# BOUNDARIES OF NSDC AND CLASSICAL T-SCHOTTKY SPACES

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## Certificate of Examination

This is to certify that the dissertation titled **Boundaries of NSDC and Classical T-Schottky Groups** submitted by **Abhijit Pant** (Reg. No. MS09002) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Prof. I. B. S. Passi    Dr. Mahender Singh    Dr. Krishnendu Gongopadhyay

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Dated: May 5, 2014

# Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Krishnendu Gongopadhyay at the Indian Institute of Science Education and Research Mohali.
This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bona fide record of original work done by me and all sources listed within have been detailed in the bibliography.

Abhijit Pant
(Candidate)
Dated: April 25 , 2014

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Krishnendu Gongopadhyay
(Supervisor)

# Acknowledgement

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Finally I would like to thank the faculty at IISER Mohali.

I would like to acknowledge that the material presented in this thesis is based on other people's work. At the beginning of every chapter we have clearly mentioned the main texts that have laid down the foundations of the chapter. My contribution is in the selection, presentation, and wherever required the elaboration of the material to make it self contained.

I would like to acknowledge that the figures present in the thesis are made in mathgv4 software and paint.

Abhijit Pant

# Contents

# Chapter 1

# Introduction

Consider $G$, a marked group with two parabolic generators, $S$ and $T$, then upto $PSL(2, \mathbb{C})$ conjugation, $G = G_\lambda = < S, T >$ where

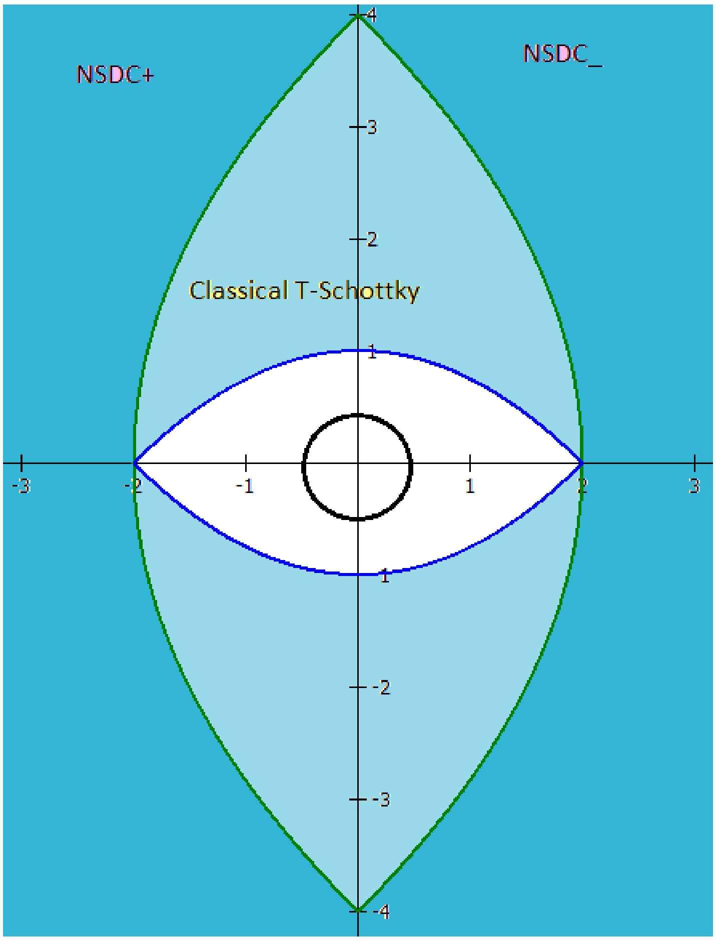
$$S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and

$$T = T_\lambda = \begin{pmatrix} 1 & 2\lambda \\ 0 & 1 \end{pmatrix}$$

for some $\lambda = |\lambda|e^{i\omega} \in \mathbb{C}$. We take $\lambda \neq 0$.
We will find conditions on $\lambda$ for which $G_\lambda$ is NSDC and Classical T-Schottky. It can be graphically depicted as in the figure: Superimposed Boundary Parabolas.

In the figure each point $\lambda \in \mathbb{C}$ corresponds to a two-generator group. The brighter colored subset of the classical T-Schottky groups comprises the non-separating disjoint circle groups (NSDC groups).




Superimposed Boundary Parabola ,(Ref: Gilman-Waterman [5] p. 13)

The white region consists of additional non-classical T-Schottky groups together with degenerate groups, isolated discrete groups and non-discrete groups. Points inside the Jørgensen circle ($|\lambda| < 1/2$) are non-discrete groups. In the thesis we will discuss only about the NSDC group and classical T-Schottky group.

The main results are discussed in chapters 4 and 5. The proofs of these results were given by Gilman and Waterman [5]. The results are:

**Theorem 1.1** *([5]) Let $G$ be a subgroup of the Möbius group generated by two parabolic elements and is parametrized by the complex number $\lambda = x + iy$. Then the boundary of the NSDC space is $\partial(NSDC) = \{(x, y) : y^2 = 16 - 8|x|\}$.*

**Theorem 1.2** *([5]) Assume that $G =< S, T >$ with tr(S) = tr(T)=2. Then*

$$|tr(ST) - 2| + |Re[tr(ST) - 2]| \geq 8$$

*$\Rightarrow G$ is discrete.*

**Theorem 1.3** *([5]) $G_\lambda$ lies on the boundary of classical T-Schottky space $\Leftrightarrow$*

$$\lambda = (2e^{\iota\omega})/(1 + |\sin\omega|)$$

*and thus eliminating $\omega \Leftrightarrow$*

$$\lambda = x + \iota y \text{ with } |y| = 1 - x^2/4.$$

However before that we shall in chapters 2 and 3, discuss some preliminary material which shall be needed in understanding our main results.

# Chapter 2

# Möbius Transformations

In this section, we will start with the formal definitions of Möbius transformations first in full generality in $\mathbb{R}^n$ and then we will restrict it to the complex plane. We will determine the general properties of these transformations and see how can we extend the action of a transformation in $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$. After this we will classify the transformations on the complex plane by two equivalent (conjugation invariant) ways: firstly by the squares of traces of the matrices they determine and secondly by their action on the upper half space of $\mathbb{R}^3$.We will further study the fixed points for these classes of transformations and determine their normal forms as matrices and prove the properties necessary for reaching our main results.

The material in this section has been referred from Beardon [2] and Maskit [7].

### 2.0.1 Möbius Transformations on $\mathbb{R}^n$

The sphere $S(a, r)$ in $\mathbb{R}^n$ is given by

$$S(a, r) = \{x \in \mathbb{R}^n : |x - a| = r\}$$

where $a \in \mathbb{R}^n$ and $r > 0$.

Definition 2.1 *The reflection (or inversion) in $S(a, r)$ is the function $\phi$ defined by*

$$\phi(x) = a + \left(\frac{r^2}{|x - a|^2}\right)(x - a). \tag{2.1}$$

In the special case of $S(0,1)(= S^{n-1})$, this reduces to

$$\phi(x) = \frac{x}{|x|^2}$$

and it is convenient to denote this by $x \mapsto x^*$ where $x^* = \frac{x}{|x|^2}$. The general reflection (2.1) may now be rewritten as

$$\phi(x) = a + r^2(x - a)^*.$$

The reflection in $S(a,r)$ is not defined when $x = a$ and this is overcome by adjoining an extra point to $\mathbb{R}^n$. We select any point not in $\mathbb{R}^n$ (for any n), label it $\infty$ and form the union

$$\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}.$$

As $|\phi(x)| \to \infty$ when $x \to a$ we define $\phi(a) = \infty$: likewise, we define $\phi(\infty) = a$. The reflection $\phi$ now acts on $\hat{\mathbb{R}}^n$ and $\phi^2(x) = x$ for all $x$ in $\hat{\mathbb{R}}^n$.Thus $\phi$ is a bijective map on $\hat{\mathbb{R}}^n$ to itself. Also, $\phi(x) = x$ if and only if $x \in S(a,r)$.

Definition 2.2 (Plane) *We define a set $P(a,t)$ to be a plane in $\hat{\mathbb{R}}^n$ if it is of the form*

$$P(a,t) = x \in \mathbb{R}^n|(x.a) = t \cup \{\infty\}$$

*where $a \in \mathbb{R}^n, a \neq 0$, $(x.a)$ is the usual scalar product and $t$ is real.*

Note that by definition, $\infty$ lies in every plane.

Definition 2.3 *The reflection $\phi$ in P(a, t) (or, in (x . a) = t) is defined as*

$$\phi(x) = x + \lambda a,$$

*where the real parameter $\lambda$ is chosen so that $\frac{1}{2}(x + \phi(x))$ is on $P(a,t)$ . The explicit formula is*

$$\phi(x) = x - 2[(x.a) - t]a^*, \tag{2.2}$$

*when $x \in \mathbb{R}^n$ and $\phi(\infty) = \infty$.*

Now we have, $\phi$ acts on $\hat{\mathbb{R}}^n$ with $\phi^2(x) = x$ for all $x$ in $\hat{\mathbb{R}}^n$ and so $\phi$ is a 1-1 onto map of $\hat{\mathbb{R}}^n$ to itself. Also, $\phi(x) = x$ if and only if $x \in P(a,t)$.

Definition 2.4 *A Möbius transformation acting in* $\mathbb{R}^n$ *is a finite composition of reflections (in spheres or planes).*

Note that each Möbius transformation is a homeomorphism of $\hat{\mathbb{R}}^n$ onto itself. The composition of two Möbius transformations is again a Möbius transformation and so is the inverse of a Möbius transformation for if $\phi = \phi_1 \dots \phi_n$ (where the $\phi_j$ are reflections) then $\phi^{-1} = \phi_n \dots \phi_1$. Finally, for any reflection $\phi$ say, $\phi^2(x) = x$ and so the identity map is a Möbius transformation.

Definition 2.5 *The group of Möbius transformations acting in* $\mathbb{R}^n$ *is called the General Möbius group and is denoted by* $GM(\hat{\mathbb{R}}^n)$*.*

Definition 2.6 *The Möbius group* $M(\hat{\mathbb{R}}^n)$ *acting in* $\hat{\mathbb{R}}^n$ *is the subgroup of* $GM(\hat{\mathbb{R}}^n)$ *consisting of all orientation-preserving Möbius transformations in* $GM(\hat{\mathbb{R}}^n)$*.*

### 2.0.2 Poincar*é* Extensions

Now, we will see how to extend the action of a Möbius transformation to a higher dimensional space. Each Möbius transformation $\phi$ acting in $\hat{\mathbb{R}}^n$ has a natural extension to a Möbius transformation $\tilde{\phi}$ acting in $\hat{\mathbb{R}}^{n+1}$ and with this one may regard $GM(\hat{\mathbb{R}}^n)$ as a subgroup of $GM(\hat{\mathbb{R}}^{n+1})$.

This extension depends on the embedding

$$x \mapsto \tilde{x} = (x_1, \dots, x_n, 0), \ x = (x_1, \dots, x_n),$$

of $\hat{\mathbb{R}}^n$ into $\hat{\mathbb{R}}^{n+1}$.

For each reflection $\phi$ acting in $\hat{\mathbb{R}}^n$, we define a reflection $\tilde{\phi}$ acting in $\hat{\mathbb{R}}^{n+1}$ as follows.

If $\phi$ is the reflection in $S(a, r)$, $a \in \hat{\mathbb{R}}^n$, then $\tilde{\phi}$ is the reflection in $S(\tilde{a}, r)$: if $\phi$ is the reflection in $P(a, t)$, then $\tilde{\phi}$ is the reflection in $P(\tilde{a}, t)$.

If $x \in \hat{\mathbb{R}}^n$ and $y = \phi(x)$, then from (2.1) and (2.2)

$$\tilde{\phi}(x_1, \ldots, x_n, 0) = (y_1, \ldots, y_n, 0) = \widetilde{\phi(x)}, \tag{2.3}$$

and in this sense we regard $\tilde{\phi}$ as an extension of $\phi$. Alternatively, we can identify $\mathbb{R}^{n+1}$ with $\mathbb{R}^n \times \mathbb{R}$ and write the above formula as

$$\tilde{\phi}(x, 0) = (\phi(x), 0).$$

Note that $\tilde{\phi}$ leaves invariant the plane $x_{n+1} = 0$ ($\hat{\mathbb{R}}^n$ and each of the half-spaces $x_{n+1} > 0$ and $x_{n+1} < 0$: these facts follow directly from (2.1) and (2.2).

As each Möbius transformation $\phi$ acting in $\hat{\mathbb{R}}^n$ is a finite composition of reflections $\phi_j$, say $\phi = \phi_1 \ldots \phi_m$, there is at least one Möbius transformation $\tilde{\phi}$, namely $\tilde{\phi}_1 \ldots \tilde{\phi}_m$, which extends the action of $\phi$ to $\mathbb{R}^{n+1}$ and which preserves

$$H^{n+1} = \{(x_1, \ldots, x_{n+1}) : x_{n+1} > 0\}$$

In fact, there can be at most one extension for if $\psi_1$ and $\psi_2$ are two such extensions, then $\psi_2^{-1}\psi_1$ fixes each point of the plane $x_{n+1} = 0$ and preserves $\phi$. Thus $\psi_1 = \psi_2$.

Definition 2.7 *The Poincaré extension of $\phi$ in $GM(\hat{\mathbb{R}}^n)$is the transformation $\tilde{\phi}$ in $GM(\hat{\mathbb{R}}^{n+1})$ .*

Observe that if $\phi$ and $\psi$ are in $GM(\hat{\mathbb{R}}^n)$ with say $\phi = \phi_1 \ldots \phi_m$ and $\psi = \psi_1 \ldots \psi_k$, then the *Poincaré* extension of $\phi\psi$ is given by

$$\begin{aligned}
\widetilde{(\phi\psi)} &= (\phi_1 \ldots \widetilde{\phi_m \psi_1} \ldots \psi_k) \\
&= \tilde{\phi}_1 \ldots \tilde{\phi}_m \tilde{\psi}_1 \ldots \tilde{\psi}_k \\
&= \tilde{\phi}\tilde{\psi}
\end{aligned}$$

so the map $\phi \mapsto \tilde{\phi}$ is an injective homomorphism of $GM(\hat{\mathbb{R}}^n)$ into $GM(\hat{\mathbb{R}}^{n+1})$.

We shall now see the action of the *Poincaré* extension $\tilde{\phi}$ in $H^{n+1}$. First, if $\tilde{\phi}$ is the reflection in the sphere $S(\tilde{a}, r)$, $a \in \mathbb{R}^n$, then by (2.7),

$$\frac{|\tilde{\phi}(y) - \tilde{\phi}(x)|}{|y - x|} = \frac{r^2}{|x - \tilde{a}||y - \tilde{a}|}$$

For the moment, let $[\tilde{\phi}(x)]_j$ denote the jth component of $\tilde{\phi}(x)$. As

$$\tilde{\phi}(x) = \tilde{a} + r^2(x - \tilde{a})^*,$$

we find that

$$[\tilde{\phi}(x)]_{n+1} = 0 + \frac{r^2 x_{n+1}}{|x - \tilde{a}|^2} \tag{2.4}$$

and this shows that

$$\frac{|y - x|^2}{y_{n+1} x_{n+1}} \tag{2.5}$$

is invariant under $\tilde{\phi}$.

The reflection $\tilde{\phi}$ in the plane $P(\tilde{a}, t)$, $a \in \mathbb{R}^n$ is a Euclidean isometry and moreover,

$$[\tilde{\phi}(x)]_{n+1} = x_{n+1} :$$

thus (2.16) is also invariant under this reflection.

We conclude that (2.16) is invariant under all *Poincaré* extensions. It is a direct consequence of this invariance that the *Poincaré* extension of any $\phi$ in $GM(\hat{\mathbb{R}}^n)$ is an isometry of the space $H^{n+1}$ endowed with the Riemannian metric $\rho$ given by

$$ds = \frac{|dx|}{x_{n+1}}$$

### 2.0.3 Möbius Transformation on Complex Plane

In this section we shall examine the action of Möbius transformations in $\hat{\mathbb{R}}^2$ and their extensions to $\hat{\mathbb{R}}^3$. We identify $\hat{\mathbb{R}}^2$ with the complex plane $\mathbb{C}$ and the algebraic structure of $\mathbb{C}$ then allows us to express the action of Möbius transformations algebraically. We shall also identify $(x, y, t)$ with the quaternion

$$x + yi + tj \tag{2.6}$$

: this enables us to express the *Poincaré* extension of a Möbius transformation in terms of the algebra of quaternions. The extended complex plane $\hat{\mathbb{C}}$ is $\mathbb{C} \cup \{\infty\}$ and this is identified with $\hat{\mathbb{R}}^2$. In terms of quaternions,

$$H^3 = \{z + tj : z \in \mathbb{C}, t > 0\}$$

and the boundary of $H^3$ in $\hat{\mathbb{R}}^3$ is $\hat{\mathbb{C}}$.

Definition 2.8 (Möbius Transformation/ Fractional Linear Transformations on $\mathbb{C}$) *: Möbius transformation is a map from $\mathbb{C}$ , the set of complex numbers, to $\mathbb{C}$ of the form $g(z) = \frac{az+b}{cz+d}$ , where a,b,c and d are complex numbers with $ad - bc \neq 0$.*

The latter condition ensures that $g$ is invertible: it also ensures that $c$ and $d$ are not both zero and the algebra of $\mathbb{C}$ then guarantees that $g$ is defined on $\mathbb{C}$ if $c = 0$ or on $\mathbb{C} - -d/c$ if $c \neq 0$. Now define $g(\infty) = \infty$ if $c = 0$ and

$$g(-d/c) = \infty, \; g(\infty) = a/c$$

if $c \neq 0$. With these definitions, g is a 1-1 map of $\hat{\mathbb{C}}$ onto itself. In addition, $g^{-1}$ is of the same form.

Definition 2.9 ($GL(2, \mathbb{C})$) *The non-singular $2 \times 2$ matrices with complex entries form a group under matrix multiplication called the General Linear Group of matrices. It is denoted by $GL(2, \mathbb{C})$.*

Definition 2.10 ($SL(2, \mathbb{C})$) *The subgroup of $GL(2, \mathbb{C})$ which contains matrices with determinant 1 is known as the Special Linear group of $2 \times 2$ matrices. It is denoted by $SL(2, \mathbb{C})$.*

Definition 2.11 ($PSL(2, \mathbb{C})$) *$PSL(2, \mathbb{C}) \cong SL(2, \mathbb{C})/\{+I, -I\}$ , where $SL(2, \mathbb{C})$ is the special linear group of $2 \times 2$ matrices with complex entries and I is the identity matrix.*

We can represent a Möbius transformation $g(z) = \frac{az+b}{cz+d}$ by a non-singular matrix as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

which acts on the sphere at infinity , $\hat{\mathbb{C}}$. Since the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$\begin{pmatrix} ta & tb \\ tc & td \end{pmatrix}$$

have the same action on $\hat{\mathbb{C}}$ the above representation determines a group isomorphism between the group of Möbius transformation and $PSL(2, \mathbb{C})$. Unless otherwise stated, we take pull backs of elements of $SL(2, \mathbb{C})$ to have positive trace. In particular for a two generator group, once the signs of the traces of the (pull backs) of the generators are chosen, the signs of the traces of all other elements of the group are determined.

Every orientation reversing conformal homeomorphism of $\mathbb{C}$ is of the form $g(z) = \frac{a\bar{z}+b}{c\bar{z}+d}, ad - bc \neq 0$ and these are called fractional reflections. We denote by $M$ the group of all fractional linear transformations and by $\tilde{M}$ the group of all fractional linear transformations and fractional reflection.

We classify the elements of $PSL(2, \mathbb{C})$ by the squares of their traces or equivalently by their action on $\hat{\mathbb{C}}$ which can be extended to an action on the upper half space of $\mathbb{R}^3$. Let $tr^2(A)$ denote $(tr(A))^2$.

Definition 2.12 (upper half plane : $H$) *The upper half space* $H^3 = \{(x, y, t)|x, y, t \in \mathbb{R}, t > 0\}$. *This space is called the* $\mathbf{hyperbolic\ three - space}$ *when equipped with the metric*

$$ds = \frac{\sqrt{dx^2 + dy^2 + dt^2}}{t}. \tag{2.7}$$

The boundary of the upper half space is the sphere at infinity denoted by $\hat{\mathbb{C}}$ ,the set of complex numbers union infinity $(\mathbb{C} \cup \infty)$.

The action of $M$ on $\hat{\mathbb{C}}$ is triply transitive; i.e., given any three distinct points $z_1, z_2, z_3$ on $\hat{\mathbb{C}}$ and any three other distinct points $w_1, w_2, w_3$, there is an element $g$ in M with $g(z_m) = w_m$.This transformation g is unique, for if we also have a fractional linear transformation $f$, with $f(z_m) = w_m$, then $f^{-1} \circ g$ has at least three fixed points, and so is the identity. To prove the above statement, it suffices to consider the case that $w_1 = 0, w_2 = 1, \ and \ w_3 = \infty$. In this case

$$g(z) = \frac{z_2 - z_3}{z_2 - z_1} \frac{z - z_1}{z - z_3} \tag{2.8}$$

We come now to the representation of $g$ in terms of quaternions. The quaternion is $z + tj$ where $z = x + iy$ and the *Poincaré* extension of $g$ is given by

$$g(z + tj) = \frac{(az + b)(\overline{cz + d}) + a\bar{c}t^2 + |ad - bc|tj}{|cz + d|^2 + |c|^2 t^2} \tag{2.9}$$

We now come to classification of Möbius transformation on the basis of fixed points in $\hat{\mathbb{C}}$.

Definition 2.13 (parabolic) *A Möbius transformation $g$ is parabolic if it has exatly one fixed point in $\hat{\mathbb{C}}$.*

The transformation $z \mapsto z + 1$ is a parabolic tranformation.
In the next few proposition we will prove certain properties of parabolic transformations and derive its normal form.

Proposition 2.14 *Every parabolic element of $M$ is conjugate to the translation $z \mapsto z + 1$.*

Proof Let $z_1$ be the fixed point of the parabolic element g. Let $z_2$ be some other point, and let $z_3 = g(z_2)$. Let $f \in M$ map this triple of points onto $\infty$, 0, and 1, respectively. Then $h = f \circ g \circ f^{-1}$ has its only fixed point at $\infty$, and maps 0 to 1.

One sees that a transformation of the form $z \mapsto az + 1$ has no finite fixed point if and only if $a = 1$.

If $g$ is parabolic, then $tr^2(g) = 4$. In fact, we can choose matrices for parabolic elements so that tr(g) = 2, by taking the positive pullback. If $g$ has its fixed point at $x$, then the matrix representation for it will be:

$$g = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

The following result gives us the *normal form* for a parabolic transformation:

Proposition 2.15 *If $g$ is parabolic with fixed point $x \neq \infty$ , then there is a unique complex number $p \neq 0$ so that*

$$g = \begin{pmatrix} 1 + px & -px^2 \\ p & 1 - px \end{pmatrix}.$$

Proof We know that there is a unique matrix with determinant 1 and trace 2 representing $g$. We write the diagonal terms as $1 + px$ and $1 - px$, and note that these determine $p$. We can then write

$$g = \begin{pmatrix} 1 + px & b \\ c & 1 - px \end{pmatrix}$$

Now, as the determinant is 1, we will have $bc = -p^2x^2$. The equation for the fixed points $z(cz + 1 - px) = (1 + px)z + b$, has $x$ as its only solution; hence $c = p$, and $b = -px^2$.

Every fractional linear transformation with two fixed points is conjugate to one with fixed points at 0 and $\infty$, such a transformation necessarily has the form $z \mapsto k^2z, k \in \mathbb{C}$. There are two special types of such transformations; the *rotations* of the form $z \mapsto e^{i\theta}z, \theta$ real, $e^{i\theta} \neq 1$ , and the *dilations* of the form $z \mapsto \lambda z, \lambda > 0, \lambda \neq 1$.

Definition 2.16 (elliptic) *Elliptic transformation is the Möbius transformation which is conjugate to a rotation.*

Definition 2.17 (loxodromic) *A non-elliptic tranfromation which has exactly two fixed points in $\hat{\mathbb{C}}$ is called a loxodromic transformation.*

Definition 2.18 (hyperbolic) *Hyperbolic transformation is the Möbius transformation which is conjugate to a dilation.*

When we extend the action of the group of Möbius transformation to the upper half space of $\mathbb{R}^3$ , we can define loxodromic elements by the fixed points on this space, and we will subclassify them into purely loxodromic and strictly loxodromic transformation. We use the term loxodromic to include both purely hyperbolic and strictly loxodromic transformations.

Now we determine normal forms for the transformations with two fixed points. We temporarily set aside the involution, or half-turn, where $k = \pm i$. For all the other transformations with two fixed points, we can distinguish between the fixed points; call one of them x and the other y. We conjugate x to 0, and y to $\infty$; then we can write $g$ in the form:

$$\frac{g(z) - x}{g(z) - y} = k^2 \frac{z - x}{z - y} \tag{2.10}$$

We choose a square root $k$; this is well defined up to multiplication by $-1$. Write $k^2 = \frac{k}{k^{-1}}$ and solve the above equation for g(z) to obtain the normal form

$$g = \frac{1}{x - y} \begin{pmatrix} xk^{-1} - yk & xy(k - k^{-1}) \\ (k - k^{-1}) & xk - yk^{-1} \end{pmatrix} \tag{2.11}$$

if $x$ and $y$ are both $\neq \infty$. If $x = \infty$, then

$$g = \begin{pmatrix} k^{-1} & y(k - k^{-1}) \\ 0 & k \end{pmatrix} \tag{2.12}$$

and if $y = \infty$, then

$$g = \begin{pmatrix} k & x(k - k^{-1}) \\ 0 & k^{-1} \end{pmatrix} \tag{2.13}$$

If $k^2 = -1$, and x and y are both $\neq \infty$, choose $k = i$, to obtain the normal form in this case:

$$g = \frac{1}{x-y} \begin{pmatrix} -i(x+y) & 2ixy \\ -2i & i(x+y) \end{pmatrix} \tag{2.14}$$

which, up to multiplication by $-1$, is symmetric in $x$ and $y$. If one of the fixed points is at $\infty$, we call the other one $x$; in this case the normal form is

$$g = \begin{pmatrix} i & -2ix \\ 0 & -i \end{pmatrix} \tag{2.15}$$

Now we discuss about the fixed points of Möbius transformation in $\hat{\mathbb{R}}^3$.

In its action on $\hat{\mathbb{C}}$, a Möbius transformation g has exactly one fixed point, exactly two fixed points or is the identity. This provides a rather primitive classification which we had seen above and we now obtain a finer classification based on the fixed points in $\hat{\mathbb{R}}^3$. This new classification is invariant under conjugation and so there is a still finer classification, namely the classification into conjugacy classes. One of our main results is that the function $tr^2$ defined below actually parametrizes the conjugacy classes.

Definition 2.19

$$tr^2(g) = \frac{tr^2(A)}{det(A)}$$

*where $A$ is the matrix corresponding to the transformation $g$.*

It is convenient to introduce certain normalized Möbius transformations. For each non-zero $k$ in $\mathbb{C}$ we define $m_k$ by

$$m_k(z) = kz \ (if k \neq 1)$$

and

$$m_1(z) = z + 1 :$$

we call these the standard forms. For future use, note that for all k (including k = 1),

$$tr^2(m_k) = k + \frac{1}{k} + 2. \tag{2.16}$$

If $g(\neq I)$ is any Möbius transformation then either $g$ has exactly two fixed points $\alpha$ and $\beta$ in $\hat{\mathbb{C}}$ or $g$ has a unique fixed point $\alpha$ in $\hat{\mathbb{C}}$ (in this case, we choose $\beta$ to be some point other than $\alpha$). Now let $h$ be any Möbius transformation with

$$h(\alpha) = \infty,\ h(\beta) = 0,\ h(g(\beta)) = 1\ if\ g(\beta) \neq \beta,$$

and observe that

$$hgh^{-1}(\infty) = \infty,$$

$$hgh^{-1}(0) = \begin{cases} 0 & \text{if } g(\beta) = \beta\ , \\ 1 & \text{if } g(\beta) \neq \beta\ . \end{cases}$$

If $g$ fixes $\alpha$ and $\beta$, then $hgh^{-1}$ fixes 0 and $\infty$ and so for some $k$ $(k \neq 1)$, we have $hgh^{-1} = m_k$. If $g$ fixes $\infty$ only then $hgh^{-1}$ fixes $\infty$ only and $hgh^{-1}(0) = 1$: thus $hgh^{-1} = m_1$,. This shows that any Möbius transformation $g$ is conjugate to one of the standard forms mk and this provides us a simple proof of of the following theorem.

Theorem 2.20 *Let $f$ and $g$ be Möbius transformations, neither the identity. Then $f$ and $g$ are conjugate if and only if $tr^2(f) = tr^2(g)$.*

Proof From the definition of $tr^2(g)$ we note that if $f$ *conjugate to* $g$ then $tr^2(f) = tr^2(g)$. Now assume that $tr^2(f) = tr^2(g)$. We know that $f$ and $g$ are each conjugate to some standard form, say $f$ *conjugate to* $m_p$ and $g$ *conjugate to* $m_q$. Thus

$$tr^2(m_p) = tr^2(f) = tr^2(g) = tr^2(m_q)$$

and using (2.32), this shows that $p = q$ or $p = 1/q$. We note that $m_p$ *conjugate to* $m_{1/p}$: this is trivial if $p = 1$ while if $p \neq 1$, we have

$$hm_ph^{-1} = m_{1/p},\ h(z) = \frac{-1}{z}.$$

We now have $f$ *conjugate to* $m_p$, $g$ *conjugate to* $m_q$ and (as $p = q$ or $p = 1/q$) $m_p$ *conjugate to* $m_q$. As conjugacy is an equivalence relation, this shows that $f$ *conjugate to* $g$ and the proof is complete.

We now study the fixed points of the standard forms in $\hat{\mathbb{R}}^3$. The action of $m_k$ in $\hat{\mathbb{R}}^3$ as given by (2.14) is

$$m_k(z+tj) = kz + |k|tj \ (k \neq 1)$$

$$m_1(z+tj) = z+1+tj,$$

and this enables one to find the fixed points of each $m_k$. Note that we are writing the upper half space in terms of quaternions $x+iy+tj$ as discussed above in (2.6). Clearly:

1. $m_1$ fixes $\infty$ but no other point in $\hat{\mathbb{R}}^3$

2. if $|k| \neq 1$, then $m_k$ fixes 0 and $\infty$ but no other points in $\hat{\mathbb{R}}^3$;

3. if $|k| = 1, k \neq 1$, then the set of fixed points of $m_k$ is

$$\{tj | t \in \mathbb{R}\} \cup \{\infty\}$$

The following definition is the extension of the definitions we had in previous part.

Definition 2.21 *Let $g$ be any Möbius transformation. We say*

1. *$g$ is parabolic if and only if $g$ has a unique fixed point in $\hat{\mathbb{C}}$ (equivalently, $g$ conjugate to $m_1$);*

2. *$g$ is loxodromic if and only if $g$ has exactly two fixed points in $\hat{\mathbb{R}}^3$ (equivalently, $g$ conjugate to $m_k$ for some $k$ satisfying $|k| \neq 1$);*

3. *$g$ is elliptic if and only if $g$ has infinitely many fixed points in $\hat{\mathbb{R}}^3$ (equivalently, $g$ conjugate to $m_k$ for some $k$ satisfying $|k| = 1, k \neq 1$).*

It is convenient to subdivide the loxodromic class by reference to invariant discs rather than invariant (fixed) points.

Definition 2.22 *Let $g$ be a loxodromic transformation. We say that $g$ is hyperbolic if $g(D) = D$ for some open disc (or half-plane) $D$ in $\hat{\mathbb{C}}$: otherwise $g$ is said to be strictly loxodromic.*

Since the classification of transformation is a conjugation invariant, we relate the type of transformation with another conjuagation invariant property , the trace , in the following proposition:

Proposition 2.23 *Let $g(\neq I)$ be any Möbius transformation. Then*
*(i) $g$ is parabolic if and only if $tr^2(g) = 4$;*
*(ii) $g$ is elliptic if and only if $tr^2(g) \in [0, 4)$;*
*(iii) $g$ is hyperbolic if and only if $tr^2(g) \in (4, +\infty)$;*
*(iv) $g$ is strictly loxodromic if and only if $tr^2(g) \notin [0, +\infty)$.*

# Chapter 3

# Fundamental Domains

This section has been referred from Maskit [7].

Definition 3.1 (Kleinian group ) *If a subgroup $G$ of $PSL(2, \mathbb{C})$ is discrete, it is called a Kleinain group.*

Definition 3.2 *We call $z \in \hat{\mathbb{C}}$ an ordinary point if it has a neighborhood $U$ such that $g(U) \cap U \neq \emptyset$ for at most finitely many $g \in G$.*

The set of ordinary points is denoted by $\Omega(G)$ and is also known as the regular set or the set of discontinuities.

### 3.0.4 Fundamental Domains

While trying to visualize a Kleinian group, the closest we can come , in general, is to draw a picture of $\Omega/G$ which somehow illustrates the action of G. The usual picture is given by a fundamental set or fundamental domain, which, roughly speaking, contains one point from each equivalence class in $Q_y$ and which, in some sense, illustrates the topology of $\Omega/G$.

Definition 3.3 *A fundamental domain $D$ for the Kleinian group (or discontinuous subgroup of $M$) $G$ is an open subset of $\Omega$ satisfying the following.*

1. *$D$ is precisely invariant under the identity in $G$.*

2. *For every $z \in \Omega$, there is a $g \in G$, with $g(z) \in D$.*

3. *The boundary of $D$ consists of limit points of $G$, and a finite or countable collection of curves; each curve lies, except perhaps for one or both of its endpoints, in $\Omega$; the intersection of the curve with $\Omega$ is called a side of $D$.*

4. *The sides are paired by $G$; that is, if $s$ is a side of $D$, then there is a side $s'$ not necessarily distinct from $s$, and there is a non-trivial element $g \in G$, called a side pairing transformation, with $g(s) = s'$. Also $(s')' = s$, and the side pairing transformation, from $s'$ to $s$, is $g^{-1}$.*

5. *If $\{s_m\}$ is a sequence of sides of $D$, then the spherical diameter, $dia(s_m) \to 0$; the sides of $D$ accumulate only at limit points.*

6. *Only finitely many translates of $D$ meet any compact subset of $\Omega$.*

We make the following observations from the definition above:

The first condition says that $D$ is disjoint from all its translates, or, equivalently, that no two points of $D$ are $G$-equivalent.

The second condition says that $\Omega \subset \bigcup g(\bar{D})$, where the union is taken over all elements of G.

We note that if there is a side $s$, and side pairing transformation $g$, with $g(s) = s$, then since the side pairing transformation from $s'$ to $s$ is $g^{-1}$, $g^{-1} = g$; that is, $g^2 = 1$.

Let $\tilde{D} = \bar{D} \cap \Omega$. The identifications of the sides induce an equivalence relation on $\tilde{D}$. An interior point is equivalent only to itself; if $x$ and $y$ lie on sides of $D$, and there is a side pairing transformation $g$, with $g(x) = y$, then $x$ and $y$ are equivalent. Let $D^*$ be $\tilde{D}$ factored by this equivalence relation.

Observe that $x$ and $y$ are equivalent points of $\bar{D}$ if and only if there is an element $g \in G$, with $g(x) = y$. Hence the projection $p$ provides a natural map of $D^*$ into $\Omega/G$ .

The endpoints of the sides that lie in $\Omega$ are called vertices. The sides of $D$ are also paired at the vertices. For each vertex $x$, and side $s$ ending at $x$, there is a unique other side $\bar{s}$, where $s$ and $\bar{s}$ both lie on the boundary of the same local component of $D$ near $x$.

The next lemma is a consequence of property 6.

Lemma 3.4 *If $x$ is a point of $D$, then there are at most finitely many points of $D$ equivalent to $x$.*

### 3.0.5 Poincar$\acute{e}$ Polyhedron Theorem

We assume that , $X$ is one of the spaces $H^n, B^n or S^n$, and $G$ is the group of isometries of $X$. We also assume that $n \geq 2$.

Assume that we are given a polyhedron $D$, where the sides of $D$ are pairwise identified by elements of $G$; our goal is to write down conditions on $D$ to guarantee that the group $G$, generated by the identifications of the sides of $D$, is discrete, and that $D$ is a fundamental polyhedron for G.

The first condition is that the sides of $D$ are paired by elements of $G$. That is, we assume that for each side $s$ of $D$, there is a side $s'$ not necessarily distinct from s, and there is an element, $g_s \in G$, satisfying the following conditions.

$(i)$ $g_s(s) = s'$.

$(ii)$ $g_{s'} = g_s^{-1}$

The isometries $g_s$ are called the side pairing transformations.

Since $s$ and $s'$ are both sides of $D$, $g_s(D)$ and $D$ either both lie on the same side of $s'$, or they lie on opposite sides. If they lie on the same side, then of course, $g_s(D) \cap D \neq 0$; this gives us our third condition:

$(iii)$ $g_s(D) \cap D = \emptyset$.

Let G be the group generated by the side pairing transformations. Observe that if there is a side $s$, with $s' = s$, then condition (ii) implies that $g_s^2 = 1$. If this occurs, the relation $g_s^2 = 1$, is called a reflection relation.

The side pairing transformations induce an equivalence relation on $\bar{D}$, where each point of $D$ is equivalent only to itself. Let $D^*$ be the space of equivalence classes, with the usual topology, so that the projection $p : \bar{D} \to D^*$ is continuous and open. If $D$ is to be a fundamental polyhedron for G, then there can be only finitely many points in each equivalence class of points of $\bar{D}$.

(iv) For every point $z \in D^*$, $p^{-1}(z)$ is a finite set.

Our next two conditions are related to the edges.The edges come in cycles; the condition above guarantees that each cycle is finite. For each edge $e = e_1$, let $e_1, \ldots, e_k$ be the ordered set of edges in the cycle containing $e$ , and let $g_1, \ldots, g_k$ be the corresponding side pairing transformations. Then the cycle transformation $h = h(e) = g_k \circ \ldots \circ g_1$ keeps $e$ invariant. $h$ depends on a choice of a side abutting $e$; if we choose the other side to start with, then we obtain $h^{-1}$ as the cycle transformation.

(v) For each edge $e$, there is a positive integer $t$ so that $h^t = 1$.

The relations in $G$, of the form $h^t = 1$, are called the cycle relations. There is essentially only one cycle relation for each equivalence class of cycles. If $e'$ is equivalent to $e$, then $h(e')$ is a conjugate of $(h(e))^{\pm 1}$.

We let $\alpha(e)$ be the angle, measured from inside $D$, at the edge e. We require

$$(vi)\ \sum_{m=1}^{k} \alpha(e_m) = 2\pi/t. \tag{3.1}$$

The conditions listed so far are sufficient to guarantee that if we look only at $D$, and those translates of $D$ that we know to about $D$, then the closures of these fit together without overlap, except along the translates of the sides, to fill out a neighborhood of $D$.

To state the last condition, we need the following construction.

We first form the group $G^*$, defined to be the abstract group generated by the side pairing transformations, and satisfying the reflection and cycle relations; we also endow $G^*$ with the discrete topology. There is an obvious homomorphism $\sigma : G^* \to G$.

We next consider the equivalence relation on $G^* \times \bar{D}$ generated by the following. The pairs $(g_1^*, x_1)$ and $(g_2^*, x_2)$ are equivalent if there is a side pairing transformation $f$ with $f(x_1) = x_2$, and if, as elements of $G^*$, $g_2^* = g_1^* \circ f^{-1}$. Let $X^*$ be $G^* \times \bar{D}$, factored by this equivalence relation. We endow $X^*$ with the usual identification topology, so that the natural projection from $G^* \times \bar{D}$ to $X^*$ is continuous.

We remark that it is not at this point clear that this equivalence relation is locallyjinite. That is, there might be infinitely many points of the form $(g_m^*, x)$ in $G^* \times \bar{D}$ which are all identified in $X*$.

There is a natural map $q : X^* \to D^*$, defined by projection on the second factor of $G^* \times \bar{D}$ followed by the projection $p$ from $D$ to $D^*$. It is easy to see that $q$ is well defined and continuous.

There is also a map $r : X^* \to X$, defined by $r(g^*, x) = \sigma(g^*)(x)$. It is easy to see that $r$ is well defined and continuous; our eventual goal is to prove that $r$ is a homeomorphism, and incidentally, that $\sigma$ is an isomorphism.

One should view $X^*$ as the set of translates of $\bar{D}$ under $G^*$, where these different translates have been sewn together at the sides so that, the map $r$ is well defined. We should also think of $X^*$ as the set of translates of $\bar{D}$ under the group $G$, where we regard overlapping, other than that given by the identifications of the sides, and the known relations of $G$ (i.e., the relations of $G^*$), as lying on different sheets over

$X$; then $r$ is the projection from this covering to $X$.

In the lemma below, we prove that $r$ is a local homeomorphism. Once we have established this, we can use $r$ to lift the local differential metric from $X$ to $X^*$; then the distance between points of $X^*$ is the infimum of the lengths of smooth paths joining them. We use this distance on $X^*$ and the projection $q$ to define a distance on $D^*$; the distance $d(z, z')$ between points of $D^*$ is the infimum of the distances $d(x, x')$, where $q(x) = z$, and $q(x') = z'$.

It is easy to see that this is the natural notion of distance on $D^*$; that is, $d(z, z') = inf \sum d(x_m, x'_m)$, where the infimum is taken over all finite sets of points $\{x_1, x'_1, \ldots, x_k, x'_k\}$, in $D$, with $p(x_1) = z$, $p(x'_m) = p(x_{m+1})$ and $p(x'_k) = z'$ .

Our last condition is
(vii) D* is complete.

Theorem 3.5 *Let $D$ be a polyhedron with side pairing transformations satisfying conditions (i) through (vii). Then $G$, the group generated by the side pairing transformations is discrete, $D$ is a fundamental polyhedron for $G$, and the reflection relations and cycle relations form a complete set of relations for $G$.*

Lemma 3.6 *Let $D$ be a polyhedron with side pairing transformations satisfying conditions (i) through (vi). Then every point $z^* \in D^*$ has a neighborhood $U$ so that $q^{-1}(U)$ is a disjoint union of relatively compact open sets $U_\alpha$, where for each $\alpha$, $r|U_\alpha$ is a homeomorphism onto a convex set.*

Proof  Notice first that $G^*$ acts as a group of homeomorphisms on $X^*$, and that $I \times D$ is a fundamental domain for this action; that is, no non-trivial translate under $G^*$ of $I \times D$ intersects it, and the union of the translates of the closure covers all of $X^*$. We also remark that this lemma asserts that the translates of $D$, under those side pairing transformations that are known to abut $D$, precisely fill out a neighborhood of $D$.

If $x$ is an interior point of $D$, then let $\delta$ be the distance from $x$ to the nearest side, and let $V$ be the ball of radius $\delta$ about $x$. Set $U = p(V)$. Since every point of $V$ is equivalent only to itself, the preimages of $V$ in $X^*$ are precisely the sets of the form $U_\alpha = g^* \times V$ these are disjoint open sets, and for each $\alpha$, $r|U_\alpha$ is a homeomorphism onto a ball of radius $\delta$.

If $x$ is an interior point of a side $s$ of $D$, then there is another side $s'$, and there is a side pairing transformation $g$ with $g(s) = s'$; set $x' = g(x)$. If $x \neq x'$, let $\delta$ be the minimum of the distance from $x$ to $x'$, the distance from $x$ to any side of $D$ other than $s$, and the distance from $x'$ to any side of $D$ other than $s'$. Let $V(V')$ be the intersection of the ball of radius $\delta/2$ about $x(x')$, with $D$. Note that $V$ and $V'$ are disjoint. Set $U = p(V)Up(V')$.

If $x = x'$ let $\delta$ be the minimum distance from $x$ to any side of $D$ other than $s$, let $V$ be the intersection of the ball of radius $\delta/2$ about $x$ with $\bar{D}$, and let $U = p(V)$.

Each connected component $U_\alpha$ of $q^{-1}(U)$ consists of the union of two half balls. If $x \neq x'$ then near $(1, x)$, these are the half balls $1 \times V$ and $g^{-1} \times V'$. If $x = x'$, these are $1 \times V$ and $g^{-1} \times V'$. Since $x'$ is the only other point of $D$ equivalent to $x$, each $U_\alpha$ is a neighborhood of a point of the form $(g^*, x)$ in $X^*$;it is clear that $r|U_\alpha$ is a homeomorphism onto a ball of radius $\delta/2$.

Next let $x = x_1$ be an interior point of an edge $e_1$. Let $\{e_1, \ldots, e_k\}$ be the cycle of edges containing $e_1$, let $h = g_k \circ \ldots g_1$ be the cycle transformation at $e_1$, and let $t$ be the order of $h$. Define the $tk$ elements of $G^*, j_1, \ldots, j_{tk}$, by

$$j_1 = g_1,$$

$$j_2 = g_2 \circ g_1,$$

$$\vdots$$

$$j_k = h,$$

$$j_{k+1} = g_1 \circ h,$$

$$j_{tk-1} = g_{k-1} \circ \ldots \circ h^{t-1},$$

$$j_{tk} = 1,$$

Let $x_{m+1} = j_m(x_1)$.
Each of the points $x_m$ lies in the intersection of two sides; let $\delta_m$ be the minimum of the distance from $x_m$ to any other side of $D$, and of the distance from $x_m$ to any point $x \neq x_m$.

Let $\delta = 1/2min(\delta_m)$, and let $V_m$ be the intersection of the ball of radius $\delta$ about $x_m$ with $\bar{D}$; observe that the sets $V_m$ are all disjoint. Set $U = \bigcup p(V_m)$. Each component $U_\alpha$ of $q^{-1}(U)$ is a union of tk "wedges". Near $(1, x)$, these wedges are the sets $(1 \times V_1), \ldots, (j_{tk-1}^{-1} \times V_{tk})$.

Each edge lies in the intersection of exactly two sides, and each side uniquely determines its side pairing transformation. It follows from condition (v) that $(1, x_1), (j_1^{-1}, x_2), \ldots, (j_{tk-1})$ is a complete set of equivalent points of $G^* \times \bar{D}$. Condition (v) also implies that the set of the form $U_\alpha$ near $(1, x)$ is a neighborhood of $(1, x)$.

Condition (vi) asserts that, in the 2-plane orthogonal to $e_1$, these $tk$ translates of $D$ fit together without overlap, and fill out a neighborhood of $x$ in that plane. It follows that $r|U_\alpha$ is a homeomorphism onto a ball of radius $\delta$.

Using the action of $G^*$ on $X^*$, we see that the same statement is true for an arbitrary point of the form $(g^*, x)$.

There are no translates of $1 \times x$ on $1 \times D$ in $X^*$, other than the obvious ones. It follows that each component $\hat{U}_\alpha$ is relatively compact in $X^*$.

We know that for each $\alpha$, $r|\hat{U}_\alpha$ is a homeomorphism. Set $U = \hat{U}_\alpha \times \hat{U}_\alpha$, and observe that for each $\alpha$, $U_\alpha$ is relatively compact in $X^*$ and $r|U_\alpha$ is a homeomorphism onto a product of discs, which is convex.

# Chapter 4

# Non Separating Disjoint Circle Groups

In this section we will prove two main results about the boundary of the NSDC space.

Theorem 4.1 *([5]) Let $G$ be a subgroup of the Möbius group generated by two parabolic elements and is parametrized by the complex number $\lambda = x + iy$. Then the boundary of the NSDC space is $\partial(NSDC) = (x, y) : y^2 = 16 - 8|x|$.*

Theorem 4.2 *([5]) If $trace(A) = 2$, $trace(B) = 2$ and $trace(AB) - 2 = 2.4d/2 - d$, where $d$ lies exterior to the NSDC teardrop, then $G =< A, B >$ is discrete.*

We begin by noting the definition of marked group.

Definition 4.3 (Marked Group) *A group with a choice of an ordered set of generators is called a marked group.*

### 4.0.6 Half Turn

Definition 4.4 (Half Turn) *An elliptic element of $PSL(2, \mathbb{C})$ of order two is called a half turn.*

If $a \in M$ has exactly two fixed points on $\mathbb{C}$, then the hyperbolic line in $H^3$ joining these points is the axis of $a$. If $a$ is elliptic, then every point on the axis $A_a$ is fixed by $a$; we also say that $a$ is the half-turn about $A_a$.

Proposition 4.5 *Let $g$ be a parabolic element of $M$, with fixed point $z$, and let $A$ be a hyperbolic line with one end point at $z$. Then there is a hyperbolic line $B \neq A$, where $B$ also has one endpoint at $z$, so that $g = b \circ a$, where $a$ is the half turn about $A$, and $b$ is the half-turn about $B$.*

Proof We can assume that $g(z) = z + 1$, and that $A$ has its other endpoint at 0; i.e., $a(z) = -z$. Then $g \circ a(z) = -z + 1$, which is a half-turn about the line with endpoints at $1/2$ and $\infty$.

Proposition 4.6 *Let $A$ and $B$ be hyperbolic lines in $(H)^3$ that do not have a common endpoint. Then there is a unique hyperbolic line $C$ orthogonal to both $A$ and $B$.*

Proof Normalize so that the half-turn a about A is the transformation $a(z) = -z$. Let b be the half-turn about B write b in the form

$$b = \frac{1}{x-y} \begin{pmatrix} -i(x+y) & 2ixy \\ -2i & i(x+y) \end{pmatrix}$$

Interpret the matrix

$$g = ab - ba = \frac{1}{x-y} \begin{pmatrix} 0 & -4xy \\ -4 & 0 \end{pmatrix}$$

as an element of $M$, and observe that $g$ is a half-turn that interchanges the endpoints of both $A$ and $B$. It follows that $g$ preserves both $A$ and $B$, so the axis of $g$ is orthogonal to both.

To prove uniqueness, suppose $C$ and $C'$ are hyperbolic lines orthogonal to both $A$ and $B$. Let $g$, $g'$ be the half-turns about $C$, $C'$, respectively. Then $h = g \circ g'$ preserves the endpoints of both $A$ and $B$. Since $h$ has four fixed points on the sphere at infinity, $h = 1$.

If $c$ and $d$ are any points in $H^3$ the upper half plane, or $\mathbb{C}_\infty$, the sphere at infinity, we let $[c, d]$ denote the hyperbolic lne in $H^3$ connecting them. The ends of [c, d] are the points where the line intersects $\mathbb{C}_\infty$.
For a and a' in $\mathbb{C}_\infty$ we let $H_{[a,a']}$ be the half-turn about the line [a, a']. The hyperbolic line, [a, a'], is the axis of the half-turn and we call a and a' the ends of the half-turn. When neither a nor a' is $\infty$, the isometric circle of $H_{[a,a']}$ is a circle in $\mathbb{C}_\infty$ with center $a + a'/2$ and radius $|a - a'|/2$. We let ISO $H_{[a,a']}$ denote the isometric circle and ISO $P_{[a,a']}$ the hyperbolic plane whose horizon is ISO $H_{[a,a']}$. We call ISO $P_{[a,a']}$ the isometric plane of the half turn. It is the hemisphere obtained by intersecting the isometric sphere of $H_{[a,a']}$ with $H^3$.
We observe that

$$H_{[a,a']}([a, a']) = [a, a'],$$

$$H_{[a,a']}(ISOH_{[a,a']}) = ISOH_{[a,a']},$$

$$H_{[a,a']}(ISOP_{[a,a']}) = ISOP_{[a,a']}.$$

More generally, whether or not $a$ or $a'$ is $\infty$, we have

Theorem 4.7 *([3]) A half-turn fixes every plane whose horizon passes through its ends.*

Proof Let $C_{a,a'}$ be any circle on the sphere at infinity passing through the improper points $a$ and $a'$. Let $P_{a,a'}$ be the plane whose horizon is $C_{a,a'}$. Then $[a, a']$ lies in $P_{a,a'}$. Let $x$ be any point in $P_{a,a'}$, x does not belong to$[a, a']$, and let $l_x$ be the perpendicular from x to $[a, a']$ with $p_x = l_x \cap [a, a']$. Then $l_x$ is contained in $P_{a,a'}$ and $H[a, a'](l_x) = l_x$. $H_{[a,a']}$ sends x to the point on $l_X$ whose hyperbolic distance from $p_x$ is the same as that of $x$ from $p_x$. Since $l_x$ is contained $Paa'$,and $p_x$ belongs to $Paa'$. Hence $H_{[aa']}(P_{aa'}) = P_{aa'}$.

Finally, we note that the matrix representing $H_{[a,a']}$ is given by

$$\frac{1}{i(a - a')} \begin{pmatrix} -(a + a') & 2aa' \\ -2 & (a + a') \end{pmatrix} \tag{4.1}$$

when neither $a$ nor $a'$ is $\infty$ and that a matrix for $H_{[a,\infty]}$ is given by

$$\begin{pmatrix} i & -2ia \\ 0 & -i \end{pmatrix} \tag{4.2}$$

### 4.0.7 Boundary of NSDC group

#### Terms and Definitions Used

If $G$ is a marked group with two parabolic generators, $S$ and $T$, then up to $PSL(2,\mathbb{C})$ conjugation, $G = G_\lambda =< S, T >$ ,where

$$S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and

$$T = T_\lambda = \begin{pmatrix} 1 & 2\lambda \\ 0 & 1 \end{pmatrix}$$

for some $\lambda = |\lambda|e^{\iota\omega} \in \mathbb{C}$. We assume that $\lambda \neq 0$.

We note that since $trS = 2, trT = 2$ and $\lambda \neq 0$,

$$tr[S,T] - 2 = trSTS^{-1}T^{-1} - 2 = 4\lambda^2;$$

$$trST^{-1} = trTS^{-1} = 2 - 2\lambda.$$

and

$$trST = tr(TS)^{-1} = 2 + 2\lambda.$$

Definition 4.8 *We now define the space of groups:*

1. *NSDC-space*

$$NSDC = \{\lambda \in \mathbb{C} | G_\lambda \ is\ NSDC\}$$

2. *Two-parabolic classical T-Schottky space*

$$\mathcal{CS} = \{\lambda \in \mathbb{C} | G_\lambda \ is\ Classical\ T-Schottky\}$$

3. *Marked two parabolic classical T-Schottky*

$$\mathcal{C}S^{PP} = \{\lambda \in \mathbb{C} | G_\lambda = \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2\lambda \\ 0 & 1 \end{pmatrix} \right\rangle is\, marked\, Classical\, T{-}Schottky\}$$

If $a \in PSL(2, \mathbb{C})$ has exactly two fixed points on $\hat{\mathbb{C}}$, then the hyperbolic line in $H$ joining these points is the axis of $a$. For a hyperbolic line $[x, y]$ with $x \neq y$,i.e distinct end points, let $H_{[x,y]}$ be the half turn about the line with ends x and y.

Let $G =< A, B >$ be a group so that $L = [n, n']$, the common perpendicular to the axis of A and B, is a proper line, then there are unique hyperbolic lines $L_A$ and $L_B$ satisfying $A = H_{L_A} H_L$ and $B = H_{L_B} H_L$. Let $a$ and $a'$ be the ends of $L_A$ so that $L_A = [a, a']$ and $b$ and $b'$ be those of $L_B$ so that $L_B = [b, b']$.

If the common perpendicular to the axes of A and B is a proper line, then we have a natural construction that associates an ordered six-tuple of complex numbers $(a, a', n, n', b, b')$ to the ordered pair of transformations. On the other hand, an ordered six-tuple in $\hat{\mathbb{C}}$, $(a, a', n, n', b, b')$ satisfying $a \neq a'$, $b \neq b'$, $n \neq n'$ determines an ordered pair of transformations.

Definition 4.9 (Ortho end) *The ortho end of the marked group $G =< A, B >$ is the six tuple of complex numbers $(a, a', n, n', b, b')$.*

Definition 4.10 (non-separating disjoint circle property) *Six points in $(a, a', n, n', b, b') \in \hat{\mathbb{C}}^6$ with $a \neq a'$, $b \neq b'$ and $n \neq n'$ have the* **non − separating disjoint circle property** (**NSDC**) *if there exist pairwise disjoint or tangent circles $C_A, C_D$ and $C_B$ (respectively) on $\mathbb{C}$, passing through a and $a'$, n and $n'$, and b and $b'$ (respectively) and no one circle separates the other two.*

Definition 4.11 (marked non-separating disjoint circle group) *A marked group $G =< A, B >$ is a marked non separating disjoint circle group (a marked NSDC group) if the ortho-end of A and B has the non-separating disjoint circle property.*

**Definition 4.12** *$G$ is a non-separating disjoint circle group, if some pair of generators for $G$ has the non-separating disjoint circle property.*

One can consider the space of NSDC groups as the set of ordered six-tuples in $\hat{\mathbb{C}}^6$ such that $(a, a', n, n', b, b')$ is the ortho end of an NSDC group. Now consider a group G which is NSDC for the ortho end $(a, a', n, n', b, b')$. We will write the matrices for $A$ and $B$, the generators of $G$, in terms of the six tuple of complex numbers. The final conditions will be derived in terms of the traces of $A$, $B$ and $[A, B]$. Since trace is conjugacy invariant, these conditions will give us the conditions on parameters for NSDC groups.

We determine the effect, a change of generators has on the non-separating disjoint circles.

**Lemma 4.13** *For a non-abelian two generator subgroup $G$ of $PSL(2, \mathbb{C})$, the following are equivalent:*

1. $G =< A, B >$ *is marked NSDC.*
2. $G =< B, A >$ *is marked NSDC*
3. $G =< A^{-1}, B^{-1} >$ *is marked NSDC*
4. $G =< A^{-1}, AB >$ *is marked NSDC*
5. $G =< A^{-1}, BA >$ *is marked NSDC*
6. $G =< A, A^{-1}B^{-1} >$ *is marked NSDC*
7. $G =< A, B^{-1}A^{-1} >$ *is marked NSDC*

**Proof** $1 \Leftrightarrow 2$

$G =< A, B >$

$A = H_{L_A} H_L$

$B = H_{L_B} H_L$

Let $L = [n, n']$ be the common perpendicular to the axes of $A$ and $B$.

$1 \Rightarrow 2$

Let $a$ and $a'$ be the ends of $L_A$ and $b$ and $b'$ be the ends of $L_B$.
Claim: The ortho end of $B$ and $A$ have NSDC property.
This follows from the fact that the ortho end of $A$ and $B$ have NSDC property as

Ortho end of $A$ and $B$ is $(a, a', n, n', b, b')$
Ortho end of $B$ and $A$ is $(b, b', n, n', a, a')$

Clearly this has NSDC property as we can have the same pairwise disjoint circles on $\hat{C}$, $C_A$, $C_D$ and $C_B$ passing through $(a, a')$, $(n, n')$, $(b, b')$ respectively and no circle separates the other two.They exist because $A$ and $B$ are NSDC. In place of $C_D$ we take $C'_D$ the circle of opposite orientation passing through $n'$ and $n$.
Thus $G =< B, A >$ is marked NSDC. We can prove $2 \Rightarrow 1$ by interchanging the roles of $A$ and $B$.

Similarly other parts can also be shown.

**Lemma 4.14** *For a non-abelian two-generator subgroup $G$ of $PSL(2, \mathbb{C})$, the following are equivalent:*

1. $G =< A, AB >$ *is marked NSDC*

2. $G =< A, BA >$ *is marked NSDC*

3. $G =< A^{-1}, A^{-1}B^{-1} >$ *is marked NSDC*

4. $G =< A^{-1}, B^{-1}A^{-1} >$ *is marked NSDC*

Proof $1 \Leftrightarrow 4$

From the above lemma we have
$G =< A, B >$ is marked NSDC $\Leftrightarrow G =< A^{-1}, B^{-1} >$ is marked NSDC. Thus, $G =< A, AB >$ is marked NSDC $\Leftrightarrow G =< A^{-1}, B^{-1}A^{-1} >$ is marked NSDC.

Similarly, $2 \Leftrightarrow 3$.

$1 \Leftrightarrow 2$
$G =< A, AB >$ is marked NSDC $\Leftrightarrow G =< A, A^{-1}B^{-1}A^{-1} >$ is marked NSDC.

(Using the $(6)^{th}$ part of the above lemma.)

$\Leftrightarrow G =< A^{-1}, A^{1}B^{-1}A^{-1}A >$ is marked NSDC.

(Using the $(5)^{th}$ part of the above lemma.)

$\Leftrightarrow G =< A^{-1}, A^{1}B^{-1} >$ is marked NSDC.

Lemma 4.15 *([5]) If $G =< A, AB >$ is marked NSDC with $A$ parabolic, then $B$ is loxodromic.*

Proof Let $A = H_A.H_N$ and $AB = H_N.H_AB$. If $C_A$, $C_N$ and $C_{AB}$ have the NSDC property, then one can write $B = H_N H_A H_N.H_{AB}$ . Therefore, $C_{AB}$ and $B(C_{AB})$ are strictly disjoint, and $B$ maps the exterior of the first circle to the interior of the second. This gives us that $B$ is neither parabolic nor elliptic. Thus $B$ is loxodromic.

Theorem 4.16 *If $S$ and $T$ are parabolic, then $G =< S, T >$ is a non-separating disjoint circle group if and only if either $< S, T >$ or $< S, T^{-1} >$ is marked NSDC.*

Proof $\Leftarrow$ If $< S, T >$ or $< S, T^{-1} >$ is marked NSDC. then $G =< S, T >$ is NSDC from the definition of NSDC group.

$\Rightarrow$ If $G =< S, T >$ is NSDC then say, there are two transformations $A$ and $B$ $\in G$ such that $G =< A, B >$ is marked NSDC. Then $A = S^{\alpha}T^{\beta}$..... $B = S^{\alpha'}T^{\beta'}$.... Using the change of generators formula and the above lemma we, get that the only possibilities are $A = S$ $B = T$ or $A = S$ $B = ST$ or $A = S$ $B = T^{-1}$ but second case is not possible as it would imply that T is loxodromic.

Since we can obtain the entries of matrices A and B from the six-tuple using the relation $A = H_{L_A}H_L$ and $B = H_{L_B}H_L$ conditions on the six tuple give us conditions on the matrices. In particular, the imposition of certain geometric configurations on the six points give us discreteness conditions.

Corresponding to a marked group with two parabolic generators, we look at the boundary of NSDC space when two pairs of points coalesce. In the six tuple $\sigma = (a, a', b, b', d, d')$ assume that $a' = b$, and $b' = d$ and then call it a four point. Clearly, if $\sigma$ is a four point NSDC, then $C_B$ is tangent to $C_A$ at $a' = b$ and to $C_D$ at $b' = d$. A six tuple of the form as described together with the circles $C_A$, $C_B$, and $C_D$ is termed a four point configuration.

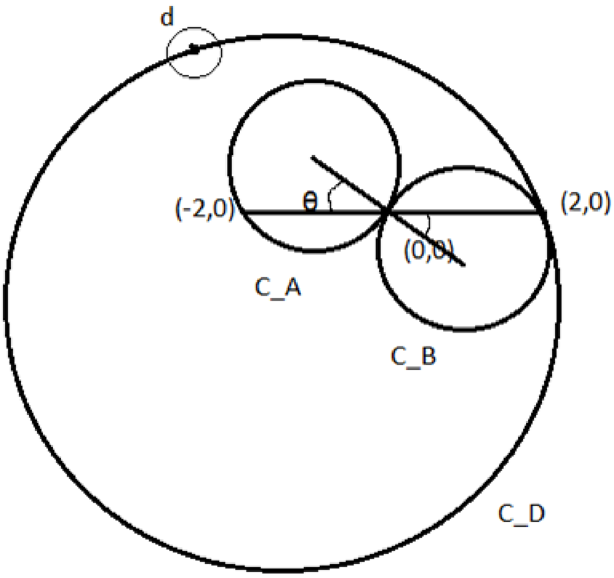
Definition 4.17 *A four point configuration corresponding to a six tuple $\sigma$ is called extreme if the configuration is NSDC but in any deleted neighbourhood of $\sigma$, there exists a four-point $\sigma_0$ which is not NSDC and $\sigma_0'$ which is NSDC.*

**MOTIVATION** :

An NSDC group generated by two parabolics involves one free parameter $d = x + iy$. The six-tuple of points are $(-2, 0, 2, d, 0, 2)$. The circles $C_A$ and $C_B$ are tangent at the point 0 and, therefore, determine an angle $\theta$ . This is the angle that the line connecting their centers makes with the positive x-axis moving in a counter-clockwise direction. Conversely, any angle $\theta$ between $\pi/4$ and $-\pi/4$ determines such a pair of tangent circles. Any circle $C_D$ passing through 2 and d will have a center with coordinates $(M, N)$ and radius $r$. If $C_D$ is tangent to $C_A$ at a point $T$, then the coordinates of the point $(M, N)$ can be computed as explicit functions of $\theta$ as can the radius r. Points on the circle $C_D$, known as the $\theta$ -circle are given by

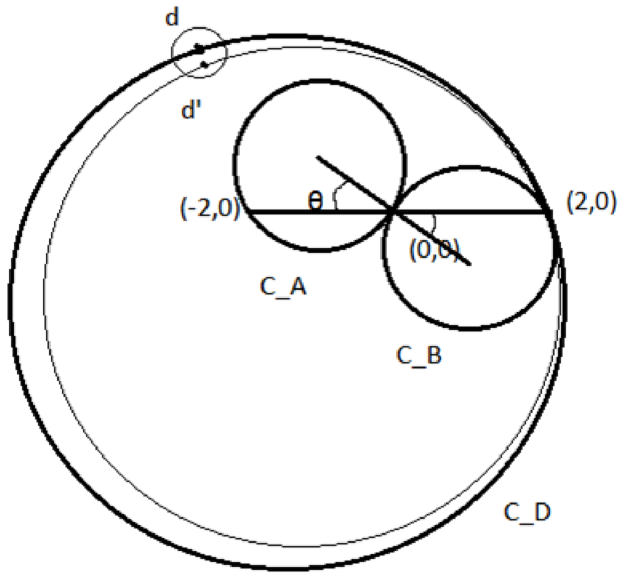
$(x, y) = (M(\theta) + r_\theta . \cos t, N(\theta) + r_\theta \sin t)$, for some real parameter $t$, $0 \leq t \leq 2\pi$.

We want to find out the extreme configurations which constitute the boundary configuration of the NSDC space. We observe that the boundary configurations are those extreme configuration which have maximal tangencies.




If $C_D$ is not required to be tangent to $C_A$ but only to pass through $d$ and be tangent to $C_B$ , then it is clear that for $d = (x, y)$ in a small circular neighborhood of $d = (x, y)$ there are circles through $(x, y)$ so that $(-2, 0, 2, d, 0, 2)$ are still NSDC. This shows that non-tangent $C_Ds$ correspond to interior points of NSDC- space.

If $C_D$ is required to be tangent to $C_A$, we have that for some $d$ near the point on $d = (x, y)$ on the $\theta$ -circle, $C_D$, this may or may not be the case.



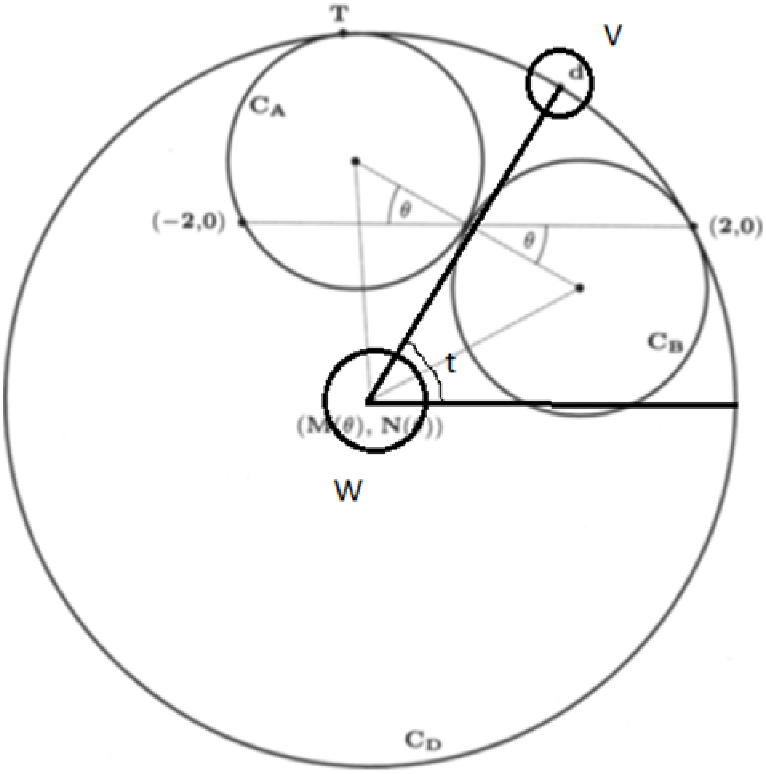

We consider all points on the $\theta$ -circle. One has $x = x(\theta, t) = M(\theta) + r_\theta \cos t$ and $y = y(\theta, t) = N(\theta) + r_\theta \sin t$.

Thus one has a map from $\mathbb{R}^2$ to itself: $(\theta, t) \mapsto (x, y)$.

Now we look at the Möbius of this map:
If Jacobian is not zero, then $f$ is an invertible one-one onto map from a small neighborhood $V$ of $d$ to a small neighborhood $W$ of $(M(\theta), N(\theta))$ and thus for any point $d' \in V$ we can find a $\theta' - circle$ with $\theta' \in W$.

Thereby making the configuration nsdc for $(-2, 0, 0, 2, 2, d')$.

Thus, the point $d$ where Jacobian of $f$ is not zero is an interior point of the nsdc space.
Therefore the boundary points are those where the Jacobian is zero.




(Ref: Gilman-Waterman [5] p. 17)

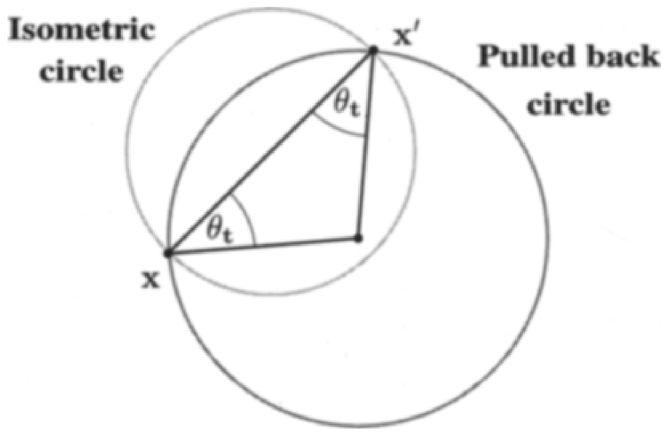
When one can calculate the Jacobian of this map to show that $(x_0, y_0)$ is a boundary point of NSDC space precisely when $x_0 = 2 - 4\frac{\sin^2\theta}{1+\sin^2\theta}$ and $y_0 = 8\frac{\sin^3\theta\cos\theta}{1+\sin^2\theta}$, using the inverse function theorem. The plot of this boundary is the tear drop.

Points $d = x + iy$ in the interior of the tear drop corresponds to (marked) groups $G_d$ that are not NSDC and points in the exterior to $G_d$ that are NSDC groups. A change of parameters maps the tear drop into a parabola and replaces the parameter $d$ by $\lambda = \frac{4d}{d-2}$. Taking marked and non-marked NSDC groups into account yields the region bounded by the two parabolas pictured in figure of superimposed parabolas as NSDC -space, that is $\{G_\lambda | G_\lambda \ or \ G_{-\lambda} \ is \ an \ NSDC \ group\}$

### 4.0.8 Pull Back Angle

The center of isometric circle of $H_{[x,x']}$ is at $(x+x')/2$ and radius is $|(x-x')/2|$. We note that any other circle passing through $x$ and $x'$ will have its centre on the perpendicular bisector of $x$ and $x'$. Thus the centre is $c_t = (x+x')/2 + it(x-x')/2$ for real number $t$.

$\theta_t$, the angle made by the radius connecting $c_t$ to $x$ or $x'$ with the line segment connecting $x$ and $x'$, is called the pull back angle of the circle.




Pull Back Angle,(Ref: Gilman-Waterman [5] p. 17)

Since our main results involve only conjugacy invariants, we can conjugate the six tuple so that $a=-2$, $a'=b=0$ and $b'=d'=2$. To reach our results we proceed first by finding three tangent circles and then determining which configurations of tangent circles are extreme. Thus we need to determine circles $C_A$, $C_B$ and $C_D$ such that $C_A$ and $C_B$ are tangent at $(0,0)$, $C_B$ and $C_D$ are tangent at $(2,0)$ and $C_D$ and $C_A$ are tangent at some point say $T$.

We note that if $C_A$ and $C_B$ are tangent at $(0,0)$, then their pull back angles must be the same but with opposite sign and thus their radii are equal.

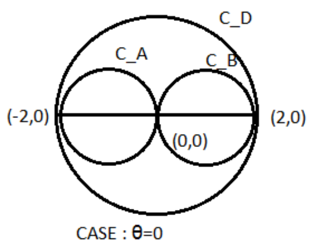
Proposition 4.18 *([5])*

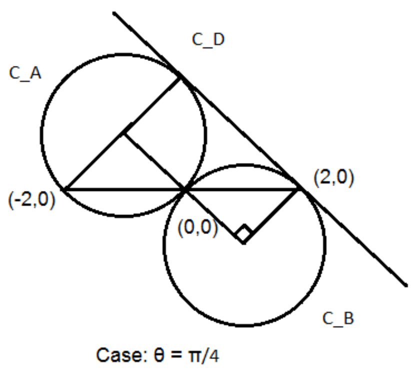
1. *Let $C_A$ and $C_B$ be pull back circles with common pull back angle $\theta$, $-\pi/4 < \theta < \pi/4$, there exists a unique circle called $\theta$ circle which is tangent to $C_A$ at $(2,0)$ and tangent to $C_B$.*

2. *When $\theta \neq \pm\pi/4$ and $\theta \neq 0$, the $\theta$ circle is given by*
   $x(\theta,t) = M(\theta) + R(\theta).\cos\theta$
   $y(\theta,t) = N(\theta) + R(\theta).\sin\theta$
   *where*
$$M(\theta) = \frac{(2.\tan\theta)}{(\tan\theta - (\tan\theta)^{-1})}$$
$$N(\theta) = \frac{2}{(tan\theta - (\tan\theta)^{-1})}$$
$$R(\theta) = \frac{2.\sqrt{(1+\tan^{-2}\theta)}}{(tan\theta - (\tan\theta)^{-1})}$$

3. *When $\theta = 0$, the circle is $(x(\theta,t), y(\theta,t)) = (2.\cos t, 2.\sin t)$ and when $\theta = \pm\pi/4$, the circle is $y = \mp(x-2)$.*

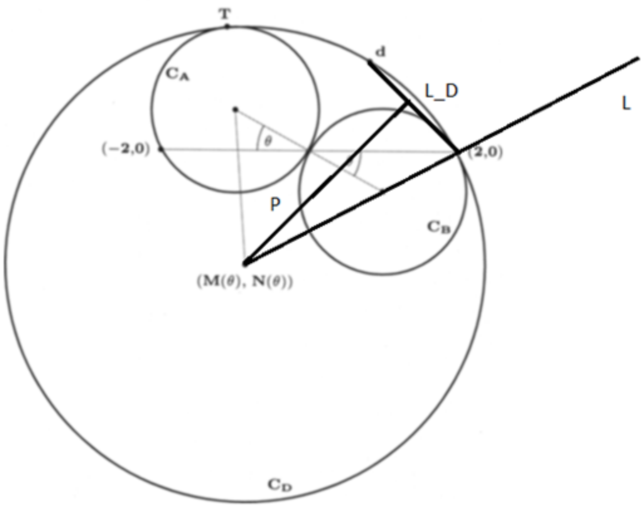
Proof (*Gilman, Waterman*) Assume the points are $A = (-2,0)$, $A' = B' = (0,0)$, $B' = D' = (2,0)$ and $D = (x,y)$. We construct a line L through $(2,0)$ with slope $\tan\theta$ and $L_D$ the line through D and $(0,0)$. The cases for $\theta = 0$ and $\theta = \pm\pi/4$ are shown in the figure below.



We will find formulas for the center and radius of the $\theta$-circle for D. In the figure D = d.
Take P to be the perpendicular bisector of the line segment connecting D and (2, 0). If L is not parallel to P then we replace D by another point $D'$ on $L_D$.

Take (M, N) to be the point of intersection of L and P. Thus (M, N) is the center of a circle passing through (2, 0) and D, and this circle will have the prescribed tangencies. We term this circle as the $\theta$ circle. We observe that the pull-back angle -$\theta$ is the same as the pull-back angle $\theta + \pi/2$.




(Ref: Gilman-Waterman [5] p. 17)

**Claim** : $M(\theta) = \frac{(2.\tan\theta)}{(\tan\theta - (\tan\theta)^{-1})}$ and $N(\theta) = \frac{2}{(tan\theta - (\tan\theta)^{-1})}$

We take $c_A$ to be the center of the circle $C_A$ and $c_B$ that of $C_B$. Then the slope of the line passing through (M, N) and (2, 0) is

1. $N/(M-2) = \tan\theta$.

2. $c_A = (-1, \tan\theta)$.

3. $c_B = (1, -\tan\theta)$.

4. Slope of the line connecting $D'$ and $D = (x, y)$ is $y/(x-2)$.

Now for $x \neq 2$ , $y \neq 0$, the perpendicular bisector passes through $(M, N)$ and $((x+2)/2, y/2)$ and its slope is $-(x-2)/y$. This yields

5. $y - 2N = -(x-2)(x+2-2M)/y$
6. The distance: Dist $[(M, N), c_B] + R_B = R_D = Dist[(M, N), c_A] + R_A$, where $R_A$, $R_B$ and $R_D$ are the radii of the circles, $C_A$, $C_B$, $C_D$.
7. As $R_A = R_B = \sec\theta$, we have Dist$[(M, N), c_A] =$ Dist $[(M, N), c_B]$.
8. $(M+1)^2 + (N - \tan\theta)^2 = (M-1)^2 + (N + \tan\theta)^2$, whence
9. $M = N\tan\theta$.

One infers from from $M/N = \tan\theta$ and $N/(M-2) = \tan\theta$ that

10. $M = M(\theta) = -2.\tan\theta^2/1 - \tan\theta^2 = -\tan 2.\theta.tan\theta$ and
11. $N = N(\theta) = -\tan 2.\theta$.

We write $R = R_D$, $R = R_\theta = \sqrt{(M-2)^2 + N^2}$, we get

12. $R(\theta) = |N/\sin\theta|$

so that any point on the $\theta$ -circle has coordinates given by

13. $(x, y) = (x(\theta, t), y(\theta, t)) = (x, y) = (M + R\cos t, N + R\sin t)$

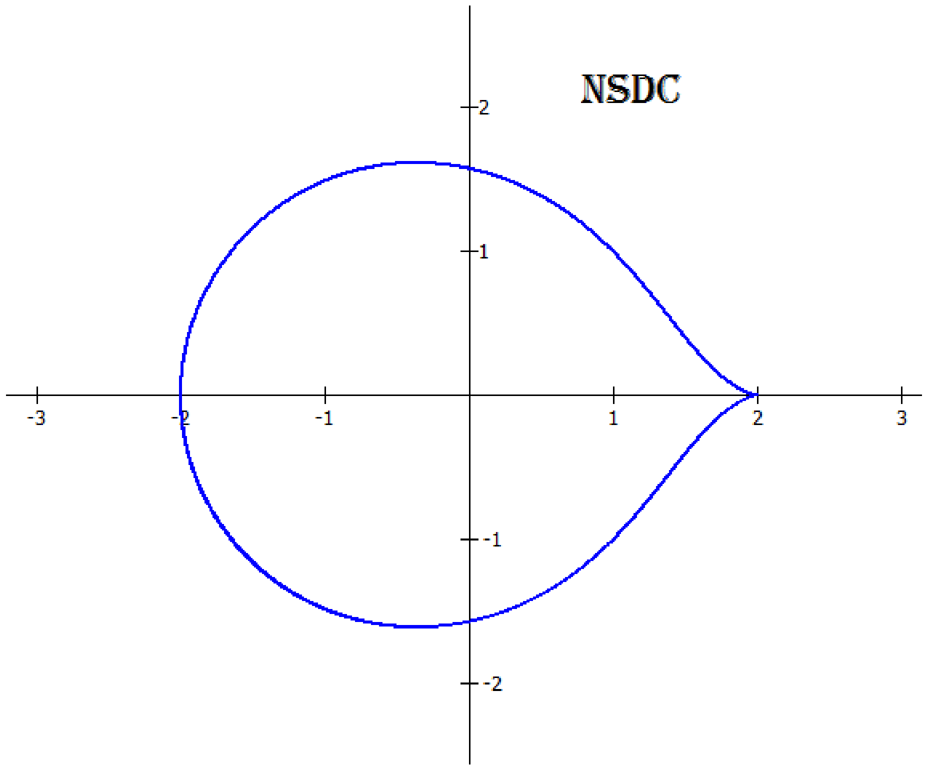
This proves our claim and hence our proposition.

Definition 4.19 *A configuration of three tangent circles in $\hat{\mathbb{C}}$ is said to be extreme at d if the three circles $C_A$, $C_B$ and $C_D$ are all tangent ( so that d is an NSDC point) and if every neighbourhood of d contains points which are NSDC and points which are not NSDC.*

Proposition 4.20 *([5]) The points in $\mathbb{C}$ where the Jacobian is zero are the extreme boundary points; there is one such point for each $\theta$ . All other points either have totally NSDC neighborhood or totally non-NSDC neighborhood. The Jacobian is non-zero except at the point $z_0 = (x_0, y_0)$ where $x_0 = 2 - 4\frac{\sin^2\theta}{1+\sin^2\theta}$ and $y_0 = 8\frac{\sin^3\theta\cos\theta}{1+\sin^2\theta}$.*

Proof We compute the partial derivatives as follows:

1. $x_\theta = M_\theta + R_\theta \cos t$ , $x_t = -R\sin t$ and $Y_\theta = N_\theta + R - \theta\sin t$, $Y_t = R\cos t$.




NSDC Teardrop, (Ref: Gilman-Waterman [5] p. 20)

We now compute the Jacobian and set it equal to zero which gives

2. $RR_\theta + R_M\theta\cos t + RN_\theta \sin t = 0.$

From $RR_\theta = M_\theta(M-2) + NN_\theta$, $\cos t = (x-M)/R$ and $\sin t = (y-N)/R$, we get the Jacobian to satisfy the following relation

3. The Jacobian is 0 if and only if $y/(x-2) = -(M_\theta/N_\theta)$.

4. We determine these partial derivatives to infer that the critical points of the Jacobian are the points that satisfy: $y/(x-2) = -\sin 2\theta$.

   Thus from $M = -tan2\theta.\tan\theta$ ,we have $M_\theta = -(2\sec 2\theta^2 \tan\theta + \sec\theta^2 \tan 2\theta)$. Hence $-M_\theta/N_\theta = 2\sec^2 2\theta \tan\theta + \sec^2\theta \tan 2\theta / -2\sec^2\theta = -\sin 2\theta$

5. We use the fact that the distance from (x, y) to (M, N) is R and $y/(x-2) = -sin2\theta$ to obtain the coordinates for extreme points .

6. If $(x_\theta, y_\theta)$ are as given in the proposition, one can calculate that $y_0/(x_0-2) = -\sin 2\theta$, thus $(x_0, y_0)$ is extreme.

Now consider the map $f(z) = 4.z/z{-}2$. This map sends the six tuple $(-2, 0, 0, 2, 2, d)$ to the six tuple $(2, 0, 0, \infty, \infty, \lambda)$ where $\lambda = 4.d/d-2$.

Extreme Four Point Configuration via $(2, 0, 0, \infty, \infty, \lambda)$.
We observe that $S = H_{[2,0]}H_{[0,\infty]} =$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

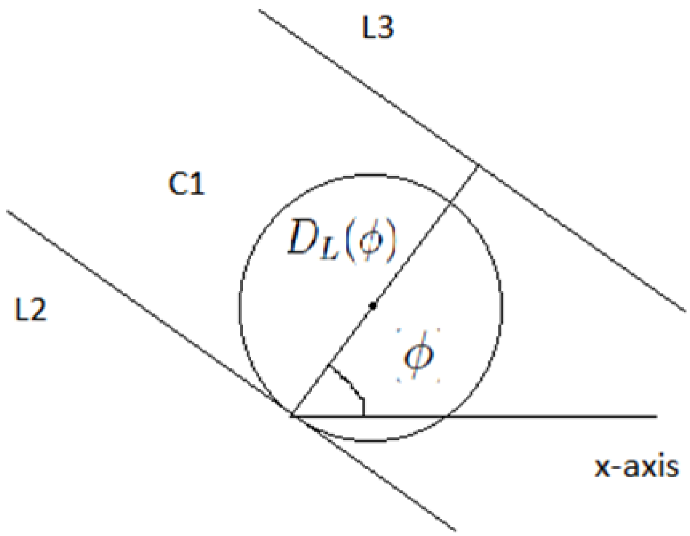
$T = H_{[0,\infty]}H_{[\infty,\lambda]}=$

$$\begin{pmatrix} 1 & -2\lambda \\ 0 & 1 \end{pmatrix}$$

Hence $G =< S, T >= G_{-\lambda}$

**Definition 4.21** *We term the marked group $G_\lambda$ of NSDC$_-$ type if the six tuple $(2, 0, 0, \infty, \infty, \lambda)$ is NSDC. We term it of NSDC$_+$ type if the six tuple $(2, 0, 0, \infty, \infty, -\lambda)$ is NSDC.*
*$G_\lambda$ is NSDC if and only if the marked is either NSDC$_-$ or NSDC$_+$*

The four point configuration for $(2, 0, 0, \infty, \infty, \lambda)$ is a circle $C_1$ through 0 and 2 along with a line $L_2$ which is tangent to $C_1$ at 0 and a line $L_3$ through $\lambda$. $C_1$ is parameterized by specifying the angle $\phi$ made by the x-axis and the radial vector through 0, which is measured in the anti-clockwise direction. Thus $(-\pi/2) < \phi < (\pi/2)$.




(Ref: Gilman-Waterman [5] p. 23)

The radius of $C_1$ is given by $R(\phi) = 1/cos\phi$.

Once we have specified $C_1$ ,the lines $L_2$ and $L_3$ are determined. Take $D_L(\phi)$ to be the signed distance between $L_2$ and $L_3$, the positive direction is taken to be the radial vector from 0 to the center of $C_1$. Now the condition that the configuration be an NSDC configuration for this six tuple reduces to the condition that $L_2$ and $L_3$ be sufficiently far apart so as to contain $C_1$ and $C_1$ actually lies between these two lines.

**Lemma 4.22** *$C_1$, $L_2$, and $L_3$ form an NSDC configuration for* $(2, 0, 0, \infty, \infty, \lambda)$

$$\Leftrightarrow D_L(\phi) \geq 2R(\phi)$$

$$\Leftrightarrow |\lambda| \cos(\omega - \phi) \cos(\phi) \geq 2.$$

Proof The condition $|D_L(\phi)| \geq 2R(\phi)$ guarantees that $L_2$ and $L_3$ are sufficiently far apart to contain $C_1$, and $D_L(\phi) > 0$ ensures that $C_1$ lies between $L_2$ and $L_3$. The equivalent condition follows from

$$D_L(\phi) = \lambda.e^{\iota\phi} \textit{ and } R(\phi) = 1/\cos(\phi)$$

$$\Rightarrow D_L(\phi) = |\lambda|(\cos\omega\cos\phi + \sin\omega\sin\phi) \textit{ and } R(\phi) = 1/\cos(\phi).$$

Theorem 4.23 *([5])* $G_\lambda$ *is* $NSDC_- \Leftrightarrow |\lambda|[1 + \cos\omega] \geq 4$
$G_\lambda$ *is* $NSDC_+ \Leftrightarrow |\lambda|[1 - \cos\omega] \geq 4$.

Proof

$$G_\lambda \textit{ is } NSDC_- \Leftrightarrow |\lambda|[\cos\omega + \cos(\omega - 2\phi)] \geq 4.$$

We maximize the left hand side as a function of $\phi$ to get the desired result; the maximum occurs when $\phi = \omega/2$.

Thus we get our main result

Corollary 4.24 *([5])* $G_\lambda$ *is NSDC* $\Leftrightarrow |\lambda|[1 + |\cos\omega|] >= 4$ *The boundary curve* $|\lambda|[1 + |\cos\omega|] = 4$ *gives the boundary of NSDC space in the polar coordinates* $(|\lambda|, \omega)$, *and this boundary curve is the piecewise-parabola whose Euclidean coordinates (x, y) satisfy* $y^2 = 16 - 8|x|$.

Proof We take $x = |\lambda|\cos\omega$ and $y = |\lambda|\sin\omega$.

Theorem 4.25 *([5]) Assume that* $G =< S, T >$ *with tr(S) = tr(T) = 2. Then G is NSDC if and only if*

$$|tr(ST) - 2| + |Re[tr(ST) - 2]| \geq 8.$$

We get the above result by substituting $\lambda = 1/2[tr(ST) - 2]$.

From the normal form for transformations with two fixed points, we get the following lemma.

Lemma 4.26 *If $\alpha \neq \beta$, $\alpha \neq \infty$, and $\beta \neq \infty$ then,*

$$H_{\alpha,\beta} = \frac{\iota}{\alpha - \beta} \begin{pmatrix} (\alpha + \beta) & -2\alpha\beta \\ 2 & -(\alpha + \beta) \end{pmatrix}.$$

### 4.0.9 Discreteness Criterion

Given any two generator subgroup $G =< A, B >$ of $PSL(2, \mathbb{C})$, one can naturally associate it to a three generator subgroup of $PSL(2, \mathbb{C})$, denoted $3G$, in which G sits as a normal subgroup of index two. Since a group and a subgroup of finite index are either simultaneously discrete or simultaneously non-discrete, one can study discreteness criteria for $G$ or for $3G$. The three- generator group, $3G$, is constructed as follows:
We define a hyperbolic line $N$ as follows. If $A$ and $B$ are both parabolic, then $N$ is the line whose ends are fixed points of $A$ and $B$; if neither $A$ nor $B$ is parabolic, we let $N$ be the common perpendicular to the axes of $A$ and $B$; if $A$ is parabolic and $B$ is not, $N$ is perpendicular to the axis $B$ through the fixed point of $A$; and if $B$ is parabolic and $A$ is not, $N$ is the perpendicular to the axis of $A$ through the fixed point of $B$.We let $H_N$ be the half turn about $N$. If the ends of $N$ are $n$ and $n'$, then $H_N = H_{[n,n']}$. Then $A$ can be factored as $A = L_A.H_N$ where $L_A$ is a half turn and $B$ can be factored as $B = H_N.L_B$ where $L_B$ is a half turn. We define the three generator group, $3G =< L_A, H_N, L_B >$.

We let $a, a'$ be the ends of $L_A$ and $b, b'$ be the ends of $L_B$.

The isometric circles of $H_N$, $L_A$ and $L_B$ pass through $n$ and $n'$, $a$ and $a'$, and $b$ and $b'$ respectively. If the isometric circles are disjoint, then one can apply the Poincaré Polyhedron theorem to the region bounded by the isometric spheres with side-pairings $L_A$, $L_B$, and $H_N$ to conclude that $3G$ is discrete.

We note that $(a, a', b, b', c, c')$ an ordered six-tuple in $\hat{\mathbb{C}}^6$ with $a \neq a'$, $b \neq b'$, $c \neq c'$ determines an ordered pair of generators $A = H_{[a,a']}.H_{[c,c']}$ and $B = H_{[c,c']}.H_{[b,b']}$ and both a two generator group $G(a, a', b, b', c, c') =< A, B >$ and a three generator group $3G(a, a', b, b', c, c') =< H_{[a,a']}, H_{[c,c']}, H_{[b,b']} >$.

**Theorem 4.27** *If $(a, a', b, b', c, c')$ has the NSDC property, then $< H_{[a,a']}, H_{[b,b']}, H_{[c,c']} >$ is discrete.*

**Proof** Let $C_A$, $C_B$ and $C_C$ be the circles containing respectively a and $a'$, b and $b'$, and c and $c'$. Apply the Poincaré Polyhedron theorem to the region bounded by $P_A$, $P_B$ and $P_C$, the planes whose horizons are $C_A$, $C_B$ and $C_C$ respectively with the side pairing transformations $H_{[a,a']}$, $H_{[b,b']}$, and $H_{[c,c']}$ to conclude discreteness using the theorem which shows that half-turns fix the corresponding planes.

**Corollary 4.28** *If $(a, a', b, b', c, c')$ has the NSDC property, then $< H_{[a,a']}.H_{[c,c']}, H_{[c,c']}.H_{[b,b']} >$ is discrete.*

**Corollary 4.29** *If the ortho end of $< A, B >$ has the NSDC property, then $< A, B >$ is discrete.*

The above result and the following computations give us the next proposition on discreteness.

By the normal form of half turn we get,

$$H_{[-2,0]} = \begin{pmatrix} -\iota & 0 \\ \iota & \iota \end{pmatrix},$$

$$H_{[0,2]} = \begin{pmatrix} -\iota & 0 \\ -\iota & \iota \end{pmatrix},$$

$$H_{[2,d]} = \frac{\iota}{2-d} \begin{pmatrix} (d+2) & -4d \\ 2 & -(d+2) \end{pmatrix}$$

and hence

$$A = H_{[-2,0]}H_{[0,2]} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix},$$

$$B = H_{[0,2]}H_{[2,d]} = \frac{1}{2-d}\begin{pmatrix} (d+2) & -4d \\ d & (-3d+2) \end{pmatrix}$$

and

$$tr(A) = tr(B) = 2$$

and

$$tr(AB) = 2 + 8d/(2-d).$$

We have shown

**Proposition 4.30** *([3]) If $tr(A) = 2, tr(B) = 2,$ and $tr(AB) - 2 = 2\frac{4d}{2-d}$ , where $d$ lies exterior to the NSDC teardrop,then $G =< A, B >$ is discrete.*

**Theorem 4.31** *([5]) Assume that $G =< S, T >$ with tr(S) = tr(T)=2. Then*

$$|tr(ST) - 2| + |Re[tr(ST) - 2]| \geq 8$$

*$\Rightarrow G$ is discrete.*

# Chapter 5

# Classical T-Schottky Groups

In this chapter we will prove the following main theorem:

Theorem 5.1 *([5]) $G_\lambda$ lies on the boundary of classical T-Schottky space $\Leftrightarrow$*

$$\lambda = (2e^{\iota\omega})/(1 + |\sin\omega|)$$

*and thus eliminating $\omega \Leftrightarrow$*

$$\lambda = x + \iota y \text{ with } |y| = 1 - x^2/4.$$

Definition 5.2 *Let $C_i$, $C_i'$, $i = 1, \ldots, n$ be a set of 2n circles in $\hat{\mathbb{C}}$ such that the interiors of the 2n circles are all pairwise disjoint. Let $F$ be the intersection of the exteriors of these circles.*

1. *$F$ is called a classical T-Schottky domain.*
2. *A classical T-Schottky pairing is a set of n Möbius transformations, $g_1$,......, $g_n$ , the side pairings, where each $g_i$ maps the exterior of $C_i$ onto the interior of $C_i'$.*
3. *The set of circles together with the side pairings is called a classical T-Schottky configuration.*
4. *The group generated by the side-pairing is called a marked classical T-Schottky group on the generators $g_1$,.......,$g_n$.*

5. *A group of Möbius transformations is a classical T-Schottky group if it is a classical marked T-Schottky group on some set of generators.*

We begin by describing the effects of marking generators.

### 5.0.10 Change of Generators

In this subsection, we begin by proving some lemmas, which will eventually lead us to the proof of Theorem 5.10 which gives us the relation between classical T-Schottky groups and marked classical T-Schottky groups for groups generated by two-parabolics.

Lemma 5.3 *For a subgroup $G$ of $PSL(2,\mathbb{C})$ ,the following are equivalent:*

1. $G =< A, B >$ *is marked classical T-Schottky;*
2. $G =< B, A >$ *is marked classical T-Schottky;*
3. $G =< A, B^{-1} >$ *is marked classical T-Schottky.*

Proof $G =< A, B >$ is marked classical T- Schottky
$\Leftrightarrow$ A, B are side pairings for a set of circles $C_1$, $C_1'$ and $C_2$, $C_2'$.
$\Leftrightarrow$ B, A are side pairings for the set of circles $C_2$, $C_2'$ and $C_1$, $C_1'$
$\Leftrightarrow G =< B, A >$ is marked classical T- Schottky.
$\Leftrightarrow$ A, $B^{-1}$ are side pairings for the set of circles $C_1$, $C_1'$ and $C_2'$, $C_2$.
$\Leftrightarrow G =< A, B^{-1} >$ is marked classical T- Schottky.

Definition 5.4 *We say that a fixed point of a parabolic transformation is represented on the boundary of a T-Schottky domain if there is an element of the group that maps the parabolic fixed point to a point on the boundary of the domain.*

Lemma 5.5 *([5]) If $G$ is a classical T-Schottky group containing a parabolic $T$ and $F$ is a classical T-Schottky domain for $G$, then the fixed point of $T$ is represented on the boundary of $F$ and two sides of $F$ are tangent at that point.*

Proof We normalize $T$ to get the fixed point of $T$ to be infinity and assume that infinity is not represented on the boundary of $F$. Since $\infty$ is a fixed point of $T$, $\infty$ cannot be an interior point of $F$ (follows as $F$ is a T-SCHOTTKY Domain). Therefore, there will be a boundary circle $C$ separating infinity from the other boundary circles.

We take $C'$ to be the boundary circle paired with this $C$ and take $g_C$ to be the side-pairing. We specify $g_C \in G$ to be the transformation which maps the exterior of $C'$ to the interior of $C$. Now consider $g_C(F)$. If $\infty$ is on the boundary of $g_C(F)$, we are done.

If not, then we note that $g_C(F)$ is also a T-Schottky domain for $G$ with $\infty$ lying exterior to it. We repeat the above process. The nested images of F accumulate at $\infty$, as it is a limit point. As $G$ contains the translation $T$, thus at some stage the image of $F$ will overlap itself.

Renormalize such that infinity lies on the boundary and notice that if only one boundary circle passes through infinity, then there is an element $g \in G$ which is non-trivial with $g(F) \cap F \neq \phi$.

Definition 5.6 *If $F$ is a classical T-Schottky domain for a group $G$, then for any $g \in G$, either $g(F) \cap F = \phi$ or $g(F) = F$. $F$ and $g(F)$ are tangent if there are circles $C$ and $D$ bounding $F$ with $C$ and $g(D)$ tangent.*

Lemma 5.7 *([5]) Let $F$ be a classical T-Schottky domain and $S$ a side pairing transformation. If $F$ and $S^N(F)$ are tangent at $\eta$ for some $N$ with $|N| \geq 2$ , then $S$ fixes $\eta$.*

Proof As $F$ is a classical T-Schottky domain, we have circles $C$ and $C'$ on the boundary of $F$ with $S(C) = C'$, where $S$ is a side pairing for the circles $C$ and $C'$. If $\eta$ is inside $F$, then $S(\eta)$ is inside $C'$, but $F$ and $S^N(F)$ are disjoint. Then we must have $\eta \in C \cap C'$. As $|N| > 2$, all $S^n(\bar{F})$ are tangent at $\eta$ for all integers $n$. The $S$ orbit of a circle through $\eta$ accumulates at a fixed point of S, so $\eta$ is such a fixed point.

Lemma 5.8 *([5]) Suppose $G =< S, T >$ is marked classical T-Schottky. Then $S^N T$ and $(ST)^N T$ are loxodromic for $|N| >= 2$.*

Proof Take $F$ to be a classical T-Schottky domain for the marked group $G$. Let $T$ pair sides $C$ and $C'$. Then we note that $S^N T$ sends the exterior of $C$ to the interior of $S^N T(C)$, thus the region bounded by these two circles forms a T-Schottky domain for the cyclic group generated by $S^N T$. Therefore, if we take $S^N T$ to be parabolic, the two circles $C$ and $S^N T(C)$ will become tangent at $\eta$. Hence $\bar{F}$ and $S^N(\bar{F})$ are tangent at $\eta$. But then by above lemma (Lemma 5.6), $\eta$ is fixed by $S$. Thus, both $C$ and the sides of $F$ paired by $S$ are tangent at $\eta$. But as the group has no elliptic elements, such a configuration is not possible. Therefore, $S^N T$ is loxodromic.

Similarly, if $(ST)^N T$ is parabolic, then $C$ and $(ST)^N T(C)$ are tangent at $\zeta$. Applying above lemma(Lemma 5.6) to the classical T-Schottky domain(for $< S, T >$) bounded by $C$ and $ST(C)$, $< ST >$, $ST$ fixes $\zeta$. Thus $C$, $T(C)$ and $ST(C)$ are tangent at $\zeta$, but this is not possible. Therefore $(ST)^N T$ must be loxodromic.

Theorem 5.9 *([5]) Suppose that $G =< A, B >$ is a marked classical T-Schottky group. If $G$ can also be generated by parabolic elements $S$ and $T$ then, up to conjugacy, interchange of generators and replacing a generator by its inverse, we have either (i) $S = A$ and $T = B$ or (ii) $S = A$ and $T = AB$.*

Proof Let $F$ be a classical T-Schottky domain for the marked group $G$. Observe that $F$ can have at most six points of tangency; that is, the four circles that bound $F$ can have a total of at most six points of tangency. If any two circles are tangent at a point, none of the other four circles can be tangent at that point, and each circle can be tangent to at most two others. Since the fixed points of both $S$ and $T$ are represented on the boundary of $F$ as points of tangency, there are words $W_1(A, B)$ and $W_2(A, B)$ with fixed points on the boundary of $F$ that are conjugate to $S$ and $T$ respectively. However, there may not necessarily be a single element of the group that conjugates $W_1(A, B)$ to $S$ and $W_2(A, B)$ to $T$. Interchanging $S$ and $T$ if necessary, we may conjugate so that the fixed point of $S$ lies on the boundary of $F$ and is fixed by the shortest possible word in $A$ and $B$. Elements of the group send points of tangency of $F$ to points of tangency of the image of $F$ and parabolic fixed points to parabolic fixed points. We observe that the generators $A$ and $B$ must each permute the parabolic fixed points on the boundary of $F$: if $A$ identifies $L$ and $M$ and $B$ identifies $C$ and $D$, we assume $p$ is a parabolic fixed lying on $L$ and consider the possibilities for its images. Since $A(L) = M$ if $A(p)$ lies on $M$, $L$ is either tangent to $C$ , $D$ or $M$ at $p$; and M is tangent at $A(p)$ to one of $L$, $C$ or $D$. Considering all possible cases of tangency points under the action of the group, we conclude that it suffices to consider the three cases in which the length of $S$ is at most three. Now if $U$ and $V$ are associated primitive elements of the free group on two generators, then the only primitive dements associated to $U$ are of the form $U^\alpha V^\epsilon U^\beta$ where $\alpha$, $\beta = 0$, $\pm 1$, $\pm 2$, ... and $\epsilon = \pm 1$. We apply this to the cases where the length of S is at most three.

Length 1. S is a word of length one in $A$ and $B$. We may assume that $S = A$ and note that, up to conjugacy by a power of $A$, we get $T = A^n B^{\pm 1}$. Then, either $n = 0$ or $n = \pm 1$. The theorem follows after a re-normalization.

Length 2. S is a word of length two in A and B. We may assume that $S = AB$ and $T = (AB)^n B$. Either n = 0 or $n = \pm 1$. If n = 0 or $n = -1$, then up to conjugacy, T has length one. If $n = 1$, then T is loxodromic. Both cases are contrary to assumption.

Length 3. $S$ is a word of length three in $A$ and $B$. We may assume that $S = A^2 B$. However, $S$ is then loxodromic, contrary to assumption.

Lemma 5.10 *([5]) If $< T, B >$ is marked classical T-Schottky with $T$ and $TB$ parabolic, then $< T, TB >$ is marked classical T-Schottky.*

Proof Normalize so that $T(\infty) = \infty$ and TB(0) = 0 and hence $B(0) = T^{-1}(0)$. A T-Schottky domain $F$ for $< T, B >$ is then bounded by L, M, C and D, where T pairs sides L and M that meet at infinity and B pairs circles C and D inside the domain bounded by L and M. 0 lies on the boundary of F. There are the following three cases.

1. 0 on L, T(0) on M. Since the Euclidean line L separates $T^{-1}(L)$ and M, we must have $B(0) \neq 0$
2. 0 is interior to the domain bounded by L and M with C and D tangent at 0. In this case, $T^{-1}(0)$ is interior to the region bounded by $T^{-1}(L)$ and $L$, and B(0) is on D. Again, $B(0) \neq T^{-1}(0)$.
3. C is tangent to M at 0, and D is tangent to L at $T^{-1}(0)$.

We have $B(0) = T^{-1}(0)$.

In case (3), we let $L_1$ be a line orthogonal to the line joining the centers of C and D and separating the interiors C and D. Such a line exists, since C and D are at worst tangent. If $M_1 = T(L_1)$ and $D_1 = T(D)$, then T pairs $L_1$ and $M_1$ and TB pairs C and $D_1$. Thus $< T, TB >$ is marked classical T-Schottky.

Theorem 5.11 *If $S$ and $T$ are parabolic, then $G =< S, T >$ is marked classical T-Schottky if and only if $G$ is classical T-Schottky.*

Proof If $G =< S, T >$ is marked classical T-Schottky then clearly $G$ is classical T-Schottky.

For the other way, we note that $G$ is classical T-Schottky if $G =< A, B >$ is marked classical T-Schottky for some generators $A, B$ in $G$. Using we get that either (i) $S = A$ and $T = B$ or (ii) $S = A$ and $T = AB$. If Case (i) holds, then we are done. If case (ii) holds then, we get that $G =< S, T >=< A, AB >$ is classical T-Schottky.

### 5.0.11 Geometry of marked two parabolic generator T-Schottky groups

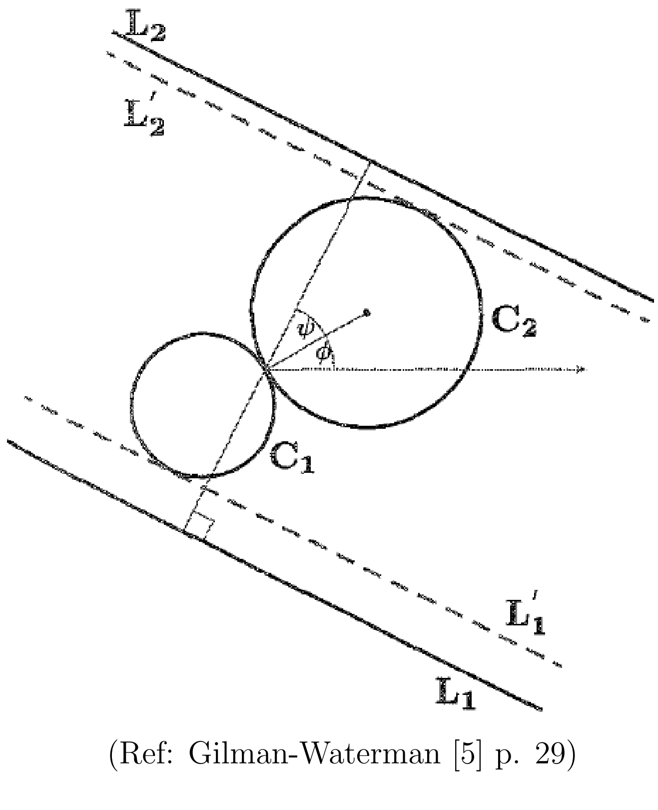
Here we consider a marked group G with two parabolic generators, S and T, and parameter $\lambda \in \mathbb{C}$. Let $G = G_\lambda$ with $\lambda = |\lambda| e^{\iota\omega}$ .

Let S be a side pairing for the circles $C_1$ and $C_2$ , and T be a translation pairing distinct lines $L_1$ to $L_2$. As S and T are parabolic, $C_1$ will be tangent to $C_2$ at 0, which is fixed by $S$. Similarly $L_1$ is tangent to $L_2$ at infinity, which is fixed by $T$.

The configuration $C_1$, $C_2$, $L_1$ and $L_2$ will determine a classical T-Schottky domain if and only if $C_1$ and $C_2$ lie in the region bounded by $L_1$ and $L_2$ thereby making $G =< S, T >$ a classical T-Schottky group. Assume 0 lies in the region bounded by $L_1$ and $L_2$.

Definition 5.12 *A classical T-Schottky group $G_\lambda$ along with its classical T-Schottky configuration is called extreme if every neighborhood of $\lambda$ contains points for which the group is classical T-Schottky and points for which it is not classical T-Schottky.*

If one of $L_1$ or $L_2$ is not tangent to either circle, then the configuration is not extreme. If $\lambda_0$ is sufficiently close to $\lambda$, one can easily find $L_1'$ paired to $L_2'$ by $T(z) = z + 2\lambda_0$ such that $C_1$, $C_2$ lie in the region bounded by $L_1'$ and $L_2'$.




(Ref: Gilman-Waterman [5] p. 29)

**The parameters $t$, $\phi$ and $\psi$**

We want to assign geometric parameters to the set of four circles.

Before proceeding ahead we will state the following lemmas:

Lemma 5.13
$$S = \begin{pmatrix} \nu & 0 \\ \nu^{-1} & \nu^{-1} \end{pmatrix}$$
*maps the circle $C_1$ : $|z - a| = r$ to the circle $C_2$: $|z - b| = \rho$ if and only if*

$$\beta = \nu^2 (r^2 - \alpha - |\alpha|^2)/r^2 - |1 + \alpha|^2$$

*and*

$$\rho = r|\nu|^2/|r^2 - |1 + \alpha|^2|.$$

*Further, $S$ maps the exterior of $C_1$ to the interior of $C_2 \Leftrightarrow |a + 1| < r$.*

Let the two circles $C_1$ and $C_2$ be tangent at 0 and assume that the line connecting their centers makes an angle $\phi$ with the x-axis. Let $\tau = te^{\iota\phi}$ and assume that the center of $C_1$ is $\alpha = -\tau$. Applying above to this situation with $\nu = 1$ so that S is parabolic, we obtain

Corollary 5.14 *If the center of $C_1$ is at $-\tau$, the center of $C_2$ is at $\tau/(\tau + \bar{\tau} - 1)$. The exterior of $C_1$ is mapped to the interior of $C_2$ if and only if $\tau + \bar{\tau} - 1 > 0$.*

Observe that by above corollary in order for the exterior of $C_1$ to be mapped to the interior of $C_2$ it is necessary that $|\phi| \leq \pi/2$.

Assume that the perpendicular to $L_1$ and $L_2$ makes an angle $\phi + \psi$ with the x-axis, $|\psi| \leq \pi/2$.

Definition 5.15 *Let $D_L = D_L(\phi, \psi)$ denote the Euclidean distance between $L_1$ and $L_2$.*

Definition 5.16 *Let $D_C = D_C(t, \phi, \psi)$ be the distance between the parallel lines $L_1'$ and $L_2'$, where $L_1'$ is tangent to $C_1$ and parallel to $L_1$ and $L_2'$ is tangent to $C_2$, with $L_1'$ and $L_2'$ chosen so that $C_1$ and $C_2$ lie between $L_1'$ and $L_2'$.*

We emphasize that the condition that we have a T-Schottky configuration is thus precisely that $D_C \leq D_L$.

In what follows, we progressively describe how best to choose the geometric parameters t, $\phi$, $\psi$. Essentially, we minimize $D_C - D_L$ as a function of t, then $\phi$, and then $\psi$.

Lemma 5.17 $D_C = t[1 + 1/(2t\cos\phi - 1)][1 + \cos\psi]$ ,$D_L = |2\lambda.e^{\iota(\phi+\psi)}|$.

Proof $D_L$ can be determined by the inner product of the vector $2\lambda$ with the perpendicular line joining the lines $l_1$ and $l_2$ that is $|2\lambda.e^{i(\phi+\psi)}|$.

Definition 5.18 $t_0 = 1/\cos\phi$.

Lemma 5.19 *For fixed $\phi$ and $\psi$ there is a T-Schottky configuration if and only if $D_C(t_0, \phi, \psi) \leq D_L(\phi, \psi)$, where $t_0 = 1/\cos\phi$. When $t = t_0$, $D_C(t_0\phi\psi) = 2(1 + \cos\psi)/\cos\phi$ , this value of $D_C$ is a minimum, and the circles have the same radius.*

Proof By above lemma

$$
\begin{aligned}
D_C &= t\left[1 + \frac{1}{(2t\cos\phi - 1)}\right](1 + \cos\psi) \\
&= 2\cos\phi\left[t + \frac{t}{(2t\cos\phi - 1)}\right]\left(\frac{1 + \cos\psi}{2\cos\phi}\right) \\
&= \left[2t\cos\phi + \frac{2t\cos\phi}{(2t\cos\phi - 1)}\right]\left(\frac{1 + \cos\psi}{2\cos\phi}\right) \\
&= \left[2t\cos\phi - 1 + \frac{2t\cos\phi}{(2t\cos\phi - 1)} + 2\right]\left(\frac{1 + \cos\psi}{2\cos\phi}\right)
\end{aligned}
$$

We minimize $D_C$ with respect to $t$

$$\frac{\partial D_C}{\partial \phi} = \left[2\cos\phi - 2\cos\phi/(2t\cos\phi - 1)^2\right] \frac{(1+\cos\psi)}{2\cos\phi}$$

For minimum we put

$$\frac{\partial D_C}{\partial \phi} = 0$$

$$\Rightarrow 2\cos\phi - \frac{2\cos\phi}{(2t\cos\phi - 1)^2} = 0$$

$$\Rightarrow (2t\cos\phi - 1)^2 = 1$$

$$\Rightarrow 2t\cos\phi - 1 = 1$$

$$\Rightarrow t\cos\phi = 1$$

$$\Rightarrow t = 1/\cos\phi$$

Thus the result follows with

$$D_C(t_0, \phi, \psi) = \frac{2(1+\cos\psi)}{\cos\phi}.$$

Definition 5.20

$$\phi_0 = \begin{cases} \frac{1}{2}(\omega - \psi) & \textit{when } |\omega - \psi| \leq \pi/2 \\ \frac{1}{2}(\omega - \psi - \pi) & \textit{when } \omega - \psi \geq \pi/2 \ . \end{cases}$$

*In both cases,*

$$|\phi_0| \leq \frac{\pi}{4}.$$

Lemma 5.21 *For fixed $\psi$, there is a T-Schottky configuration if and only if $D_L(\phi_0, \psi) \geq D_C(t_0, \phi_0, \psi)$ , where $\phi_0$ satisfies the above definition.*

Proof By above lemma , there is a T-Schottky configuration if and only if

$$|2\lambda.e^{\imath(\phi+\psi)}| \geq 2(1+\cos\psi)/\cos\phi$$

Further,

$$|2\lambda.e^{\iota(\phi+\psi)}| \geq 2(1+\cos\psi)/\cos\phi$$

if and only if

$$|\lambda||\cos(\omega-\psi)+cos(\omega-\psi-2\phi)| \geq 2(1+\cos\psi)$$

Now maximizing the left hand side as a function of $\phi$, we get

$$|\lambda|^2|sin(\omega-\psi-2\phi)| = 0$$

This gives us the following as $|\phi| \leq \pi/2$,

$$\phi = \begin{cases} \frac{1}{2}(\omega-\psi) & \text{when } |\omega-\psi| \leq \pi/2 \\ \frac{1}{2}(\omega-\psi-\pi) & \text{when } \omega-\psi \geq \pi/2 \ . \end{cases}$$

and the result follows.

Utilizing the above, we explicitly characterize classical T-Schottky groups on two parabolic generators.

Theorem 5.22 *([5]) $G_\lambda$, $0 \leq \omega < \pi$, is marked classical T-Schottky if and only if*

$$|\lambda|(1+\sin\omega) \geq 2.$$

*If the above inequality holds, then a T-Schottky configuration may be obtained by choosing*

$$\psi = \pm(\pi/2)$$

$$\phi = \omega/2 - \pi/4$$

*and*

$$t = |1+\iota\tan(\omega/2-\pi/4)|.$$

*Further, the two choices of triples above, $(\pi/2, \phi, t)$ and $(-\pi/2, \phi, t)$, are the only choices of parameters $(\psi\phi t)$ guaranteed to give a classical T-Schottky configuration for every $\omega (0 \leq \omega < \pi)$,*

*If $\omega = 0$, then every classical T-Schottky configuration can be obtained by choosing some $\psi$, then setting $\phi = \psi_0$.*

Proof From the proof of above lemma, $G_\lambda$, is marked classical T-Schottky if and only if

$$|\lambda|[1 + |\cos(\omega - \psi)|] \geq 2(1 + \cos\psi)$$

But

$$|\lambda|[1 + |\cos(\omega - \psi)|] \geq 2(1 + \cos\psi)$$

$$\Leftrightarrow \begin{cases} |\lambda|\cos^2(\omega/2 - \psi/2) \geq 2\cos^2(\psi/2) & -\pi/2 \leq \omega - \psi \leq \pi/2 \ ; \\ \textit{or} & \\ |\lambda|\sin^2(\omega/2 - \psi/2) \geq 2\cos^2(\psi/2) & \pi/2 \leq \omega - \psi \leq 3\pi/2 \ . \end{cases}$$

$$\Leftrightarrow \begin{cases} |\lambda|[\cos(\omega/2) + \sin(\omega/2) tan(\psi/2)]^2 \geq 2 & -\pi/2 \leq \omega - \psi \leq \pi/2 \ ; \\ \textit{or} & \\ |\lambda|[\sin(\omega/2) - \cos(\omega/2) tan(\psi/2)]^2 \geq 2 & \pi/2 \leq \omega - \psi \leq 3\pi/2 \ . \end{cases}$$

Provided $\omega \neq 0$, the maximum of the left hand side as a function of $\psi$ occurs precisely when $\psi = \pm(\pi/2)$.

If $\omega = 0$ then the left hand side is independent of $\psi$ so any $\psi$ may be chosen.

As a consequence, we can explicitly describe the boundary of classical T-Schottky space with two parabolic generators simply as a portion of a parabola We have $x + \iota y = |\lambda| e^{\iota\omega}$.

Theorem 5.23 *([5]) $G_\lambda$ lies on the boundary of classical T-Schottky space $\Leftrightarrow$*

$$\lambda = (2e^{\iota\omega})/(1 + |\sin\omega|)$$

*and thus eliminating* $\omega \Leftrightarrow$

$$\lambda = x + \iota y \text{ with } |y| = 1 - x^2/4.$$

Finally, we put above theorem into a conjugacy invariant form,

Theorem 5.24 *([5]) If* $G =< S, T >$ *with tr(S) = tr(T) = 2, then G is classical T-Schottky if and only if*

$$|tr(ST) - 2| + |Im[tr(ST)]| \geq 4.$$

Proof The result follows by recalling that $\lambda = (1/2)[tr(ST) - 2]$.

This translates to the following sufficient condition for discreteness.

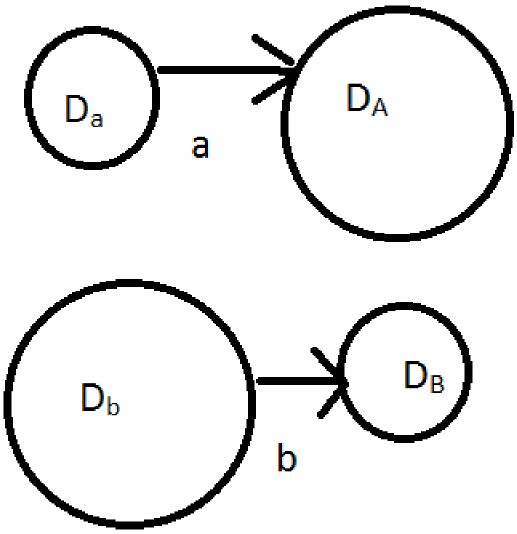
Corollary 5.25 *([5]) If* $G =< S, T >$ *with* $tr(S) = tr(T) = 2$,

$$|tr(ST) - 2| + |Im[tr(ST)]| \geq 4$$

*implies that G is discrete.*

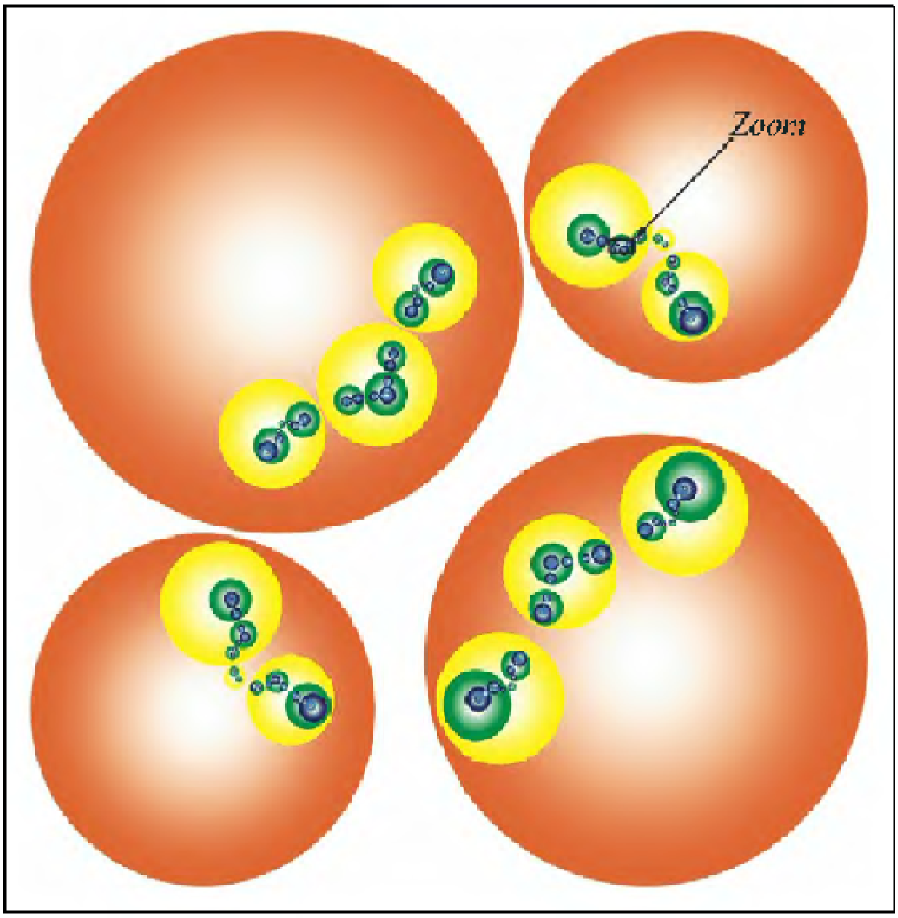
## 5.1 A Visual Representation of Schottky Groups

Let us take a pair of disks $D_A$ and $D_a$ such that the Möbius transformation $a$ maps the exterior of $D_A$ to the inside of $D_a$. Similarly we take another pair of disks $D_B$ and $D_b$ and a transformation $b$.



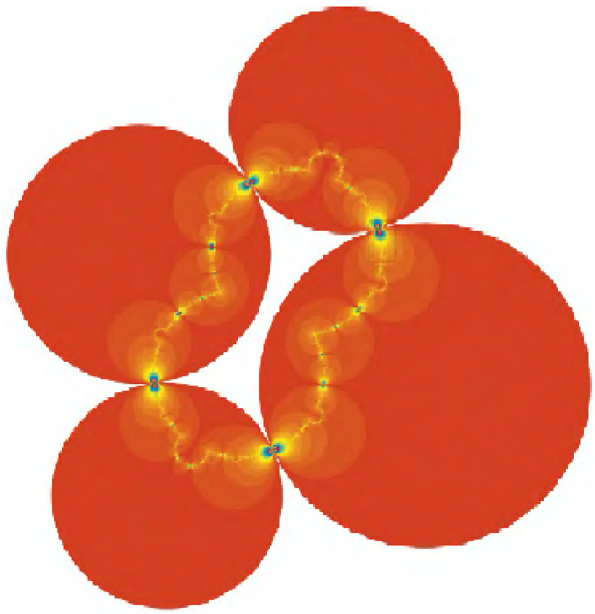

Transformations $a$ and $b$.

Now we apply the transformations $a\ b\ a^{-1}\ b^{-1}$ to get this beautiful picture. The wonderful feature of this picture is that no matter how much we zoom in the essentials of this pattern repeat.




Reference: Indra's Pearls, [6]

Now we bring about a small variation. We bring the Schottky Circles together so that they touch or kiss.
Inside each circle there are three further circles tangent to each other so that we get a succession of chains each contained within the other. The chain at a given level is like a bead necklace. The limiting curve is named Indra's Necklace.



INDRA'S NECKLACE, Reference: Indra's Pearls, [6]

# Conclusion

We have described in this thesis the boundary of the space of non-separating disjoint circle groups and of Classical T-Schottky groups having two parabolic generators.

As a result of this description we have also derived a criterion for discreteness of groups generated by two parabolic generators seen as a subgroup of PSL(2, $\mathbb{C}$).

# Bibliography


[1] Ahlfors,L, *Complex Analysis* , McGraw-Hill, 1966.

[2] Beardon, A.F, *The Geometry of Discrete Groups*; Springer, 1983.

[3] Gilman J, *A Discreteness condition for Subgroups of PSL(2,* $\mathbb{C}$); Contemporary Mathematics, Vol 211, 261-267, 1997.

[4] Gilman J, *Boundaries for Two- Parabolic Schottky Groups*; London Mathematical Society Lecture Note Series 329, Cambridge University Press, Cambridge, 2005.

[5] Gilman J, Waterman P, *Classical Two Parabolic Schottky Groups*; Journal D' Analyse Mathematique ,VOL. 98, 1-42, 2006 .

[6] Mumford D., Series C., Wright D. *Indra's Pearls*; Cambridge University Press, 2002.

[7] Maskit, B *Kleinian Groups*; Springer, 1988.