Simplicial Objects

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A dissertation submitted for the partial fulfilment of
BS-MS dual degree in Science

Department of Mathematical Sciences
Indian Institute of Science Education and Research Mohali
April 2014

Certificate of Examination

This is to certify that the dissertation titled **"Simplicial Objects"** submitted by **Mr. Sampat Kumar Sharma** (Reg. No. MS09112) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Alok Maharana	Dr. Mahender Singh	Prof. I.B.S. Passi (Supervisor)

Dated: April 25, 2014

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. I.B.S. Passi at the Indian Institute of Science Education and Research Mohali.
This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Sampat Kumar Sharma
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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. I.B.S. Passi
(Supervisor)

Acknowledgement

I would like to express my sincere gratitude to my supervisor, Prof. I.B.S. Passi, for his kind supervision and guidance in completing my dissertation. Discussions with him and his guidance helped me in developing better understanding of the subject.

I would also like to thanx my friends and family members for continuous support and encouragement.

Sampat Kumar Sharma

Contents

Abstract

This project consists of two parts, Part one consists of study of simplicial homotopy theory, In particular, Dold-Kan correspondence between category $Simp(\mathcal{A})$ of simplicial objects in abelian category and category $Ch_{\geq 0}$ of non-negative chain complexes. Part two is a study of dimension subgroups.

Introduction

A fundamental developement in mathematics during the last century has been the simplicial homotopy theory and its applications in various areas of subject.
In Chapter 1, we have given some basic definitions and properties of simplicial objects and simplicial maps.

In Chapter 2, first of all we construct a simplicial set from a topological space which gives rise to a functor from category of topological spaces to category of simplicial sets. We next, have given conditions for two n-simplexes being homotopic. It comes out that being homotopic is an equivalence relation on K_n. After that geometric realisation of a simplicial set is given which gives rise to a functor from the category of simplicial sets to the category of topological spaces. The main result of this Chapter is that singular simplex functor is adjoint to geometric realisation functor which provides an equivalence between the category of Kan-complexes and category of CW-complexes.

The main result of Chapter 3 is Dold-Kan correspondence which gives an equivalence between the category Simp($\mathcal{A}$) of simplicial objects in an abelian category $\mathcal{A}$ and the category of chain complexes in $\mathcal{A}$. Using the Dold-Kan correspondence, construction of Eilenberg-Maclane spaces of all type is given.

In Chapter 4, we study dimension subgroups, including the counter-example of a group G due to Rips 1972, showing that $D_4(G) \neq \gamma_4(G)$.

Chapter 1

Simplicial objects

In this Chapter we will give some basic definitions which will serve as a base for the simplicial homotopy theory.

Definition 1.1 *A category $\mathcal{C}$ is a collection of objects together with*
(i) a class of disjoint sets, denoted by $Hom(A,B)$ for each pair $A, B \in \mathcal{C}$, (an element $f \in Hom(A,B)$ is called a morphism from A to B.
(ii) For each triple (A,B,D) of objects in $\mathcal{C}$ there is a map

$$Hom(B,D) \times Hom(A,B) \longrightarrow Hom(A,D)$$

which satisfies two axioms.
***(a) Associativity**: if $f : A \to B,\ g : B \to D,\ h : D \to E$ are morphisms then*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

***(b) Identity morphism**: For every object B in $\mathcal{C}$, there exists a morphism $I_B : B \longrightarrow B$ such that for any*

$$f : A \longrightarrow B, \quad g : B \longrightarrow D$$

$$I_B \circ f = f,\ g \circ I_B = g.$$

Example 1.2 **Δ-Category**:

Objects of this Δ-Category are sequence of integers $\{0, 1, \cdots, n\},\ n \geq 0$ denoted by $[n]$ and morphism between any two objects $[n]$ and $[m]$ is a non-decreasing map

$$\alpha : [n] \longrightarrow [m]$$

i.e., $\alpha(i) \leq \alpha(j) \ \ for \ i < j$.
Define face map δ_i and degeneracy map σ_i in Δ as follows:
$\delta_i : [n] \longrightarrow [n+1]$ by

$$\delta_i(j) = j \ \ if \ j < i$$

$$\delta_i(j) = j + 1 \ \ if \ j \geq i.$$

$\sigma_i : [n] \longrightarrow [n-1]$ by

$$\sigma_i(j) = j \ \ if \ j \leq i$$

$$\sigma_i(j) = j - 1 \ \ if \ j > i.$$

Remark 1.3 For example, when we compute the face and degeneracy maps, then it comes out that there are two distinct face maps from [0] to [1] and one degeneracy map from [1] to [0]. There are three distinct face maps from [1] to [2] and two distinct degeneracy maps from [2] to [1].

Lemma 1.4 *[8, page 7] Every morphism $\alpha : [n] \longrightarrow [m]$ in Δ can be written uniquely in the following form*

$$\alpha = \delta_{i_1} \cdots \delta_{i_s} \sigma_{j_1} \cdots \sigma_{j_t}$$

with $0 \leq i_s < \cdots i_1 \leq m$ and $0 \leq j_1 < \cdots j_t \leq n$. It follows that α has a unique epi-monic factorization, $\alpha = \delta\sigma$, where $\delta = \delta_{i_1} \cdots \delta_{i_s}$ and $\sigma_{j_1} \cdots \sigma_{j_t}$

Proof Let $i_s < \cdots i_1$ be elements of $[m]$ not in $\alpha([n])$. Let $j_1 < \cdots j_t$ be the elements of $[n]$ such that $\alpha(j_i) = \alpha(j_i + 1)$. Then $\alpha = \delta_{i_1} \cdots \delta_{i_s} \sigma_{j_1} \cdots \sigma_{j_t}$. In particular, α factorizes as

$$[n] \twoheadrightarrow [p] \hookrightarrow [m]$$

where $p = n - t = m - s$.

Definition 1.5 *A simplicial object K in a category $\mathcal{C}$ is a contravariant functor*

$$K : \Delta \longrightarrow \mathcal{C}.$$

It may be noted that a simplicial object is essentialy in a category $\mathcal{C}$ is given by sequence of objects in $\mathcal{C}$, $\{K_n\}_{n\geq 0}$ along with face maps

$$d_i : K_n \longrightarrow K_{n-1}$$

and degeneracy maps

$$s_i : K_n \longrightarrow K_{n+1}$$

satisfying the following simplicial identities

$$d_i d_j = d_{j-1} d_i \ for \ i < j,$$
$$s_i s_j = s_{j+1} s_i \ for \ i \leq j,$$
$$d_i s_j = s_{j-1} d_i \ for \ i < j,$$
$$d_i s_i = Id = d_{i+1} s_i,$$
$$d_i s_j = s_j d_{i-1} \ for \ i > j+1.$$

Example 1.6 The $n - simplex$ $\Delta[n]$ is a simplicial object with

$$\Delta[n]_k = \{(i_0, \cdots, i_k) | 0 \leq i_0, \leq \cdots, \leq i_k \leq n, \ k \leq n\}.$$

face map $d_j : \Delta[n]_k \longrightarrow \Delta[n]_{k-1}$ and degeneracy map $s_j : \Delta[n]_k \longrightarrow \Delta[n]_{k+1}$ is given by

$$d_j(i_0, \cdots i_k) = (i_0, \cdots i_{j-1}, i_{j+1}, \cdots i_k),$$
$$s_j(i_0, \cdots i_k) = (i_0, \cdots i_j, i_j, \cdots i_k).$$

Definition 1.7 *A cosimplicial object in a category $\mathcal{C}$ is a sequence of objects, $K^n \in \mathcal{C}$ together with maps $d^i : K^n \longrightarrow K^{n+1}$, $s^i : K^n \longrightarrow K^{n-1}$, which satisfy cosimplicial identities:*

$$d^j d^i = d^i d^{j-1} \ for \ i < j \qquad for \ 0 \leq i \leq n,$$

$$s^j s^i = s^i s^{j+1} \; for \; i \leq j \qquad for \; 0 \leq i \leq n,$$

$$s^j d^i = d^i s^{j-1} \; for \; i < j \qquad for \; 0 \leq i \leq n,$$

$$s^i d^i = Id = s^i d^{i+1} \qquad for \; 0 \leq i \leq n,$$

$$s^j d^i = d^{i-1} s^j \; for \; i > j+1 \qquad for \; 0 \leq i \leq n.$$

Definition 1.8 *Let K be a simplicial object in category $\mathcal{C}$. Then an element $x \in K_n$ is said to be an n-simplex.*

Definition 1.9 *An n-simplex $x \in K_n$ is said to be degenerate if there exists $y \in K_{n-1}$ such that $x = s_i y$ for some i, $0 \leq i \leq n$.*

Definition 1.10 *A simplicial object K is said to be reduced if K_0 has only one element.*

Definition 1.11 *A simplicial map $f : K \longrightarrow L$ between two simplicial objects in a category $\mathcal{C}$ consists of a collection of maps,i.e., $\{f_n\}$, $f_n : K_n \longrightarrow L_n$ for $n \geq 0$, such that the maps f_n are compatible with face and degeneracy map*

$$f_n d_i = d_i f_{n+1} \qquad for \; 0 \leq i \leq n$$

$$f_n s_i = s_i f_{n-1} \qquad for \; 0 \leq i \leq n$$

Cartesian product: The cartesian product of two simplicial objects K and L in a category $\mathcal{C}$ is defined as follows:

$$(K \times L)_n = K_n \times L_n, \; for \; n \geq 0,$$

$$d_i(x, y) = (d_i(x), d_i(y)), \; for \; 0 \leq i \leq n,$$

$$s_i(x, y) = (s_i(x), s_i(y)) \; for \; 0 \leq i \leq n.$$

Example 1.12 : Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$ be an object. Let $K = \{K_n\}$ with $K_n = A$. Define face and degeneracy maps on A to be the identity maps on A; these face and degeneracy maps satisfies simplicial identities, and so $K = \{K_n\}$ becomes a simplicial object.

Chapter 2

Simplicial objects in topology

In this Chapter we will develop the simplicial homotopy theory. Using simplicial homotopy theory we will see equivalence between Kan-complexes and CW-complexes which allows us to work with simplicial homotopy theory in place of classical homotopy theory.

Definition 2.1 *We define simplicial set to be a simplicial object in the category of* ***Sets****.*

Notations Category of simplicial sets is denoted by Set_Δ and the category of topological spaces is denoted by Top.

2.1 Construction of simplicial set from a topological space

Let X be a topological space and

$$\Delta^n = \{(t_0, \cdots, t_n) \mid \Sigma_{i=0}^n t_i = 1, t_i \geq 0,\ t_i \in \mathbb{R}\}$$

be the standard n-simplex.
Define:-

$$Sing_n(X) = \{\gamma : \Delta^n \longrightarrow X \mid \gamma \textit{ is a continuous map}\}$$

to be the collection of all singular n-simplices. We define the face and degeneracy map as follows:

$$(d_i\gamma)(t_0, \cdots, t_{n-1}) = \gamma(t_0, \cdots, t_{i-1}, 0, t_i, \cdots t_{n-1}) \quad 0 \leq i \leq n-1$$

$$(s_i\gamma)(t_0, \cdots, t_{n+1}) = \gamma(t_0, \cdots, t_{i-1}, t_i + t_{i+1}, \cdots t_{n+1}) \quad 0 \leq i \leq n$$

So, it remains to show that these face and degeneracy map satisfy simplicial identities.

$$\begin{aligned}(s_i s_j \gamma)(t_0, \cdots, t_{n+2}) &= (s_j \gamma)(t_0, \cdots, t_{i-1}, t_i + t_{i+1}, \cdots t_{n+2}) \\ &= \gamma(t_0, \cdots, t_{i-1}, t_i + t_{i+1}, \cdots t_j, t_{j+1} + t_{j+2} \cdots t_{n+2}) \\ &= (s_{j+1} s_i \gamma)(t_0, \cdots, t_{n+2}).\end{aligned}$$

Thus,

$$s_i s_j = s_{j+1} s_i, \quad for\ i \leq j.$$

Therefore $Sing(X)$ becomes a simplicial set.
In particular, $Sing$ defines a functor from category of topological spaces to category of simplicial sets.

$$Sing : Top \to Set_\Delta$$

2.2 Kan complexes

Definition 2.2 *Collection of* $n, (n-1)$*-simplices,*

$$x_0, \cdots, x_{k-1}, -, x_{k+1}, \cdots x_n$$

satisfying compatibility condition if

$$d_i x_j = d_{j-1} x_i, \ \forall\ i < j,\ i, j \neq k.$$

Definition 2.3 *A simplicial set* K *is said to be a Kan-complex if, for every collection of* $n, (n-1)$*-simplices,*

$$x_0, \cdots, x_{k-1}, -, x_{k+1}, \cdots x_n$$

satisfying compatibility condition there exists an n*-simplex* y *such that*

$$d_i y = i, \ \forall\ i \neq k.$$

Notation λ_k^n is the standard n-simplex Δ^n with the interior and the k^{th} face removed.

Lemma 2.4 *[7] A simplicial set is a Kan-complex if and only if, for any* $0 \leq k \leq n$*, and any simplicial morphism* $\lambda_k^n \to K$ *can be extended to a morphism* $\Delta^n \to K$*.*

Example 2.5 0-simplex Δ^0 is a Kan-complex.

Definition 2.6 *A map $f : K \to L$ of simplicial sets is said to be a Kan fibration if for every collection of n, $(n-1)$-simplices*

$$x_0, \cdots, x_{k-1}, -, x_{k+1}, \cdots x_n$$

of K which satisfy the compatibility condition and for any n-simplex $y \in L_n$ satisfying

$$d_i y = f(x_i), \ \forall \ i \neq k$$

there exists an n-simplex $x \in K_n$ such that

$$f(x) = y, \ \text{and } d_i x = x_i, \ \forall \ i \neq k.$$

Lemma 2.7 *[7] Let L be a simplicial set generated by single element l_0 Then a map $f : K \to L$ is a Kan-fibration if and only if K is a Kan-complex.*

Proof Let $f : K \to L$ be a Kan fibration So we need to show that K is a Kan-Complex. Suppose

$$x_0, \cdots, x_{k-1}, -, x_{k+1}, \cdots x_n$$

are n, $(n-1)$-simplices satisfying compatibility condition and y is an n-simplex of L such that

$$d_i y = f(x_i), \ \forall \ i \neq k$$

Since f is a Kan-fibration so there exists an n-simplex $x \in K_n$ such that $d_i x = x_i, \ \forall \ i \neq k$ and $f(x) = y$. Thus K is a Kan-complex.
Conversely, supppose K is a Kan complex. Now just using the definition we get the desired result.

Definition 2.8 *A simplicial group is a simplicial object in the category of groups.*

Lemma 2.9 *[10] A simplicial group G is a Kan-Complex.*

Proof Suppose

$$x_0, \cdots, x_{k-1}, , x_{k+1}, \cdots x_{n+1}$$

be $n+1$, n-simplices satisfying compatibility condition. Our aim is to find $g \in G_{n+1}$ such that

$$d_i g = x_i, \ \forall \ i \neq k.$$

We will proceed by induction on r to find $g_r \in G_{n+1}$ such that $d_i g_r = x_i, \ \forall \ i \leq r, \ i \neq k$. Set $g_{-1} = 1 \in G_{n+1}$. Suppose we have found $g_{r-1} \in G_{n+1}$ such that

$$d_i g_{r-1} = x_i \ \forall \ i \leq r-1, \ i \neq k.$$

Now we will find g_r if $r = k$, then we will take $g_r = g_{r-1}$.
If $r \neq k$, Consider the element $y = {x_r}^{-1}(d_r g_{r-1}) \in G_n$. Since we have $d_i g_{r-1} = x_i$ for $i \leq r-1$.

$$\begin{aligned} d_i y &= d_i(x_r)^{-1} d_i d_r g_{r-1} \\ &= (d_i x_r)^{-1} d_{r-1} d_i g_{r-1} \\ &= (d_i x_r)^{-1} d_{r-1} x_i \\ &= (d_i x_r)^{-1} d_i x_r = 1. \end{aligned}$$

Thus we have $d_i y = 1, \ \forall \ i < r, \ i \neq k$.

Now $d_i(s_r y) = s_{r-1}(d_i y) = 1, \ for \ i < r, \ i \neq k$.
Now take $g_r = g_{r-1}(s_r y)^{-1}$

$$d_i g_r = d_i g_{r-1} d_i (s_r y)^{-1} = d_i g_{r-1} = x_i, \ \forall \ i < r, i \neq k$$
$$d_r g_r = d_r(g_{r-1}(s_r y)^{-1}) = (d_r g_{r-1})(d_r(s_r y)^{-1}) = x_r$$

Thus simplicial group G becomes a Kan-complex.

2.3 Simplicial homotopy theory

Definition 2.10 *Let K be a simplicial set. Then we say two n-simplices $x, x' \in K_n$ are homotopic if*

$$d_i x = d_i x', \ \forall \ 0 \leq i \leq n,$$

and there exist an element $y \in K_{n+1}$ with the property that $d_n y = x, \ d_{n+1} y = x'$ and $d_j y = s_{n-1} d_j x = s_{n-1} d_j x' \ \forall \ 0 \leq j < n$.

Remark 2.11 In above definition the $(n+1)$-simplex y is called a homotopy from x to x', and we write $x \sim x'$.

Lemma 2.12 *[1, page 115] Let K be a simplicial set which satisfies the Kan condition. Then the relation of being homotopic is an equivalence relation on K_n, $\forall\ n \geq 0$*

Proof (i) **Reflexive**:- Let $x \in K_n$ *and* $y = s_n x \in K_{n+1}$
Since $d_i x = d_i x,\ \forall\ 0 \leq i \leq n$
and $d_n y = d_n s_n x = x,\ d_{n+1} y = d_{n+1} s_n x = x$
and $d_j y = d_j s_n x = s_{n-1} d_j x$. Thus we have $x \sim x$

(ii) **Symmetric and Transitive**:- Let $x, x', x'' \in K_n$ *Such that* $x' \sim x$ *and* $x'' \sim x'$. Suppose y' is a homotopy from x' to x and y'' is a homotopy from x'' to x'.

$$d_i x' = d_i x = d_i x'' \quad \forall\ 0 \leq i \leq n.$$

Since y' is a homotopy from x' to x. So,

$$d_i y' = \begin{cases} s_{n-1} d_i x'; & for\ 0 \leq i < n \\ x'; & for\ i = n \\ x; & for\ i = n+1 \end{cases}$$

Since y'' is a homotopy from x'' to x. So,

$$d_i y'' = \begin{cases} s_{n-1} d_i x''; & for\ 0 \leq i < n \\ x''; & for\ i = n \\ x; & for\ i = n+1 \end{cases}$$

Now we want to construct a homotopy from x'' to x'. For $0 \leq j < n$, Choose $z_j = s_{n-1} s_{n-1} d_j x'$. Now using simplicial identities for $0 \leq j < n$, we have $s_{n-1} s_{n-1} d_j = d_j s_n s_n$. Thus, for $0 \leq i < j < n$,

$$
\begin{aligned}
d_i z_j &= d_i s_{n-1} s_{n-1} d_j x' \\
&= d_i d_j s_n s_n x' \\
&= d_{j-1} d_i s_n s_n x' \\
&= d_{j-1} s_{n-1} s_{n-1} d_i x' \\
&= d_{j-1} z_i.
\end{aligned}
$$

For $0 \leq j < n$,

$$d_{n+1} z_j = d_{n+1} s_{n-1} s_{n-1} d_j x' = s_{n-1} d_n s_{n-1} d_j x' = s_{n-1} d_j x'$$

$$d_n z_j = d_n s_{n-1} s_{n-1} d_j x' = s_{n-1} d_j x'$$

Since y' and y'' are homotopies from x' to x and x'' to x. So we have $n+2$, $(n+1)$-simplices

$$z_0, z_1, \cdots, z_{n-1}, -, y', y''$$

which satisfies the compatibility condition. Since K is a Kan-complex so there exists an $(n+2)$-simplex z such that

$$d_j z = z_j \quad for\ 0 \leq j < n-1$$

$$d_{n+1} z = y', \quad d_{n+2} z = y''.$$

Now using simplicial identities we can check that $(d_n z)$ is a homotopy from x'' to x'. If we take $x'' = x$ then we see that relation is symmetric. Thus being homotopic is an equivalence relation.

2.3.1 Homotopy groups

Let K be a simplicial set and take a 0-simplex $k_o \in K_0$ and consider L to be the sub simplicial set generated by k_o. Thus for every $n \geq 0$ there is exactly one element in L_n which is

$$k_o^n = s_{n-1} \cdots s_0 k_o.$$

If K is a Kan-complex then (K, k_0) is said to be a Kan-pair.

Definition 2.13 *Let (K, k_0) be a Kan pair. Then*

$$\pi_n(K, k_0) = \{x \in K_n | d_i x = k_0^{n-1}\}/\sim$$

where $\sim$ is an equivalence relation defined earlier.

Remark 2.14

$$\pi_0(K, k_0) = K_0/\sim$$

$\pi_0(K)$ is called path connected component of K. K is said to be path connected if there is only one class in $\pi_0(K)$.
We will define composition of two elements in $\pi_n(K, k_0)$[7]. Let $[a], [b] \in \pi_n(K, k_0)$. Let x and y denotes representatives for the classes $[a]$,$[b]$ respectively. Then the following $n+1$, n-simplices

$${k_0}^n, \cdots, k_0^n, x, -, y$$

satisfy compatibility condition. Since K is a Kan-complex so there exists an $(n+1)$-simplex z such that

$$d_{n+1}z = y, d_{n-1}z = x \textit{ and } d_i z = k_0^n, \ \forall\ 0 \leq i < n-1.$$

So, we define $[a] * [b]$ to be the equivalence class of $d_n z$. Suppose there is another $z' \in K_{n+1}$ which satisfy $d_{n+1}z' = y,\ d_{n-1}z' = x,\ \textit{and } d_i z' = {k_0}^n,\ \forall\ 0 \leq i < n-1$. Then look at the $n+2$, $(n+1)$-simplices

$${k_0}^{n+1}, \cdots, k_0^{n+1}, s_n d_{n-1} z, -, z, z'.$$

These simplices satisfy compatibility condition. Since K is a Kan-complex, So there exists $\omega \in K_{n+2}$ with the property that $d_{n+1}w = z, d_{n-1}\omega = s_n d_{n-1} z$,and $d_i\omega = k_0^{n+1},\ \forall\ 0 \leq i < n-1$.
Claim $d_n\omega$ is a homotopy from $d_n z$ to $d_n z'$.
Justification

$$d_i(d_n z) = d_{n-1}(d_i z) = d_{n-1}{k_0}^n = k_0^{n-1}$$

$$d_i(d_n z') = d_{n-1}(d_i z') = k_0^{n-1}.$$

Thus $d_i(d_n z) = d_i(d_n z'),\ \forall 0 \leq i \leq n-1$.

$$d_n(d_n\omega) = d_n(d_{n+1}\omega) = d_n z$$

$$d_{n+1}(d_n\omega) = d_n(d_{n+2}\omega) = d_n z'$$

$$d_i(d_n\omega) = d_{n-1}(d_i\omega) = d_{n-1}k_0^{n+1} = k_0^n$$

$$s_{n-1}d_i(d_n z) = s_{n-1}d_{n-1}(d_i z) = k_0^n$$

$$Thus\ s_{n-1}d_i(d_n z) = d_i(d_n\omega)\ \forall\ 0 \le i < n-1.$$

Thus $d_n\omega$ is a homotopy from $d_n z$ to $d_n z'$. so $d_n z \sim d_n z'$.
Suppose instead of y, we pick y' as a representative of $[b]$. Then $[a] * [b] = [d_n z']$ such that $d_i z' = k_0^n,\ \forall\ 0 \le i < n-1$. and $d_{n-1}z' = x,\ d_{n+1}z' = y'$ Since y' and y belongs to the same homotopy class so there is a homotopy ω from y' to y. Consider $n+2$, $(n+1)$-simplices

$$k_0^{n+1}, \cdots, k_0^{n+1}, s_{n-1}x, z'-, \omega.$$

These simplices satisfy compatibility condition. Since K is a Kan-complex, so there exists $u \in K_{n+2}$ with the property that

$$d_i u = k_0^{n+1}$$

$$d_{n-1}u = s_{n-1}x$$

$$d_n u = z''$$

$$d_{n+2}u = \omega$$

Now take the $(n+1)$-simplex $\alpha = d_{n+1}u$

$$d_i\alpha = d_i(d_{n+1}u) = d_n(d_i u) = k_0^n\ \forall\ 0 \le i < n$$

$$d_{n-1}\alpha = d_{n-1}(d_{n+1}u) = d_n(d_{n-1}u) = d_n s_{n-1}x = x\ \forall\ 0 \le i < n$$

$$d_n\alpha = d_n z',\ d_{n+1}\alpha = d_{n+1}(d_{n+1}u) = d_{n+1}(d_{n+2}u) = y.$$

Thus our choice of representative is independent. Thus composition is well defined.

Lemma 2.15 *[7, page 15] Let (K, k_0) be a Kan pair, then the set $\pi_n(K, k_0)$ forms a group for $n \ge 1$, where composition is defined above.*

Proof Existence of Identity

$$[{k_0}^n] \; is\; the\; left\; identity\; of\; \pi_n(K, k_0).$$

Let

$$[a] \in \pi_n(K, k_0)$$

Claim $[k_0^n] * [a] = [a]$
Justification Consider the $(n+1)$, n-simplices

$${k_0}^n, \cdots, {k_0}^n, -, a$$

These simplices satisfy compatibility condition. Choose $z = s_n a$. Then we have $d_i z = {k_0}^n \;\forall\; 0 \leq i \leq n-1$, $d_{n+1} z = a$
Thus $[k_0^n] * [a] = [d_n z] = [a]$.
Associativity:-
Let x, y, z be the representatives of $[a], [b], [c] \in \pi_n(K, k_0)$ respectively.
Suppose that $[a] * [b] = [d_n \omega]$ and $[b] * [c] = [d_n \omega']$.
Consider the $n+1$, n-simplices

$$k_0^n, \cdots, k_0^n, d_n\omega, -, z$$

satisfy the compatibility condition and K is a Kan-complex so there exist $u \in K_{n+1}$ such that

$$d_i u = k_0^n, \;\forall\; 0 \leq i < n-1$$

$$d_{n-1} u = d_n \omega$$

$$d_{n+1} u = z.$$

Thus $[d_n u] = [d_n \omega] c = ([a] * [b]) * [c]$
Now consider $n+2$, $(n+1)$-simplices

$${k_0}^{n+1}, \cdots, k_0^{n+1}, \omega, -, u, \omega'$$

which satisfy compatibility condition and K is a Kan-complex so there exists $v \in K_{n+2}$ with the property that

$$d_i v = k_0^{n+1} \; 0 \leq i < n-1$$

$$d_{n-1}v = \omega$$

$$d_{n+1}v = u$$

$$d_{n+2}v = \omega'$$

.

$$d_{n-1}(d_n v) = d_{n-1}(d_{n-1}v) = d_{n-1}(\omega) = x$$

$$d_{n+1}(d_n v) = d_n(d_{n+2}v) = d_n(\omega') = x$$

$$d_i(d_n v) = d_{n-1}(d_i v) = k_0^n$$

Thus $[d_n(d_n v)] = [a] * [d_n\omega'] = [a] * [bc]$

$$[a] * ([b] * [c]) = [a] * [d_n w'] = [d_n(d_n v)] = [d_n d_{n+1} v] = [d_n u] = [a * b] * c.$$

Thus the composition is associative.
Using the fact that K is a Kan-complex we can prove the existence of inverse of every element. Thus the set $\pi_n(K, k_0)$ is a group and $\pi_n(K, k_0)$ is called nth Simplicial homotopy group with respect to 0-simplex k_0.

Definition 2.16 *A simplicial set K which satisfy compatibility condition is said to be contractible if all of its simplicial homotopy groups are trivial.*

2.3.2 Relative homotopy

Let K be a Kan-complex and $L \subset K$ be its sub Kan-complex. Pick a 0-simplex $l_0 \in L_0$. Then the triple (K, L, l_0) is said to be a Kan-triple.

Definition 2.17 *Let K be a simplicial set and L be a sub-simplicial set. Two n-simplices $x, x' \in K_n$ are said to be homotopic relative to L if*

$$d_i x = d_i x', \ \forall\ 1 \le i \le n,$$

$$d_0 x \sim d_0 x' \ in\ L$$

and there exist an $(n+1)$-simplex $\omega \in K_{n+1}$ such that

$$d_0\omega = y,\ d_n\omega = x,\ d_{n+1}\omega = x'$$

$$d_i\omega = s_{n-1}d_ix = s_{n-1}d_ix' \ \ \forall 1 \le i \le n.$$

Here $y \in K_n$ is a homotopy from d_0x to d_0x' and $\omega \in K_{n+1}$ is said to be relative homotopy from x to x' and we write $x \sim_L x'$.

Definition 2.18 *Let (K, L, l_0) be a Kan-triple. Then we define*

$$\pi_n(K, L, l_0) = \{x \in K_n |\ d_0x \in L_{n-1},\ d_ix = {l_0}^{n-1},\ \forall\ 1 \le i \le n\}/\sim_L.$$

First of all we will define composition of elements of $\pi_n(K, L, l_0)$[7]. Take $[a]$,$[b] \in \pi_n(K, L, l_0)$ for $n \ge 2$.
Let x and y be representatives of the classes $[a]$, $[b]$ respectively. Since $[a]$,$[b] \in \pi_n(K, L, l_0)$ so $d_0x,\ d_0y\ \in L_{n-1}$ and since $\pi_{n-1}(L_{n-1}, l_0)$ is a group. So

$$[d_0x][d_0y] = [d_{n-1}z]\ for\ z \in L_n$$

$$d_iz = l_0^{n-1}\ \forall 0 \le i \le n-3$$

$$d_{n-1}z = d_0x$$

$$d_nz = d_0y$$

Consider $n+1$, n-simplices

$$z, l_0^n, \cdots, l_0^n, x, -, y$$

. One can check that these n-simplices are satisfying compatibility condition and since K is a Kan-complex so there exists a $u \in K_{n+1}$ such that

$$d_iu = l_0^n,\ \forall 1 \le i < n-1$$

$$d_{n-1}u = x$$

$$d_{n+1}u = y$$

$$d_0u = y$$

So we define $[a][b] = [d_nu]$.

Lemma 2.19

$$\pi_n(K, L, l_0)$$

is a group for $n \ge 2$.

Theorem 2.20 *[8, Theorem 3.7] Let (K, L, l_0) be a Kan-triple. Then there is a long exact sequence*

$$\cdots \rightarrow \pi_{n+1}(K, L, l_0) \xrightarrow{d} \pi_n(L, l_0) \xrightarrow{i} \pi_n(K, l_0) \xrightarrow{j} \pi_n(K, L, l_0) \rightarrow \cdots$$

where $d[x] = [d_0 x]$ and maps i and j are maps induced from inclusion.

Proof We need to prove that $Im(d) = Ker(i)$. Let $[x] \in \pi_{n+1}(K, L, l_0)$ i.e., $x \in K_{n+1}$ such that $d_0 x \in L_n$ and $d_i x = l_0^n \ \forall\ 1 \leq i \leq n+1$. Now consider $n+2$, $(n+1)$-simplices

$$-, {l_0}^{n+1}, \cdots, l_0^{n+1}, x.$$

These $(n+1)$-simplices are satisfying compatibility condition, since (K, L, l_0) is a Kan-triple so there exists an $(n+2)$-simplex ω such that

$$d_i \omega = l_0^{n+1},\ \forall\ 1 \leq i \leq n+1$$

$$d_{n+2} \omega = x.$$

Claim: $d_0 \omega$ is a homotopy from l_0^n to $d_0 x$.
Justification:

$$d_i l_0^n = d_i d_0 x = l_0^{n-1},\ \forall\ 0 \leq i \leq n.$$

$$d_n(d_0 \omega) = d_0(d_{n+1} \omega) = d_0 l_0^{n+1} = l_0^n$$

$$d_{n+1}(d_0 \omega) = d_0(d_{n+2} \omega) = d_0 x$$

$$d_i(d_0 \omega) = l_0^n \ for\ 1 \leq j < n.$$

$$s_{n-1}(d_i l_0^n) = l_0^n$$

$$s_{n-1}(d_i d_0 x) = l_0^n$$

So $d_i(d_0 \omega) = s_{n-1}(d_i l_0^n) = s_{n-1}(d_i x) \ for\ 1 \leq j < n$.

Thus $[d_0 x] \sim [l_0^n]$, so $i[d_0 x] = [l_0^n]$.
Thus $iod = I_{\pi_n(K, l_0)}$. Thus $Im(d) \subseteq Ker(i)$.
Let $[a] \in Ker(i)$. Suppose y is the representative of the class $[a]$. Since $y \in Ker(i)$, so $i[y] = l_0^n$. Since i is an inclusion map thus $l_0^n \sim y$.
Let ω be the homotopy from y to l_0^n. Consider $n+2$, $(n+1)$-simplices

$$\omega, {l_0}^{n+1}, \cdots, l_0^{n+1}, -.$$

These $(n+1)$-simplices satisfy compatibility condition so there exists an $(n+2)$-simplex u such that

$$d_i u = l_0^{n+1}, \ \forall\ 1 \leq i \leq n+1$$

$$d_0 u = \omega.$$

$d_0(d_{n+2}u) = d_{n+1}(d_0 u) = d_{n+1}\omega = y.$
This implies that $[y] = d[d_{n+2}u]$. Thus $[y] \in Im(d)$. $\Longrightarrow Ker(i) \subseteq Im(d)$.

Definition 2.21 *Let K and L be simplicial objects in category C. Then two simplicial maps f and g are simplicially homotopic if there exist a collection of morphisms $\{h_i\}$, where*

$$h_i : K_n \longrightarrow L_{n+1} \quad for\ 0 \leq i \leq n.$$

with the property that

$$d_0 h_0 = f, \quad d_{n+1} h_n = g$$

$$d_i h_j = h_{j-1} d_i \quad for\ i < j$$

$$d_i h_j = d_i h_{i-1} \quad for\ i = j \neq 0$$

$$d_i h_j = h_j d_{i-1} \quad for\ i > j+1$$

and

$$s_i h_j = h_{j+1} s_i \quad for\ i \leq j$$

$$s_i h_j = h_j s_{i-1} \quad for\ i > j$$

The collection $\{h_i\}$ is said to be homotopy from f to g and we write $f \simeq g$.

Proposition 2.22 *[16] Let C be a category of sets or an abelian category. Let K and L be simplicial objects in C, and $f, g : K \longrightarrow L$ two simplicial maps. For $i = 0, 1$, let $\epsilon_i : K \longrightarrow K \times \Delta[1]$ be the induced map by $\delta_i : [0] \longrightarrow [1]$ in Δ. Then there is a one to one correspondence between simplicial homotopies from f to g and simplicial maps $F : K \times \Delta[1] \longrightarrow L$.*

2.4 Geometric realisation of a simplicial set

In this section we will define Geometric realisation functor and then we will see the equivalence between Kan-complex and CW-complex.

Definition 2.23 *A point $P_n = (t_0, \cdots, t_n) \in \Delta^n$ is said to be an interior point of Δ^n if either $n = 0$ or $0 < t_i < 1, \ \forall\ i$.*

The notion of geometric realisation was given by Milnor[9]. The geometric realisation of $\mid K \mid$ of K is a topological space obtained from the disjoint union

$$\mid K \mid = (\sqcup_n K_n \times \Delta^n)/\sim$$

where the set K_n is viewed as a topological space with discrete topology. Equivalence relation is given as follows

$$(d_i x, p) \sim (x, \delta_i p); \quad (x, p) \in K_n \times \Delta^{n-1}$$
$$(s_i x, p) \sim (x, \sigma_i p); \quad (x, p) \in K_{n-1} \times \Delta^n$$

where $d_i,\ s_i$ are face and degeneracy maps of K respectively and $\delta_i,\ \sigma_i$ are the maps of $\Delta - category$.

Definition 2.24 *An element $(x, p) \in \sqcup(K_n \times \Delta^n)$ is said to be non-degenerate if x is non-degenerate and p is an interior point of Δ^n).*

Proposition 2.25 *[7, page 20] Each element $(x, p) \in \sqcup(K_n \times \Delta^n)$ is equivalent to a unique non-degenerate element of $\sqcup(K_n \times \Delta^n)$.*

Proof If x_n is non-degenerate, then we are done. If x_n is degenerate then there exists $x_{n-1} \in K_{n-1}$ such that $x_n = s_i x_{n-1}$. So in this way each element x_n can be written in the form

$$s_{j_r} \cdots s_{j_1} x_{n-r}$$

with $0 \leq j_1 \leq \cdots j_r \leq n$ and $x_{n-r} \in K_{n-r}$ is non-degenerate. Similarly for each $p_n \in \Delta^n$ can be written uniquely in the form

$$\delta_{i_q} \cdots \delta_{i_1} p_{n-q}$$

where $0 \leq 1_1 \leq \cdots i_q \leq n$ and p_{n-q} is in interior of Δ^{n-q}.
Define morphisms f and g as follows

$$f : \sqcup(K_n \times \Delta^n) \longrightarrow \sqcup(K_n \times \Delta^n)$$

by $f(x_n, p_n) = (K_{n-r}, \sigma_{j_1} \cdots \sigma_{j_r} p_n)$ and

$$g : \sqcup(K_n \times \Delta^n) \longrightarrow \sqcup(K_n \times \Delta^n)$$

by $g(x_n, p_n) = (\delta_{i_1} \cdots \delta_{i_q} x_n, p_{n-q})$
where $x_n = s_{i_r} \cdots s_{i_1} k_{n-r}$ with k_{n-r} non-degenerate and $p_n = \delta_{i_q} \cdots \delta_{i_1} u_{n-q}$ with u_{n-q} in the interior of Δ^{n-q}, look at the map $f \circ g$, this composition map takes each element to a unique non-degenerate element.

Theorem 2.26 *[9, page 358] The geometric realisation of a simplicial set K is a CW-complex with one n-cell for each non-degenerate n-simplex of K.*

$\mid - \mid$ associates each simplicial set K to a topological space $\mid K \mid$, which gives rise to a functor $\mid - \mid : Sets_\Delta \longrightarrow Top$.
Any simplicial morphism $f : K \longrightarrow L$ induces a morphism $\mid f \mid : \mid K \mid \longrightarrow \mid L \mid$ which maps $\mid x_n, s_n \mid$ *to* $\mid f(x_n), s_n \mid$.

Definition 2.27 *Let C and D be two categories and let $S : C \longrightarrow D$ and $T : D \longrightarrow C$ be covariant functors then the pair (S, T) is said to be an adjoint pair if there is a bijection from the functor $Hom_D(S(-), -)$ to the functor $Hom_C(-, T(-))$.*

Theorem 2.28 *[1, page 120] The singular simplex functor $Sing : Top \longrightarrow Sets_\Delta$ and the geometric realisation functor $\mid - \mid : Sets_\Delta \longrightarrow Top$ are adjoint. Further, for a simplicial set K and a topological space X, there is a one to one correspondence between homotopy classes of continuous maps $\mid K \mid \longrightarrow X$ and homotopy classes of simplicial maps $K \longrightarrow Sing(X)$. In particular, $\pi_i(X, x_0) = \pi_i(Sing(X), Sing(x_0))$.*

Chapter 3

Simplicial objects in homological algebra

In this chapter we will work on abelian categories. Through out the Chapter, $\mathcal{A}$ is an abelian category and A will denote a simplicial object in abelian category $\mathcal{A}$. We will prove equivalence between category Simp($\mathcal{A}$) and category of chain complexes $Ch_{\geq 0}(\mathcal{A})$ in $\mathcal{A}$ which enables us to construct Eilenberg-Maclane spaces of all type. Let $\mathcal{A}$ be an abelian category and A be a simplicial object in category $\mathcal{A}$. We will denote category of simplicial objects by $Simp(\mathcal{A})$.

Definition 3.1 *A category $\mathcal{C}$ is additive if folowing conditions are satisfied:*
(i) It has a zero object.
(ii) Every hom-set $Hom(A, B)$ has an addition, endowing it with the structure of an abelian group, and such that composition of morphisms is bilinear.
(iii) all finitary biproducts exists.

Definition 3.2 *An additive category $\mathcal{A}$ is said to be an abelian category if the following conditions are satisfied:*
(i) Every morphism in $\mathcal{A}$ has kernel and cokernel.
(ii)Every monomorphism is the kernel of its cokernel.
(iii)Every epimorphism is the co-kernel of its kernel.

Definition 3.3 *Let A be a simplicial object in category $\mathcal{A}$. Then the associated chain complex $C(A)$ of A is the complex*

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots C_0 \to 0$$

with $C_n = A_n$ *and differentials defined as*

$$\partial_n = \Sigma_{i=0}^{n}(-1)^i d_i : C_n \longrightarrow C_{n-1}, \qquad n \geq 0$$

where d_i *are the face maps of simplicial object* A*. Since* A *is a simplicial object, thus by using simplicial identities, we get*

$$\partial_n \circ \partial_{n+1} = 0 \ \ \forall\ n \geq 0.$$

Thus $C(A)$ *is a well-defined chain complex.*

Definition 3.4 *The normalised chain complex of a simplicial object* A *is a chain complex with*

$$N_n(A) = \bigcap_{i=0}^{n-1} Ker(d_i : A_n \longrightarrow A_{n-1})\ \forall\ n \geq 0$$

and differentials are defined as

$$\partial_n = (-1)^n d_n\ \forall\ n \geq 0.$$

Remark 3.5 Normalised chain complex forms a functor from category of simplicial objects to category of non-negative chain complexes

$$N : Simp(\mathcal{A}) \longrightarrow Ch_{\geq 0}(\mathcal{A})$$

For a given simplicial morphism,

$$f : A \longrightarrow B;\ \ A, B\ \in Simp(\mathcal{A}),$$

we have a collection of maps

$$f_n : A_n \longrightarrow B_n.$$

Then $N(f) : N(A) \longrightarrow N(B)$ is a morphism of chain complexes

$$N(f)_n = f_n\ |_{N_n(A)}: N_n(A) \longrightarrow N_n(B).$$

Lemma 3.6 *[16] Let A be a simplicial object in category $\mathcal{A}$. Let $N(A)$ be associated normalised chain complex and $C(A)$ is associated chain complex and suppose $D(A)$ is degenerate sub-complex of $C(A)$ generated by image of degeneracy maps s_i, i.e.*

$$D_n(A) = \Sigma_{i=0}^{n} s_i(C_{n-1}(A)),$$

then $C(A) = N(A) \oplus D(A)$.

Proof Let $y \in N_n(A) \bigcap D_n(A)$. Suppose to the contrary that $y \neq 0$. Since $y \in D_n(A)$ thus

$$y = \Sigma s_i(x_i).$$

Let i be the smallest integer such that $s_i(x_i) \neq 0$. Then $d_i(y) = x_i \neq 0$, which is a contradiction since $y \in N_n(A)$. Thus $y = 0$.
Now let $y \in C_n(A)$, if $y = 0$ then we are done.

If $y \neq 0$ and $d_k y = 0 \ \ \forall\ 0 \leq k < n$. Then $y \in N_n(A)$ and again $y = y + 0$.
Now suppose for some $k < n$, $d_k(y) \neq 0$. Now look at

$$y' = y - s_k d_k(y), \quad s_k d_k(y) \in D_n(A)$$

for $i < n$

$$d_i(y') = d_i(y) - s_{k-1} d_{k-1} d_i(y) = 0$$

which $\implies y' \in N_n(A)$.
Thus $y \in N(A) + D(A)$.
Since $C(A) = N(A) \oplus D(A)$, thus $N(A) \simeq C(A)/D(A)$.

Definition 3.7 *For a given simplicial object A in an abelian category $\mathcal{A}$, we define*

$$\pi_n(A) = H_n(N(A)).$$

Theorem 3.8 *[16, Theorem 8.3.8] Let A be a simplicial object in category $\mathcal{A}$. Then for all $n \geq 0$,*

$$\pi_n(A) = H_n(C(A)).$$

Example 3.9 Classifying space:- Let G be a group, now construct simplicial set BG as follows

$$BG_0 = \{1\} \quad BG_n = G^n$$

Define face and degeneracy maps as follows

$$\begin{aligned} d_i(g_1, \cdots, g_n) &= (g_2, \cdots, g_n); \quad if\ i = 0, \\ d_i(g_1, \cdots, g_n) &= (g_1, \cdots, g_i g_{i+1}, \cdots g_n); \quad 0 < i < n, \\ d_i(g_1, \cdots, g_n) &= (g_1, \cdots, g_{n-1}); \quad i = n, \\ s_i(g_1, \cdots, g_n) &= (g_1, \cdots, g_i, 1, g_{i+1}, \cdots g_n). \end{aligned}$$

Claim: BG is a simplicial set.
Justification:

$$\begin{aligned} d_i d_j(g_1, \cdots, g_n) &= d_i(g_1, \cdots, g_i \cdots, g_j g_{j+1}, \cdots, g_n); \quad for\ 0 < i < j < n \\ &= (g_1, \cdots, g_i g_{i+1} \cdots, g_j g_{j+1}, \cdots, g_n) \\ &= d_{j-1} d_i. \end{aligned}$$

Thus BG is a simplicial set.

$$N_1(BG) = Ker(d_0 : G_1 \to \{1\}) = G$$

$$N_n(BG) = \bigcap_{i=0}^{n-1} Ker(d_i : BG_n \longrightarrow BG_{n-1})$$

for $i = 0$, $d_0 : BG_n \longrightarrow BG_{n-1}$,

$$Kerd_0 = \{(g_1, \cdots, g_n) \in BG_n \mid g_i = 1 \quad for\ i \geq 2\}$$

$$Kerd_n = \{(g_1, \cdots, g_n) \in BG_n \mid g_i = 1 \quad for\ i \leq n-1\}$$

So $N_n(BG) = \{1\} \quad for\ n \neq 1$.
So normalised chain complex of BG

$$\cdots \to 1 \cdots \to G \xrightarrow{\partial_1} 1$$

So $\pi_1(BG) = H_1(N(BG)) = G$ and $\pi_n(BG) = 1 \quad for\ n \neq 1$.

Definition 3.10 *A connected topological space X is said to be an Eilenberg-Maclane space of type $K(G,\ n)$ if*

$$\pi_i(X) = \begin{cases} G & if\ i = n \\ 1 & i \geq 1,\ i \neq n \end{cases}$$

Remark 3.11 BG is an Eilenberg space of type $K(G, 1)$.

3.1 The Dold-Kan Correspondence

Definition 3.12 *If we have two categories $\mathcal{C}$ and $\mathcal{D}$, then equivalence of categories consists of a functor $F : \mathcal{C} \to \mathcal{D}$, a functor $G : \mathcal{D} \to \mathcal{C}$ and two isomorphisms $\epsilon : FG \to I_{\mathcal{D}}$ and $\eta : GF \to I_{\mathcal{C}}$. Here $FG : \mathcal{D} \to \mathcal{D}$ and $GF : \mathcal{C} \to \mathcal{C}$, denote the respective compositions of F and G, and $I_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ and $I_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ denote the identity functors on $\mathcal{C}$ and $\mathcal{D}$, assigning each object and morphism to itself.*

Theorem 3.13 *[4] Let $\mathcal{A}$ be an abelian category and N be the normalised chain functor*

$$N : Simp(\mathcal{A}) \longrightarrow Ch_{\geq 0}(\mathcal{A})$$

then
N is an equivalence of category of simplicial objects $Simp(\mathcal{A})$ with category of chain complexes $Ch_{\geq 0}(\mathcal{A})$.

Proof **(a)**: First of all we will construct a functor

$$K : Ch_{\geq 0}(\mathcal{A}) \longrightarrow Simp(\mathcal{A})$$

define for $C \in Ch_{\geq 0}(\mathcal{A})$

$$K_n(C) = \bigoplus_{\eta} C_{\eta}[p]$$

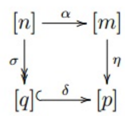
be the finite direct sum of $C_\eta[p]$ and η runs over all the surjections

$$\eta : [n] \longrightarrow [p], \quad for\ p \leq n,\ and\ C_\eta[p] = C_p$$

Now for a given morphism $\alpha : [n] \longrightarrow [m]\ in\ \Delta$, we define

$$K(C)(\alpha) : K_m(C) \longrightarrow K_n(C)$$

by its restriction to each summand C_η.



[7, page 30]

Since $\eta \circ \alpha : [n] \longrightarrow [p]$ be a morphism and every morphism in $\Delta - category$ has a unique epimonic factorisation, So let $\epsilon \circ \eta'$ be the unique epimonic factorisation of $\eta \circ \alpha$.
If $p = q$ then $\eta \circ \alpha = \eta'$. Then $K(C)(\alpha, \eta)$ is a natural map sending C_p to C_p.
If $p = q + 1$, and $\epsilon = d_p$. Then we define

$$K(C)(\alpha, \eta) = d : C_p \longrightarrow C_{p-1} \subseteq K_n(C)$$

. where d is a differential of chain-complex. Otherwise we define $K(C)(\alpha, \eta) = 0$.
Now our next step is to show composition of morphisms.
Let $\alpha : [l] \longrightarrow [m] : \beta : [m] \longrightarrow [n]$ be two composable morphisms in Δ. Now consider the map

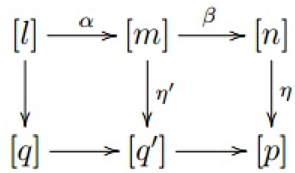
$$K(C)(\beta \circ \alpha) : K_n(C) \longrightarrow K_l(C).$$

For any $\eta : [n] \twoheadrightarrow [p],\ p \leq n$, look at the restriction $K(C)(\beta \circ \alpha)$.

If $q \leq p - 2$ then the way we defined $K(C)(\alpha)$, it comes out that

$$K(C)(\beta \circ \alpha, \eta) = 0 = K(C)(\alpha, \eta') \circ K(C)(\beta, \eta)$$

If $q = p - 1$ then $\epsilon = d_p$ and $K(C)(\beta \circ \alpha) = d_p$.
If $q' = p - 1$ then $K(C)(\alpha, \eta') \circ K(C)(\beta, \eta) = Id \circ d_p$.
If $q' = p$ then $K(C)(\alpha, \eta') \circ K(C)(\beta, \eta) = d_p \circ Id$.



[7, page 30]

In any of above cases we have

$$K(C)(\beta \circ \alpha, \eta) = K(C)(\alpha, \eta') \circ K(C)(\beta, \eta)$$

Now if $q = p$, then we have $q = q' = p$ and $\eta = \eta'$.So,

$$K(C)(\beta \circ \alpha, \eta) = Id = K(C)(\alpha, \eta') \circ K(C)(\beta, \eta)$$

Thus K is a functor

$$K : Ch_{\geq 0}(\mathcal{A}) \longrightarrow Simp(\mathcal{A}).$$

Claim: K is inverse to N.
Justification: For any surjection
$\eta : [n] \twoheadrightarrow [p],\ p \leq n$ in Δ there is a unique epi-monic factorisation of η. So $\eta = \sigma_{i_1} \cdots \sigma_{i_t}$.
If $n \neq p$, then we have

$$C_\eta = (s_{i_t} \cdots s_{i_1}) C_{Id_p}$$

where s_{i_j} are the degeneracy maps of simplicial object $K(C)$, which implies $C_\eta \in DK(C)$.
Now consider the case where $n = p$

$$d_i \mid_{C_{Id_n}} = K(C)(\partial_i, Id_n) = d, \quad if\ i = n$$
$$= 0, \quad else$$

Thus $N_n(K(C)) = C_{Id_n} = C_n$. So it proves that N is inverse to K.
Now it remains to prove that

$$KN \approx Id_{Simp(\mathcal{A})}.$$

For any simplicial object $A \in Simp(\mathcal{A})$, we define

$$\psi_n : K_n N(A) \longrightarrow A_n$$

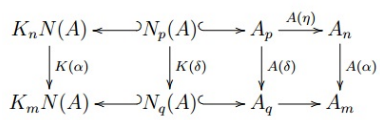
where ψ_n is defined on its restrictions to summand $K_nN(A)$.
For each

$$\eta : [n] \longrightarrow [p], \quad for\ p \leq n$$

the corresponding summand is $N_\eta(A) = N_P(A)$ which is a subobject of A_p.
$\psi_n \mid_{N_\eta(A)}$ is defined to be the composition

$$N_\eta(A) = N_P(A) \hookrightarrow A_p \longrightarrow A_n.$$

For any given map $\alpha : [m] \longrightarrow [n]\ in\ \Delta$, let $\delta \circ \sigma$ be the unique epimonic factorization of $\eta \circ \alpha$.
Then the diagram



[7, page 31]

Commutes and ψ is a simplicial map, which is natural in A.
Now our aim is to prove that ψ_n is an isomorphism.

$$\psi_0 : K_0N(A) \longrightarrow A_0$$

So $K_0(N(A)) = N_0(A) = A_0$. Thus $\psi_0 : A_0 \longrightarrow A_0$ is an isomorphism.
Now suppose ψ_k is an isomorphism for all $k < n$. Since $A_n = N_n(A) \bigoplus D_n(A)$, So $N_n(A)$ is in the image of ψ_n. Because the way we have defined ψ_n, it comes out that $\psi_n(N_n(A))$ is simply the inclusion $N_n(A) \hookrightarrow A_n$. Now take some $z \in D_n(A)$
Thus $z = s_i x$ for some $1 \leq i \leq n-1;\ x \in A_{n-1}$. Since ψ_{n-1} is an isomorphism, so $x \in Im(\psi_{n-1})$ which implies that $z = s_i x \in (\psi_n)$, Thus the map ψ_n is surjective.
Claim: ψ_n is injective.
Justification: Let $(x_\eta) \in K_n(N(A))$ such that $\psi_n(x_\eta) = 0$.
Now $x_\eta \in N_\eta(A) \quad for\ \eta : [n] \twoheadrightarrow [p]$.
If $p = n$, Then $(x_\eta) = x_{Id} = 0$ as $\psi_n(N_n(A))$ is simply the inclusion $N_n(A) \hookrightarrow A_n$.
Now suppose $p < n$ then there exist $\epsilon : [p] \longrightarrow [n]\ such\ that\ \epsilon \circ \eta = Id_n\ and\ \eta \circ \epsilon = Id_p$.
$K(\epsilon, \eta)(x_\eta) = x_{Id_p} \in K_p(N(A))$ and since $\psi_p(x_{Id_p}) = 0$ and ψ_p is an isomorphism so x_{Id_p}

$= 0$, It follows that $x_\eta = 0$, and thus ψ_n is injective. Thus

$$KN \approx Id_{Simp(\mathcal{A})}.$$

Remark 3.14 Under the correspondence on N, simplicial homotopic maps in $Simp(\mathcal{A})$ correspondence to chain homotopic maps in $Ch_{\geq 0}(\mathcal{A})$.

Construction of Eilenberg-Maclane spaces

Dold-Kan correspondence enables us to construct Eilenberg-Maclane spaces of all type. Let X be a simplicial set in abelian category and let (A, n) be a chain complex concentrated in degree n, where A is an abelian group, Then by Dold-Kan correspondence we have,

$$Hom_{Ch\geq 0}(N(X), (A, n)) \cong Hom_{Simp(\mathcal{A})}(X, K(A, n))$$

$$\pi_i(K, (A, n)) = H_i((A, n), \mathbb{Z}) = \begin{cases} A & if\ k = n \\ 1 & otherwise \end{cases}$$

Thus $K(A, n)$ is an Eilenberg-Maclane space of type n.

Till now we have simplicial homotopy theory and have proved that, we can use simplicial homotopy theory in place of classical homotopic theory. Simplicial homotopy will be helpful to survey the dimension quotient problem in group theory. The main result of chapter 3 is Dold-Kan correspondence which enables us to construct Eilenberg-Maclane spaces of all type n.

Chapter 4

Dimension subgroups

In this chapter, we will survey some results on dimension subgroups. Using group theoretical properties we will see that for all groups second and third term of dimension series are equal to the second and the third term of lower central series respectively. We will also see that exponent of fourth dimension quotient is 1 or 2.

Definition 4.1 *Let G be a group and R be a commutative ring with identity. The group ring $R[G]$ of G over R is given by formal sums*

$$\sum \alpha(g)g, \qquad \alpha(g) \in R,\ g \in G$$

with only finitely many $\alpha(g)$ being non-zero. The addition and multiplication are defined as follows:

$$\sum_{\alpha(g)\in R, g\in G} \alpha(g)g + \sum_{\beta(g)\in R, g\in G} \beta(g)g \quad = \quad \sum \{\alpha(g)+\beta(g)\}g$$

$$\{\sum_{\alpha(g)\in R, g\in G} \alpha(g)g\}\{\sum_{\beta(h)\in R, h\in G} \beta(h)h\} = \sum_{x\in G}\{\sum_{gh=x} \alpha(g)\beta(h)\}x \tag{4.1}$$

Above two operations makes $R[G]$ a ring.
If $R = \mathbb{Z}$ then $\mathbb{Z}[G]$ is called integral group ring of G.

Definition 4.2 *The map*

$$\epsilon : \mathbb{Z}[G] \longrightarrow \mathbb{Z}$$

$$\sum \alpha(g)g \mapsto \sum \alpha(g)$$

is called the augmentation map and $Ker\epsilon = \Delta(G)$ *is called the augmentation ideal of* $\mathbb{Z}[G]$.

Definition 4.3 *Let* G *be a group and define:*

$$D_n(G) = \{g \in G \mid g - 1 \in \Delta^n\}$$

then, we gets a sequence

$$G = D_1(G) \supseteq D_2(G) \supseteq \cdots$$

of normal subgroups of G *with the property that*

$$[D_n(G), D_m(G)] \subseteq D_{n+m}(G) \quad n, m \in \mathbb{N}$$

The sequence $\{D_n(G)\}_{n\geq 1}$ *which we have obtained is called dimension series of a group* G.

Definition 4.4 *Let* G *be a group and define:*

$$\gamma_1(G) = G, \gamma_n(G) = [G, \quad \gamma_{n-1}(G)], \; n \in \mathbb{N}.$$

We get

$$\gamma_1(G) \supseteq \gamma_2(G) \supseteq \cdots$$

a series of subgroups of G *with the property that*

$$[\gamma_n(G), \quad \gamma_m(G)] \subseteq \gamma_{n+m}(G), \; n, m \in \mathbb{N}$$

The sequence of subgroups which we have obtained is called lower central series of a group G.

Remark 4.5

$$\gamma_n(G) \subseteq D_n(G) \quad for \quad all \quad n \in \mathbb{N}.$$

Definition 4.6 *A filtration of* $\Delta(G)$ *is a sequence*

$$\Delta(G) = I_1 \supseteq I_2 \supseteq \cdots$$

of ideals of $\mathbb{Z}G$ *with the property that*

$$I_n I_m \subseteq I_{n+m}$$

So, clearly $\{\Delta^n\}_{n=1}^{\infty}$ *is a filtration of* $\Delta(G)$.

For any ideal I of ℤG, we define

$$\partial(I) = \{x \in G : x - 1 \in I\}.$$

Definition 4.7 *A sequence*

$$G = H_1 \supseteq H_2 \supseteq \cdots$$

is called an N-series for G if

$$[H_n \quad H_m] \subseteq H_{n+m}(G) \quad n, m \in \mathbb{N}.$$

Remark 4.8 (i) If $\{ I_n\}_{n=1}^{\infty}$ is a filtration for $\Delta(G)$ then $\{\partial(I_n)\}_{n=1}^{\infty}$ is an N-series for G.
(ii) Lower central series is most rapidly decreasing N-series for G.

Lazard's Problem For a given group G and a given N-series $\{H_n\}_{n=1}^{\infty}$ of G, does there always exist a filtration $\{I_n\}_{n=1}^{\infty}$ of $\Delta(G)$ such that

$$\partial(I_n) = H_n; \ \forall \ n \in \mathbb{N}$$

Definition 4.9 *Let G be a group and $\{H_n\}_{n=1}^{\infty}$ be a N-series for G. If there exist a filtration $\{I_n\}_{n=1}^{\infty}$ of $\Delta(G)$ such that*

$$\partial(I_n) = H_n; \ \forall \ n \in \mathbb{N}$$

then $\{I_n\}_{n=1}^{\infty}$ is called Lazard filtration of $\Delta(G)$ relative to given N-series.

Definition 4.10 *Let G be a group and $\{H_n\}_{n=1}^{\infty}$ be an N-series for G, then this N-series induces a weight function on G*

$$\omega(x) = \begin{cases} k, & if \ x \in H_k \backslash H_{k+1} \\ \infty, & if \ x \in \cap_k H_k \end{cases}$$

Definition 4.11 *Let G be a group and $\{H_n\}_{n=1}^{\infty}$ be an N-series for G. Define Λ_k to be the span over ℤ of the product*

$$(g_1 - 1)(g_2 - 1) \cdots (g_s - 1) \ \text{ with the property that } \sum_{i=1}^{s} \omega(g_i) \geq k.$$

Clearly $\Lambda_1 = \Delta(G)$. Note that each Λ_k is an ideal of $\mathbb{Z}G$ and $\Lambda_i\Lambda_j \subseteq \Lambda_{i+j}\ \forall\ i, j \geq 0$. Thus $\{\Lambda_n\}_{n=1}^{\infty}$ forms a filtration of $\Delta(G)$ and this filtration is called canonical filtration of $\Delta(G)$ induced by the N-series $\{H_n\}_{n=1}^{\infty}$.

Lemma 4.12 *[5] Let G be a group and $\{H_n\}_{n=1}^{\infty}$ be an N-series for G, if there exists a Lazard filtration $\{I_n\}_{n=1}^{\infty}$ of $\Delta(G)$ then $\{\Lambda_n\}_{n=1}^{\infty}$ is the smallest Lazard filtration.*

Proof Since $\{I_n\}_{n=1}^{\infty}$ is a Lazard filtration of $\Delta(G)$ relative to N-series $\{H_n\}_{n=1}^{\infty}$. So we have

$$\partial(I_n) = H_n \quad for\ all\ n.$$

Let $g_1, g_2, \cdots g_s, \sum_{j=1}^{s} \omega(g_j) \geq k$. Then

$$(g_1 - 1)(g_2 - 1)\cdots(g_s - 1) \in I_{\omega(g_1)} I_{\omega(g_2)} \cdots I_{\omega(g_s)} \subseteq I_k.$$

Hence $\Lambda_k \subseteq I_k$ for all k. If $x \in H_k$ then $\omega(x) \geq k$ and so $x - 1 \in \Lambda(k)$. Thus

$$H_k \subseteq \partial(\Lambda_k) \subseteq \partial(I_k) = H_k.$$

Theorem 4.13 *[5] Let G be a group and $\{H_n\}_{n=1}^{\infty}$ be an N-series for G and $\{\Lambda_n\}_{n=1}^{\infty}$ be the canonical filtration of $\Delta(G)$ relative to given N-Series, then*

1. $\partial(\Lambda_2) = H_2$
2. $\partial(\Lambda_3) = H_3$.

Remark 4.14 By above two theorems, we conclude, in particular, that $D_2(G) = \gamma_2(G)$ and $D_3(G) = \gamma_3(G)$. But problem comes at fourth level and Rips gave first counterexample.

Example 4.15 [Rips:72][13]
Let G be a group with generators

$$a_0, a_1, a_2, a_3, b_1, b_2, b_3, c$$

and defining relations

$$b_1^{64} = b_2^{16} = b_3^{4} = c^{256} = 1.$$

$$[b_2,\ b_1] = [b_3,\ b_1] = [b_3,\ b_2] = [c,\ \ b_1] = [c,\ b_2] = [c,\ b_3] = 1.$$

$$a_0^{64} = b_1^{32},\ a_1^{64} = b_2^{-4}b_3^{-2},\ a_2^{16} = b_1^4 b_3^{-1},\ a_3^4 = b_1^2 b_2.$$

$$[a_1,\ \ a_0] = b_1c^2,\ [a_2,\ \ a_0] = b_2c^8,\ [a_3,\ \ a_0] = b_3c^{32},$$

$$[a_2,\ \ a_1] = c_1,\ [a_3,\ \ a_1] = c^2,\ [a_3,\ \ a_2] = c^4,$$

$$[b_1,\ \ a_1] = c^4,\ [b_2,\ \ a_2] = c^{16},\ [b_3,\ \ a_3] = c^{64}.$$

$$[b_i,\ \ a_j] = 1,\ \ \ if\ \ \ i \neq j$$

$$[c,\ \ a_i] = 1,\ \ \ for\ i = 0, 1, 2, 3$$

Then $\gamma_4(G) = 1$, while the element

$$[a_1,\ \ a_2]^{128}[a_1,\ \ a_3]^{64}[a_2,\ \ a_3]^{32} = c^{128} \in D_4(G)\ and\ c^{128} \neq e.$$

Since we have seen a group G in which

$$D_4(G) \neq \gamma_4(G).$$

Thus in general, the answer to Lazard's problem is **NO**.

Structure of $D_4(G)$(see[13]) Let G be a nilpotent group of class 3 given by its pre-abelian presentation

$$< x_1, x_2 \cdots x_m \mid x_1^{d(1)}\xi_1, \cdots x_k^{d(k)}\xi_k, \xi_{k+1}, \cdots \gamma_4(< x_1, x_2 \cdots x_m >) >$$

with $k \leq m,\ d(i) > 0,\ d(k) \mid \cdots d(2) \mid d(1)\ and\ \xi_i \in \gamma_2(< x_1, x_2 \cdots x_m >)$

Then $D_4(G)$ consists of all elements of the form

$$\omega = \prod_{1 \leq i < j \leq k} [x_i^{d(i)},\ \ x_j]^{a_{ij}},\ \ \ a_{ij} \in \mathbb{Z}$$

such that, $d(j) \mid \binom{d(i)}{2} a_{ij},\ \ \ (1 \leq i < j \leq m)$ and

$$y_l = \prod_{1 \leq i < l} x_i^{-d(i)a_{il}} \prod_{l < j \leq k} x_j^{d(l)a_{lj}} \in \gamma_2(G)^{d(l)}\gamma_3(G)\ \ \ for\ 1 \leq l \leq k.$$

Remark 4.16 If in structure of $D_4(G)$, $m \leq 3$, then

$$D_4(G) = \gamma_4(G)[12].$$

Theorem 4.17 *[12]*

Let $G =< X \mid r_1, r_2 >$ be a 2-relator group then $D_4(G) = \gamma_4(G)$.

Proof G has a pre-abelian presentation of the form

$$G =< x_1, x_2, \cdots x_n, \cdots \mid x_1^{d(1)}\xi_1, x_2^{d(2)}\xi_2, \xi_3, \cdots >$$

with $\xi_i \in \gamma_2 < x_1, \cdots >$ *and* $d(2) \mid d(1)$.

Then modulo $\gamma_4(G)$, the group $D_4(G)$ consists of the elements of the form

$$\omega = [x_1^{d(1)}, \quad x_2]^{a_{12}},$$

such that, $d(2) \mid \binom{d(1)}{2} a_{12}$,

and

$$y_2 = x_1^{-d(1)a_{12}} \in \gamma_2(G)^{d(2)}\gamma_3(G).$$

Therefore, modulo $\gamma_4(G)$, for some $z \in \gamma_2(G)$, we have

$$\omega = [x_1^{d(1)a_{12}}, \ x_2] = [y_2^{-1}, \ x_2] = [z^{-d(2)}, \ x_2] = [z, \ x_2^{-d(2)}] = 1.$$

Example 4.18 **4-generator, 3-relator**: Let G be a group defined by the presentation $< x_1, x_2, x_3, x_4 \mid x_1^4[x_4, \ x_3]^2[x_4, \ x_2] = 1, x_2^{16}[x_4, \ x_3]^4[x_4, \ x_1]^{-1} = 1, \ x_3^{64}[x_4, \ x_2]^{-4}[x_4, \ x_1]^{-2} = 1 >$.

then $\omega = [x_1, \ x_2^{32}][x_1, \ x_3^{64}][x_2, \ x_3^{128}] \in D_4(G) \diagdown \gamma_4(G)$.

Lemma 4.19 *[12] Let G be a group. If $x_1, x_2, x_3 \in G$ and there exists $\xi_j \in \gamma_2(G), \ j = 1, \cdots 6$ and $\eta_i \in \gamma_3(G)$, such that*

$$x_1^4 = \xi_1, \ x_2^{16} = \xi_2, \ x_2^{32}x_3^{64} = \xi_4^4\eta_1$$

$$x_1^{-32}x_3^{128} = \xi_5^{16}\eta_2, \ x_1^{-64}x_2^{-128} = \xi_6^{64}\eta_3,$$

then

$$\omega = [x_1 \ \ x_2^{32}][x_1 \ \ x_3^{64}][x_2 \ \ x_3^{128}] \in D_4(G).$$

Proof Since $\gamma_2(G) \subseteq 1 + \Delta^2(G)$, we have

$$1 - \omega \equiv \alpha_1 + \alpha_2 + \alpha_3 \ mod \Delta^4(G)$$

where $\alpha_1 = (1 - [x_1, \ x_2^{32}]), \ \alpha_2 = (1 - [x_1, \ x_3^{64}]), \ \alpha_3 = (1 - [x_2, \ x_3^{128}])$. Now working Modulo $\Delta^4(G)$, we have

$$\alpha_1 \equiv (1 - x_2^{32})(1 - x_1) - (1 - x_1^{32})(1 - x_2)$$

$$\alpha_2 \equiv (1 - x_3^{64})(1 - x_1) - (1 - x_1^{64})(1 - x_3)$$

$$\alpha_3 \equiv (1 - x_3^{128})(1 - x_2) - (1 - x_2^{128})(1 - x_3)$$

therefore,

$$\alpha_1 + \alpha_2 + \alpha_3 \equiv 0$$

and hence $\omega \in D_4(G)$.

Remark 4.20 Simplicial methods are helpful in study of group rings. For the details see ([13]).

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