Geometric Group Theory

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Certificate of Examination

This is to certify that the dissertation titled **Geometric Group Theory** submitted by **Mr. Amit Ranjan** (Reg. No. MS09014) for the partial fulfilment of BS-MS dual degree programme of the institute, has been examined by the thesis committee duly appointed by the institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: May 2, 2014

Declaration

The work presented in this dissertation has been carried out by me under the guidance of **Prof. I. B. S. Passi** at Indian Institute of Science Education and Research Mohali.This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. This is a bonafide record of original study done by me and all sources listed within have been detailed in the bibliography.

Amit Ranjan
Dated: May 2, 2014

In my capacity as the supervisor of the candiadte's project work, I certify that the above statements made by the candidates are true to the best of my knowledge.

I. B. S. Passi

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Contents

Abstract

In this report we study hyperbolic groups and some of their properties. We also study growth of groups, in particular, Grigorchuk's example of group of intermediate growth.

Introduction

In this THESIS we study two topics in Geometric Group Theory namely, Hyperbolic Groups and Growth of Groups.

A geodesic metric space is defined to be hyperbolic if all geodesic triangles in it are thin in a particular sense. A group is defined to be hyperbolic if its Cayley graph is a hyperbolic metric space. In Chapter 1 we study the algorithm, known as Dehn algorithm which efficiently solves the word problem for groups and discuss the result that hyperbolic groups have a solvable word problem.

In Chapter 2 we study the growth of groups, in particular, Grigorchuk's example of a group whose growth is intermediate, i.e., neither polynomial nor exponential.

Chapter 1

Hyperbolic groups

1.1 Cayley graph

Definition 1.1 (Cayley graph) . *Let G be a group and let $S \subset G$ be a generating set of G. Then the* Cayley graph *of G with respect to the generating set S is the graph $Cay(G, S)$ whose*

- *set of vertices is G, and whose*
- *set of edges is*

$$\{\{g,\ g \cdot s\} \mid g \in G, s \in S \setminus \{e\}\} \tag{1.1}$$

i.e., two vertices in a Cayley graph *are adjacent if and only if they differ by an element of the generating set in question. Here e denotes the identity element of G.*

Example 1.2 (Cayley graphs).

- The Cayley graphs of the additive group $\mathbb{Z}$ with respect to the generating sets $\{1\}$ and $\{2, 3\}$ respectively are illustrated in Figure 1.1. Notice that, when looking at these two graphs from far away, they seem to have the same global structure, namely they look like the real line; in more technical terms, these graphs are quasi-isometric with respect to the corresponding word metrics .

- The Cayley graph of the additive group $\mathbb{Z}^2$ with respect to the generating set $\{(1, 0), (0, 1)\}$ looks like the integer lattice in $\mathbb{R}^2$, see Figure 1.2; When viewed from far away, this Cayley graph looks like the Euclidean plane.

- The Cayley graph of a free group F with respect to a free generating set S is a tree.

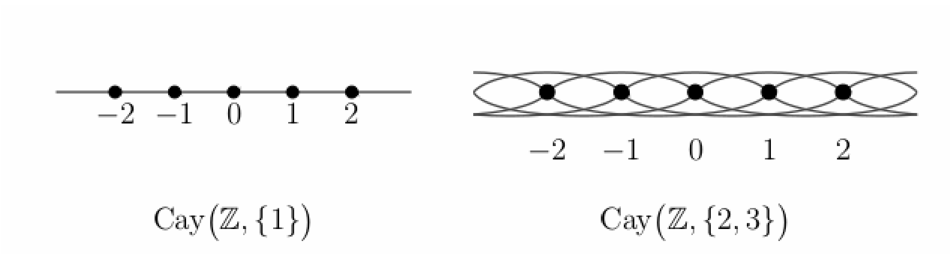


Figure 1.1: Cayley graph of the additive group $\mathbb{Z}$.

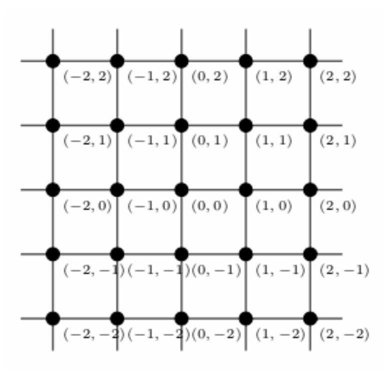


Figure 1.2: Cayley graph of $\mathbb{Z} \times \mathbb{Z}$

1.2 Quasi-isometry

Definition 1.3 (Quasi-isometry) *Let $f : X \longrightarrow Y$ be a map between metric spaces (X, d_X) and (Y, d_Y). The map f is a* quasi-isometric embedding *if there are constants $c \in \mathbb{R}_{>0}$, $b \in \mathbb{R}_{\geq 0}$ such that*

$$\frac{1}{c}d_X(x, x^{'}) - b \leq d_Y(f(x), f(x^{'})) \leq cd_X(x, x\prime) + b \ \ \forall \ x, \ x' \in X.$$

A map $f' : X \longrightarrow Y$ has a finite distance *from f if there is a constant $c \in R_{\geq 0}$ with*

$$d_X(f(x), f^{'}(x)) \leq c \ \ \forall \ x \in X.$$

The map f is a quasi-isometry *if there is a quasi-isometric embedding for which there is a quasi-inverse i.e., a quasi isometric embedding $g : X \longrightarrow Y$ such that gof and fog have finite distance from the identity maps Id_Y and Id_X respectively.*
Two metric spaces X and Y are quasi isometric *if there exists a quasi-isometry $X \longrightarrow Y$; in this case we write $X \sim_{QI} Y$.*

Definition 1.4 (Metric on a graph) *Let $G(V, E)$ be a connected graph. Then the map $V \times V \longmapsto \mathbb{R}_{\geq 0}$*
$(v, w) \longrightarrow min\{m \in \mathbb{N} \mid$ there is a path of length m connecting v and w $\}$
is a metric on V.

Definition 1.5 (Word metric) *Let G be a group and let $S \subset G$ be a generating set. The* word metric *d_S on G with respect to S is the metric on G associated with the Cayley graph $Cay(G, S)$. In other words*
$d_S(g, h) = min\{n \in \mathbb{N} \mid \exists \ s_1, ..., s_n \in S \cup S^{-1}, \ g^{-1}h = s_1...s_n\}, \ g, h \in G.$

Definition 1.6 (Quasi-isometry type of finitely generated groups) *The group G is quasi-isometric to a metric space X if for some (and hence all) finite generating sets S of G the metric space (G, d_S) and X are quasi-isometric. We write $G \sim_{QI} X$ if G and X are quasi-isometric.*

Example 1.7 If $n \in \mathbb{N}$, then $\mathbb{Z}^n$ is quasi-isometric to the euclidean space $\mathbb{R}^n$ because the inclusion $\mathbb{Z}^n \longrightarrow \mathbb{R}^n$ is a quasi-isometric embedding with quasi-dense image.

Remark Let G be a group and let $S \subset G$ be a generating set. Then S is finite if and only if the word metric d_S on G is proper in the sense that all balls of finite radius in (G, d_S) are finite.

Definition 1.8 (Geodesic (respectively Quasi-geodesic)) *Let (X,d) be a metric space and let $c, b \in \mathbb{R}_{>0}$*

- *then a* geodesic *(resp.* quasi-geodesic*) in X is a isometric (resp. quasi-isometric) embedding $\gamma : I \longrightarrow R$, where $I = [t, t'] \subset \mathbb{R}$ is some closed interval. The point $\gamma(t)$ is the starting point of γ and $\gamma(t')$ is the end point of γ.*

- *The space X is* geodesic *(respectively (c, b)* quasi-geodesic *if for all $x, x' \in X$ there exists a (c,b) quasi geodesic in X with start point x and end point x'.*

Example 1.9 If G = (V, E) is a connected graph, then the associated metric on V turns V into a (1,1) geodesic space because the distance between two vertices is realised as the length of some path in the graph G, and any path in the graph G of shortest distance between two vertices is a (1,1) quasi-geodesic.
In particular, if G is a group and S is a generating set of G then (G, d_S) is (1,1) quasi-geodesic space.

1.3 Hyperbolic spaces

Definition 1.10 (δ-slim geodesic triangle) *Let (X,d) be a metric space. A geodesic triangle in X is a triple $(\gamma_0, \gamma_1, \gamma_2)$ consisting of geodesics $\gamma_j : [0, l_j] \to X$ in X such that*

$$\gamma_0(l_0) = \gamma_1(0),\ \gamma_1(l_1) = \gamma_2(0),\ \gamma_2(l_2) = \gamma_0(0).$$

A geodesic triangle $(\gamma_0, \gamma_1, \gamma_2)$ *is* δ-slim *if*

$$im\gamma_0 \subset B_\delta^{X,d}(im\gamma_1 \cup im\gamma_2),$$
$$im\gamma_1 \subset B_\delta^{X,d}(im\gamma_0 \cup im\gamma_2),$$
$$im\gamma_2 \subset B_\delta^{X,d}(im\gamma_0 \cup im\gamma_1).$$

Definition 1.11 (δ Hyperbolic space) *Let X be a metric space and let $\delta \in \mathbb{R}_{\geq}0$. We say that X is δ-* Hyperbolic *if X is geodesic and if all geodesic triangles in X are δ- slim. The space X is* hyperbolic *if there exists a $\delta \in \mathbb{R}_{\geq}0$ such that X is δ hyperbolic.*

Example 1.12 (i) Any geodesic metric space X of finite diameter is $diam(X)$-hyperbolic.
(ii) $\mathbb{R}$ is a 0-hyperbolic.

Example 1.13 $\mathbb{R}^2$ is not hyperbolic because for any $\delta \in \mathbb{R}_{g\geq 0}$, the Euclidean triangle with vertices $(0,0),(0,3\delta)$ and $(3\delta,0)$ is not δ- slim

1.4 Quasi-hyperbolic space

Definition 1.14 (δ -slim quasi-geodesic triangle) *Let (X,d) be a metric space, and let $c, b \in \mathbb{R}_{\geq 0}$. A* quasi-geodesic triangle *in X is a triple $(\gamma_0, \gamma_1, \gamma_2)$ consisting of quasi-geodesics $\gamma_j : [0, L_j] \to X$ in X such that*

$$\gamma_0(l_0) = \gamma_1(0),\ \gamma_1(l_1) = \gamma_2(0),\ \gamma_2(l_2) = \gamma_0(0).$$

A (c, b)-quasi-geodesic triangle $(\gamma_0,\ \gamma_1,\ \gamma_2)$ consists of (c,b)-quasi-geodesicis $\gamma_j : [0, L_j] \to X$ such that

$$\gamma_0(l_0) = \gamma_1(0),\ \gamma_1(l_1) = \gamma_2(0),\ \gamma_2(l_2) = \gamma_0(0)$$

A quasi-geodesic triangle *$(\gamma_0, \gamma_1, \gamma_2)$ is δ-*slim *if*

$$im\gamma_0 \subset B_\delta^{X,d}(im\gamma_1 \cup im\gamma_2),$$

$$im\gamma_1 \subset B_\delta^{X,d}(im\gamma_0 \cup im\gamma_2),$$
$$im\gamma_2 \subset B_\delta^{X,d}(im\gamma_1 \cup im\gamma_1).$$

Definition 1.15 (δ quasi-hyperbolic space) *Let X be a metric space. Let $\delta \in \mathbb{R}_{\geq 0}$. We say that X is δ* quasi-hyperbolic *if X is quasi-geodesic and if all quasi-geodesic triangles in X are δ-slim. The space X is* quasi-hyperbolic *if there exists a $\delta \in \mathbb{R}_{\geq 0}$ such that X is δ quasi-hyperbolic.*

Proposition 1.16 *(see [6, CL]) (Quasi-isometry invariance of quasi-hyperbolicity). Let (X, d_X) and (Y, d_Y) be two metric spaces.*

1. *If Y is quasi-geodesic and if there exists a quasi-isometric embedding $X \to Y$, then X is also quasi-geodesic.*
2. *If Y is quasi-hyperbolic and if there exists a quasi-isometric embedding $X \to Y$, then X is also quasi-hyperbolic.*
3. *In particular, if X and Y are quasi-isometric, then X is quasi-hyperbolic if and only if Y is quasi-hyperbolic.*

If (X, d) be a geodesic space, then X is hyperbolic if and only if X is quasi-hyperbolic. It turns out that hyperbolicity is a quasi-isometry invariant in the class of geodesic spaces (see [6, CL]).

1.5 Hyperbolic groups

Definition 1.17 (Hyperbolic group) *A finitely generated group G is said to be* hyperbolic group *if for some (and hence any) finite generating set S of G, the Cayley graph $Cay(G, S)$ is hyperbolic.*

Example 1.18 (Hyperbolic groups)

- All finite groups are hyperbolic because the associated metric spaces have finite diameter.

- The group $\mathbb{Z}$ is hyperbolic.
- Any free group of rank n is hyperbolic.

Example 1.19 (Non hyperbolic groups)

- $\mathbb{Z}^2$ is not hyperbolic.

It is known that hyperbolic groups are finitely presented.

Definition 1.20 *A group is said to have solvable word problem if there exists an algorithm that can decide in finite no. of steps whether a given word in the set of generators represents the identity element of the group.*

Gromov ([5, MGro]) has shown that hyperbolic groups have solvable word problem.

Chapter 2

Growth of Groups

Let G be a finitely generated group, generated by $x_1, \cdots, x_d$, say. Each element $x \in G$ can be written as a word in the generators, i.e., as a product $y_1 \cdots y_n$, where each y_i is either one of the generators or its inverse. The number n is called the length of the word. The identity element is represented by the empty word, which has length zero. In general, a given element $x \in G$ can be represented by many words. Out of all of these, we choose one of minimal length (this word is not necessarily unique) and call this the length $l(x)$ of x. For an integer $n \geq 0$, we write $a_G(n)$ for the number of elements $x \in G$ of length n, and $s_G(n)$ for the number of words of length at most n, i.e., $s_G(n) = \sum_{i=0}^{n} a_G(i)$. We term $a_G(n)$ and $s_G(n)$ the growth functions of G. More specifically, $a_G(n)$ is the strict growth function and $s_G(n)$ is the cumulative growth function of G. Our interest is in these two functions, their properties, and their relationship with the structure and properties of G. The subscript G will be often omitted, if it is clear from the context which group is meant.

Definition 2.1 *A* growth function *is a non decreasing function* $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

Definition 2.2 (Dominance) *A growth function f_2 dominates a growth function f_1, written $f_1 \prec f_2$, if there exist constants $d \geq 1$, $c \geq 0$ such that*

$$f_1(k) \leq d f_2(ck) \; \forall \; k$$

We say that two growth functions f_1 and f_2 are equivalent, written $f_1 \sim f_2$ if $f_1 \prec f_2$ and $f_2 \prec f_1$.

Proposition 2.3 *Let G be a finitely generated group. If S and T are two finite generating sets for G,then the growth functions with respect to S and T are equivalent. That is, $a_G(n,S) \sim a_G(n,T)$.*

So for each group G, the growth type, defined to be the equivalence class of one of its growth functions is distinct and independent of any generating set. A word of length $m+n$ can be written as a product of two words one of length m and one of length n, so that $a_G(m+n,S) \leq a_G(m,S)a_G(n,S)$. Therefore, $\omega(G,S) := \lim_{n\to\infty} \sup a_G(n,S)^{1/n}$ exists and is finite. For the same reasons, $\phi(G,S)$ $:= \lim_{n\to\infty} \sup s_G(n,S)^{1/n}$ exists, and it is clear that $\phi(G,S) \geq \omega(G,S)$. Suppose that the group G is infinite, then $a(n,S) \geq 1$ for all n, and hence $\omega(G,S) \geq 1$.

For any given $\epsilon > 0$, we have $a(n,S) \geq (\omega(G,S)+\epsilon)^n$, if n is large enough. therefore $s(n,S) \geq A + n(\omega(G,S)+\epsilon)^n$, for some constant A, and it follows that $\phi(G,S) \leq \omega(G,S)$. Thus $\omega(G,S) = \phi(G,S)$.

We must consider how $\omega(G,S)$ depends on S. Since $a_G(n,S) \sim a_G(n,T)$. for two finite generating sets S and T, then $a_G(n,S) \prec a_G(n,T)$ and $a_G(n,T) \prec a_G(n,S)$ so that

$$\omega(G,S) = \lim_{n\to\infty} \sup a_G(n,S)^{1/n} \leq \lim_{n\to\infty} \sup a_G(n,T)^{1/n} = \omega(G,T),$$

$$\omega(G,T) = \lim_{n\to\infty} \sup a_G(n,T)^{1/n} \leq \lim_{n\to\infty} \sup a_G(n,S)^{1/n} = \omega(G,S).$$

Hence if $\omega(G,S) > 1$ for one finite generating set, then it will be for any other. This shows that ω is well defined globally for each group

Definition 2.4

1. *If $\omega(G) > 1$, then G has* exponential growth*, and if $\omega(G) = 1$ then* subexponential growth*, of G (rather, of (G,X), where X is the relevant generating set of G).*

2. *A group G has* polynomial growth*, if there exist numbers c and s such that $s_G(n) \leq cn^s$, for all n.*

3. *If a group G has polynomial growth, its degree is defined by* $d(G) = inf(s|$ *such that* $s_G(n) \leq cn^s)$.

4. *A group G has* intermediate growth, *if its growth is neither exponential nor polynomial.*

The first Grigorchuk group is a finitely generated group which provided the first example of a finitely generated group of intermediate growth. The group was originally constructed by Grigorchuk in a 1980 paper and he then proved in 1984 that this group has intermediate growth, thus providing an answer to an important open problem which was posed by John Milnor in 1968 about the existence of a finitely generated group of intermediate growth. Originally, Grigorchuk's has constructed a group G of Lebesgue-measure-preserving transformations on the unit interval, later on a simpler descriptions of G were found and now it is usually presented as a group of automorphisms of the infinite regular binary rooted tree. We will try to describe this example in both way.

Rostislav Grigorchuk construction [2, Grigorchuk],[1, Mann] Let us consider transformations of the open unit interval $(0, 1)$. We remove from this interval all points whose coordinate is rational with denominator a power of 2. Let E denote the identity transformation, and let P denote the transformation which interchange the two halves $(0, 1/2)$ and $(1/2, 1)$ with each other, which means that a point x is mapped either to $x+1/2$ or to $x-1/2$. We can write the unit interval as the disjoint union of countably many subintervals $(1 - 1/2^{n-1}, 1 - 1/2^n), (n = 1, 2, \cdots)$. Group Γ is generated by four transformations a, b, c and d. Here a is just the interchange of interval i.e; is P is applied to the full interval. The other three generators will apply to each of the subintervals above by either E or P, as follows: b applies P to each of the first two subintervals and then E to the third one, and again repeats the pattern PPE periodically. The generator c applies similarly the periodic pattern PEP, and d applies the pattern EPP .

Although initially the Grigorchuk group was defined as a group of Lebesgue measure-preserving transformations of the unit interval, at present this group is usually given by its realization as a group of automorphisms of the infinite regular binary rooted tree T_2. The tree T_2 is realized as the set Σ^* of all (including the empty string) finite strings in the alphabet $\Sigma = \{0, 1\}$. The empty string ϕ is the root vertex of T_2 and for a vertex x of T_2 the string $x0$ is the left child of x and the string $x1$ is the right child of x in T_2. The group of all automorphisms $Aut(T_2)$ can thus be thought of as the group of all length-preserving permutations σ of Σ^* that also respect the initial segment relation, that is such that whenever a string x is an initial segment of a string y then $\sigma(x)$ is an initial segment of $\sigma(y)$. The Grigorchuk group G is then defined as the subgroup of $Aut(T_2)$ generated by four specific elements a, b, c, d of $Aut(T_2)$, that is $G =< a, b, c, d >\leq Aut(T_2)$, where the automorphisms a, b, c, d of T_2 are defined recursively as follows:

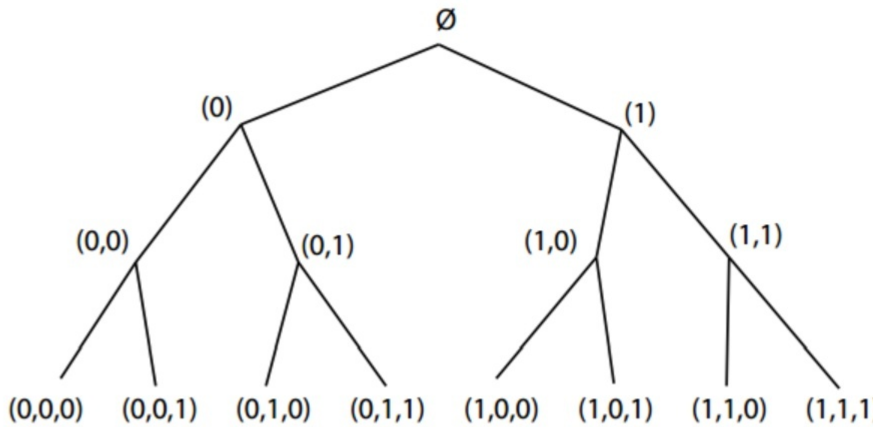
- $a(0x) = 1x,\ a(1x) = 0x$ for every x in Σ^*;

- $b(0x) = 0a(x),\ b(1x) = 1c(x)$ for every x in Σ^*;

- $c(0x) = 0a(x),\ c(1x) = 1d(x)$ for every x in Σ^*;

- $d(0x) = 0x,\ d(1x) = 1b(x)$ for every x in Σ^*.

Thus a swaps the right and left branch trees $T_L = 0\Sigma^*$ and $T_R = 1\Sigma^*$ below the root vertex ϕ and the elements b, c, d can be represented as: $b = (a, c)$, $c = (a, d)$, $d = (1, b)$. Here $b = (a, c)$ means that b fixes the first level of T_2 (that is, it fixes the strings 0 and 1) and that b acts on T_L exactly as the automorphism a does on T_2 and that b acts on T_R exactly as the automorphism c does on T_2. The notation $c = (a, d)$ and $d = (1, b)$ is interpreted similarly, where 1 in $d = (1, b)$ means that d acts on T_L as the identity map does on T_2. Of the four elements a, b, c, d of $Aut(T_2)$ only the element a is defined explicitly and the elements b, c, d are defined inductively (by induction on the length $|x|$ of a string x in Σ^*), that is, level by level.



Theorem 2.5 *[1, page no 96 Mann],[2, Grigorchuk]* Γ *has intermediate growth.*

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