X are not unique.)

2. Let $f, g : X \to Y$ be a pair of morphisms in $\mathcal{M}$. A *left homotopy* from f to g consists of a choice of a cylinder object X as in (1), and a morphism $H : Cyl(X) \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$. If such a homotopy exists, we say that f is *left homotopic* to g.

3. By dualizing (1), we obtain the notion of a *path object* for any given object of $\mathcal{M}$.

4. By dualizing (2), we obtain the notion of a *right homotopy* between two morphisms in $\mathcal{M}$.

5. If a pair of maps is both left and right homotopic, we say that it is *homotopic.*

6. A morphism $f : X \to Y$ in $\mathcal{M}$ is said to be a *homotopy equivalence* if there exists a morphism $g : Y \to X$ such that $g \circ f$ is homotopic to 1_X and $f \circ g$ is homotopic to 1_Y.

Proposition A.9. *Let $\mathcal{M}$ be a model category. Let X be a cofibrant object and Y be a fibrant object. Then, a pair of maps $f, g : X \to Y$ is left homotopic if and only if it is right homotopic. Moreover, in this case, being left (and hence right) homotopic is an equivalence relation on $Mor_{\mathcal{M}}(X, Y)$.*

The above result tells us that for any two objects X and Y of $\mathcal{M}_{cf}$, homotopy is an equivalence relation on the set $Mor_{\mathcal{M}}(X, Y)$. Let us denote the set of equivalence classes by $Mor_{\mathcal{M}}(X, Y)/\sim$. Then, it is easy to check homotopy classes of morphisms are well-behaved under composition. Indeed, we can construct a category $\mathcal{M}_{cf}/\sim$, the objects of which are the cofibrant-fibrant objects of $\mathcal{M}$ and the set of morphisms from an object X to an object Y is the set $Mor_{\mathcal{M}}(X, Y)/\sim$. Clearly, the image of $f \in Mor_{\mathcal{M}}(X, Y)$ maps to an isomorphism in $\mathcal{M}_{cf}/\sim$ if and only if it is an homotopy equivalence.

It is easy to see that any homotopy equivalence is a weak equivalence. Thus, the functor $\mathcal{M}_{cf} \to Ho(\mathcal{M}_{cf})$ factors as $\mathcal{M}_{cf} \to \mathcal{M}_{cf}/\sim \to Ho(\mathcal{M}_{cf})$.

Theorem A.10 (See [15, Prop. 1.2.8 and Cor. 1.2.9]). *Let $\mathcal{M}$ be a model category. Let X and Y be cofibrant-fibrant objects of $\mathcal{M}$. Then a morphism $f : X \to Y$ is a weak equivalence if and only if it is a homotopy equivalence. The functor $\mathcal{M}_{cf} \to Ho(\mathcal{M}_{cf})$ is an isomorphism of categories.*

The above result is a generalization of the classical result of Whitehead, which states that a continuous map between CW-complexes is a weak equivalence if and only if it is homotopy equivalence.

We end this section by describing the appropriate notion of functors between model categories. To require functors between model categories to preserve the model structure is too restrictive in practice. Instead, it is more appropriate to consider adjoint pairs such that the left adjoint preserves "half" of the model structure and the right adjoint preserves the other half.

Definition A.11. Let $\mathcal{M}$ and $\mathcal{N}$ be model categories. We say that a functor $F : \mathcal{M} \to \mathcal{N}$ is a *left Quillen functor* if it is a left adjoint and preserves cofibrations and trivial cofibrations. Similarly, we say that a functor $G : \mathcal{N} \to \mathcal{M}$ is a *right Quillen functor* if it is a right adjoint and preserves fibrations and trivial fibrations.

It is easy to verify that if F is a left Quillen functor, its right adjoint is a right Quillen fucntor and vice versa. If F is a left Quillen functor and G is its adjoint, we say that the pair (F, G) is a *Quillen pair.*

Let $Q : \mathcal{M} \to \mathcal{M}$ is a cofibrant approximation functor. Then, if (F, G) are as above, it can be proved that F takes weak equivalences between cofibrant objects into weak equivalences. Thus, the functor $F \circ Q$ takes weak equivalences to weak equivalences, and hence induces a functor $LF : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$, which we call as the *left derived functor of F*. Similarly, if $R : \mathcal{N} \to \mathcal{N}$ is a fibrant approximation functor, $G \circ R$ induces a functor $RG : \mathrm{Ho}(\mathcal{N}) \to \mathrm{Ho}(\mathcal{M})$. One can show that (LF, RG) is an adjoint pair (see [15, Lemma 1.3.10]). If this pair actually gives an equivalence between $\mathrm{Ho}(\mathcal{M})$ and $\mathrm{Ho}(\mathcal{N})$, we say that (F, G) is a *Quillen equivalence.*

A.2 Simplicial sets

We first define the *cosimplicial indexing category* $\mathbf{\Delta}$.

- For any non-negative integer n, let $[n]$ denote the ordered set $\{0 < 1 < \ldots < n\}$. The set of objects of $\mathbf{\Delta}$ is $\{[n] | n \in \mathbb{Z}_{\geq 0}\}$.
- The morphisms from $[m]$ to $[n]$ in $\mathbf{\Delta}$ are the order-preserving maps.

The opposite category $\mathbf{\Delta}^{op}$ is called as the *simplicial indexing category.*

Definition A.12. Let $\mathcal{C}$ be any category.

1. A *cosimplicial object in* $\mathcal{C}$ is a functor $\mathbf{\Delta} \to \mathcal{C}$. The category of cosimplicial objects in $\mathcal{C}$ is denoted by $\mathbf{\Delta}\mathcal{C}$. Given a cosimplicial object $X : \mathbf{\Delta} \to \mathcal{C}$, the object $X([n])$ will be written as X^n.
2. A *simplicial object in* $\mathcal{C}$ is a functor $\mathbf{\Delta}^{op} \to \mathcal{C}$. The category of simplicial objects in $\mathcal{C}$ is denoted by $\mathbf{\Delta}^{op}\mathcal{C}$. Given a simplicial object $X : \mathbf{\Delta}^{op} \to \mathcal{C}$, the object $X([n])$ will be written as X_n.

To begin with, we are particularly interested in the case $\mathcal{C} = \mathbf{Set}$. A simplicial object in **Set** is called a *simplicial set.* If X is a simplicial set, the set X_n is called the *set of n-simplices* of X.

For any non-negative integer n, we define the *standard n-simplex* $\Delta^n \in Obj(\mathbf{\Delta}^{op}\mathbf{Set})$ be the functor $Mor_{\mathbf{\Delta}}(-, [n])$, which is a contravariant functor from $\mathbf{\Delta}$ to **Set** and hence may be interpreted as a covariant functor from $\mathbf{\Delta}^{op}$ to **Set**. The functor $[n] \mapsto \Delta^n$ is a covariant functor from $\mathbf{\Delta}$ to $\mathbf{\Delta}^{op}\mathbf{Set}$ and so is a cosimplicial object in $\mathbf{\Delta}^{op}\mathbf{Set}$. We denote this cosimplicial object by Δ. Using Yoneda's lemma, it is easy to see that for any simplicial set X, there is a natural bijection $X_n \xrightarrow{\sim} Mor_{\mathbf{\Delta}^{op}\mathbf{Set}}(\Delta^n, X)$.

Note that there are precisely two morphisms from $[0]$ to $[1] = \{0 < 1\}$, one which carries 0 to 0 and the other which carries 0 to 1. The corresponding morphisms $\Delta^0 \to \Delta^1$ will be denoted by i_0 and i_1 respectively and called the *endpoints* of Δ^1.

In order to define a model structure on $\mathbf{\Delta}^{op}\mathbf{Set}$, we first want to define what are the We now wish to define a functor from $\mathbf{\Delta}^{op}$ to **Top**, called the *geometric realization functor*. For this, we first need to define a cosimplicial object in **Top**, which we will denote by $|\Delta|$.

Let e_0^n be the element $(0, \ldots, 0) \in \mathbb{R}^n$. For $1 \leq i \leq n$, let e_i^n be the element of $\mathbb{R}^n$ having i-th coordinate equal to 1 and all other coordinates equal to 0. Let

$$|\Delta|^n := \{\sum_{i=0}^{n} t_i e_i^n | 0 \leq t_i \leq 1 \text{ and } \sum_{i=0}^{n} t_i = 1\}.$$

Thus, $|\Delta|^n$ is the convex hull of $e_0^n, \ldots, e_n^n$. If $f : [m] \to [n]$ is a morphism in $\mathbf{\Delta}$, we define $|\Delta|(f) : |\Delta|^m \to |\Delta|^n$ by

$$|\Delta|(f)(\sum_{i=0}^{m} t_i e_i^m = \sum_{i=0}^{n} t_i e_{f(i)}^n).$$

It is easy to see that $|\Delta|(g \circ f) = |\Delta|(g) \circ |\Delta|(f)$. Thus, $|\Delta|$ is a cosimplicial object in **Top**.

For any topological space X, we define $Sing(X)$ to be a simplicial set by setting by $Sing(X)_n = Mor_{\mathbf{Top}}(|\Delta|^n, X)$. For any $f : [m] \to [n]$, the corresponding map $Sing(X)(f) : Mor_{\mathbf{Top}}(|\Delta|^n, X) \to Mor_{\mathbf{Top}}(|\Delta|^m, X)$ is given by precomposing with $|\Delta|(f) : |\Delta|^m \to |\Delta|^n$. This gives a functor $Sing : \mathbf{Top} \to \mathbf{\Delta}^{op}\mathbf{Set}$. It can be proved that this functor has a left adjoint, which we write as $X \mapsto |X|$ for any simplicial set X. One can verify that $|\Delta^n| = |\Delta|^n$. The functor $X \mapsto |X|$ is called the *geometric realization* functor.

As an example, observe that $|\Delta^0|$ is just a point, $|\Delta^1|$ is actually isomorphic to the interval $[0, 1]$ and the geometric realizations of the endpoint morphisms $i_0, i_1 : \Delta^0 \to \Delta^1$ correspond to the endpoints of the interval $|\Delta^1|$.

Given any simplicial set X, we define $\pi_0(X)$ to be equal to $\pi_0(|X|)$, i.e. the set of path connected components of the topological space $|X|$. Given any $x \in X_0$, we view it as a morphism $x : \Delta^0 \to X$. Applying the geometric realization functor, we get $|x| : |\Delta^0| \to |X|$. Since $|\Delta^0| \cong *$ (a point), we may identify $|x|$ with a point of the space $|X|$. For any integer $i > 0$, we define $\pi_i(X, x)$ to be equal to the group $\pi_i(|X|, |x|)$.

A morphism $f : X \to Y$ of simplicial sets is said to be a *weak equivalence* if and only if it induces a bijection $\pi_0(X) \to \pi_0(Y)$ and a group isomorphisms $\pi_i(X, x) \to \pi_i(Y, f(x))$ for any $x \in X_0$. In other words, f is a weak equivalence if and only if $|f| : |X| \to |Y|$ is a weak equivalence in the model category of topological spaces.

A morphism $f : X \to Y$ of simplicial sets is a *cofibration* if and only if it is a monomorphism, i.e. $f_n : X_n \to Y_n$ is an injection for every n. Thus, every simplicial set is a cofibrant object.

A *fibration* of simplicial sets is a morphism that has the right lifting property with respect to every morphism that is both a cofibration as well as a weak equivalence. Fibrations of simplicial sets are also called as *Kan fibrations* and fibrant simplicial sets are called as *Kan complexes*.

It can be proved that this gives a model structure on **Top** (see [15, Chapter 3]). Also, the pair $(| - |, Sing(-))$ gives a Quillen equivalence between $\mathbf{\Delta}^{op}\mathbf{Set}$ and **Top**.

A.3 Simplicial model categories

We will restrict ourselves to a very brief sketch of the notion of simplicial model categories. A precise discussion may be found in [14, Section 9.1]. We also refer the reader to a more general discussion in [15, Section 4.2] regarding model categories enriched over monoidal model categories.

A *simplicial category* is a category $\mathcal{C}$, equipped with a rule that associates to each pair of objects X and Y, a simplicial set $Map(X, Y)$, called the *simplicial mapping space* from X to Y. This is required to satisfy the following requirements:

- The association $(X, Y) \mapsto Map(X, Y)$ is required satisfy all the properties satisfied by sets of morphisms. In other words, there exists an associative composition rule for mapping spaces, an identity map $i_X : \Delta^0 \to Map(X,X)$ and the analogues of the left and right unit laws hold.

- For every pair of objects (X,Y), we have a bijection $Map(X,Y)_0 \cong Mor_{\mathcal{C}}(X,Y)$ which is compatible with the composition rules.

Example A.13. A basic example is the category of simplicial sets. For simplicial sets X and Y, we define the simplicial set $Map(X,Y)$ by $Map(X,Y)_n = Map(X \times \Delta^n, Y)$. Given a morphism $f : [m] \to [n]$ in $\mathbf{\Delta}$, the corresponding map $Map(X,Y)_n \to Map(X,Y)_m$ is induced by the map $\Delta(f) : \Delta^m \to \Delta^n$. Given simplicial sets K and L, we define $K \otimes L := K \times L$ and $K^L := Map(L,K)$.

A *simplicial model category* is a model category $\mathcal{M}$ that is also equipped with the structure of a simplicial category, along with bifunctors

$$\mathcal{M} \times \mathbf{\Delta}^{op}\mathbf{Set} \to \mathcal{M}, \qquad (Z,K) \mapsto Z \times K$$

and

$$(\mathbf{\Delta}^{op}\mathbf{Set})^{op} \times \mathcal{M} \to \mathcal{M}, \qquad (K,Z) \mapsto Z^K$$

satisfying the following axioms:

1. There are natural isomorphisms

$$Map(X \otimes K, Y) \cong Map(K, Map(X,Y)) \cong Map(X, Y^K).$$

   Note that the mapping space in the center is from the simplicial category structure on the category of simplicial sets while the mapping spaces on the left and right are from the simplicial category structure on $\mathcal{M}$.

2. Given a cofibration $i : A \to B$ and a fibration $p : X \to Y$ in $\mathcal{M}$, the map of simplicial sets

$$Map(B,X) \to Map(A,X) \times_{Map(A,Y)} Map(B,Y)$$

   is a fibration. It is a trivial fibration if either i or p is a weak equivalence.

The second condition above can viewed as an enhanced version of the lifting axiom in the definition of a model category (condition (4) in Definition A.1). Indeed, it can be interpreted to mean that in a diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

where i is a cofibration, p is a fibration and at least one of i or p is a weak equivalence, a lifting morphism $B \to Y$ not only exists but is also unique up to homotopy equivalence (see [14, Prop. 9.6.1]).

We note a consequence of the second condition and the adjointness relation in the first condition.

Lemma A.14 ((see [14, Proposition 9.3.8]))**.** *If $j : L \to K$ is an inclusion (i.e. cofibration) of simplicial sets and $p : X \to Y$ is a fibration in $\mathcal{M}$, then the morphism $X^K \to X^L \times_{Y^L} Y^K$ is a fibration in $\mathcal{M}$. It is a trivial fibration if either j or p is a weak equivalence.*

A.4 Simplicial sheaves

Let T be a Grothendieck site and let $Shv(T)$ denote the category of sheaves of sets on T. A *simplicial sheaf* on T is a sheaf taking values in the category of simplicial sets. It is easy to see that such a sheaf may also be viewed as a simplicial object in the category $Shv(T)$. Thus, we will denote the category of simplicial sheaves by $\Delta^{op}Shv(T)$. We will describe a model structure on this category, called the *locally injective model structure.*

Let $\mathcal{X}$ denote an object of $\boldsymbol{\Delta}^{op}Shv(T)$. Let $\pi_0(\mathcal{X})$ denote the sheafification of the presheaf $U \mapsto \pi_0(\mathcal{X}(U))$. This is called the sheaf of (simplicially) connected components of $\mathcal{X}$.

Let V be any object of T and let x be any element of $\mathcal{X}(V)$. Then, for any positive integer i, let $\pi_i(\mathcal{X}|_V, x)$ denote the sheafification of the presheaf $(U \mapsto$

$V) \mapsto \pi_i(\mathcal{X}(U), x|_U)$ on the site T/V.

We say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ is a *weak equivalence* if the following two conditions hold:

- The induced morphism $\pi_0(f) : \pi_0(\mathcal{X}) \to \pi_0(\mathcal{Y})$ is an isomorphism.
- For any object V of T, any element $x \in \mathcal{X}(V)$, and any integer $i > 0$, the induced morphism $\pi_i(\mathcal{X}|_V, x) \to \pi_i(\mathcal{Y}, f(x))$ is an isomorphism of group sheaves on the site T/V.

If T is a site with enough points, it can be easily seen that $f : \mathcal{X} \to \mathcal{Y}$ is a weak equivalence if and only if f induces a weak equivalence between the stalks of $\mathcal{X}$ and $\mathcal{Y}$ at all points of T.

A morphism in $\mathbf{\Delta}^{op}Shv(T)$ is said to be a *cofibration* if it is a monomorphism. A morphism is said to be a *fibration* if it has the right lifting property with respect to all cofibrations that are also weak equivalences. It can be proved that this defines a model structure on $\mathbf{\Delta}^{op}Shv(T)$. We refer to [16, Cor. 2.7] for the proof.

In fact, $\mathbf{\Delta}^{op}Shv(T)$ has the structure of a simplicial model category. For any simplicial set K, the sheaf associated to K will also be denoted by K. Then, for any two simplicial sheaves $\mathcal{X}$ and $\mathcal{Y}$, we define $Map(\mathcal{X}, \mathcal{Y})$ to be the simplicial set such that its set of n-simplicies is given by

$$Map(\mathcal{X}, \mathcal{Y})_n := Mor_{\mathbf{\Delta}^{op}Shv(T)}(\mathcal{X} \times \Delta, \mathcal{Y}).$$

If K is a simplicial set and $\mathcal{X}$ is a simplicial sheaf, $K \otimes \mathcal{X}$ is just the product $K \times \mathcal{X}$ of the constant sheaf associated to K with $\mathcal{X}$. The object $\mathcal{X}^K$ is simply $Map(K, \mathcal{X})$.

A.5 Left Bousfield localizations

Let $\mathcal{M}$ be a model category and let A be a class of morphisms in $\mathcal{M}$. We would like to modify the model structure on $\mathcal{M}$ in such a way that the morphisms in A

also become weak equivalences. This is analogous to the notion of localization of categories. However, depending on whether we require the "localization functor" to be a left or right Quillen functor, we obtain two different notions of localization.

Given a model category $\mathcal{M}$ and a class of maps $\mathcal{A}$, a *left localization* of $\mathcal{M}$ with respect to $\mathcal{A}$ is a model category $\mathcal{L}$ along with a left Quillen functor $j : \mathcal{M} \to \mathcal{L}$ such that the following properties hold:

(a) The left derived functor of j maps the images of the elements of $\mathcal{A}$ in $\mathrm{Ho}(\mathcal{M})$ to isomorphisms in $\mathrm{Ho}(\mathcal{L})$.

(b) If $\phi : \mathcal{M} \to \mathcal{N}$ is a left Quillen functor from $\mathcal{M}$ into any other model category $\mathcal{N}$ such that the left derived functor of ϕ maps the images of elements of A in $\mathrm{Ho}(\mathcal{M})$ to isomorphisms in $\mathrm{Ho}(\mathcal{N})$, then ϕ factors uniquely as $\phi = \widetilde{\phi} \circ j$ where $\widetilde{\phi}$ is a left Quillen functor.

A *right localization of* $\mathcal{M}$ *with respect to* $\mathcal{A}$ may be defined in an analogous manner.

We focus on a specific kind of left localization, called the *left Bousfield localization*. This notion and the related results that we quote below make sense in a very general setting (see [14, Chapter 3]). However, we restrict ourselves to the case of a simplicial model category since that is sufficient for our requirements.

Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{A}$ be a class of morphisms in $\mathcal{M}$. Let $C : \mathcal{M} \to \mathcal{M}$ be a cofibrant approximation functor and let $F : \mathcal{M} \to \mathcal{M}$ a fibrant approximation functor.

We say that an object Z is $\mathcal{A}$-*local* if for any morphism $f : U \to V$ in A, the morphism of simplicial sets $Map(C(V), F(Z)) \to Map(C(U), F(Z))$ is a weak equivalence. This is equivalent to saying that the map $Mor_{\mathrm{Ho}(\mathcal{M})}(V, Z) \to Mor_{\mathrm{Ho}(\mathcal{M})}(U, Z)$ is a bijection for any morphism $f : U \to V$ in $\mathcal{A}$.

A morphism $h : X \to Y$ in $\mathcal{M}$ is said to be an $\mathcal{A}$-local equivalence if for any $\mathcal{A}$-local object Z the morphism of simplicial sets $Map(C(Y), F(Z)) \to Map(C(X), F(Z))$ is a weak equivalence. This is equivalent to saying that the map $Mor_{\mathrm{Ho}(\mathcal{M})}(Y, Z) \to$

$Mor_{\mathrm{Ho}(\mathcal{M})}(X, Z)$ is a bijection for any $\mathcal{A}$-local object Z. It follows from the definition that the elements of $\mathcal{A}$ are all $\mathcal{A}$-local equivalences.

A *left Bousfield localization* of $\mathcal{M}$ with respect to $\mathcal{A}$ is a model category structure $L_{\mathcal{A}}\mathcal{M}$ on the underlying category of $\mathcal{M}$ such that the following conditions hold:

(a) The weak equivalences in $L_{\mathcal{A}}\mathcal{M}$ are precisely the $\mathcal{A}$-local equivalences.

(b) The cofibrations of $L_{\mathcal{A}}(\mathcal{M})$ are precisely the cofibrations of $\mathcal{M}$.

(c) The fibrations of $L_{\mathcal{A}}(\mathcal{M})$ are the morphisms which have the right lifting property with respect to cofibrations that are $\mathcal{A}$-local equivalences.

A left Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{A}$, if it exists, is actually a left localization of $\mathcal{M}$ with respect to $\mathcal{A}$ (see [14, Prop. 3.3.18]).

If a left Bousfield localization $L_{\mathcal{A}}(\mathcal{M})$ of $\mathcal{M}$ with respect to $\mathcal{A}$ exists, then the identity functor of the underlying category of $\mathcal{M}$ is a left Quillen functor from the model category $\mathcal{M}$ to the model category $L_{\mathcal{A}}(\mathcal{M})$. The right adjoint of this functor, which is the identity functor itself, is a right Quillen functor from $L_{\mathcal{A}}(\mathcal{M})$ to $\mathcal{M}$. It follows immediately from this that if an object X is fibrant in $L_{\mathcal{A}}(\mathcal{M})$, then it is an $\mathcal{A}$-local object which is fibrant in $\mathcal{M}$. The converse of this statement requires a small technical condition.

A model category $\mathcal{M}$ is *left proper* if every pushout of a weak equivalence along a cofibration is a weak equivalence.

Fact A.15. *Let $\mathcal{M}$ be a left proper simplicial model category. Let $\mathcal{A}$ be a class of morphisms in $\mathcal{M}$ such that the left Bousfield localization $L_{\mathcal{A}}(\mathcal{M})$ exists. Then, an object Z is fibrant in $L_{\mathcal{A}}(\mathcal{M})$ model structure if and only if it is $\mathcal{A}$-local and fibrant in $\mathcal{M}$ (see [14, Prop. 3.4.1].*

To prove the existence of a Bousfield localization requires us to impose certain technical conditions on the model category $\mathcal{M}$. Since we do not need the general result for our purposes, we will not discuss this question. (See [14, Chapter 4] for one such existence theorem.) We restrict ourselves to quoting the following:

Fact A.16 (see [24, Theorem 2.5, page 71])**.** *Let T be a small Grothendieck site and let $\mathcal{A}$ be a set of morphisms in $\mathbf{\Delta}^{op}Shv(T)$. Then the left Bousfield localization of $\mathbf{\Delta}^{op}Shv(T)$ exists.*

Appendix B

The Nisnevich topology

The Nisnevich topology is a Grothendieck topology for schemes. We review some basic properties of this topology. The main reference for this material is [24].

B.1 Nisnevich coverings

Definition B.1. Let X be a scheme. A *Nisnevich covering* of X is a family of étale morphisms $\{p_i : U_i \to X\}_i$ such that for any point x of X, there exists an index j and a point $u \in U_j$ such that $p_j(u) = x$ and the induced homomorphism $\kappa(x) \to \kappa(u)$ is an isomorphism of fields.

We now fix a base scheme S which is noetherian and of finite dimension. We consider the category Sm/S of smooth schemes of finite type over S. It is easy to check that Nisnevich coverings constitute a Grothendieck pretopology on Sm/S. The corresponding Grothendieck topology is called the *Nisnevich topology.* This topology is strictly finer than the Zariski topology and strictly coarser than the étale topology.

Recall that for any scheme X over S, the presheaf $h_X(-) := Mor_{Sm/S}(-, X)$ represented by X is a sheaf for the étale topology. As the étale topology is finer than the Nisnevich topology, it follows that h_X is a Nisnevich sheaf. This gives us a fully faithful embedding $Sm/S \to Shv(Sm/S)$, $X \mapsto h_X$.

B.2 Points for the Nisnevich topology

Let X be a smooth scheme of finite type over S and let x be a point of X. A *Nisnevich neighbourhood of x in X* is an étale morphism of pointed schemes $p : (U, u) \to (X, x)$ such that the induced morphism $\kappa(x) \to \kappa(u)$ is a field isomorphism. If $p : (U, u) \to (X, x)$ and $q : (V, v) \to (X, x)$ are two Nisnevich neighbourhoods of x in X, a morphism from p to q is defined to be a morphism $\phi : (U, u) \to (V, v)$ of pointed schemes, such that $p = q \circ \phi$. The Nisnevich neighbourhood of x in X form a cofiltered category. Given any Nisnevich sheaf $\mathcal{F}$, we define $\mathcal{F}_x = \varinjlim \mathcal{F}(U)$ where the direct limit is over all Nisnevich neighbourhoods $(U, u) \to (X, x)$ of x in X. We call $\mathcal{F}_x$ as the *stalk of $\mathcal{F}$ at x*. The functor $Shv(Sm/S) \to \mathbf{Set}$ is a point of the topos $Shv(Sm/S)$. This topos has enough points.

The above construction of stalks is a special case of a more general definition, which extends any presheaf on Sm/S to essentially smooth schemes. A scheme over S is said to be *essentially smooth* if it is the inverse limit of a cofiltered system of smooth schemes S-schemes with transition maps that are étale and affine. In general, an essentially smooth S-scheme is not on object of Sm/S. However, given a presheaf $\mathcal{F}$ on Sm/S, we define its value on an essentially smooth scheme $X = \varprojlim X_\alpha$ (where $(X_\alpha)_\alpha$ is a cofiltered system of smooth S-schemes as described above), we define $\mathcal{F}(X) := \varinjlim \mathcal{F}(X_\alpha)$.

Given a smooth scheme X over S and a point x of X, it is easy to see that the inverse limit of all the Nisnevich neighbourhoods of x in X is an essentially smooth scheme. In fact, as we see in Appendix C, it is precisely the henselian local scheme $\operatorname{Spec} \mathcal{O}^h_{X,x}$. Thus, the ring $\mathcal{O}^h_{X,x}$, which is the henselization of the usual local ring $\mathcal{O}_{X,x}$, is the appropriate notion of the "local ring at a point" for the Nisnevich topology.

B.3 Nisnevich sheaves

An important property of the Nisnevich topology is that Nisnevich sheaves can be described in terms of a special class of coverings.

Definition B.2. Let X be a scheme. An *elementary Nisnevich cover* of X is a pair of morphisms $(p_1 : U \to X, p_2 : V \to X)$ such that

(a) p_1 is an open immersion,

(b) p_2 is an étale morphism, and

(c) if $Z = X \backslash p_1(U)$, the morphism $V \times_X Z \to Z$ induced by p_2 is an isomorphism.

It is clear that the pair of maps in an elementary Nisnevich cover actually do constitute a Nisnevich covering of X. For an elementary Nisnevich cover of X as above, the square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p_2 \\ U & \xrightarrow{p_1} & X \end{array}$$

is called an *elementary distinguished square.*

Fact B.3 (see [24, Prop. 1.4, page 96])**.** *A presheaf $\mathcal{F}$ on Sm/S is a Nisnevich sheaf if and only if for any elementary Nisnevich cover $(p_1 : U \to X, p_2 : V \to X)$, the square*

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}(U \times_X V) \end{array}$$

is cartesian.

If $\mathcal{F}$ is a Nisnevich sheaf, the fact that the above square is cartesian follows from the definition of a sheaf. The converse, however, is non-trivial.

We will use the above fact to construct sections of Nisnevich sheaves. Suppose $\mathcal{F}$ is a Nisnevich sheaf and $(p_1 : U \to X, p_2 : V \to X)$ are a pair of morphisms which constitute a Nisnevich covering of X. To construct a section of $\mathcal{F}$ over X, we need to pick a section $\sigma \in \mathcal{F}(U)$ and a section $\tau \in \mathcal{F}(V)$ such that the following three conditions hold:

(a) The images of σ under restriction along two projection morphisms $\mathcal{F}(U) \rightrightarrows \mathcal{F}(U \times_X U)$ are the same.

(b) The images of τ under restriction along two projection morphisms $\mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_X V)$ are the same.

(c) The image of σ under $\mathcal{F}(U) \to \mathcal{F}(U \times_X V)$ is equal to the image of τ under $\mathcal{F}(V) \to \mathcal{F}(U \times_X V)$.

However, if (p_1, p_2) is an elementary Nisnevich cover as in Definition B.2, we are only required to check (c). This makes it very easy to construct sections in such situations.

Appendix C

Henselization

Given a variety X and a point x on X, a Zariski neighbourhood of x in X is an open subset of X containing x. These neighbourhoods form a cofiltered system (where the morphisms are just inclusions) and the inverse limit of this system is the spectrum of the local ring $\mathcal{O}_{X,x}$. As we discussed in B.2, we can perform an analogous construction using Nisnevich neighbourhoods of a point and thus obtain the notion of an "infinitesimal neighbourhood of x at X", which happens to be the spectrum of the henselization of the local ring $\mathcal{O}_{X,x}$.

More generally, we can talk about the infinitesimal neighbourhood of a Z in X, where Z is a closed subscheme of X. In the context of the Zariski topology, this gives us the notion of *Zariski pairs*. In the context of the Nisnevich topology, we have the corresponding notion of *henselian pairs*.

C.1 Henselian pairs

In this section, a *pair* will mean an ordered pair of the form (R, I) where R is a ring and I is an ideal in R. A morphism of pairs $(R, I) \to (S, J)$ is a homomorphism $\phi : R \to S$ such that $\phi(I) \subset J$.

Given a pair (R, I), an étale R-algebra $R \to S$ is said to be a *Nisnevich neighbourhood of* (R, I) if $R/I \to S/IS$ is an isomorphism. (This is called as an *étale neighbourhood of* (R, I) in [27].) The category of such R-algebras is a a filtered subcategory of the category of all R-algebras. Let R^h denote the direct limit of the Nisnevich neighbourhoods of (R, I) (computed in the category of all R-algebras). We say that R^h is the *henselization of* R at I. If I^h denote the ideal IR^h, we will also say that (R^h, I^h) is the *henselization of* (R, I).

We say that a pair (R, I) is a *henselian pair* if the canonical morphism $(R, I) \to (R^h, I^h)$ is an isomorphism of pairs. It can be proved that henselization is a functor from the category of all pairs into the full subcategory of henselian pairs. It is the left adjoint of the inclusion functor from the category of henselian pairs into the category of pairs ([29, Lemma 15.12.1]). In particular, for a fixed pair (R, I), the morphism $(R, I) \to (R^h, I^h)$ is an initial object in the category of all morphisms $(R, I) \to (S, J)$ from (R, I) into a henselian pair (S, J).

For a pair (R, I), let $\widehat{R}$ denote the I-adic completion of R and let $\widehat{I}$ denote the ideal $I\widehat{R}$. Then the pair $(\widehat{R}, \widehat{I})$ is henselian. Thus, the canonical morphism $(R, I) \to (\widehat{R}, \widehat{I})$ factors as $(R, I) \to (R^h, I^h) \to (\widehat{R}, \widehat{I})$.

Fact C.1. *The canonical morphism* $(R, I) \to (R^h, I^h)$ *induces an isomorphism* $\widehat{R} \to \widehat{R^h}$ *(see [29, Lemma 15.12.2]). Here* $\widehat{R^h}$ *is the* I*-adic (or equivalently* I^h*-adic) completion of* R^h*.*

A little more can be said in the noetherian case:

Fact C.2 (see [29, 15.12.4])**.** *Let* (R, I) *be a pair such that* R *is noetherian. Then if* R^h *is the henselization of* R *at* I*, the canonical homomorphism* $R \to R^h$ *is a flat extension. Also,* R^h *is noetherian and* $R^h \to \widehat{R}$ *is a faithfully flat extension.*

If R is a local ring with maximal ideal $\mathfrak{m}$, the henselization of R at $\mathfrak{m}$ will be called as the *henselization of* R. If the pair $(R, \mathfrak{m})$ is a henselian pair, we say that R is a *henselian local ring*. In this special case, Fact C.3 implies the following:

Fact C.3 (See [12, Theorem 18.5.11, (c)])**.** *Let* $U = \operatorname{Spec} R$ *where* R *is a henselian local ring. Let* u *be the closed point of* U*. Let* $f : V \to U$ *be a finite morphism.*

Then, V is the disjoint union of the schemes $\operatorname{Spec} \mathcal{O}_{V,v}$ *where v ranges over all the points of $f^{-1}(u)$.*

C.2 Henselian power series

Let R be a commutative ring and let $R[t]$ denote the polynomial ring in one variable with coefficients in R. Then, the henselization of $R[t]$ at the ideal $\langle t \rangle$ will be called as the ring of *henselian power series* and will be denoted by $R\{t\}$.

First, let us take R to be a noetherian ring. Then, Fact C.2 implies that $R\{t\}$ injects into $R[[t]]$ and we will identify $R\{t\}$ with its image in $R[[t]]$. Since $R\{t\} \to R[[t]]$ is faithfully flat, it is easy to see that an element $f \in R\{t\}$ is a unit if and only if it is a unit in $R[[t]]$, i.e. if and only if its image under the quotient map $R[[t]] \to R[[t]]/\langle t \rangle \cong R$ is a unit in R.

The results stated in the previous paragraph continue to hold even when R is not noetherian. Indeed, the functor $R \to R\{t\}$ commutes with filtered colimits. So one can express R as a filtered colimit of its finitely generated $\mathbb{Z}$-algebras and generalize the above statements to the non-noetherian setting. (See [9, Subsection 2.1.2].)

If $(R, \mathfrak{m})$ is a local ring, a polynomial $f(t) \in R[t]$ is said to be a *Weierstrass polynomial* if it is of the form $f(t) = t^d + a_{d-1}t^{d-1} + \ldots + a_0$ where $a_i \in \mathfrak{m}$ for all i.

Fact C.4 ([9, Proposition 3.1.2])**.** *Let $(R, \mathfrak{m})$ be a henselian local ring. Let $f \in R\{t\} \backslash \mathfrak{m}R\{t\}$. Then f can be uniquely factored as $f = P \cdot u$ where P is a Weierstrass polynomial and u is a unit in $R\{t\}$. Also, the natural homomorphism $R[t]/\langle P \rangle \to R\{t\}/\langle f \rangle$ is an isomorphism.*

We will require the following easy consequence of this result.

Lemma C.5. *Let R be a henselian local ring with maximal ideal $\mathfrak{m}$. Let I be a proper ideal of $R[t]$ such that the following conditions hold:*

1. *The homomorphism $R \to R[t]/I$ is a finite extension.*

2. The only prime ideal of $R[t]$ containing I and $\mathfrak{m}R[t]$ is $\langle \mathfrak{m}, t\rangle$.

Then, the homomorphism $R[t]/I \to R\{t\}/IR\{t\}$ is an isomorphism.

Proof. Let $Z = \operatorname{Spec} R[t]/I$, which we view as a closed subscheme of $\operatorname{Spec} R[t]$. Let x_0 be the closed point of $\operatorname{Spec} R$ and let y_0 be the point of $\operatorname{Spec} R[t]$ corresponding to the ideal $\langle \mathfrak{m}, t\rangle$.

Let $\pi : Z \to \operatorname{Spec} R$ be the morphism corresponding to the R-algebra homomorphism $R \to R[t]/I$. According to condition (1), π is a finite morphism. Thus, if $z \in Z$ is any point, there exists a point z_0 in its closure such that $\pi(z_0) = x_0$. By (2), the closed subscheme Z and the fibre $\pi^{-1}(x_0)$ have only the point y_0 in common. Thus, we see that every point of Z lies in the closure of y_0.

By Fact C.3, we see that Z is isomorphic to $\operatorname{Spec} \mathcal{O}_{Z,y_0}$. Thus, we see that if $S = R[t]_{\langle \mathfrak{m},t\rangle}$, then the homomorphism $R[t]/I \to S/IS$ is an isomorphism.

The ring $R\{t\}$ is a local ring with maximal ideal $\langle \mathfrak{m}, t\rangle$. Thus, the canonical homomorphism $R[t] \to R\{t\}$ induces a local homomorphism $S \to R\{t\}$. This homomorphism is flat, and since it is a local homomorphism, it is faithfully flat. Thus, as $IS \not\subset \mathfrak{m}S$, we see that $IR\{t\} \not\subset \mathfrak{m}R\{t\}$. Let f be an element of $IR\{t\}\backslash\mathfrak{m}R\{t\}$. Then, by Fact C.4 $f = u \cdot p$ where u is a unit in $R\{t\}$ and p is a Weierstrass polynomial. Since $S \to R\{t\}$ is a faithfully flat extension, $IS = S \cap IR\{t\}$. Thus, $p \in IS$.

By Fact C.4, the ring homomorphism $R[t]/pR[t] \to R\{t\}/pR\{t\}$ is an isomorphism. Thus, it follows that the homomorphism $S/pS \to R\{t\}/pR\{t\}$ is surjective. It is also injective since it is a faithfully flat extension. Thus, it is an isomorphism.

As $pS \subset IS$ and $pR\{t\} \subset IR\{t\}$, it follows that $S/IS \to R\{t\}/IR\{t\}$ is an isomorphism. This completes the proof. □

C.3 The approximation property

Henselization and completion are two different notions of an "infinitesimal neighbourhood" of a closed subscheme. "Approximation theorems" allow us to solve problems in the completion of a ring, and then use it to obtain a solution in the henselization.

A noetherian local ring $(R, \mathfrak{m})$ is said to be an *approximation ring* if for any finite system of polynomial equations with coefficients in R, the set of solutions in R is dense, with respect to the $\mathfrak{m}$-adic topology, in the set of solutions in the $\mathfrak{m}$-adic completion $\widehat{R}$.

We note the following two results of Popescu.

Fact C.6 (See [26, Theorem 1.3])**.** *Excellent henselian local rings are approximation rings.*

Fact C.7 (See [[26, Corollary 3.5])**.** *Let* $(R, \mathfrak{m})$ *be an approximation ring. Then* $R\{t\}$ *is an approximation ring.*

In order to apply these results to the henselizations of local rings at points of varieties, we recall the following:

Fact C.8 (See [12, Corollary 18.7.6])**.** *Let* k *be a field. Let* X *be a variety over* k *and let* x *be a point of* X*. Then, the ring* $\mathcal{O}^h_{X,x}$ *is an excellent ring.*

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