Classification of Quaternionic Möbius transformations

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Certificate of Examination

This is to certify that the dissertation titled **Classification of Quaternionic Möbius transformations** submitted by **Mr.Varun Singh** (Reg.No.MS07028) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated : May 08 2012

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Krishnendu Gongopadhyay at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Varun Singh
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In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Krishnendu Gongopadhyay
(Supervisor)

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I am indebted to my thesis supervisor Dr. Krishnendu Gongopadhyay for his continuous encouragement and support. I deeply appreciate his way of working, start from basis, and must say that it has had a great influence on me. I owe much more to him than I can write here.

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Contents

Abstract

In this thesis our aim is to classify the quaternionic Möbius transformations. After identifying them with the isometries of the 1-dimensional quaternionic hyperbolic space, we use their conjugacy invariants to obtain the classification. Along the way, classification of isometries of the 2-dimensional real hyperbolic space is also discussed.

Almost all this work is available in the literature in one form or the other. I have learnt a great deal of the subject matter by discussing it with Dr. Gongopadhyay. Also, I have been highly influenced by the book *Geometry of Discrete Groups* by Alan Beardon and Jvyäskylä lecture notes by John Parker.

Notation

$\mathbf{H}^2$	:	$\{x + iy \in \mathbb{C} : y > 0\}$.
$\mathcal{R}e(z)$	:	Real part of z.
$\mathcal{I}m(z)$	:	Imaginary part of z.
$\mathbb{C}$	:	Complex plane.
$\mathbb{R}$	:	Real line.
$Isom(\mathbf{H}^2)$	:	Isometry group of $\mathbf{H}^2$.
$\mathbf{D}^2$	:	$\{z \in \mathbb{C} :\mid z \mid< 1\}$.
$\mathbb{H}$	:	Division ring of real quaternions.
σ	:	Quaternionic determinant.
τ	:	Quaternionic trace.
H	:	Hermitian matrix.
$\mathbb{B}$	:	$\{z \in \mathbb{H} :\mid z \mid< 1\}$.

Chapter 1

Introduction

Hyperbolic geometry was created in the first half of the nineteenth century in order to prove the dependence of Euclid's fifth postulate on the first four. Euclid wrote his famous Elements around 300 B.C. In the first book of elements Euclid develops plane geometry starting with basic geometric terms, five common notions concerning with magnitudes and five postulates. In modern language those can be stated as follows.

Let X be a set. L and C are sets of certain subsets of X. We call the elements of X as points, the elements of L as lines and the elements of C as circles. Then $\mathbb{E} = (X, \mathrm{L}, \mathrm{C})$ satisfies the following postulates:

(1) For all $A, B \in X$, $A \neq B$ there exists a unique $l \in \mathrm{L}$ such that $A \in l$.

(2) Given a $l \in \mathrm{L}$ there exists at least three points which do not belongs to l.

(3) For all l, $m \in \mathrm{L}$ we have either $l \cap m$ = a single point or $l \cap m = \phi$. If $l \cap m = \phi$ then l and m are called parallel lines.

(4) For a ordered pair (A, B) of points, there exits a unique $C \in \mathrm{C}$ with centre A and passing through B.

(5) There is an intuitive notion of angle between two lines. Euclid's classical fourth postulate says that all right angle are equal.

(6) (Modern version of Euclid's classical Fifth Postulate) For $l \in \mathrm{L}$ and $P \in X$ such that P does not belong to l, there exits a unique $m \in \mathrm{L}$ such that $P \in m$ and l is parallel to m.

In other words Euclid's classical fifth postulate can be stated as follows : Through a point outside a given line there is one and only one line parallel to the given line. For two thousand years mathematician attempted to establish Euclid's Fifth postulate from the other simple postulates. In each case one reduced the proof of the fifth postulate to the conjugation of the other postulates with an additional postulate which proved to be equivalent to the fifth postulate. As an incidence how much people tried it we note the reference of Gottingen Mathematician Kastner (1719 - 1800) who directed a thesis of student Klugel (1739 - 1812) which considered approximately thirty proof attempts for the parallel postulate !
Decisive progress came in the 19th century, when mathematician abandoned the effort to find a contradiction in the denial of the fifth postulate and instead workout carefully and completely the consequence of such a denial. It was found that a coherent theory arises if instead one assume that

Given a line and a point not on it, there is more than one line going through the given point that is parallel to the given line.

Unusual consequences of this change came to be recognized as fundamental and surprising properties of non-Euclidean geometry : geodesics were not straight lines, but curved; the sum of angles of a triangle were not equal to π and so forth.
Gauss began his meditation on the theory of parallel about 1792. After trying to prove the fifth postulate over twenty years, Gauss discovered that the denial of the fifth postulate leads to a new strange geometry which he called 'non-Euclidean geometry'. After investing its properties for over ten years and discovering no inconsistencies, Gauss was fully convinced of its consistency. In a letter to F.A Taurinus in 1824, he wrote:

The assumption that the sum of three angles of a triangle is smaller than 180 degrees leads to a geometry which is quiet different from our (Euclidean) geometry, but which is in itself completely consistence.

Gauss's assumption that the sum of the angle of a triangle is less than 180 degree is equivalent to the denial of Euclid's fifth postulate. Unfortunately Gauss never published his results on non-Euclidean geometry. Only a few year passed before non-Euclidean geometry was rediscovered independently by Nikolai Lobachevsky

and J.Bolyai. Lobachevsky published the first account of non-Euclidean geometry in 1829 in a paper entitled *on the principles of geometry.* A few year later, in 1932, Bolyai published an independent account of non-Euclidean geometry in a paper entitled *the absolute science in space.*
Gauss, Bolyai, Lobachevsky developed non-Euclidean geometry axiomatically on a synthetic basis. They didn't prove the consistency of their geometries. The basis necessary for an analytic study of hyperbolic non-Euclidean geometry was laid by Leonhard Euler, Gaspard Monge, and Gauss in their investigation of curved surfaces. Later on many people tried to find an analytic model of hyperbolic geometry where these five postulates can be proved and mathematician found several of them.

We shall consider in this exposition two of the most famous analytic models of the hyperbolic geometry which are known as Upper half space model and Poincaȑe disk model respectively. Our aim is to classify the isometries in these models.

Chapter 2

Hyperbolic Plane Geometry

To study the 2 - dimensional hyperbolic space, firstly we will use the upper-half space model.

$$\mathbf{H}^2 = \{x + iy : y > 0\}$$

and this is equipped with the metric

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y} \ .$$

Definition 2.0.1 A Geodesics is a path which minimizes the length between two points in hyperbolic plane. The geodesics in $\mathbf{H}^2$ are semi circles and straight lines orthogonal to the real axis $\mathbb{R}$.

Any two points on $\mathbf{H}^2$ can be joined by a unique geodesic and distance between those points is measured along geodesic.
To each piecewise continuously differentiable curve in $\mathbf{H}^2$, say $\gamma : [a, b] \longrightarrow \mathbf{H}^2$, $\gamma = \{z(t) = x(t) + iy(t) : t \in [a, b], y(t) > 0\}$

$$length(\gamma) = \int_a^b \frac{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}}{y(t)} = \int_a^b \frac{\frac{|dz|}{|dt|}}{y(t)} dt.$$

Definition 2.0.2 Hyperbolic distance ρ between two points z, w $\in \mathbf{H}^2$ can be defined as

$$\rho(z, w) = inf(length(\gamma))$$

where the infimum is taken over all γ which join z to w in $\mathbf{H}^2$.
It is clear that ρ is non-negative, symmetric and satisfies the Triangle Inequality

$$\rho\left(z, w\right) \leq \rho\left(z, u\right) + \rho\left(u, w\right).$$

Let z_0 and z_1 be two points in $\mathbf{H}^2$ with same x-coordinate and their y-coordinate be y_0 and y_1 respectively. Then length of the vertical segment γ joining z_0 and z_1 is

$$\int_\gamma ds = \left| \int_{y_0}^{y_1} \frac{dy}{y} \right| = \left| log\frac{y_1}{y_0} \right|.$$

Suppose l is a another different path from z_0 to z_1 then length of l is

$$= \left| \int_{y_0}^{y_1} \frac{1}{y} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \right|$$

$$\geq \left| \int_{y_0}^{y_1} \frac{1}{y} \left[\left(\frac{dy}{dt}\right)^2\right]^{\frac{1}{2}} dt \right|$$

$$= \left| \int_{y_0}^{y_1} \frac{1}{y} dy \right|.$$

Thus the length of l is greater than than length of γ. So, γ is the shortest path between P_0 and P_1. Hence all vertical lines in $\mathbf{H}^2$ are geodesics.

Isometries of $\mathbf{H}^2$

Definition 2.0.3 A transformation g : $\mathbf{H}^2 \longmapsto \mathbf{H}^2$ is an isometry of $\mathbf{H}^2$ if for any z, $w \in \mathbf{H}^2$, $\rho(z, w) = \rho(g(z), g(w))$.
The reflection in a line $x = c$ is given by

$$r(z) = 2c - \overline{z}$$

is an isometry of $\mathbf{H}^2$.
If $z = x + iy$ then $dz = dx + idy$. And

$$\frac{|dz|}{\mathcal{I}m(z)} = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

For $w = 2c - (x - iy)$, $dw = -(dx - idy)$ and $|dw| = \sqrt{dx^2 + dy^2} = |dz|$. This implies

$$\mathcal{I}m(z) = \mathcal{I}m(w), hence \ \frac{|dw|}{\mathcal{I}m(w)} = \frac{|dz|}{\mathcal{I}m(z)}.$$

This shows that the reflections are isometries of $\mathbf{H}^2$.

Definition 2.0.4 An Inversion ψ of $\mathbb{C}$ with respect to the circle $S(a, r) = \{z \in \mathbb{C} : | z - a |= r\}$ is defined by

$$\psi(z) = a + \left(\frac{r}{| z - a |}\right)^2 (z - a).$$

An inversion ψ of $\widehat{\mathbb{C}}$ in $S(0, 1)$ is an isometry of $\mathbf{H}^2$. Inversion in $S(0, 1)$ is

$$\psi : \widehat{\mathbb{C}} \longmapsto \widehat{\mathbb{C}}$$

where $\psi(z) = \frac{1}{\overline{z}}$. Suppose $\psi(z) = \frac{1}{\overline{z}} = w$, then

$$dw = -\overline{dz}(\overline{z})^{-2}$$

and

$$dz = -\overline{dw}(\overline{w})^{-2}$$

so,

$$ds^2 = \frac{| dw |^2}{(\mathcal{I}m(w))^2} = \frac{4dw\overline{dw}}{(w - \overline{w})^2}\ .$$

On substituting the value, we get

$$ds^2 = \frac{4dz\overline{dz}}{(z - \overline{z})^2}$$

$$ds^2 = \frac{| \, dz \, |^2}{(\mathcal{I}m(z)^2)} \ .$$

Hence, inversion in $S(0,1)$ is an isometry in $\mathbf{H}^2$.

Theorem 2.0.5 A subset L of $\mathbf{H}^2$ is a geodesic if and only if L is the intersection of H^2 with either a vertical straight line or a circle orthogonal to the real line.

Proof Suppose that there is a geodesic l of $\mathbf{H}^2$ which is neither a vertical line nor a semi circle orthogonal to real axis. Then, take any two points say P and Q on l. Then through P, Q there will be either a vertical line or a semi circle with center on the real line. For, if P and Q are not joined by vertical lines, join them by straight line. Take perpendicular bisector of PQ. Suppose it cuts the real axis at the point C. Then semi circle with centre C and radius CP or CQ will satisfy the assertion. But then the length of PQ along the above semi circle or vertical length will be greater than the length of PQ along l, which cannot be possible as we have shown above earlier. Hence l cannot be any other path than a vertical line, or a semi circle of above type.

$\square$

Corollary 2.0.6 *Any two points z and w in $\mathbf{H}^2$ can be joined by a unique geodesic and the hyperbolic distance between z, $w \in \mathbf{H}^2$ is equal to the hyperbolic length of the unique geodesic segment connecting these two points.*

2.1 The Isometry Group

Lemma 2.1.1 *Two matrices $P, Q \in SL(2, \mathbb{R})$ induces the same isometry if and only if $P = \pm\, Q$.*

Proof Here P, $Q \in SL(2,\mathbb{R})$ where $P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$P = \eta Q \ .$$

Where $\eta = \pm 1$. Case(1): If $\eta = 1$, then $a = A$, $b = B$, $c = C$, $d = D$ which implies for all $z \in \mathbb{C}$

$$\frac{az+b}{cz+d} = \frac{Az+B}{Cz+D} \ .$$

Case(2): If $\eta = -1$, then transformation representing both P and Q :

$$\frac{Az+B}{Cz+D} = \frac{-az-b}{-cz-d} \ ,$$

$$\frac{Az+B}{Cz+D} = \frac{(-1)az+b}{(-1)cz+d} \ ,$$

$$= \frac{az+b}{cz+d} \ .$$

Therefore both the isometry are same.
Conversely, Assume $P = Q$, then

$$\frac{Az+B}{Cz+D} = \frac{az+b}{cz+d}$$

for all $z \in \mathbf{H}^2$. Further,

$$(Az+B)(cz+d) = (Cz+D)(az+b),$$

$$Acz^2 + Adz + Bcz + Bd = aCz^2 + aDz + bCz + bD.$$

These polynomial are equal, and monomial of same degree have the same coefficient. This implies $Ac = aC$, $Bd = bD$ and $Ad+Bc = aD+bC$. Suppose η_1 and η_2 satisfies following

$$\frac{A}{a} = \frac{C}{c} = \eta_1 \quad and \quad \frac{B}{b} = \frac{D}{d} = \eta_2 \ .$$

Then

$$\eta_1 ad + \eta_2 bc = aD + bC = \eta_2 ad + \eta_1 bc.$$

Above equation can be rewritten as

$$\eta_1(ad - bc) = \eta_2(ad - bc).$$

Since $ad - bc = 1$, so $\eta_1 = \eta_2$. Therefore there is a η such that

$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c} = \frac{D}{d} = \eta.$$

This can be represented by the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \eta \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Equating determinant we get $\eta^2 = 1$ and so $\eta = \pm 1$.

$\square$

Lemma 2.1.2 $PSL(2, \mathbb{R}) \subset Isom(\mathbf{H}^2)$.

Proof We know that the translation $z \longrightarrow z + b$, $b \in \mathbb{R}$ and the dilations $z \longrightarrow az$, $a \in \mathbb{R}$, $a > 0$ are isometries of $\mathbf{H}^2$. Thus all transformations of the form $z \longrightarrow az + b$ a,$b \in \mathbb{R}$ $a \neq 0$, are isometries. Also the maps $\iota : z \longmapsto \frac{1}{z}$ and $\eta : z \longmapsto -\overline{z}$ are in $Isom(\mathbf{H}^2)$. Thus the maps of the form $z \longmapsto \frac{1}{cz+d}$, c,$d \in \mathbb{R}$, $c \neq 0$ are in $Isom(\mathbf{H}^2)$. It follows that all the linear fractional transformations of the form

$$S : z \longmapsto \frac{az + b}{cz + d},\ a, b, c, d \in \mathbb{R}$$

are in $Isom(\mathbf{H}^2)$ provided with some conditions on a, b, c, d which follow from the relation : $\mathcal{I}m \left(\frac{az+b}{cz+d}\right) > 0$. Now,

$$\mathcal{I}m \left(\frac{az + b}{cz + d}\right) = \frac{\mathcal{I}m(z)}{|cz + d|^2} det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which is > 0 if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0\ .$$

Thus $z \longmapsto \frac{az+b}{cz+d}$ is an isometry of $\mathbf{H}^2$ if and only if $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. Since the map ι is also in $Isom(\mathbf{H}^2)$, the map

$$\iota \circ S : z \longmapsto \frac{c\overline{z} + d}{a\overline{z} + b}$$

is in $Isom(\mathbf{H}^2)$ if and only if

$$det \begin{pmatrix} c & d \\ a & b \end{pmatrix} = -det \begin{pmatrix} a & b \\ c & d \end{pmatrix} < 0.$$

The group

$$M^+(1) = \{z \longmapsto \frac{az+b}{cz+d} : ad - bc > 0\}$$

may be identified with the group $SL(2,\mathbb{R})$ in a natural way: just divide the numerator and denominator of the linear fractional by $\sqrt{ad-bc}$. After this identification we see that $SL(2,\mathbb{R})$ acts on $\mathbf{H}^2$ as a subgroup of isometries under the action :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d} \ .$$

Note that for a given matrix A in $SL(2,\mathbb{R})$, A and $-A$ produce the same isometry under the above mention action. Hence $M^+(1)$ is isomorphic to $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\}$.

$\square$

Theorem 2.1.3 Any transformation in $PSL(2,\mathbb{R})$ maps geodesics onto geodesics in $\mathbf{H}^2$.

Proof Let $T \in PSL(2,\mathbb{R})$ be a transformation, where z and w be two distinct points in $\mathbf{H}^2$. From lemma 2.1.2 we know that T is an isometry of $\mathbf{H}^2$ and we also know that for $[z,w]$ a closed segment on the geodesic joining distinct points z and w in $\mathbf{H}^2$, $\rho(z,w) = \rho(z,\zeta) + \rho(\zeta,w)$ if and only if $\zeta \in [z,w]$ i.e. $T(\zeta) \in [T(z),T(w)]$. So it implies T maps segment $[z,w]$ onto segment $[Tz,Tw]$ and hence geodesics onto geodesics.

$\square$

Definition 2.1.4 The action of group G on set X is called *transitive* if X is non empty and for any $x, y \in X$ there exists a $g \in G$ such that $gx = y$.

Lemma 2.1.5 *$PSL(2,\mathbb{R})$ acts transitively on* $\mathbf{H}^2$.

Proof Let $ai + b \in \mathbf{H}^2$, $a > 0$. Then consider the transformation $T(z) = \frac{\frac{a}{\sqrt{a}}z + \frac{b}{\sqrt{a}}}{\frac{1}{\sqrt{a}}}$ induced by the

$$A_T = \begin{pmatrix} \frac{a}{\sqrt{a}} & \frac{b}{\sqrt{a}} \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix}$$

on $\mathbf{H}^2$. Applying A_T on i, $T(i) = ai + b$.
Now choosing a suitable element from $PSL(2, \mathbb{R})$ we can send any $z \in \mathbf{H}^2$ to i. On composing that with A_T we can see $PSL(2, \mathbb{R})$ acts transitively on $\mathbf{H}^2$.

□

Lemma 2.1.6 *$PSL(2, \mathbb{R})$ acts transitively on the set of all geodesics in $\mathbf{H}^2$.*

Proof It suffices to show that any geodesic l can be mapped onto the y-axis $x = 0$ by the action of an appropriate element of $PSL(2, \mathbb{R})$. If l is perpendicular to the real line, suppose it intersects the real line at the point c. Then the element of $PSL(2, \mathbb{R})$ corresponding to either $z \longmapsto z + c$, or , $z \longmapsto z - c$ does the job. If l is a semi circle perpendicular to the real axis, let the point of intersections with the real line be a, b, $b > a$. Then the element of $PSL(2, \mathbb{R})$ corresponding to the isometry $z \longmapsto \frac{z-b}{z-a}$ maps l onto the line $x = 0$.

□

Lemma 2.1.7 *$PSL(2, \mathbb{R})$ acts triply transitively on $\widehat{\mathbb{R}}$.*

Proof Suppose a, b, c in $\mathbb{R}$ are mutually distinct. We assume $a > b > c$.
$Case(1)$. Let $a \neq \infty$. Let

$$A = \begin{pmatrix} a - c & -b(a - c) \\ a - b & -c(a - b) \end{pmatrix}$$

Consider the action of $\frac{A}{|A|}$ on $\mathbf{H}^2$:

$$z \longmapsto \frac{(a - c)z - b(a - c)}{(a - b)z - (a - b)c} .$$

Then $Aa = 1,\ Ab = 0,\ Ac = \infty$.
$Case(2)$. Let $a = \infty$.
The transformation $z \longmapsto \frac{z-b}{z-c}$ maps (∞, b, c) onto $\{1, 0, \infty\}$. Thus any triplet $\{a, b, c\} \subset \widehat{\mathbb{R}}$ is mapped onto $\{1, 0, \infty\}$ by a suitable element of $PSL(2, \mathbb{R})$.

$\square$

Definition 2.1.8 Given four distinct points z_1, z_2, z_3, z_4 in $\mathbb{C}$, we define the cross ratio of z_1, z_2, z_3, z_4 to be

$$[z_1, z_2; z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$

Theorem 2.1.9 Let z, w be two distinct points in $\mathbf{H}^2$. Let p_z and p_w be the end points of geodesic on $\widehat{\mathbb{R}}$. Then

$$\rho(z, w) = \mid \ln(p_z, q_w; z, w) \mid$$

where $(p_z, q_w; z, w)$ denote the cross ratio of p_z, q_w, z, w.

Proof Consider when $z = ia$, $w = ib$, i.e. both points lie on the y-axis. Then $\{p_z, q_w\} = \{0, \infty\}$. Let γ be the segment of the y-axis joining ia and ib. Then

$$\rho(z, w) = length(\gamma) = \mid \int_a^b \frac{dy}{y} \mid = \mid \ln \frac{b}{a} \mid.$$

Note that $(0, \infty; z, w) = \frac{a}{b}$. Suppose σ is some other arc from ia to ib, then

$$\begin{aligned} \rho(z, w) &= \mid \int_\sigma \frac{1}{y} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \mid \\ &\geq \mid \int_a^b \frac{1}{y} \left[\left(\frac{dy}{dt}\right)^2\right]^{\frac{1}{2}} dt \mid \\ &= \mid \ln \frac{b}{a} \mid. \end{aligned}$$

Therefore $\rho(z,w) = \mid \ln(0,\infty,z,w) \mid$ where $z = ia$, $w = ib$.
Suppose z, w are points other than pure imaginary. Then there is a unique geodesic joining them. Let $A \in PSL(2,\mathbb{R})$ be such that, for $a, b \in \mathbb{R}$, $Az = ia$, $Aw = ib$, $Ap_z = 0$, $Aq_w = \infty$. Since linear fractional transformation preserve the cross ratio, we have

$$(0,\infty; ia, ib) = (Ap_z, Aq_w; Az, Aw) = (p_z, q_w; z, w).$$

Therefore $\rho(z,w) = \mid \ln(p_z, q_w; z, w) \mid$.

$\square$

Definition 2.1.10 $PSL^*(2,\mathbb{R}) = SL^*(2,\mathbb{R})/\{\pm I\}$ where $SL^*(2,\mathbb{R})$ is a group of real matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $det(g) = \pm 1$.

Theorem 2.1.11 The group $Isom(\mathbf{H}^2)$ is generated by fractional linear transformation in $PSL(2,\mathbb{R})$ together with $z \longmapsto -\overline{z}$, and is isomorphic to $PSL^*(2,\mathbb{R})$.

Proof Let ϕ be a isometry of $\mathbf{H}^2$, then ϕ maps geodesic to geodesic. Let I denote the positive Imaginary Axis, then $\phi(I)$ is a geodesic.
and there exists

$$g(z) = \frac{az+b}{cz+d},\ a,b,c,d \in \mathbb{R}\ \ and \quad ad - bc = 1$$

such that $g(\phi(I)) = I$.
For if $g(\phi(i)) \neq i$ then compose $g\phi$ with

$$\sigma : z \longmapsto kz$$

If this composition does not leave $(0,i)$ and (i,∞) invariant. Now, apply

$$z \longmapsto \frac{-1}{z}$$

Hence, $g\phi$ fixes each point of I. Let $z = x + iy \in \mathbf{H}^2$ and $g\phi(z) = u + iv$. Then for all positive t,

$$\rho(z,it) = \rho(g\phi(z), g\phi(it)),$$

$$= \rho(g\phi(z), it).$$

Using,

$$\sinh\left[\frac{1}{2}\rho(z,w)\right] = \frac{\mid z-w \mid}{2(Im(z)Im(w))^{\frac{1}{2}}}$$

we get

$$\frac{|z-it|}{2(yt)^{\frac{1}{2}}} = \frac{\mid u+iv-it \mid}{2(vt)^{\frac{1}{2}}}$$

$$\frac{x^2+(y-t)^2}{y} = \frac{u^2+(v-t)^2}{v}$$

$$v\left[\frac{x^2}{t^2}+\left(\frac{y}{t}-1\right)^2\right] = \left[\frac{u^2}{t^2}+\left(\frac{v}{t}-1\right)^2\right]y$$

Now as $t \longrightarrow \infty$

$$v = y.$$

We have

$$x^2+(y-t)^2 = u^2+(v-t)^2$$

Therefore $u = \pm x$. Thus $g\phi(z) = z$ or $-\overline{z}$. If $g\phi(z) = z$,

$$\phi(z) = g^{-1}(z).$$

This implies $\phi \in PSL(2,\mathbb{R})$. And when $g\phi(z) = -\overline{z}$,

$$\phi(z) = \frac{a\overline{z}+b}{c\overline{z}+d},\ with\ ad-bc = -1.$$

This proves the theorem.

□

2.2 The Poincaré Disk Model

Up to this point, we have focused our attention on developing the upper half model of the hyperbolic plane. There are number of other models of the hyperbolic plane. One of the most useful among those other model is the Poincaré disk model $\mathbf{D}^2$.

The underlying space of the Poincaré model of the hyperbolic plane is the open unit disk

$$\mathbf{D}^2 = \{z \in \mathbb{C} : \ \mid z \mid < 1\}$$

in the complex plane $\mathbb{C}$. Since upper half space and $\mathbf{D}^2$ are both disks on the Riemann sphere $\widehat{\mathbb{C}}$, there exits a map

$$f : \mathbf{H}^2 \longmapsto \mathbf{D}^2$$

such that $f(z) = \frac{z-i}{z+i}$. Note that f maps $\mathbf{H}^2$ onto the unit disk $\mathbf{D}^2$ and $f^{-1} : \mathbf{D}^2 \longmapsto \mathbf{H}^2$ is given by

$$f^{-1}(z) = \frac{i(1+z)}{1-z}.$$

The metric on $\mathbf{D}^2$ is given by $d(z, w) = \rho(f^{-1}(z), f^{-1}(w))$.
Let $v = f^{-1}(z) = \frac{i(1+z)}{1-z}$, then

$$\mathcal{I}m(v) = \frac{1- \mid z \mid^2}{\mid 1-z \mid^2}$$

Therefore

$$ds = \frac{\mid dv \mid}{\mathcal{I}m(v)} = \frac{2 \mid dz \mid}{1- \mid z \mid^2}.$$

Definition 2.2.1 Isometry group of $\mathbf{D}^2$ is defined as a set $\{f\varphi f^{-1} : \varphi \in \text{Isom}\ (\mathbf{H}^2)\}$, with composition of maps as its operator.

Lemma 2.2.2 *For any orientation preserving isometry of* $\mathbf{H}^2$, $\varphi(z) = \frac{az+b}{cz+d}$, *where* $a, b, c, d \in \mathbb{R}$, *and* $ad - bc = 1$, *we have*

$$f\varphi f^{-1}(z) = \frac{a'z - b'}{\overline{b'}z - a'}, \mid a' \mid^2 - \mid b' \mid^2 = 1$$

which is also an orientation preserving isometry .

Proof For $\varphi \in Isom(\mathbf{H}^2)$, $\varphi(z) = \frac{az+b}{cz+d}$, $ad - bc = 1, \quad a, b, c, d \in \mathbb{R}$.
Isometry for $\mathbf{D}^2$ is defined as $f\varphi f^{-1}(z)$.

$$\begin{aligned}
f\varphi f^{-1} &= f\varphi\left(\frac{i(1+z)}{1-z}\right) \\
&= f\left(\frac{(a\frac{i(1+z)}{1-z} + b)}{(c\frac{i(1+z)}{1-z} + d)}\right) \\
&= \frac{z(ia-b) + (b+ia) - iz(ic-d) - i(d+ic)}{z(ia-b) + (b+ia) + iz(ic-d) - i(d+ic)} \\
&= \frac{[(c-b) + i(a-d)z] + [(b+c) + i(a-d)]}{[i(a-d) - (b-c)]z + [i(a+d) + (b-c)]} \\
&= \frac{a'z - b'}{\overline{b'}z - \overline{a'}} .
\end{aligned}$$

Where $a' = (c-b) + i(a-d)$, $b' = -(b+c) + i(a-d)$.

□

Lemma 2.2.3 *For any orientation reversing isometry $\varphi(z) = \frac{a\overline{z}+b}{c\overline{z}+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc = -1$*

$$f\varphi f^{-1} = \frac{a'\overline{z} - b'}{\overline{b'}z - \overline{a'}}$$

with $\mid a' \mid^2 - \mid b' \mid^2 = -1$ which is also an orientation reversing isometry.

Theorem 2.2.4 The Isometries of the Poincaré disk model of the hyperbolic plane are given by

$$\begin{aligned}
&\frac{a'z - b'}{\overline{b'}z - \overline{a'}}\ , \quad \mid a' \mid^2 - \mid b' \mid^2 = 1, \\
&\frac{a'\overline{z} - b'}{\overline{b'}z - \overline{a'}}\ , \quad \mid a' \mid^2 - \mid b' \mid^2 = 1.
\end{aligned}$$

Thus the matrix group

$$SU(1,1) = \left\{ \begin{pmatrix} a' & b' \\ \overline{b'} & \overline{a'} \end{pmatrix} : \mid a' \mid^2 - \mid b' \mid^2 = 1 \right\}$$

acts as a group of orientation preserving isometries of $\mathbf{D}^2$.

□

2.3 Classification of Isometries

Theorem 2.3.1 (Brouwer's Fixed Point) Let $f : \overline{\mathbf{D}^n} \longmapsto \overline{\mathbf{D}^n}$ be any continuous map. Then f has at least one fixed point, where $\overline{\mathbf{D}^n} = \{x \in \mathbb{R}^n : x_1^2 + x_2^2 + x_3^2 + ... + x_n^2 \leq 1\}$.

Let $\varphi \in Isometry(\mathbf{D}^2)$. Using Brouwer fixed point theorem, we can say that φ has at least one fixed point on the closure of the disk given by the disk along with its boundary circle. Then we have two possibilities: either φ has a fixed point in $\mathbf{D}^2$ or the fixed point lie on the boundary circle.
We claim that any isometry of the hyperbolic plane can have at most two fixed points on the boundary. To verify it we pass on to the upper half space model. There an isometry is $f : z \longrightarrow \frac{az+b}{cz+d}$ suppose $f(z) = z$ i.e.

$$z = \frac{az+b}{cz+d}\ .$$

above equation can have at most two solution two solutions, we conclude that the isometries of $\mathbf{H}^2$ can have at-most two fixed points on the boundary.
So we can divide the isometries of the hyperbolic space in three mutually disjoint classes based on their fixed points. We classify isometry of hyperbolic space on the basis of their fixed points as
1) *elliptic* if it fixes a point on the hyperbolic space.
2) *parabolic* if it fixes no point in the hyperbolic space but fixes a unique point on the circle at the infinity of the hyperbolic space.
3) *hyperbolic* if it fixes no point in the hyperbolic space but fixes two points on the circle at the infinity of hyperbolic space.

Lemma 2.3.2 *Any elliptic isometry of the hyperbolic space has a unique fixed point.*

Proof Consider the upper half space model of $\mathbf{H}^2$. The orientation preserving isometries of $\mathbf{H}^2$ are precisely

$$z \longmapsto \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1.$$

Note that

$$\phi(z) = \frac{az+b}{cz+d} = z \ ,$$

$$cz^2 + dz - az - b = 0 \ .$$

An isometry is elliptic if it fixes a point in the hyperbolic space, then the above quadratic have both roots as complex conjugates. And since upper half space consists of points above the real axis so it has a unique fixed point on the space.

$\square$

Lemma 2.3.3 *An element* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *of* $PSL(2,\mathbb{R})$ *is*
(1) *elliptic if and only if* $\mid traceA \mid < 2$,
(2) *parabolic if and only if* $\mid traceA \mid = 2$,
(3) *hyperbolic if and only if* $\mid traceA \mid > 2$.

Proof If an element $A \in PSL(2,\mathbb{R})$ fixes a point in the plane if

$$cz^2 + (d-a)z - b = 0.$$

From above equation three cases arises
(1) when both roots are real and distinct,
(2) when roots are real and repeated,
(3) when roots are complex conjugate.
From the definition we learnt that an isometry is elliptic if it fixes a point in hyperbolic space, which invoke case (3) when roots of $cz^2 + dz - az - b = 0$ are complex conjugate. So

$$(d-a)^2 - 4bc < 0 \ ,$$

$$(d+a)^2 - 4(ad-bc) < 0 \ ,$$

$$\mid d-a \mid < 2 \ .$$

Hence $\mid traceA \mid < 2$.
An isometry is parabolic if it fixes no point in hyperbolic space but fixes a unique point on boundary. So

$$(d-a)^2 - 4bc = 0\ ,$$

$$(d+a)^2 - 4(ad-bc) = 0\ ,$$

$$\mid d-a \mid = 2\ .$$

Hence $\mid traceA \mid = 2$.
An isometry is hyperbolic if it fixes no point in the hyperbolic space but fixes 2 points on the boundary. So

$$(d-a)^2 - 4bc > 0\ ,$$

$$(d+a)^2 - 4(ad-bc) > 0\ ,$$

$$\mid d-a \mid > 2\ .$$

Hence $\mid traceA \mid > 2$.

$\square$

Conjugacy classes of isometries
Here we will show what are the conjugacy class of isometries of $\mathbf{H}^2$.

Lemma 2.3.4 *If φ is an orientation preserving isometry of $\mathbf{H}^2$. Then φ is elliptic if and only if it is conjugate in $PSL(2,\mathbb{R})$ to a unique element of form*

$$T_\theta = \frac{\cos\theta z + \sin\theta}{-\sin\theta + \cos\theta}, 0 \leq \theta \leq 2\pi\ .$$

Proof If φ is orientation preserving elliptic isometry. Suppose b be its fixed point. Since $PSL(2, \mathbb{R})$ acts transitively on $\mathbf{H}^2$, we can conjugate φ in $PSL(2, \mathbb{R})$ to an elliptic element which fixes i.

$$\varphi(z) = \frac{az + b}{cz + d}$$

then

$$\varphi(i) = \frac{ai + b}{ci + d}$$

from above equation we get $-c + di = ai + b$, on equating real and imaginary part we get

$$d = a, \quad b = -c.$$

Therefore

$$\varphi(z) = \frac{az + b}{-bz + a}, \ a^2 + b^2 = 1, \quad a, b \in \mathbb{R}$$

this implies

$$\varphi(z) = \frac{\cos\theta z + \sin\theta}{-\sin\theta z + \cos\theta}, 0 \leq \theta \leq 2\pi \ .$$

So, corresponding element in $PSL(2, \mathbb{R})$ is $A(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ with $0 \leq \theta \leq 2\pi$. Note that if $\pi \leq \theta \leq 2\pi$, then $A(\theta) = A(-\varphi)$, where $\varphi = 2\pi - \theta$ and $0 \leq \varphi \leq \pi$. But in $PSL(2, \mathbb{R})$ there is no element which conjugates $A(\varphi)$ and $A(-\varphi)$. If $\varphi \neq \theta$, then $A(\theta)$ and $A(\varphi)$ are not conjugate to each other since eigenvalue of their corresponding matrices are different.
So we can say every elliptic element of $PSL(2, \mathbb{R})$ is conjugate to a unique element of the form $A(\theta)$, $0 \leq \theta \leq 2\pi$.
Conversely, every element conjugate to the form of $T(\theta)$ is elliptic.

$\square$

Lemma 2.3.5 *An Orientation preserving isometry of $\mathbf{H}^2$ is parabolic if and only if it is conjugate to one of the following isometries in $PSL(2, \mathbb{R})$:*

$$T_1 : z \longmapsto z + 1 \ ,$$

$$T_2 : z \longmapsto z - 1 \ .$$

Proof If φ is a orientation preserving isometry of $\mathbf{H}^2$ which is parabolic, then it should fixes a unique point on boundary. Suppose it fixes point w.
Now, let (p,q) be two point on $\widehat{\mathbb{R}}$, then map $z \longmapsto \frac{z-p}{z-q}$ maps (p,q) onto $(0,\infty)$. Suppose q be ∞,then map $z \longmapsto z-p$ maps (p,∞) onto $(0,\infty)$. Thus any two point on $\widehat{\mathbb{R}}$ can be mapped onto $(0,\infty)$ by a suitable element of $PSL(2,\mathbb{R})$.
Hence we get one element $f \in PSL(2,\mathbb{R})$, such that $f(w) = \infty$ then

$$f\varphi f^{-1} = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{R} \qquad \textit{and} \;\; ad-bc = 1 \; .$$

Here, $f\varphi f^{-1}(\infty) = \infty$, which implies $c = 0$ and $ad = 1$.
Therefore

$$f\varphi f^{-1} = a^2 z + 1 \; .$$

We claim that $a = 1$, otherwise, $f\varphi f^{-1}$ has another fixed point $\frac{ad}{1-a^2}$ on $\widehat{\mathbb{R}}$ which is not possible by hypothesis. Therefore $f\varphi f^{-1}(z) = z + t$, $t \in \mathbb{R}$, $t \neq 0$.
Now take $g \in PSL(2,\mathbb{R})$ such that $g(z) = \frac{z}{|t|^2}$. Then $g(f\varphi f^{-1})g^{-1}(z) = z \pm 1$. Converse is clear since conjugate of parabolic will be again a parabolic.

$\square$

From above lemma we can say that every parabolic element of $PSL(2,\mathbb{R})$ is conjugate to a transformation

$$z \longmapsto z+1 \;\; or \;\; z \longmapsto z-1$$

And their corresponding matrices are $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

Lemma 2.3.6 *An orientation preserving isometry of* $\mathbf{H}^2$ *is hyperbolic if and only if it is conjugate to a unique element* ϕ_λ, $\lambda > 1$ *in* $PSL(2,\mathbb{R})$, *where* $\phi_\lambda = \lambda z$.

Proof Let φ be an orientation preserving hyperbolic isometry of $\mathbf{H}^2$. Let $\varphi(p) = p$ and $\varphi(q) = q$, where p, $q \in \widehat{\mathbb{R}}$, $p \neq q$.
Since $PSL(2,\mathbb{R})$ acts transitively on $\widehat{\mathbb{R}}$, there exits η in $PSL(2,\mathbb{R})$ such that $\eta(p) = 0$ and $\eta(q) = \infty$. Therefore

$$\eta\varphi\eta^{-1}(0) = 0 \quad \textit{and} \quad \eta\varphi\eta^{-1}(\infty) = \infty. \tag{2.1}$$

Also there will be $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 1$ and

$$\eta\varphi\eta^{-1}(z) = \frac{az + b}{cz + d} \ .$$

From equation (2.1) it follows $b = c = 0$. Therefore, $\eta\varphi\eta^{-1}(z) = \frac{az}{d}$, $ad = 1$. That is $\eta\varphi\eta^{-1} = a^2 z$, a is non zero real number.
Thus every orientation preserving isometry will be conjugate in $PSL(2, \mathbb{R})$ to an element of the form $\phi_\lambda = \lambda z$, $\lambda > 0$.
Thus in $PSL(2, \mathbb{R})$ any hyperbolic element will be conjugate to an matrix of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$.
We denote this matrix also by ϕ_λ. Note that ϕ_λ is conjugate to $\phi_{\lambda^{-1}}$, in $PSL(2, \mathbb{R})$ by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
Again if $\lambda \neq k$ then ϕ_λ is not conjugate to ϕ_k since the eigenvalue of corresponding matrices are not the same.
This shows that any hyperbolic isometry is conjugate to a unique element of form ϕ_λ, $\lambda > 1$. Conversely, if any isometry of $\mathbf{H}^2$ is conjugate to any such ϕ_λ, then it is hyperbolic.

□

Thus we now classified all kind of orientation preserving isometries of the hyperbolic plane upto their conjugacy.
And we have a result

Theorem 2.3.7 The conjugacy class in the $PSL(2, \mathbb{R})$ are represented precisely by the following types:

$$A(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, 0 \leq \theta \leq 2\pi$$

$$T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ \ and \ \ T_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\varphi_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \ .$$

For each of the above elements we get a unique conjugacy class in $PSL(2,\mathbb{R})$.

Chapter 3

Quaternionic Möbius transformation

3.1 The Quaternions

We denote $\mathbb{H}$ as the division ring of the real quaternions. Elements of $\mathbb{H}$ have the form $z = z_0 + z_1i + z_2j + z_3k$ where $z_i \in \mathbb{R}$ and holds

$$i^2 = j^2 = k^2 = ijk = -1,$$

$$ij = k, \quad ji = -k,$$

$$jk = i, \quad kj = -i,$$

$$ki = j, \quad ik = -j.$$

Conjugate of z is defined as $\overline{z} = z_0 + z_1i + z_2j + z_3k$.
We define real part of z to be $\mathcal{R}e(z) = (z + \overline{z})/2 = z_0$ and imaginary part to be $\mathcal{I}m(z) = (z - \overline{z})/2 = z_1i + z_2j + z_3k$.
Modulus of z is defined as

$$\mid z \mid = \sqrt{\overline{z}z} = \sqrt{z_0^2 + z_1^2 + z_2^2 + z_3^2}\ .$$

Inverse of a quaternion z is defined as $z^{-1} = \overline{z} \mid z \mid^{-2}$.

Let $p, q \in \mathbb{H}$, then some properties of quaternions are :
1) $\overline{(pq)} = \overline{q}\overline{p}$.
2) $(pq)^{-1} = q^{-1}p^{-1}$.
3) $| pq | = | q || p |$.

Remark Unlike multiplication of real or complex number, multiplication of quaternions are non commutative.

Definition 3.1.1 Two quaternions p and q are said to be similar if there exist non zero quaternion $r \in \mathbb{H}$ such that $p = rqr^{-1}$. And $\{rqr^{-1} : r \in \mathbb{H} - \{0\}\}$ is the similarity class of q.

Here we define another map

$$T_r : \mathbb{H} \longmapsto \mathbb{H} \;\; by \;\; T_r(z) = rzr^{-1}.$$

And we have $| T_r(z) | = | r | | z || r^{-1} | = | z |$. Further

$$\overline{T_r(z)} = \overline{r}^{-1}\overline{z}\,\overline{r} = r\overline{z}r^{-1} = T_r(\overline{z}).$$

Lemma 3.1.2 *Two quaternions p and q are similar if and only if $| p | = | q |$ and $\mathcal{R}e(p) = \mathcal{R}e(q)$.*

Proof If p and q are similar then there is non zero quaternion r so that $q = T_r(p)$.

$$| q |=| T_r(p) |=| rpr^{-1} |=| r || r^{-1} || p |=| p |$$

and $\mathcal{R}e(p) = \mathcal{R}e(q)$.

Conversely, suppose $p \in \mathbb{H}$. Since $| p | = | q |$ and $\mathcal{R}e(p) = \mathcal{R}e(q)$, we can consider

$$q = \mathcal{R}e(p) + | \mathcal{I}m(p) | \, i = \mathcal{R}e(p) + \sqrt{| p |^2 - \mathcal{R}e(p)^2} \; i$$

which satisfy above condition. We claim that we can find a non zero $r \in \mathbb{H}$ so that $T_r(q) = rqr^{-1} = p$. The result for general quaternions similar to z will follow by composition.
Let

$$r = \mathcal{I}m(p) + \mid \mathcal{I}m(p) \mid i.$$

Here $p = p_0 + p_1 i + p_2 j + p_3 k$ and let $y = \mathcal{I}m(p) = p_1 i + p_2 j + p_3 k$.
This implies $r = y + \mid y \mid i$. Further

$$\mid r \mid^2 = (y + \mid y \mid i)(-y - \mid y \mid i) = 2 \mid y \mid^2 + 2 \mid y \mid p_1 .$$

Hence,

$$\begin{aligned} r(\mid y \mid i)r^{-1} &= (y + \mid y \mid i)(\mid y \mid i)(-y - \mid y \mid i)/(2 \mid y \mid^2 + 2 \mid y \mid p_1) \\ &= (- \mid y \mid yiy + 2 \mid y \mid^2 y + \mid y \mid^3 i)/(2 \mid y \mid^2 + 2 \mid y \mid p_1) \\ &= (2 \mid y \mid^2 y + 2 \mid y \mid p_1 y)/(2 \mid y \mid^2 + 2 \mid y \mid p_1) \\ &= y. \end{aligned}$$

Since $r\mathcal{R}e(p)r^{-1} = \mathcal{R}e(p)$ we then have $r(\mathcal{R}e(p) + \mid \mathcal{I}m(p) \mid i)r^{-1} = p$.

□

3.2 Quaternionic matrices and Möbius transformations

Here we define quaternionic vector space in the same way as we define real vector space and complex vector space. Quaternionic right vector space means scalers act by right multiplication.

Definition 3.2.1 A Quaternionic right vector space is defined as a set V with operations addition and right scaler multiplication such that for every $\mathbf{u}$, $\mathbf{v} \in V$ and $t \in \mathbb{H}$ we have $\mathbf{u} + \mathbf{v}$ and $\mathbf{u}t$ in V. It is an abelian group under addition and scaler multiplication is associative and distributive.

Definition 3.2.2 Two dimensional quaternionic right vector space is defined as

$$\mathbb{H}^2 = \left\{ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : v_1, v_2 \in \mathbb{H} \right\}$$

with the operations

$$\mathbf{v} + \mathbf{w} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}, \qquad \mathbf{v}t = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} t = \begin{pmatrix} v_1 t \\ v_2 t \end{pmatrix} .$$

Linear map acts on $\mathbb{H}^2$ as

$$P : \mathbb{H}^2 \longmapsto \mathbb{H}^2$$

$$P\mathbf{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{pmatrix}$$

as left multiplication by 2×2 matrices with quaternion entries.

For each 2×2 matrix

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with quaternion entries we have σ and τ as quaternionic determinant and quaternionic trace respectively as follows :

$$\tau_P = \begin{cases} cac^{-1} + d & when \;\; c \neq 0, \\ bdb^{-1} + a & when \;\; c = 0, b \neq 0, \\ (d-a)a(d-a)^{-1} + d & when \;\; b = c = 0, a \neq d, \\ a + \overline{a} & when \;\; b = c = 0, a = d. \end{cases}$$

$$\sigma_P = \begin{cases} cac^{-1}d - cb & when \;\; c \neq 0, \\ bdb^{-1}a & when \;\; c = 0, b \neq 0, \\ (d-a)a(d-a)^{-1}d & when \;\; b = c = 0, a \neq d, \\ a\overline{a} & when \;\; b = c = 0, a = d. \end{cases}$$

Remark Quaternionic determinant and quaternionic trace are conjugation invariant.

For 2×2 quaternionic matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we define its real determinant to be $\sqrt{\alpha}$ where

$$\alpha_P = \mid a \mid^2 \mid d \mid^2 + \mid b \mid^2 \mid c \mid^2 - 2\mathcal{R}e[a\bar{c}d\bar{b}].$$

Observe that in each case $\alpha = \mid \sigma \mid^2$.

Definition 3.2.3 $SL(2, \mathbb{H}) : \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{H}, \alpha = 1 \right\}$.

And the group $SL(2, \mathbb{H})$ is generated by matrices of the form

$$D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where $\mid \lambda \mid\mid \mu \mid = 1$.

Define right projective vector map $\mathbb{P} : \mathbb{H}^2 - \{0\} \longmapsto \widehat{\mathbb{H}} = \mathbb{H} \cup \infty$ by

$$\mathbb{P} : \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \longmapsto \begin{cases} v_1 v_2^{-1} & if \quad v_2 \neq 0 \\ \infty \end{cases}$$

Standard lift of any $v \in \widehat{\mathbb{H}}$ to $\mathbb{H}^2$ is defined as

$$v \longmapsto \mathbf{v} = \begin{pmatrix} v \\ 1 \end{pmatrix} \quad for \ v \in \mathbb{H}, \quad \infty \longmapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Matrices in $SL(2, \mathbb{H})$ acts on $\widehat{\mathbb{H}}$ by left multiplication on the standard lift followed by right projection. For point $v \in \mathbb{H}$ this is

$$P(v) = \mathbb{P}P\mathbf{v},$$

$$= \mathbb{P} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix},$$

$$= \mathbb{P}\begin{pmatrix} av+b \\ cv+d \end{pmatrix} ,$$

$$= (av+b)(cv+d)^{-1} .$$

$$P(\infty) = \mathbb{P}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = ac^{-1} .$$

This is quaternionic Möbius transformation. The map

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto P(v) = (av+b)(cv+d)^{-1}$$

from $SL(2,\mathbb{H})$ to the group of quaternionic Möbius transformation is a surjective homomorphism.

For $a \in \mathbb{R}$ consider map

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \longmapsto (av)(a)^{-1}$$

$$= ava^{-1}$$

$$= v .$$

Therefore kernel of the group is real multiple of identity matrix. We identify a group of quaternionic Möbius transformation with $PSL(2,\mathbb{H}) = SL(2,\mathbb{H})/\{\pm I\}$.

Lemma 3.2.4 *Let* $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{H})$ *and suppose either* $b = 0$ *or* $c = 0$. *Then* σ *and* τ *are both real if and only if either* a *and* d *are both real or else* a *and* d *are similar and* $\sigma = |\, a\,|^2$ *and* $\tau = 2\mathcal{R}e[a]$.

Proof If a and d are both real then clearly σ and τ are both real. Suppose that σ and τ are both real but a and d are not both real.

Case(1) If $b = 0$ and $c \neq 0$ then our claim is $cac^{-1} = \overline{d}$. Since $b = 0$, $\sigma = cac^{-1}d$ and $\tau = cac^{-1} + d$. On computing we get

$$\tau = \sigma\overline{d} \mid d \mid^{-2} + d.$$

On equating non real parts of the equation we see that $\sigma = \mid d \mid^2$. Hence

$$\mid \sigma \mid = \frac{\overline{d}}{d^{-1}} \ ,$$

$$cac^{-1}d = \frac{\overline{d}}{d^{-1}} \ ,$$

$$cac^{-1} = \overline{d} \ .$$

Here a is similar to $\overline{d}$ and hence to d. Also, $\sigma = \mid d \mid^2 = \mid a \mid^2$ and $\tau = \overline{d} + d = 2\mathcal{R}e(d) = 2\mathcal{R}e(a)$.

Case (2) If $c = 0$, $b \neq 0$, then our claim is $bdb^{-1} = \overline{a}$.

Here $\sigma = bdb^{-1}a$ and $\tau = bdb^{-1} + a$. Further on computing we get

$$\tau = \sigma \overline{a} \mid a \mid^{-2} + a \ .$$

On equating non real part of equation we get

$$\mid \sigma \mid = \mid a \mid^2 \ ,$$

$$\mid \sigma \mid = \frac{\overline{a}}{a^{-1}} \ ,$$

$$bdb^{-1} = \overline{a} \ .$$

Here d is similar to $\overline{a}$ and hence to a. Also $\sigma = \mid a \mid^2$ and $\tau = 2\mathcal{R}e(d) = 2\mathcal{R}e(a)$.

Case(3) If $b = c = 0$ and $a \neq d$. Then our claim is $(d-a)a(d-a)^{-1} = \overline{d}$.

Here $\sigma = (d-a)a(d-a)^{-1}d$ and $\tau = (d-a)a(d-a)^{-1} + d$.

On computing we get

$$\tau = \sigma d^{-1} + d \ ,$$

$$\tau = \sigma \overline{d} \mid d \mid^{-2} + d \ .$$

On equating non real part of equation we get

$$\mid \sigma \mid = \frac{\overline{d}}{d^{-1}} \ ,$$

$$(d-a)a(d-a)^{-1} = \overline{d} \ .$$

Here a is similar to $\overline{d}$ and hence to d. Also, $\sigma = \mid a \mid^2$ and $\tau = 2\mathcal{R}e(a)$. The proof of the converse implication in the lemma is straight forward.

$\square$

Proposition 3.2.5 *Given $P \in SL(2, \mathbb{H})$, if $\sigma = \sigma_p$ and $\tau = \tau_p$ are real then they are preserved under conjugation in $SL(2, \mathbb{H})$.*

Proof For $P \in SL(2, \mathbb{H})$, σ and τ are real. And we know that $SL(2, \mathbb{H})$ is generated by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

where $\mid \lambda \mid \mid \mu \mid = 1$.

Let any of these generator be Q. Then, $R = QPQ^{-1}$ where $R = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$.

It suffices to show that for each choice of Q, we have $\sigma_R = \sigma_P$ and $\tau_R = \tau_P$.

Case (1) when Q = $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ we have $QPQ^{-1} =$

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} = \begin{pmatrix} \lambda a \lambda^{-1} & \lambda b \mu^{-1} \\ \mu c \lambda^{-1} & \mu d \mu^{-1} \end{pmatrix}.$$

If $c \neq 0$ then

$$\sigma_R = \mu(cac^{-1} + d)\mu^{-1} \ .$$

Here since σ_P is real $(cac^{-1} - cb)$ and μ^{-1} can commute, implies

$$\sigma_R = \sigma_P.$$

And

$$\tau_R = \mu(cac^{-1} + d)\mu^{-1}$$

again since τ_P is real $(cac^{-1} + d)$ and μ^{-1} can commute

$$\tau_R = \tau_P \ .$$

Suppose that $c = 0$. From previous lemma $a' = a$ and $d' = d$. Otherwise $\sigma_R = \mid a' \mid^2 = \mid a \mid^2 = \sigma_P$ and $\tau_R = 2\mathcal{R}e[d] = 2\mathcal{R}e[a] = \tau_P$.

In case (2) when $Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ we have $R = QPQ^{-1}$

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+c & b-a+d-c \\ c & d-c \end{pmatrix}.$$

If $c \neq 0$ then

$$\sigma_R = c(a+c)c^{-1}(d-c) - c(b-a+d-c),$$

$$= cac^{-1}d - cb,$$

$$\sigma_R = \sigma_P.$$

If $c = 0$ then $a' = a$ and $d' = d$ and result follows from previous lemma. In case (3) when $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ we have

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

If $b \neq 0$ and $c \neq 0$ then, using previous lemma we have

$$\sigma_R = bdb^{-1}a - bc = \sigma_P ,$$

$$\tau_R = bdb^{-1} + a = \tau_P .$$

Suppose that $b = 0$ or $c = 0$. If a and d are both real then $\sigma_R = da = \sigma_P$ and $\tau_R = d + a = \tau_P$. Otherwise, $\sigma_R = |a|^2 = |d|^2 = \sigma_P$ and $\tau_R = 2\mathcal{R}e[d] = 2\mathcal{R}e[a] = \tau_P$. $\square$

Lemma 3.2.6 *Let $f(v) = (av + b)d^{-1}$ where a and d are similar and not real. Then f has a fixed point in $\mathbb{H}$ if and only if $bd = \overline{a}b$. Moreover, if $bd = \overline{a}b$ then f is conjugate to $g_0(v) = ava^{-1}$ and if $bd \neq \overline{a}b$ then f is conjugate to $g_1(v) = (av+1)a^{-1}$.*

Proof Suppose f has a fixed point z. Then $f(z) = z$ for $z \in \mathbb{H}$.

$$(az+b)d^{-1} = z \ ,$$

$$az + b = zd \ ,$$

$$b = zd - az \ .$$

Since, a and d are similar

$$\mid a \mid^2 = \mid d \mid^2 \quad and \ \mathcal{R}e(a) = \mathcal{R}e(d).$$

$$i.e. \ a + \overline{a} = d + \overline{d} \ .$$

Now

$$\begin{aligned} bd - \overline{a}b &= (zd - az)d - \overline{a}(zd - az) \\ &= zd^2 + \mid a \mid^2 z - (a + \overline{a})zd \\ &= zd(d + \overline{d}) - (d + \overline{d})zd \\ &= 0. \end{aligned}$$

Now, conversely assume $bd = \overline{a}b$ then $b = \overline{a}bd^{-1}$. Set $z = (\overline{a} - a)^{-1}b$ then

$$\begin{aligned} f(z) &= (a(\overline{a} - a)^{-1}b + b)d^{-1} \\ &= (\overline{a} - a)^{-1}(a + \overline{a} - a)bd^{-1} \\ &= (\overline{a} - a)^{-1}b \\ &= z. \end{aligned}$$

For second part, conjugating by a diagonal map if necessary, we may always suppose $d = a$. When $ba = \overline{a}b$, conjugating so that 0 is a fixed point gives the result. When $ba - \overline{a}b \neq 0$ it is easy to check that $(ba - \overline{a}b)$ commutes with a. Conjugating by $h(z) = ((a - \overline{a})z + b)(ba - \overline{a}b)^{-1}$ gives the result.

□

Theorem 3.2.7 A quaternionic Möbius transformation is conjugate to a real Möbius transformation if and only if σ and τ are both real.

Proof If a quaternionic Möbius transformation is conjugate to a real Möbius transformation then σ and τ are real using Proposition 3.2.5.
Conversely, given $f(z) = \frac{az+b}{cz+d}$ a quaternionic Möbius transformation suppose that its σ and τ are both real. If a and d are real but b is not real then we replace f by a conjugate transformation which fixes ∞. This implies $c = 0$. From Proposition 3.2.5 we can say σ and τ remain real. On applying lemma 3.2.4, where if a and d are real but b is not real then we conjugate f by the map $g(z) = b^{-1}z$, to obtain the real möbius transformation $gfg^{-1}(z) = (az+1)d^{-1}$.
When a and d are not real but similar with $\sigma = \mid a \mid^2$ and $\tau = 2\mathcal{R}e[a]$. If $b \neq 0$ then

$$\sigma = bdb^{-1}a$$

$$\sigma a^{-1} = bdb^{-1}.$$

Using

$$\sigma a^{-1} = \overline{a}$$

we get

$$bdb^{-1} = \overline{a}\ .$$

Applying lemma 3.2.6, we get f has fixed point v in $\mathbb{H}$. After replacing f by another conjugate transformation we get that $v = 0$. This implies $b = 0$. Again applying final conjugation by map of the form $z \longmapsto uz$ for $u \in \mathbb{H}$, we ensure that $a \neq d$. This implies that for $a = x + \mu y$ and $d = x + \nu y$ for real numbers x and y and distinct purely imaginary unit quaternions μ and ν. We see that

$$\begin{pmatrix} \mu & 1 \\ 1 & -\nu \end{pmatrix} \begin{pmatrix} x+\mu y & 0 \\ 0 & x+\nu y \end{pmatrix} \begin{pmatrix} \mu & 1 \\ 1 & -\nu \end{pmatrix}^{-1} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Thus here f is conjugate to a real Möbius transformation.

$\square$

3.3 Quaternionic Hermitian forms

Let $P = (p_{ij})$ be a $m \times n$ quaternionic matrix, we define conjugate transpose of a quaternionic matrix be $P^* = (\overline{p}_{ji})$. A $n \times n$ quaternionic matrix H is Hermitian if and only if $H^* = H$.

Definition 3.3.1 A Quaternionic Hermitian form is defined on the quaternionic right vector space $\mathbb{H}^k$ as a map $\langle .,. \rangle = \mathbb{H}^k \times \mathbb{H}^k \longmapsto \mathbb{H}$ by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* H \mathbf{z}.$$

For all $\mathbf{z}, \mathbf{z}_1, \mathbf{z}_2, \mathbf{w}$ column vectors in $\mathbb{H}^k$ and $\lambda \in \mathbb{H}$ following satisfies

$$\langle \mathbf{z}_1 + \mathbf{z}_2, \mathbf{w} \rangle = \mathbf{w}^* H(\mathbf{z}_1 + \mathbf{z}_2) = \mathbf{w}^* H \mathbf{z}_1 + \mathbf{w} H \mathbf{z}_2 = \langle \mathbf{z}_1, \mathbf{w} \rangle + \langle \mathbf{z}_2, \mathbf{w} \rangle,$$
$$\langle \mathbf{z}\lambda, \mathbf{w} \rangle = \mathbf{w}^* H(\mathbf{z}\lambda) = (\mathbf{w}^* H \mathbf{z})\lambda = \langle \mathbf{z}, \mathbf{w} \rangle \lambda,$$
$$\langle \mathbf{w}, \mathbf{z} \rangle = \mathbf{z}^* H \mathbf{w} = \mathbf{z}^* H^* \mathbf{w} = (\mathbf{w}^* H \mathbf{z})^* = \overline{\langle \mathbf{z}, \mathbf{w} \rangle}\ .$$

Using last property we conclude that $\langle \mathbf{z}, \mathbf{z} \rangle = \overline{\langle \mathbf{z}, \mathbf{z} \rangle}$. So, for all $\mathbf{z} \in \mathbb{H}^k$, $\langle \mathbf{z}, \mathbf{z} \rangle \in \mathbb{R}$. We define subsets V_-, V_+ and V_0 as

$$V_+ = \{\mathbf{z} \in \mathbb{H}^k : \langle \mathbf{z}, \mathbf{z} \rangle > 0\}$$
$$V_- = \{\mathbf{z} \in \mathbb{H}^k : \langle \mathbf{z}, \mathbf{z} \rangle < 0\}$$
$$V_0 = \{\mathbf{z} \in \mathbb{H}^k - \{0\} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\}\ .$$

Consider $\mathbb{H}^2$ with a quaternionic hermitian form of signature (1, 1). There are two standard quaternionic Hermitian forms. We call these the first and second Hermitian forms. For $\mathbf{z}, \mathbf{w} \in \mathbb{H}^2$. Let $\mathbf{z}$, $\mathbf{w}$ be the column vectors $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ respectively. The first Hermitian form is defined as:

$$\langle \mathbf{z}, \mathbf{w} \rangle_1 = \mathbf{w}^* H_1 \mathbf{z} = \overline{w}_1 z_1 - \overline{w}_2 z_2\ .$$

It is given by Hermitian matrix H_1 :

$$H_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ .$$

For second Hermitian form, H_2 is :

$$H_2 = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix} \ .$$

Here we introduce an involution $*$ on $\mathbb{H}$.

$$z^* = (z_0 + z_1 i + z_2 j + z_3 k)^* = z_0 + z_1 i + z_2 j - z_3 k \ .$$

Second Hermitian form is defined as

$$\langle \mathbf{z}, \mathbf{w} \rangle_2 = \mathbf{w}^* H_2 \mathbf{z} = \overline{w}_2 k z_1 - \overline{w}_1 k z_2 = k(w_2^* z_1 - w_1^* z_2) \ .$$

Also

$$\langle \mathbf{z}\lambda, \mathbf{z}\lambda \rangle = \overline{\lambda} \mathbf{z}^* H \mathbf{z} \lambda = \overline{\lambda} \langle \mathbf{z}, \mathbf{z} \rangle \lambda = \mid \lambda \mid^2 \langle \mathbf{z}, \mathbf{z} \rangle \ .$$

Therefore the map $\mathbb{P} : \mathbb{H}^2 - \{0\} \longrightarrow \widehat{\mathbb{H}}$

$$\mathbb{P} : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{cases} z_1 z_2^{-1} & if \ z_2 \neq 0 \\ \infty & if \ z_2 = 0 \end{cases}$$

respects division of $\mathbb{H}^2$ into V_+, V_-, V_0.

Here we define Hyperbolic 4 - space $\mathbf{H}^4$ to be $\mathbb{P}V_-$ and its ideal boundary $\partial \mathbf{H}^4$ to be $\mathbb{P}V_0$. In first Hermitian form

$$\begin{aligned} \langle \mathbf{z}, \mathbf{z} \rangle &= \overline{z}_1 z_1 - \overline{z}_2 z_2 \\ &= \overline{z} z - 1 \\ &= \mid z \mid^2 - 1. \end{aligned}$$

In $V_- = \{\mathbf{z} \in \mathbb{H}^2 : \langle \mathbf{z}, \mathbf{z} \rangle < 0\}$ where $\langle \mathbf{z}, \mathbf{z} \rangle < 0$, implies for first Hermitian form $\mid z \mid^2 - 1 < 0$. So

$$\mid z \mid^2 < 1 \ .$$

We define quaternionic unit ball to be $\mathbb{B} = \{z \in \mathbb{H} : \mid z \mid < 1\}$. We can easily deduce that $\mathbb{P}(V_-) = \mathbb{B}$. Hyperbolic metric on $\mathbf{H}^4 = \mathbb{P}(V_-)$ is defined by

$$ds^2 = \frac{-4}{\langle \mathbf{z}, \mathbf{z} \rangle^2} det \begin{pmatrix} \langle \mathbf{z}, \mathbf{z} \rangle & \langle d\mathbf{z}, \mathbf{z} \rangle \\ \langle \mathbf{z}, d\mathbf{z} \rangle & \langle d\mathbf{z}, d\mathbf{z} \rangle \end{pmatrix}, \cosh^2\left(\frac{\rho(z,w)}{2}\right) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle}$$

For first Hermitian from H_1, it comes out to be

$$ds^2 = \frac{4dz\overline{dz}}{(1- \mid z \mid^2)^2} \ .$$

Definition 3.3.2 A symplectic transformation P is defined to be an automorphism $P : \mathbb{H}^{1,1} \longmapsto \mathbb{H}^{1,1}$ which is a right linear bijection such that $\langle P\mathbf{z}, P\mathbf{w} \rangle = \langle \mathbf{z}, \mathbf{w} \rangle$ for all $\mathbf{z}$, $\mathbf{w}$ in $\mathbb{H}^{1,1}$.
Those symplectic transformation which preserve a given Hermitian form are denoted by $Sp(H)$.
Since, P preserve the form we have

$$\mathbf{w}^* P^* H P \mathbf{z} = (P\mathbf{w})^* H (P\mathbf{z}) = \langle P\mathbf{z}, P\mathbf{w} \rangle = \langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* H \mathbf{z}.$$

Therefore we have $P^*HP = H$. If H is non degenerate then it is invertible and this translates to an easy formula for the inverse of P

$$P^{-1} = H^{-1} P^* H \ .$$

Proposition 3.3.3 *Let P be a 2×2 quaternionic matrix in $Sp(H_1)$. Then*

$$\mid a \mid = \mid d \mid, \mid b \mid = \mid c \mid, \mid a \mid^2 - \mid c \mid^2 = 1, \quad \overline{a}b = \overline{c}d, \quad a\overline{c} = b\overline{d}.$$

Proof Here, given first Hermitian matrix $H_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We have

$$P^{-1} = H_1^{-1} P^* H_1$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \overline{a} & -\overline{c} \\ -\overline{b} & \overline{d} \end{pmatrix}.$$

Using $P^{-1}P = I = PP^{-1}$ we get

$$P^{-1}P = \begin{pmatrix} \mid a \mid^2 - \mid c \mid^2 & \overline{a}b - \overline{c}d \\ \overline{d}c - \overline{b}a & \mid d \mid^2 - \mid b \mid^2 \end{pmatrix}$$

and

$$PP^{-1} = \begin{pmatrix} \mid a \mid^2 - \mid b \mid^2 & \overline{b}d - \overline{a}c \\ \overline{c}a - \overline{d}b & \mid d \mid^2 - \mid c \mid^2 \end{pmatrix}.$$

Equating both matrices we get $\overline{a}b = \overline{c}d$, $a\overline{c} = b\overline{d}$ and $\mid a \mid^2 - \mid c \mid^2 = 1 = \mid a \mid^2 - \mid b \mid^2$. Also $\mid d \mid^2 - \mid b \mid^2 = 1 = \mid d \mid^2 - \mid c \mid^2$. Thus we have $\mid a \mid = \mid d \mid$ and $\mid b \mid = \mid c \mid$.

□

3.4 Fixed Points and eigenvalues

Suppose $P \in SL(2, \mathbb{H})$ and $\mathbf{v} \in \mathbb{H}^2$ is non zero column vector such that $P\mathbf{v} = \mathbf{v}y$ where quaternion y is the right eigenvalue of matrix P then $\mathbf{v}$ is called right eigenvector of P. Let $\mathbf{w} = \mathbf{v}u^{-1}$, where $u \neq 0$, then $P\mathbf{w} = P(\mathbf{v}u^{-1}) = (P\mathbf{v})u^{-1} = \mathbf{v}yu^{-1} = \mathbf{w}(uyu^{-1})$. Therefore all quaternions similar to y are also eigenvalue of P. Now Suppose $Q \in SL(2, \mathbb{H})$ we see from equation $QPQ^{-1}(Q\mathbf{v}) = Q\mathbf{v}y$ that conjugate matrices have the same right eigenvalues.

Suppose v is a fixed by $P(z) = (az + b)(cz + d)^{-1}$ and so $\mathbf{v}$, the standard lift of v, is an eigenvalue of P. Then we have

$$\frac{av + b}{cv + d} = v$$

$$vcv + vd - av - b = 0.$$

It can easily seen

$$P\mathbf{v} = \mathbf{v}y$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} v \\ 1 \end{pmatrix} \begin{pmatrix} cv + d \end{pmatrix} .$$

In order to show that every quaternionic Möbius transformation has a right eigenvalue. Suppose $c \neq 0$. Then the right eigenvalue y corresponding to the fixed point v is $y = cv + d$ where

$$vcv + vd - av - b = 0.$$

Substituting $v = c^{-1}(y - d)$ in the above equation we get

$$y^2 - (cac^{-1} + d)y + cac^{-1}d - cb = 0.$$

Hence y satisfy quaternionic characteristic polynomial

$$y^2 - \tau y + \sigma = 0. \tag{3.1}$$

Using $\overline{y}y = | y |^2$ and $y + \overline{y} = 2\mathcal{R}e(y)$ we get

$$y^2 - (y + \overline{y})y + \overline{y}y = y^2 - 2\mathcal{R}e(y)y + | y |^2 = 0.$$

Now subtracting these two equation we get

$$(\sigma - | y |^2) = (\tau - 2\mathcal{R}e(y))y.$$

Case (1) When $\tau = 2\mathcal{R}e(y)$ then $\sigma = | y |^2$. So the coefficient of quaternionic characteristic are all real. And we can get a right eigenvalue from it.
Case (2) When $\tau \neq 2\mathcal{R}e(y)$ then multiply by $(\tau - 2\mathcal{R}e(y))^{-1}$ from left. We get

$$y = (\tau - 2\mathcal{R}e(y))^{-1}(\sigma - | y |^2).$$

Substituting the above equation in $2\mathcal{R}e(y) = y + \overline{y}$ and $\overline{y}y = | y |^2$ we get following:

$$| y |^2 = (\tau - 2\mathcal{R}e(y))^{-1}(\sigma - | y |^2)(\overline{\sigma} - | y |^2)(\overline{\tau} - 2\mathcal{R}e(y))^{-1}$$

and

$$2\mathcal{R}e(y) = (\tau - 2\mathcal{R}e(y))^{-1}(\sigma - | y |^2) + (\overline{\sigma} - | y |^2)(\overline{\tau} - 2\mathcal{R}e(y))^{-1}.$$

On multiplying $(\tau - 2\mathcal{R}e(y))$ from left and $(\overline{\tau} - 2\mathcal{R}e(y))$ from right we get

$$(\sigma - \mid y \mid^2)(\overline{\sigma} - \mid y \mid^2) = \mid y \mid^2 (\tau - 2\mathcal{R}e(y))(\overline{\tau} - 2\mathcal{R}e(y)),$$

$$(\sigma - \mid y \mid^2)(\overline{\tau} - 2\mathcal{R}e(y)) + (\tau - 2\mathcal{R}e(y))(\overline{\sigma} - \mid y \mid^2) = 2\mathcal{R}e(y)(\tau - 2\mathcal{R}e(y))(\overline{\tau} - 2\mathcal{R}e(y)).$$

Here we define some quantities :

$$\delta = \mathcal{R}e(\sigma\overline{\tau}),$$

$$\gamma = \mid \tau \mid^2 + 2\mathcal{R}e(\sigma),$$

$$\beta = \mathcal{R}e(\tau).$$

Now using the above definition and on expanding the equation we get

$$1 + \mid y \mid^4 = (\gamma - 4\beta\mathcal{R}e(y) + 4\mathcal{R}e(y)^2) \mid y \mid^2, \tag{3.2}$$

$$2\delta - 2\beta \mid y \mid^2 + 4 \mid y \mid^2 \mathcal{R}e(y) = (\gamma - 4\beta\mathcal{R}e(y) + 4\mathcal{R}e(y)^2)2\mathcal{R}e(y). \tag{3.3}$$

Here we got the equation whose coefficient are all in term of reals. Now multiplying equation (3.2) by $2\mathcal{R}e(y)$ and equation (3.3) by $\mid y \mid^2$ and subtracting from each other leads to

$$2\mathcal{R}e(y)(1 - \mid y \mid^4) = (2\delta - 2\beta \mid y \mid^2) \mid y \mid^2 .$$

Using this, we can eliminate $2\mathcal{R}e(y)$ to obtain

$$\gamma \mid y \mid^2 (1 - \mid y \mid^4)^2 + 4 \mid y \mid^4 (\delta - \beta \mid y \mid^2)(\delta \mid y \mid^2 - \beta) = (1 + \mid y \mid^4)(1 - \mid y \mid^4)^2.$$

Dividing equation by $4\mid y \mid^6$ gives

$$\gamma(\mid y \mid^2 - \mid y \mid^{-2})^2/4 + \delta^2 + \beta^2 - \delta\beta(\mid y \mid^2 + \mid y \mid^{-2}) = (\mid y \mid^2 + \mid y \mid^{-2})(\mid y \mid^2 - \mid y \mid^{-2})^2/4.$$

Define $T = (\mid y \mid^2 + \mid y \mid^{-2})/2$, here $T \geq 1$ and equality holds at $\mid y \mid^2 = 1$.
Further rearranging the terms we get $T^2 - 1 = (\mid y \mid^2 - \mid y \mid^{-2})^2/4$ and substituting

it in the above equation we get

$$2T^3 - \gamma T^2 + 2(\beta\delta - 1)T + (\gamma - \delta^2 - \beta^2) = 0.$$

We define cubic $q(t)$:

$$q(t) = 2t^3 - \gamma t^2 + 2(\delta\beta - 1)t + (\gamma - \delta^2 - \beta^2).$$

Observe that $q(1) = -(\delta - \beta)^2 \leq 0$ and so $q(t) = 0$ has at least one root in the interval $[1,\infty)$. Moreover, $\mid y \mid = 1$ implies that $q(1) = -(\delta - \beta)^2 = 0$ and so $\delta = \beta$. Let T be a root of $q(t)$ at least 1, then $\mid y \mid^4 - 2T \mid y \mid^2 + 1 = 0$ and so

$$\mid y \mid^2 = T \pm \sqrt{T^2 - 1}\ .$$

Also dividing equation(3.2) by $\mid y \mid^2$ and substituting for T, it gives

$$2T = \mid y \mid^2 + \mid y \mid^{-2} = \gamma - 4\delta\mathcal{R}e(y) + 4\mathcal{R}e(y)^2\ .$$

Hence

$$2\mathcal{R}e(y) = \beta \pm \sqrt{\beta^2 - \gamma + 2T}\ .$$

Here we choose $T > (\gamma - \beta^2)/2$ so that $2\mathcal{R}e(y)$ is defined.

Theorem 3.4.1 Let y be root of $y^2 - \tau y + \sigma = 0$.
(1)If $\beta T \geq \delta$ then $\mid y \mid^2 = T \pm \sqrt{T^2 - 1}$ and $2\mathcal{R}e[y] = \beta \pm \sqrt{2T - \gamma + \beta^2}$.
(2)If $\beta T < \delta$ then $\mid y \mid^2 = T \pm \sqrt{T^2 - 1}$ and $2\mathcal{R}e[y] = \beta \mp \sqrt{2T - \gamma + \beta^2}$.

Proof We have seen if y is a root of $y^2 - \tau y + \sigma = 0$ then

$$\mid y \mid^2 = T \pm \sqrt{T^2 - 1}$$

and

$$2\mathcal{R}e(y) = \beta \pm \sqrt{\beta^2 - \gamma + 2T}\ .$$

In each of these equation there is a choice of sign, so write it as

$$| y |^2 = T + \varepsilon_1\sqrt{T^2-1}, \quad 2\mathcal{R}e(y) = \beta + \varepsilon_2\sqrt{\beta^2 - \gamma + 2T}.$$

On computing $(T^2-1)(\beta^2-\gamma+2T))$, we get

$$\begin{aligned}(T^2-1)(\beta^2-\gamma+2T)) &= 2T^3 + T^2\beta^2 - T^2\gamma - \beta^2 + \gamma - 2T \\ &= q(T) + \beta^2T^2 + \delta^2 - 2\delta\beta T \\ &= q(T) + (\beta T - \delta)^2 \\ &= (\beta T - \delta)^2.\end{aligned}$$

Also we know that $y = (\tau - 2\mathcal{R}e(y))^{-1}(\sigma - | y |^2)$. On substituting value of $| y |^2$ and $2\mathcal{R}e(y)$ in it, we get

$$y = (\tau - \beta - \varepsilon_2\sqrt{\beta^2-\gamma+2T}\ )^{-1}(\sigma - T - \varepsilon_1\sqrt{T^2-1}\ )\ .$$

Using $\beta = \mathcal{R}e(\tau)$

$$\begin{aligned}y &= (\mathcal{I}m(\tau) - \varepsilon_2\sqrt{\beta^2-\gamma+2T}\ )^{-1}(\sigma - T - \varepsilon_1\sqrt{T^2-1}\ ) \\ &= \frac{(-\mathcal{I}m(\tau) - \varepsilon_2\sqrt{\beta^2-\gamma+2T})(\sigma - T - \varepsilon_1\sqrt{T^2-1})}{2(T - \mathcal{R}e\sigma)}\ .\end{aligned}$$

As $y + \overline{y} = 2\mathcal{R}e(y)$, substituting the value of y and $\overline{y}$ from above we get :

$$4\mathcal{R}e(y)(T - \mathcal{R}e(\sigma)) = -\mathcal{I}m(\tau)\sigma + \overline{\sigma}\mathcal{I}m(\tau) + 2\varepsilon_2(T - \mathcal{R}e(\sigma))\sqrt{\beta^2-\gamma+2T} + 2\varepsilon_1\varepsilon_2 \mid \beta T - \delta \mid\ .$$

Now,

$$\begin{aligned}\mathcal{I}m(\tau)\sigma - \overline{\sigma}\mathcal{I}m(\tau) - 2\mathcal{R}e(\sigma)\beta &= \mathcal{I}m(\tau)\sigma - \overline{\sigma}\mathcal{I}m(\tau) - 2\mathcal{R}e(\sigma)\mathcal{R}e(\tau) \\ &= -2\mathcal{R}e(\sigma\overline{\tau}) \\ &= -2\delta.\end{aligned}$$

Therefore

$$2\varepsilon_1\varepsilon_2 \mid \beta T - \delta \mid =$$

$$2(\beta+\varepsilon_2\sqrt{\beta^2-\gamma+2T})(T-\mathcal{R}e(\sigma))+\mathcal{I}m(\tau)\sigma-\overline{\sigma}\mathcal{I}m(\tau)-2\varepsilon_2(T-\mathcal{R}e(\sigma))\sqrt{\beta^2-\gamma+2T}\ .$$

$$2\varepsilon_1\varepsilon_2 \mid \beta T-\delta \mid = 2(\beta T-\delta)\ .$$

Thus if $\beta T > \delta$ we have $\varepsilon_1\varepsilon_2 = 1$ and two choices of sign must be same i.e. either $\varepsilon_1 = \varepsilon_2 = 1$ or $\varepsilon_1 = \varepsilon_2 = -1$.
If $\beta T < \delta$, then we have $\varepsilon_1\varepsilon_2 = -1$.

$\square$

Therefore we have shown that every $P \in SL(2,\mathbb{H})$ with $c \neq 0$ has a right eigenvalue y. Then $v = c^{-1}(y-d)$ is a fixed point. Thus for $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{H})$ we can conjugate it to a upper triangular form :

$$\begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} cv+d & -c \\ 0 & a-vc \end{pmatrix}.$$

Theorem 3.4.2 Let $Q = \begin{pmatrix} cv+d & -c \\ 0 & a-vc \end{pmatrix}$ where $Q \in SL(2,\mathbb{H})$ then $y_1 = cv+d$ and $y_2 = a-vc$ are right eigenvalue of Q.

Proof As we know if y is right eigenvalue of $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{H})$, then it will be right eigenvalue of RPR^{-1}, where $R \in SL(2,\mathbb{H})$. So here $y_1 = cv+d$ is a right eigenvalue of Q, where $Q = RPR^{-1}$ for some $R \in SL(2,\mathbb{H})$.
For $y_2 = a-vc$, substituting value of v from y_1 we get

$$\begin{aligned} y_2 &= a - c^{-1}(y_1-d)c \\ &= c^{-1}(cac^{-1}+d-y_1)c \\ &= c^{-1}(\tau-y_1)c. \end{aligned}$$

Using

$$\mid y_1 \mid^2 = T+\varepsilon_1\sqrt{T^2-1} \ \ and \ \ 2\mathcal{R}e(y_1) = \beta+\varepsilon_2\sqrt{2T-\gamma+\beta^2}\ .$$

We get $2\mathcal{R}e(y_2) = 2\mathcal{R}e(\tau - y_1) = 2\beta - 2\mathcal{R}e(y_1) = \beta - \varepsilon_2\sqrt{\beta^2 - \gamma + 2T}$. From equation $y_1^2 - \tau y_1 + \sigma = 0$ we get $\sigma = y_1(\tau - y_1)$. Therefore

$$| y_2 |^2 = | \tau - y_1 |^2 = | \sigma |^2 / | y_1 |^2 = T - \varepsilon_1\sqrt{T^2 - 1} \ .$$

Therefore y_2 is a right eigenvalue of Q.

□

Consider $P = \begin{pmatrix} cv + d & -c \\ 0 & a - vc \end{pmatrix}$ we now re-explain term of diagonal element of matrix P. We denote $cv + d$ with y_1 and $a - vc$ with y_2. Both are representatives for similar classes of eigenvalue. For matrix P, its $c = 0$, $a = y_1$, and $d = y_2$. Substituting these values in β, γ, δ we have modified definitions:
$\beta = \mathcal{R}e[y_1] + \mathcal{R}e[y_2]$,
$\gamma = | y_1 |^2 + | y_2 |^2 + 4\mathcal{R}e[y_1][y_2]$,
$\delta = | y_1 |^2 \ \mathcal{R}e[y_2] + | y_2 |^2 \ \mathcal{R}e[y_1]$.

3.5 Classification of Quaternionic Möbius transformations

Lemma 3.5.1 *Let $P(z) \in Sp(H)$ for some quaternionic Hermitian form H and suppose y is the right eigenvalue of H. Then $\overline{y}^{-1}$ is also an eigenvalue of P.*

Proof We know $Sp(H)$ preserve the given Hermitian form H. So, $P^*HP = H$ and

$$P^{-1} = H^{-1}P^*H.$$

Suppose y is an eigenvalue of P, then $\overline{y}$ will be eigenvalue for P^*. And since P^{-1} and P^* are conjugate(via H), therefore P^{-1} have the same eigenvalue $\overline{y}$. Therefore $\overline{y}^{-1}$ is also an eigenvalue of P.

□

Now we will classify the quaternionic Möbius transformation on the basis of number of fixed points.

Definition 3.5.2 Let $P \in PSp(H_1)$ be a quaternionic Möbius transformation preserving quaternionic unit ball $\mathbb{B}$ then
(1) $P(z)$ is *elliptic* it fixes at least one fixed point inside $\mathbb{B}$;
(2) $P(z)$ is *parabolic* if it fixes no point inside $\mathbb{B}$ but exactly one point on the boundary of $\mathbb{B}$;
(3) $P(z)$ is *loxodromic* if it fixes no point inside $\mathbb{B}$ but fixes exactly two points on the boundary of the $\mathbb{B}$.

Here is another classification into k - simple transformation

Definition 3.5.3 $P \in SL(2,\mathbb{H})$ is said *simple* if it is conjugate to an element of $SL(2,\mathbb{R})$.
And $P \in PSL(2,\mathbb{H})$ is $k - simple$ if it can be written as a product of k simple matrices and no fewer.

Proposition 3.5.4 *A map in* $PSL(2,\mathbb{H})$ *is* $3 - simple$ *if and only if* $\beta \neq \delta$.

Proof Suppose g_1, $g_2 \in PSL(2,\mathbb{H})$ be two simple maps. Where

$$g_1(z) = \frac{a_1z + b_1}{c_1z + d_1} \quad and \quad g_2(z) = \frac{a_2z + b_2}{c_2z + d_2}\ .$$

Let $g = g_1g_2$ then $g(z) = \frac{az+b}{cz+d}$. Conjugating if necessary, we suppose that $c = c_1a_2 + d_1c_2 = 0$.
Here $\sigma_1 = c_1a_1c_1^{-1}d_1 - c_1b_1 = 1$ and $\sigma_2 = c_2a_2c_2^{-1}d_2 - c_2b_2 = 1$. From this we get $b_1 = a_1c_1^{-1}d_1 - c_1^{-1}$ and $b_2 = a_2c_2^{-1}d_1 - c_2^{-1}$. Now substituting value of b_1 and b_2 in $a = a_1a_2 + b_1c_2$ and $d = c_1b_2 + d_1d_2$ we get

$$a = -c_1^{-1}c_2,$$

$$d = -c_1 c_2^{-1}.$$

In definition of β and δ we substitute these values and thus we get

$$\beta - \delta = \mathcal{R}e[a + d] - \mathcal{R}e[\mid a \mid^2 \overline{d} + \mid d \mid^2 \overline{a}]$$
$$= 0.$$

Conversely, if g is 3-simple then it is necessarily loxodromic and we conjugate g so that it assume the form $g(z) = (\lambda u)z(\lambda^{-1}v)d^{-1}$, where $\lambda > 1$ and u and v are unit quaternions that are not similar. One can check that that $\beta \neq \delta$.

□

So, Now we can classify quaternionic Möbius transformations.

Theorem 3.5.5 Let $P(z)$ be a quaternionic Möbius transformation with $\alpha = 1$, then
(a) If $\sigma = 1$ and $\tau \in \mathbb{R}$ then $P(z)$ is 1 - simple, $\beta = \delta$, $\gamma = \beta^2 + 2$ and the following holds.

(1) If $0 \leq \beta^2 < 4$ then $P(z)$ is *elliptic.*

(2) If $\beta^2 = 4$ then $P(z)$ is *parabolic.*

(3)If $\beta^2 > 4$ then $P(z)$ is *loxodromic.*

(b) If $\beta = \delta$ and either $\tau \in \mathbb{R}$ or $\sigma \neq 1$, then $P(z)$ is 2 - simple and following holds.

(1) If $\gamma - \beta^2 < 2$ then $P(z)$ is *elliptic.*

(2) If $\gamma - \beta^2 = 2$ then $P(z)$ is *parabolic.*

(3) If $\gamma - \beta^2 > 2$ then $P(z)$ is *loxodromic.*

(c) If $\beta \neq \gamma$ then $P(z)$ is 3-simple *loxodromic.*

Proof The map is 1-simple if and only if $\sigma = 1$ and $\tau \in \mathbb{R}$. Otherwise, two cases will arises either $\beta = \delta$ or $\beta \neq \delta$. If $\beta \neq \delta$ then from proposition 3.5.4 we can say g will be 3-simple. And if $\beta = \delta$ then g will be 2-simple. This completes the classification into (a), (b) and (c).
In case (a), we have $\beta^2 = \tau^2$ and further classification into (1), (2), and (3) can be easily be retrieved from usual classification of real Möbius transformation. In case (b), if $P(z)$ is *elliptic* then we conjugate $P(z)$ such that it is of the form $P(z) = azd^{-1}$, for unit quaternion a and d. This map satisfies $\gamma - \beta^2 < 2$. If $P(z)$ is *parabolic* then we conjugate $P(z)$ such that it is of the form $P(z) = (az+1)a^{-1}$. This map satisfies $\gamma - \beta^2 = 2$. Finally, if $P(z)$ is *loxodromic* then we conjugate $P(z)$ such that it is of the form $P(z) = (\lambda u)z(\lambda^{-1}v)d^{-1}$, where $\lambda > 1$ and u and v are unit quaternions. Since, $\beta = \delta$ we find that u and v are similar. This means that $\gamma - \delta^2 > 2$.

□

Further Development
In this thesis after reviewing the classical approach to classify orientation-preserving isometries of $\mathbf{H}^2$, we have given a complete account of a generalization of this approach in the 4 -dimensional hyperbolic geometric setting. The approach we presented here is essentially following Cao-Parker-Wang [**?**] and Parker [2]. However, we note that there are several other attempts to classify orientation-preserving isometries of the four and five dimensional hyperbolic spaces using quaternionic Möbius transformations. Between two and four, there is the three dimensional hyperbolic space. Following similar approach as in the two dimensional case, one may obtain the same type of algebraic classification for orientation-preserving isometries of $\mathbf{H}^3$ [1]. Gongopadhyay-Kulkarni [**?**] extended the classical classification of orientation-preserving isometries of $\mathbf{H}^3$ to the orientation-reversing isometries. Also Gongopadhyay-Kulkarni identified the full group of isometries of $\mathbf{H}^3$ with two copies of $GL(2,\mathbb{C})$. Under this identification $GL(2,\mathbb{C})$ acts on $\mathbf{H}^3$ by the complex Möbius

transformations. Using this action, Gongopadhyay-Kulkarni refined the classical trichotomy of the dynamical types of the isometries and classified those algebraically. In higher dimensions, orientation-reversing isometries of the hyperbolic space are not so well-studied. So from now on, we will only refer to the orientation-preserving isometries unless mentioned otherwise.

Around the same time Cao-Parker-Wang announced their work, T. Kido [**?**] also used quaternionic Möbius transformations to classify isometries of $\mathbf{H}^4$. Unfortunately, Kido's preprint never appeared, possibly until very recently. An attempt to identify the full group of isometries of $\mathbf{H}^5$ with the group of all quaternionic Möbius transformations was made by Wilker[**?**]. Gongopadhyay re-proved Wilker's main result in [4] and identified the group of orientation-preserving isometries of $\mathbf{H}^5$ with the group $PGL(2, \mathbb{H})$, where $GL(2, \mathbb{H})$ is the group of all 2×2 invertible matrices over the quaternions $\mathbb{H}$. Using this identification, Gongopadhyay [4] classified the orientation-preserving isometries of $\mathbf{H}^5$. The idea is to embed $\mathbb{H}$ into the matrix ring $M_2(\mathbb{C})$ and then to use the conjugacy invariants of the complex representations of the isometries for obtaining the algebraic classification. As a consequence of this approach, Gongopadhyay [**?**] also obtained a classification of the isometries of $\mathbf{H}^4$ which is different from the ones by Cao-Parker-Wang or Kido. Further, Gongopadhyay also demonstrated the usefulness of the centralizers, up to conjugacy, in the classification problem of isometries. There are also other authors who attempted the classification problem. However, among all these works, the classification of Gongopadhyay and Parker-Short are complete in the sense that these authors were able to take into account of all possible dynamical types of the isometries. On the other hand, all other works considered only the classical ellipto-parabolic-hyperbolic trichotomy. However, the approach taken by all these authors are mutually disjoint and each classification has its advantages, as well as disadvantages. Finally we would like to remark that the idea of Gongopadhyay in [4] has found fruitful applications in the context of complex and quaternionic hyperbolic geometries also, for example see [**?**].

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