Modular Dynamical System and Modular Flow

Atul Verma

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Certificate of Examination

This is to certify that the dissertation titled **"Modular Flows and Modular Dynamics"** submitted by **Atul Verma** (Reg. No. MS09034) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Mahender Singh   Dr. Krishnendu Gongopadhyay   Prof. Kapil H. Paranjape
(Supervisor)

Dated: May 1, 2014

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. Kapil H. Paranjape at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions.

Atul Verma
(MS09034)

Dated: May 1, 2014

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Kapil H. Paranjape
(Supervisor)

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Chapter 1

Introduction

1.1 Background

The study of periodic orbits in dynamical systems is a basic problem with a long history. In variety of examples one would like to find periodic orbits and to study their general structure. This is of great importance, for instance, in taking the semi-classical limit of dynamical systems. We are interested in the case of flows in three dimensional manifolds. Periodic orbits of such a flow is an embedding of S^1 into the 3-manifold, hence the flow is 3-dimensional, periodic orbits are called knots and the collection of these orbits forms a link, which is often non-trivial. Thus one can ask which knot type arises as orbits for a certain flow [13].

We start with the first introduction to the topic of flows on homogeneous spaces and proceed further to finding knots in S^3.

1.2 Homogeneous Spaces

Let G be a locally compact second countable group, for example:

1. $\mathbb{R}$.

2. The general linear group $GL(n, \mathbb{R})$ consisting of all non singular $n\times\ n$ matrices with real entries.

3. Closed subgroups of known locally compact groups, quotient of locally compact groups by closed normal subgroups.

Let S be a locally compact Hausdorff space. A left *action* of G on S is a continuous map $(x,s) \mapsto xs$ from $G \times S$ to S such that:

1. $s \mapsto xs$ is a homeomorphism of S for each $x \in G$ and,

2. $x(ys) = (xy)s$ for all $x,y \in G$ and $s \in S$.

A space equipped with an action of G is called a *G-space*. A *G-space* is **transitive** if for every $s,t \in S$ there exists $x \in G$ such that $xs = t$.

The standard examples of transitive *G-spaces* are the quotient spaces G/H (where H is a closed subgroup of G), on which G acts by left multiplication,

$$H = \left\{ g \in G \mid gx_0 = x_0 \right\}$$

is the stabilizer of x_0.

By **homogeneous space** we shall mean a transitive *G-space* S that is isomorphic to quotient space G/H that is, if S is a transitive *G-space* pick $s_0 \in S$, define $\phi : G \mapsto S$ by $\phi(x) = xs_0$, and let

$$H = \left\{ x \in G \mid xs_0 = s_0 \right\}.$$

Then H is a closed subgroup of G and ϕ is a continuous surjection of G onto S that is constant on the left cosets of H. Hence ϕ induces a continuous bijection $\Phi : G/H \to S$ such that $\Phi \circ q = \phi$, where q is the natural quotient map. The only additional thing needed to identify S with G/H is the continuity of Φ^{-1}, which is always not the case, for example, consider the case where $G = \mathbb{R}$ with discrete topology, acting by translation on $\mathbb{R}$ with the usual topology. But it is valid if it is σ-compact.

Proposition 1.1. *In the above context if G is σ-compact then Φ is a homeomorphism.*

Proof. It is sufficient to show that ϕ maps open sets in G to open sets in S. Suppose U is open in G and $x_0 \in G$; pick a compact symmetric neighbourhood V of 1 such that $x_oVV \subset U$. Since G is σ- compact set, there is a countable set $\{y_n\} \subset G$ such that the sets y_nV cover G. Then $S = \bigcup_1^\infty \phi(y_nV)$ are all homeomorphic to $\phi(V)$ since $s \mapsto y_ns$ is a homeomorphism of S, and they are compact and hence closed. By the Baire category theorem for locally compact Hausdorff spaces, $\phi(V)$ must have an interior point, say $\phi(x_1)$ $(x_1 \in V)$. But then $\phi(x_0)$ is an interior point of $\phi(x_0x^{-1}V)$, and $x_0x^{-1}V \subset x_0VV \subset U$, so $\phi(x_0)$ is an interior point of $\phi(U)$. Thus $\phi(U)$ is open [8]. ☐

Thus if the map Φ is a homeomorphism, we shall identify S with G/H. Henceforth, we consider homogeneous spaces G/H, where G is an arbitrary locally compact group and H is an arbitrary closed subgroup. The question here is to address whether there is G-invariant Radon measure on G/H, that is, a Radon measure λ such that $\lambda(xE) = \lambda(E)$ for every x $\in G$. The answer is not always affirmative. We obtain a necessary and sufficient condition for the existence of invariant measure [8].

Suppose that $f : X \mapsto \mathbb{R}$ is a real-valued function whose domain is an arbitrary set X. The support of f, denoted by supp(f), is the set of points in X where f is non-zero

$$supp(f) = \{x \in X \mid f(x) \neq 0\}.$$

The support of f is the smallest subset of X with the property that f is zero on its complement. Functions with compact support on a topological space X are those whose support is a compact subset of X.

Let G be a locally compact group, we denote the space of compactly supported continuous functions on G by $C_c(G)$, we set

$$C_c^+(G) = \{f \in C_c(G) : f \geq 0 \text{ and } f \neq 0\}.$$

1.3 Measures on homogeneous spaces

Definition 1.2. *A **Haar measure** on G, a locally compact Hausdorff topological group is a non-zero Radon measure $\mu : \Sigma \to [0, \infty)$, with Σ a σ-algebra containing all Borel subsets of G, such that*

1. $\mu(G) = 1$,
2. $\mu(\alpha S) = \mu(S)$ *for all* $\alpha \in G$, $S \in \Sigma$. *Here* $\alpha S = \{\alpha\, t \mid t \in S\}$.

Theorem 1.3 (Uniqueness Theorem). *If λ and μ are Haar measures then there exists $c \in (0, \infty)$, such that $\lambda = c\mu$.*

Proof. For detailed proof see [8]. □

Definition 1.4. *If G is as above with left Haar measure μ. We will now examine the extent to which μ fails to be a right in variant. If, for $x \in G$, we define $\mu_x(E) = \mu(Ex)$, then μ_x is again a left Haar measure, by the associative law: $y(Ex) = (yE)x$. By the uniqueness theorem there is there is a $\triangle(x) > 0$ such that $\mu_x = \triangle(x)\mu$, and $\triangle(x)$ is independent of the original choice of μ. The function $\triangle : G \mapsto (0, \infty)$ thus defined is called the **modular function** of G.*

Proposition 1.5. *If K is any compact subgroup of G then $\triangle|K \equiv$ 1.*

Thus $\mathbb{R}^n/\mathbb{Z}^n$ admits an $\mathbb{R}^n$-invariant measure, and $\mathbb{R}^n\backslash$ (0) admits a measure invariant under the action of group $SL(n, \mathbb{R})$ of n $\times$ n with determinant 1. In particular if G is a unimodular locally compact group and Γ is a discrete subgroup of G then G/Γ admits a G-invariant measure, if the quotient G the quotient G/Γ is compact then automatically the measure is finite, but in general the invariant measure may not be finite. These observations apply in particular to $G = SL(n.\mathbb{R})$ [8].

Theorem 1.6. *Every locally compact group G possesses a left Haar measure μ*

Proof. We will start by constructing μ as a linear functional on $C_c(G)$. imagine a function $\phi \in C_c^+(G)$ that is bounded by 1, equals 1 on a small open set and is supported on a very slightly larger open set U. If $f \in C_c^+(G)$ is sufficiently slow varying so that it is essentially constant on the left translates of U, f can be well approximated by a linear combination of left translates of ϕ. $f \approx \sum c_j L_{x_j}\phi$. If μ were a Haar measure on G, we would have $\int f d\mu \approx (\sum c_j) \int \phi d\mu$. This approximation will get better and better if support of ϕ shrinks to a point, and if we introduce a normalization to cancel out the factor of $\int \phi d\mu$ on the right we will obtain $\int f d\mu$ as a limit of the sums $\sum c_j$. To understand clearly consider the case $G = \mathbb{R}$: ϕ is essentially the characteristic function of a small interval, $f \approx \sum c_j L_{x_j}\phi$ is approximation of f by step functions and $\approx (\sum c_j) \int \phi d\mu$ is essentially a Riemann sum for $\int f d\mu$. To be more precise if $f, \phi \in C_c^+(G)$, we define $(f : \phi)$ to be the infimum of all finite sums $\sum_1^n c_j$ such that $f \leq \sum_1^n c_j L_{x_j}\phi$ for some $x_1, \ldots, _n \in G$. This makes sense because the support of f can be covered by some finite number N of left translates of the set where $\phi \geq \frac{1}{2}\|\phi\|_{sup}$ and it follows that $(f : \phi) \leq 2N\|f\|_{sup}/\|\phi\|_{sup}$. □

What follows, G is a locally compact group with left *Haar* measure $d\mu$, H is a closed subgroup of G with left *Haar* measure $d\xi : G \mapsto G/H$, q is the canonical quotient map $q\mu = \mu H$ and $\triangle_G$ and $\triangle_H$ are the modular functions of G and H. We define a map $P : C_c(G) \mapsto C_c\ (G/H)$ by

$$Pf(\mu H) = \int_H f(\mu\xi) d\xi.$$

This is well defined by the left -invariance of $d\mu$: if $\{y = \mu\eta\}$ with $\eta \in H$ then

$$\int_H \mathrm{f}(y\xi)\, d\xi = \int_H \mathrm{f}(\mu\xi)\, d\xi.$$

Pf is continuous. Moreover, if $\phi \in C_c(G/H)$ we have

$$P[(\phi \circ q)\mathrm{f}] = \phi P\mathrm{f}.$$

We now show that P maps $C_c(G)$ onto $C_c(G/H)$ [8].

Lemma 1.7. *If $E \subset G/H$ is compact, there exists a compact $K \subset G$ with $q(K) = E$.*

Proof. Pick an open neighbourhood V of 1 in G with compact closure. Since q is an open map, the sets $q(xV)$ $(x \in G)$ are an open cover of E, so there is finite subcover qx_jV $(j = 1, \ldots, n)$. Let $K = q^{-1}E \cap \bigcup_1^n x_j\overline{V}$. Since $q^{-1}(E)$ is closed, K is compact and $q(K) = E$ [8]. □

Lemma 1.8. *If $F \subset G$, there exists* f ≥ 0 *in $C_c(G)$ such that P*f $= 1$ *on F.*

Proof. Let E be a compact neighbourhood of F in G/H, and by using lemma 1.3 we can obtain a compact set $K \subset G$ such that $q(K) = E$ Choose non-negative $g \in C_c(G)$ with $g \geq$ 0on K and $\phi \in C_c(G)$ supported in E such that $\phi = 1$ on F and set

$$f = \frac{\phi \circ q}{P_g \circ q}$$

with the understanding that the fraction is zero wherever the numerator is zero. f is a continuous function since $P_g \geq 0$ on support ϕ, its support is contained in support g and $Pf = (\phi/P_g)P_g = \phi$ [8]. □

Proposition 1.9. *If $\phi \in C_c(G/H)$, there exists* f $\in C_c(G)$ *such that P*f $= \phi$ *and* f ≥ 0 *if $\phi \geq 0$.*

Proof. If $\phi \in C_c(G/H)$, by lemma 1.8 there exists $g \geq 0$ in $C_c(G)$ such that Pg $= 1$ on support ϕ. Let $=$ f $(\phi \circ q)g$. We have Pf $= \phi(P$g $= \phi)$. □

The following condition describes a necessary and sufficient condition for G/H to admit a measure invariant under the action of G; by a measure we shall mean a Radon measure, namely a measure defined on all Borel subsets which assigns finite measure to every compact set. For any closed subgroup H of G, including G itself, we denote by $\triangle_H$ the modular homomophism of H. Then we have the following:

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Theorem 1.10. *G/H admits a G-invariant measure if and only if $\triangle_G(h) = \triangle_H(h)$ for all $h \in H$; when it exits the invariant measure is unique up to scaling. In this case, μ is unique up to a constant factor, and if this factor is suitably chosen we have*

$$\int_G \mathrm{f}(x)dx = \int_{G/H} P\mathrm{f} d\mu = \int_{G/H}\int_H \mathrm{f}(x\xi)d\xi d\mu(xH).$$

Proof. [8] Suppose a G-invariant measure μ exists. Then $\mathrm{f} \mapsto \int P\mathrm{f}\ d\mu$ is a nonzero left invariant positive linear functional on $C_c(G)$, so $\int P\mathrm{f}\ d\mu = \mathrm{c}\int \mathrm{f(x)dx}$ for some c > 0 by the uniqueness of Haar measure on G. In view of the above proposition, this formula completely determines μ, so μ is unique up-to the arbitrary constant factor in Haar measure. Replacing μ by $c^{-1}\mu$ we may assume c $= 1$, so that our condition given in theorem satisfies. This being the case if $\eta \in H$ and $\mathrm{f} \in C_c(G)$ we have

$$\begin{aligned}\triangle_G(\eta)\int_G \mathrm{f}x dx &= \int_G \mathrm{f}(x\eta^{-1})dx\\ &= \int_{G/H}\int_H \mathrm{f}(x\xi\eta^{-1})d\xi d\mu xH\\ &= \triangle_H(\eta)\int_{G/H}\int_H \mathrm{f}(\mathrm{x}\xi)d\xi d\mu xH\\ &= \triangle_H(\eta)\int \mathrm{f}(x)dx\end{aligned}$$

so that $\triangle_G(\eta) = \triangle_H(\eta)$.

Conversely, suppose $\triangle_G \mid H = \triangle_H$. We claim that if $\mathrm{f} \in C_c(G)$ and $P\mathrm{f} = 0$ then $\int P\mathrm{f(x)}\ dx = 0$. Indeed, by lemma 1.3 there exists $\phi \in C_c(G)$ such that $P\phi = 1$ on $q(supp\mathrm{f})$. We have

$$0 = P\mathrm{f}(xH) = \int \mathrm{f}(x\xi)d\xi = \int \mathrm{f}(x\xi^{-1})\triangle_H(\xi^{-1})d\xi = \int \mathrm{f}(x\xi^{-1})\triangle_G(\xi^{-1})d\xi,$$

so

$$\begin{aligned}0 &= \int_G\int_H \phi(x)\mathrm{f}(x\xi^{-1})\triangle_G(\xi^{-1})d\xi dx\\ &= \int_H\int_G \phi(x)\mathrm{f}(x\xi^{-1})\triangle_G(\xi^{-1})dx d\xi\\ &= \int_H\int_G \phi(x\xi)\mathrm{f}(x)dx d\xi\\ &= \int_G P\phi(xH)\mathrm{f}(x)dx = \int_G \mathrm{f(x)}dx.\end{aligned}$$

This means that if Pf $= P$g then $\int_G$ f $= \int_G$ g. Thus it follows from the proposition that the map Pf $\mapsto \int_G$ f is a well defined G-invariant positive linear functional on $C_c(G/H)$. The associated Radon measure is then the desired measure μ. □

Corollary 1.11. *If H is compact, G/H admits a G-invariant Radon measure.*

Proof. To prove this we use the proposition given below. For detailed proof see [8]. □

1.4 Introduction to Dynamical Systems

Definition 1.12. *Let G be a locally compact group. A closed subgroup Γ of G is called a lattice in G if Γ is discrete and G/Γ admits a finite G-invariant measure.*

For example: In G=$SL(n,\mathbb{R})$, $\Gamma = SL(n,\mathbb{Z})$ is a lattice in G.

1.4.1 Flows

Let G be a locally compact group and Γ a lattice in G. For a closed subgroup H of G the H-action on G/Γ is called the flow induced by H on G/Γ. We are interested in actions of cyclic subgroups (equivalently of elements of G), or one-parameter flows, namely actions induced by (continuous) one-parameter subgroups g_t where $g_t \in G$ for all t $\in \mathbb{R}$.

1.4.2 Example

$G = SL(2,\mathbb{R})$, Γ a lattice in G, and

$$H = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right\}.$$

This corresponds to what is called the geodesic flow associated with the surface $\mathbb{H}^2/\Gamma$ where $\mathbb{H}^2$ is the Poincare upper half-plane. In particular when $\Gamma = SL(2,\mathbb{Z})$ it corresponds to the geodesic flow associated with the modular surface.

We can show that every element of G is either conjugate to a diagonal or an upper triangular unipotent matrix, or is contained in a compact subgroup of G; in the latter case it acts as a "rotation" of the plane, with respect to a suitable choice of the basis.

1. **Case I** when a element of $SL(2,\mathbb{R})$ is conjugate to a diagonal matrix $D = \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$: Take $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a element of $SL(2,\mathbb{R})$, then if g is conjugate to a then $gag^{-1} \in SL(2,\mathbb{R})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} adr - bc/r & -abr + ab/r \\ cdr - dc/r & -bcr + ad/r \end{pmatrix},$$

   the determinant of above matrix turns out to be $(ad - bc)^2$, which is 1.

2. **Case II** when a element of $SL(2,\mathbb{R})$ is conjugate to a triangular matrix $T = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, taking g as above and calculating as in above case

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - acx - bc & -ab + a^2x + ba \\ cd - c^2x - dc & -bc + acx + ad \end{pmatrix},$$

   the determinant of T also turns out to be $(ad - bc)^2 = 1$.

3. **Case III** In this case where the conjugate has to be a compact subgroup of G we can choose $C = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ as a representative, then similarly as above

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a\cos \end{pmatrix}$$

$$= \begin{pmatrix} ad\cos\theta + ac\sin\theta + bd\sin\theta - bc\cos\theta & -ab\cos\theta - a^2\sin\theta - b^2\sin\theta + ab\cos\theta \\ cd\cos\theta + c^2\sin\theta + d^2\sin\theta - dc\cos\theta & -bc\cos\theta - dc\sin\theta - bd\sin\theta + d^2\cos\theta \end{pmatrix},$$

   the determinant of C also turns out be $(ad - bc)^2 = 1$.

Chapter 2

Lattice Invariants

In this chapter we lay the foundations to study flow on homogeneous spaces taking $Sl(2,\mathbb{R})\backslash SL(2,\mathbb{Z})$ as an example.

Definition 2.1. *A Lattice L is the collection of all integer linear combination of a pair of linearly independent vectors,i.e*

$$L = \left\{ m \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + n \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} : m, n \in \mathbb{Z} \right\}$$

Another interpretation of the lattice L can be the image of multiplication by a matrix as follows:

$$\begin{pmatrix} m \\ n \end{pmatrix} \mapsto m \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + n \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}$$

such that the condition of linear independence is not violated, i.e.the determinant of the matrix is non zero $x_1x_2 - y_1y_2 \neq 0$ This lattice can also be given by a different matrix

$$\begin{pmatrix} kx_1 + lx_2 & mx_1 + nx_2 \\ ky_1 + ly_2 & my_1 + ny_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} k & l \\ m & n \end{pmatrix}$$

where $k, l, m, n \in \mathbb{Z}$ such that $kn - ml = \pm 1$

2.1 Group Theoretic Approach

Another approach to understand the structure of lattices can be in terms of group theory. We know that the collection of $GL(2,\mathbb{R})$ of $2\times$ 2 matrices with non-zero determinant forms a group. The subset $GL(2,\mathbb{Z})$ of $GL(2,\mathbb{R})$ which consists of matrices with integer entries and the determinant as ± 1 is a subgroup. As it is the group of invertible matrices over ring of integers,since determinant is multiplicative therefore only invertible integers are ± 1. So if we look at the definition of lattices above, we can identify the collection of lattices with the set $\Big\{GL(2,\mathbb{R})/GL(2,\mathbb{Z})\Big\}$ which is in-fact a *coset space* [9].

2.2 Fundamental Parallelogram

Let u, v, be the vectors in the plane. Then $0, u, v, u+v$ form the vertices of a fundamental parallelogram. The area of this fundamental parallelogram of the lattice L is then

$$Area = l_u . l_v . \sin\theta,$$

θ is the angle between the vectors $u = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$.

$$= |u|.|v|. \left[1 - \left(\frac{\vec{u}.\vec{v}}{|u|.|v|}\right)^2\right]^{1/2}$$

$$= \Big[|u|^2.|v|^2 - |\vec{u}.\vec{v}|^2\Big]^{1/2}$$

$$= \Big[(x_1^2+y_1^2)(x_2^2+y_2^2) - 2x_1x_2y_1y_2 + x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2\Big]^{1/2}$$

$$= x_1y_2 - x_2y_1$$

which is the determinant of the matrix in the previous section. The interesting thing to note is that the area of the any parallelogram not just the fundamental parallelogram is always the determinant of the matrix of the vectors taken. Below is the discussion of the argument. Given P be the parallelogram we have to show the area of this parallelogram is the same as the area of the fundamental parallelogram.

$$P = \Big\{(ax_1 + bx_2, ay_1 + by_2) : 0 \leq a, b \leq 1\Big\}.$$

So, let us take

$$u = ax_1 + bx_2$$

$$v = ay_1 + by_2$$

i.e.

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix},$$

As seen earlier the same lattice can also be given by different matrix as

$$\begin{pmatrix} ax_1 + bx_2 & cx_1 + dx_2 \\ ay_1 + by_2 & cy_1 + dy_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

And as we have just seen the area of the parallelogram is related to the determinant of the matrix therefore we have $Area = det(H).det(B)$

$$H = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}, B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

but $detB$ is ± 1. Hence the area of the parallelogram is $|x_1y_2 - x_2y_1|$, which is same as the area of the parallelogram.
The subset $SL(2,\mathbb{R})$ of $GL(2,\mathbb{R})$ consisting of matrices of determinant 1 is also a subgroup. Moreover $SL(2,\mathbb{Z})$ which is the intersection of $SL(2,\mathbb{R})$ and $GL(2,\mathbb{Z})$ the two subgroups of $GL(2,\mathbb{R})$ consists of 2×2 matrices with integer entries and determinant 1. Thus we can identify the collection of lattices whose fundamental parallelogram has area 1 with the *coset space* $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$, and call this space as space of *lattices of co-area 1.*

2.3 Lattices in Gaussian Plane

So far we have discussed lattices in plane, now we identify the plane with the Gaussian complex plane. A Lattice L is then the collection of all integer linear combinations of a pair of complex numbers ω_1,ω_2; here ω_2 is not in the line $\mathbb{R}\omega_1$ which passes through ω_1. We can re-write this condition as $(\omega_2/\omega_1) \notin \mathbb{R}$; or simply saying that the imaginary part of (ω_2/ω_1) is non-zero. We can express (ω_1, ω_2) in terms of their real and imaginary parts as follows

$$\omega_1 = x_1 + y_1\sqrt{-1}\, and\, \omega_1 = x_1 + y_1\sqrt{-1}.$$

Now we have real and imaginary part of (ω_2/ω_1) as

$$\frac{x_1x_2 + y_1y_2}{x_1^2 + y_1^2}$$

and

$$\frac{x_1y_2 - y_1x_2}{x_1^2 + y_1^2}$$

respectively, as explained above and earlier the condition for linear independence is same as the determinant of the matrix to be non zero of the lattice L :

$$L = \Big\{a\omega_1 + b\omega_2 : a, b \,\in\, \mathbb{Z},\, \omega_1, \omega_2 \,\in\, \mathbb{C}\Big\}$$

$$= a\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + b\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

which is x_1x_2 - $y_1y_2 \neq 0$ [9].

2.4 Characterization of Lattices

The set of lattices in $\mathbb{C}$ can be identified as a integral vector space spanned by the pairs of complex numbers ω_1,ω_2 with $\mathrm{Im}(\omega_2/\omega_1) > 0$. The set of all lattices L can be identified with the set of all such pairs

$$L = \{(\omega_1, \omega_2)|Im(\omega_1/\omega_2) > 0\}$$

where we quotient by the action of modular group $SL(2, \mathbb{Z})$. Let $\lambda \in \mathbb{C}^{\times} = \mathbb{C}$ - 0 act on L by

$$\lambda : (\omega_1, \omega_2) \mapsto (\lambda\omega_1, \lambda\omega_2).$$

Now we can identify $L/\mathbb{C}^{\times}$ with $\mathbb{H}$ by the map

$$(\omega_1, \omega_2) \mapsto z = \omega_1/\omega_2.$$

Thus we have,
[**Note**] After we quotient by $SL(2, \mathbb{Z})$ the map $(\omega_1, \omega_2) \mapsto (\omega_1/\omega_2)$, gives a bijection

between $M/\mathbb{C}^{\times}$ and $\mathbb{H}/G$
where G is $SL(2,\mathbb{Z})/\pm I$. Let F be a complex-valued function on M, the space of lattices. We say F is of weight 2k if $F(\lambda L) = \lambda^{-2k} F(L)$ for all lattices $L \in M$, for all $\lambda \in \mathbb{C}^{\times}$. In particular if $\lambda = \lambda\ (\omega_1, \omega_2)$, the lattice generated by (ω_1, ω_2) , we may write F as a function of the basis elements and

$$F(\lambda\omega_1, \lambda\omega_2) = \lambda^{-2k} F(\omega_1, \omega_2)$$

and set $\lambda = \omega_2$ shows that $\omega_2^{2k} F(\omega_1, \omega_2)$ depends only on $z = (\omega_1/\omega_2)$, so we may rewrite

$$F(\omega_1, \omega_2) = \omega_2(\omega_1/\omega_2)\, for\, some\, F : \mathbb{H} \mapsto \mathbb{C}$$

and then $F(z)$ will be a modular function of weight 2k in terms of $z \in \mathbb{H}$.
We may now begin with an Eisenstein series $\mathbf{E}_{2k}(\omega_1, \omega_2)$ associated to any lattice L which is indeed a modular form and $SL(2,\mathbb{Z})$invariant.

$$\mathbf{E}_{2k}(L) = \sum_{\omega \,\in\, L\backslash 0} \frac{1}{\omega^{2k}}$$

where k > 1 to ensure the convergence. In terms of basis ω_1, ω_2 for lattice can be written as

$$\mathbf{E}_{2k}(\omega_1, \omega_2) = \sum_{(a,b)\neq 0} \frac{1}{(a\omega_1 + b\omega_2)^{2k}}$$

which is related to $\omega_{2^{2k}}$ as

$$\mathbf{E}_{2k}(L) = \sum_{(a,b)\neq(0,0)} \frac{1}{(az+b)^{2k}}$$

Now we show that the Eisenstein series is convergent for k > 1.

Since, the series $\mathbf{E}_k(L) = 0$ for odd integers k since the terms $\frac{1}{\omega^k}$ and $\frac{1}{\omega^{-k}}$ cancel out each other. Therefore we are only interested in Eisenstein series of even weight. Thus for any lattice L the sum $\sum_{\omega\in L}$ converges absolutely for k > 2.
To prove $|\sum \frac{1}{|\omega^k|}|$ converges let F be a fundamental parallelogram for lattice L and D be the length of longer diagonal of F. Then $|z| \le D\ \forall\ z \in F$. Let $\omega = a_1\omega_1 + a_2\omega_2 \in L$ s.t. $|\omega| \ge 2D$; $a_i \in \mathbb{Z}$. If x_1 and x_2 are real numbers such that

$$a_i \le x_i \le a_1 + 1$$

then ω and $x_1\omega_1 + x_2\omega_2$ differ by an element of F. So

$$|a_1\omega_1 + a_2\omega_2| + D \geq |x_1\omega_1 + x_2\omega_2|$$

$$|a_1\omega_1 + a_2\omega_2| \geq |x_1\omega_1 + x_2\omega_2| - D \geq |x_1\omega_1 + x_2\omega_2| - \frac{1}{2}|a_1\omega_1 + a_2\omega_2|$$

Since $|\omega| \geq 2D$,

$$|a_1\omega_1 + a_2\omega_2| \geq \frac{2}{3}|x_1\omega_1 + x_2\omega_2|$$

$$|a_1\omega_1 + a_2\omega_2| + D - |a_1\omega_1 + a_2\omega_2| - 2D \geq |x_1\omega_1 + x_2\omega_2|$$

$$D \leq |x_1\omega_1 + x_2\omega_2|$$

Now on comparing the sum to the integral

$$\sum_{|\omega|\geq 2D} \frac{1}{|\omega|^k} \leq \iint_{|x_1\omega_1+x_2\omega_2|\geq D} \frac{(3/2)^k}{|x_1\omega_1 + x_2\omega_2|^k} dx_1 dx_2$$

Now change the variable as $u + \iota v = x_1\omega_1 + x_2\omega_2$. Then

$$(\frac{3}{2})^k \frac{1}{areaofF} \iint \frac{1}{(u^2+v^2)^{k/2}} dudv$$

If $\omega_1 = a_1 + b_1 \; and \; \omega_2 = a_2 + b_2$, then we know the area of the fundamental parallelogram with ω_1 and ω_2 as basis is the determinant $|a_1b_2 - a_2b_1|$. So

$$(\frac{3}{2})^k \frac{1}{|a_1b_2 - a_2b_1|} \int_{\theta=0}^{2} \pi \int_{r=D}^{\infty} \frac{1}{r^k} r \, drd\theta \leq \infty$$

Thus the sum converges for $\omega \geq 2D$. Since there are only finitely many ω with $|\omega| < 2D$ we have sown the sum converges for $k \geq 4$. Continuing with this work Weierstrass was able to show that:

1. A certain "discriminant"

$$\triangle(L) = 49\mathbf{E}_6(L^2) - \mathbf{E}_4(L^3)$$

   is non-zero for every lattice.

2. Conversely, given any pair of complex numbers (a, b) so that $\triangle(a, b) = 49b^2$ - $20a^3 \neq 0$,there is a *unique* lattice L such that $(a, b) = (\mathbf{E}_4(L), \mathbf{E}_6(L))$.

Let us explain about this discriminant, for this one should know about elliptic curves.

Definition 2.2. *An **elliptic curve** over a field k is a non-singular projective algebraic curve E of genus 1 over k with a chosen base point $\mathrm{O} \in E$.*

Suppose $E = \mathbb{C}/L$ is an elliptic curve over $\mathbb{C}$, viewed as a quotient of $\mathbb{C}$ by a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, with $\omega_1/\omega_2 \in \mathbb{H}$. The Weierstrass $\wp$-function of the lattice L is

$$\wp = \wp_L(u) = \frac{1}{u^2} + \sum_{k=4,6,8,\ldots} (k-1)E_k(\omega_1/\omega_2)u^{k-2},$$

where the sum is over even integers, $k \geq 4$ and

$$E_k(z) = \sum_{m,n\in\mathbb{Z}}^{*} \frac{1}{(mz+n)^k}.$$

. The star on top of the sum symbol means that for each z the sum is over all $m, n \in \mathbb{Z}$ such that $mz + n \neq 0$ It satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - 60E_4(\omega_1/\omega_2)\wp - 140E_6(\omega_1/\omega_2).$$

If we set $x = \wp$ and $y = \wp^{'}$, the above is an (affine) equation of the form $y^2 = ax^3+bx+c$ for an elliptic curve that is complex analytically isomorphic to $\mathbb{C}/L$ for why the cubic has distinct roots). The discriminant of the cubic

$$4x^3 - 60E_4(\omega_1/\omega_2)x - 140E_6(\omega_1/\omega_2)$$

is $16\triangle(\omega_1/\omega_2)$, where

$$\triangle(z) = (60E_4(z))^3 - 27(140E_6(z))^2. \ z \ = \omega_1/\omega_2.$$

Definition 2.3. *A generalized Weierstrass equation over k is an equation of the form*

$$E : Y^2Z + a_1XYZ + a_3$$

Definition 2.4. *For a Weierstrass equation as above, define the following quantities:*

$$b_2 = a_1^2 + 4a_2, \quad b_4 = 2a_4 + a_1a_3$$

$$b_6 = a_3^2 + 4a_6, \quad b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2$$

$$\triangle = -b_8b_2^2 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6$$

Then $\triangle$ is the discriminant of the generalized Weierstrass equation.

Definition 2.5. *Two lattices L and $L^{'}$ in $\mathbb{C}$ are said to be homothetic if there exists $\lambda \in \mathbb{C}^{*} = \mathbb{C} - 0$ with $\lambda L = L$ and we write $L \sim L^{'}$. If we denote the set of all lattices by $\mathbb{L}$, the relation $\sim$ is an equivalence relation on $\mathbb{L}$, so that we have a notion of homothety (equivalence) class of lattices.*

To prove the lattice is unique we need to define the j function and study its properties.

Definition 2.6. *Let $\tau \in \mathbb{H}$ and consider the lattice $L_{\tau} = \tau\mathbb{Z} + \mathbb{Z}$, the j-function $j(\tau)$ is defined to be*

$$j(\tau) = j(L_{\tau})$$

Theorem 2.7. *The lattices L_{τ} and $L_{\tau}^{'}$, τ, $\tau \in \mathbb{H}$, are homothetic if and only if there exist $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ such that $\tau^{'} = \dfrac{a\tau + b}{c\tau + d}$.*

We will need to proof the lemma given below to prove this theorem.

Lemma 2.8. *Let $L \subset \mathbb{C}$ be a lattice and let ω_1 , ω_2 and $\omega_1^{'}$, $\omega_2^{'}$ be two oriented bases for L, then*

$$\omega_1^{'} = \omega_1 + b\omega_2 \text{ and } \omega_2^{'} = c\omega_1 + d\omega_2$$

for some

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Proof. Write $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $L = \mathbb{Z}\omega_1^{'} + \mathbb{Z}\omega_2^{'}$. The assumption of the lemma means that $L = L^{'}$. Then $L \subset L^{'}$ implies that there exist $a^{'}$, $b^{'}$, $c^{'}$, $d^{'} \in \mathbb{Z}$ such that

$$\omega_1 = a\omega_1 + \omega_2 \text{ and } \omega_2^{'} c\omega_1 + d\omega_2$$

and similarly $L \subset L^{'}$ implies that

$$\omega_1^{'} = a\omega_1 + b\omega_2 \text{ and } \omega_2^{'} = c\omega_1 + d\omega_2$$

for some $a,\ b,\ c,\ d \in \mathbb{Z}$.

Substituting $\omega_1^{'}$ and $\omega_2^{'}$ in the first set of equations we have that

$$\omega_1 = a^{'}\omega_1 + a^{'}b\omega_2 + b^{'}c\omega_1 + b^{'}d\omega_2$$

$$\omega_2 = c^{'}\omega_1 + c^{'}b\omega_2 \ + \ d^{'}c\omega_1 + d^{'}d\omega_2$$

so that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a^{'} & b^{'} \\ c^{'} & d^{'} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

Since ω_1, ω_2 are linearly independently over $\mathbb{R}$ we must have that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^{'} & b^{'} \\ c^{'} & d^{'} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Remark 2.9. *We will assume that the angle from ω_2 to ω_1 is positive and between 0 and π.*

using this remark,

$$0 < Im\left(\frac{\omega_1^{'}}{\omega_2^{'}}\right) = Im\left(\frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2}\right) = Im\left(\frac{a\left(\frac{\omega_1}{\omega_2}\right) + b}{c\left(\frac{\omega_1}{\omega_2}\right) + d}\right)$$

so if we let $\tau = \frac{\omega_1}{\omega_2} = s + it$ for some $s,\ t \in \mathbb{R}$ we have that

$$\frac{a\tau + b}{c\tau + d} = \left(\frac{a\tau + b}{c\tau + d}\right)\left(\frac{c\overline{\tau} + d}{c\overline{\tau} + d}\right) = \left(\frac{(ac|\tau|^2 + (ad + bc)s + bd) + ((ad - bc)t)i}{|c\tau + d\tau|^2}\right)$$

Hence

$$0 < Im\left(\frac{\omega_1^{'}}{\omega_2^{'}}\right) = \frac{(ad - bc)Im\left(\frac{\omega_1}{\omega_2}\right)}{\left|c\left(\frac{\omega_1}{\omega_2}\right) + d\right|^2}$$

which implies that ad-bc $>$ 0.

Finally since the determinant is multiplicative and det $\begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$, have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ and hence result follows.

□

Proof. We now prove theorem 2.7 using lemma 2.8

$$L_\tau \textit{ and } L_\tau^{'} \textit{ are homothetic} \iff \mathbb{Z}\tau \ + \ \mathbb{Z} \ = \ \mathbb{Z}\lambda\tau \ + \ \mathbb{Z}\lambda \textit{ for some } \lambda \in \mathbb{C}^*$$

$$\iff \tau^{'} \ = \ a\lambda\tau \ + \ b\lambda \ and \ 1 \ = \ c\lambda\tau \ + d\lambda \ for \ some \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ \in \ SL(2,\mathbb{Z})$$

$$\implies \tau^{'} \ = \ \frac{a\tau \ + \ b}{c\tau \ + \ d}.$$

Conversely if $\tau^{'} = \frac{a\tau+b}{c\tau+d}$ then choosing $\lambda = c\tau \ + \ d$ we have

$$\lambda L_{\tau}^{'} = \mathbb{Z}(a\tau \ + \ b) \ + \ \mathbb{Z}(c\tau + d) \ = \ \mathbb{Z}\tau \ + \ \mathbb{Z} = L_{\tau}$$

so that L_{τ} *and* $L_{\tau}^{'}$ *are homothetic.* □

Theorem 2.10. *1. $j(\tau)$ is a holomorphic function on $\mathbb{H}$*

2. If τ and $\tau^{'}$ lie in $\mathbb{H}$, then $j(\tau) = j(\tau^{'})$ if and only if $\tau^{'} = \gamma\tau$ for some $\gamma \in SL(2,\mathbb{Z})$.

Proof. 1. Since $\triangle(\tau) = 0$ in $\mathbb{H}$, it suffice to show that g_2 and g_3 are holomorphic in $\mathbb{H}$. Now, both g_2 and g_3 are given by series of the form

$$\Sigma_{(a,b)\neq(0,0)} \frac{1}{(a \ + \ b \ \tau)^k}$$

with $k > 2$, we have seen that sum converges absolutely.For the sum to define a holomorphic function we only need to verify that the sum converges uniformly on compact subsets of $\mathbb{H}$. A very detailed proof can be found in.

2. using theorem 2.7, if τ,$\tau^{'} \in \mathbb{H}$, then L_{τ}, $L_{\tau}^{'}$ are homothetic if and only if $\tau = y\tau$ for some $y \in SL(2,\mathbb{Z})$. Also from Theorem 1.3.2 we have that $j(\tau) = j(\tau^{'})$ if and only if L_{τ} are $L_{\tau}^{'}$ are homothetic. Combining both results we are done [4]. □

Remark 2.11. *From the above theorem, since*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ \in \ SL(2,\mathbb{Z}) \ and \ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ * \tau \ = \tau \ + \ 1,$$

we must have that $j(\tau) = j(\tau + 1)$, that is the j-function is $\mathbb{Z}$-periodic. A $\mathbb{Z}$-periodic holomorphic map that takes $\tau \mapsto q = e^{2\pi i\tau}$ takes $\mathbb{H} \mapsto D^{'}$, where $D^{'} = D\backslash 0$ and $D = q \ \in \ \mathbb{C} \ : \ |q| \ < \ 1$ is the open complex unit disk. Indeed, each $\tau \ \in \ \mathbb{H}$ is mapped onto a unique point q in $D^{'}$, but each $q \in D$ is the image of infinitely many points in $\mathbb{H}$. However we see that if τ and τ map onto q, then $e^{2\pi i\tau} = e^{2\pi i\tau'}$, that is $e^{(2\pi i\tau\ -\ \tau')} = 1$, so that τ and τ' differ by an integer.

Hence, we have a well defined function $g\ :\ D\ \mapsto\ \mathbb{C}$ with $j(\tau) = g(e^{2\pi i\tau})$. Since j is holomorphic on $\mathbb{H}$, g is holomorphic on D so that g has a Laurent series expansion

$$g(q)\ =\ \Sigma_{n\in\mathbb{Z}}a_n q^n\ for\ q\ \in D'.$$

Finally, $j(\tau)$ can hence be expressed as a q-expansion $j(\tau) = \Sigma_{n\in\mathbb{Z}}a_n q^n$.

An elliptic curve E defined over $\mathbb{C}$ has the form

$$E(\mathbb{C})\ = y^2\ =\ 4x^3\ -\ Ax\ - B$$

then the discriminant in this form is $\triangle\ =\ A^3 - 27B^2\ \neq 0$. This form of discriminant will be used to prove the theorem below only.

Proposition 2.12. *The* ***q****-expansion of* $j(\tau)$ *is*

$$j(\tau)\ = \frac{1}{q}\ +\ \Sigma_{n\in\mathbb{Z}}a_n q^n = \frac{1}{q}\ + 744 + \ldots$$

where $q\ = e^{2\pi i\tau}$ *and the coefficients* $an \in \mathbb{Z}$.

Proof. We will only give a sketch of the proof. Indeed, very detailed proofs can be found in [12]. As a first step, they show that

$$E_k(z) = 2\zeta(k) + 2\cdot\frac{(2\pi i)^k}{(k-1)!}\cdot\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n.$$

where $\zeta(Z) = \sum_{n=1}^{\infty}$ is the Riemann-zeta function and $\sigma_k(n) = sum_{d/n}d^k$. Then, with the standard values

$$\zeta(4) = \frac{\pi^4}{490}\ and\zeta(6) = \frac{\pi^6}{945}$$

we have that

$$g_2\tau\ =\ 60E_4\ =\ \frac{4\pi^4}{3}(1\ + 240A)\ =\ (2\pi)^4\left(\frac{1}{2}\ +\ 20A\right)$$

$$g_3\tau\ =\ 60E_6\ =\ \frac{8\pi^6}{27}(1\ + 504B)\ =\ (2\pi)^6\left(\frac{1}{216}\ +\ \frac{7}{3}B\right)$$

where $A = \sum_{n=1}^{\infty} \sigma_3 n q^n$ and $B = \sum_{n=1}^{\infty} \sigma_5 n q^n$. Now with the help of these expressions we can compute

$$(2\pi)^{-12}\triangle \;=\; \left(\frac{1}{12} \;+\; 20A\right)^3 \;-\; 27\left(\frac{1}{216} \;+\; \frac{7}{3}B\right)^2$$

$$= \left(\frac{1}{1728} \;+ \frac{5}{12} \;+ 100A^2 \;+\; 8000A^3\right) \;-\; \left(\frac{1}{1728} \;- \frac{7}{12} \;+ 147B^2\right)$$

$$= \sum_{n=1}^{\infty} \left(\textstyle\sum_{d/n} \frac{(5d^3 \;+\; 7d^5)}{12}\right) q^n \;+\; \sum_{n\geq 1} c_n q^n \;(c_n) \;\in\; \mathbb{Z}$$

But since

$$5d^3 \;+\; 7d^5 = d^3(5 \;+\; 7d^2) \;\equiv\; \begin{cases} d^3(d^2-1) \equiv 0(mod3) \\ d^3(1 \;-\; d^2) \equiv\; 0(mod4) \end{cases}$$

we have that $12/5d^3 \;+\; 7d^5$ so that

$$(2\pi)^{-12}\triangle \;=\; q \;+\; \sum_{n\geq 1} k_n q^n \;(k_n) \;\in \mathbb{Z}$$

Hence

$$j(\tau) \;=\; \frac{1728g_2^3}{\triangle} \;= \frac{(\frac{1}{20A})^3}{(2\pi)^{-12}\triangle} \;=\; \frac{1 \;+\; \sum_{n\geq 1} d_n q^n}{q \;+\; \sum_{n\geq 1} d_n q^n} \;(k_n, d_n \;\in \mathbb{Z})$$

and after division the result follows [4]. □

Theorem 2.13. *$j : \mathbb{H} \mapsto \mathbb{C}$ is surjective.*

Proof. Suppose that $c \in \mathbb{C}$ and $j(\tau) \neq c$ for all $\tau \in \mathbb{H}$. Consider the integral

$$\frac{1}{2\pi i}\int_{\gamma} \frac{j'(\tau)}{j(\tau)-c} d\tau$$

for a contour γ. Since by the previous theorem $j(\tau)$ is holomorphic on $\mathbb{H}$ we have that $\frac{j'(\tau)}{j(\tau)-c}$ is also holomorphic on $\mathbb{H}$ in view of our assumption on c. So from *Residue Theorems* we expect this integral to be equal to zero. We prove that this is not the case. Consider γ to be the contour containing an arc of the unit circle from $\frac{(-1+\iota 3)}{2}$ to $\frac{(1+\iota 3)}{2}$, two vertical segments up to any height greater than 1, and a horizontal segment with $Im(\tau) = M$ for some $M > 0$. As we have seen in Remark 1.3.8, $j(\tau) =$

$j(\tau\ +\ 1)$. Therefore, the integrals over the left and right vertical parts of π are the same, except that they are in opposite directions, so they cancel each other.

Similarly $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{Z})$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} * \tau\ = -\frac{1}{\tau}$ implies that $j(\tau) = j(-\frac{1}{\tau})$. But,

$$-\ \frac{1}{\tau} = -\ \frac{1}{\tau}\frac{\overline{\tau}}{\overline{\tau}}\ =\ -\ \frac{\overline{\tau}}{|\tau|^2}.$$

So for any τ on the arc of the unit circle since $|\tau| = 1$ we have

$$j(\tau) = j(-\overline{\tau})$$

So again, integrating from $(-1+i(3))/2$ to i is equal to integrating from i to $(1+i(3))/2$ in the reverse direction. So they cancel out as well. So we are left with

$$\frac{1}{2\pi i}\int_{\gamma}\frac{j^{'}(\tau)}{j(\tau)-c}d\tau\ =\ \frac{1}{2\pi i}\int_{\delta M}\frac{j^{'}(\tau)}{j(\tau)-c}d\tau$$

where δM is the horizontal segment. We make a change of coordinates $q = e^{2\pi i\tau}$. So if $\tau\ =\ a\ +\ iM$ we have that

$$q = e^{2\pi i(a+iM)} = e^{-2\pi M}e^{2\pi ia}.$$

Hence, as τ varies on the horizontal segment, that is $-\frac{1}{2} \geq\ a \geq \frac{1}{2}$, q varies around a circle K of radius $e^{-2\pi M}$ about $q = 0$ in the negative direction. Also since $|q| = e^{-2\pi M}$ we see that $q \mapsto 0$ as $M \mapsto \infty$. Hence, every point above the segment δM are mapped inside K. Now since by assumption $\frac{j^{'}(\tau)}{j(\tau)-c}$ is holomorphic on $\mathbb{H}$, it does not have any poles in K, except possibly at $q = 0$. Since,

$$j(\tau)\ =\ \frac{1}{q}\ +\ \ldots$$

we have that

$$j^{'}(\tau)\ =\ \left(-\tfrac{1}{q^2}\ +\ \ldots\right)\frac{dq}{d\tau}$$

and since $q\ =\ e^{2pii\tau}$ we have $\frac{dq}{d\tau}\ =\ e^{2\pi iq}$ and d$\tau = \frac{dq}{2\pi iq}$. Therefore

$$\frac{1}{2\pi i}\int_{\delta M}\frac{j^{'}(\tau)}{j(\tau)-c}d\tau\ =\ \frac{1}{2\pi i}\oint\frac{-\frac{1}{q^2}\ +\ \ldots}{\frac{1}{q}\ +\ \ldots}$$

$$= \frac{1}{2\pi i} \oint \frac{1}{q} + \ldots dq$$
$$= 1.$$

This contradiction shows that $j(\tau) = c$ for some $\tau \in \mathbb{H}$ and hence j surjects [4]. $\square$

Using definition 2.6 and theorem 2.10 and 2.13 we will prove the result mentioned above.

Theorem 2.14. *Let $E(\mathbb{C})$ be an elliptic curve over $\mathbb{C}$ given by*

$$y^2 = 4x^3 - g_2x - g_3,\ g_2, g_3 \ \in \mathbb{C},\quad g_2^3 - 27g_3^2 \ \neq \ 0.$$

Then there is a unique lattice $L \subset \mathbb{C}$ such that

$$g_2 = g_2(L) \text{ and } g_3 = g_3(L).$$

Proof. Since $j : \mathbb{H} \mapsto \mathbb{C}$ is surjective and $g_2^3 \ - \ 27g_3^2 \ \neq \ 0$ there exists γ $\mathbb{H}$ such that

$$j(\tau) = 1728 \frac{g_2^3}{g_2^3 \ - \ 27g_3^2},$$

that is

$$\frac{g_2^3(L_\tau)}{g_2^3(L_\tau) \ - \ 27g_3^2(L_\tau)} = \frac{g_2^3}{g_2^3 \ - \ 27g_3^2}$$

As we stated earlier that there exists $\lambda \in \mathbb{C}^*$ such that

$$g_2 = \lambda^{-4} g_2(L_\tau) = g_2(\lambda L_\tau)$$

$$g_3 = \lambda^{-6} g_3(L_\tau) = g_3(\lambda L_\tau)$$

Hence, λL_τ is the desired lattice and this proves existence. Suppose that there exists L and $L^{'}$ such that

$$g_2 = g_2(L) = g_2(L^{'}) \text{ and } g_3 = g_3(L) = g_3(L^{'})$$

Then we have that *j(L)* and *j(SL)*$^{'}$ and using theorem it must be that L and $L^{'}$ are homothetic. Hence there exists $\alpha \in \mathbb{C}^*$ such that $L = \alpha L^{'}$. But then,

$$g_2(L^{'}) = \alpha^{-4} g_2(L_\tau) = g_2(\alpha L_\tau)$$

$$g_3(L^{'}) = \alpha^{-6} g_3(L_\tau) = g_3(\lambda L_\tau)$$

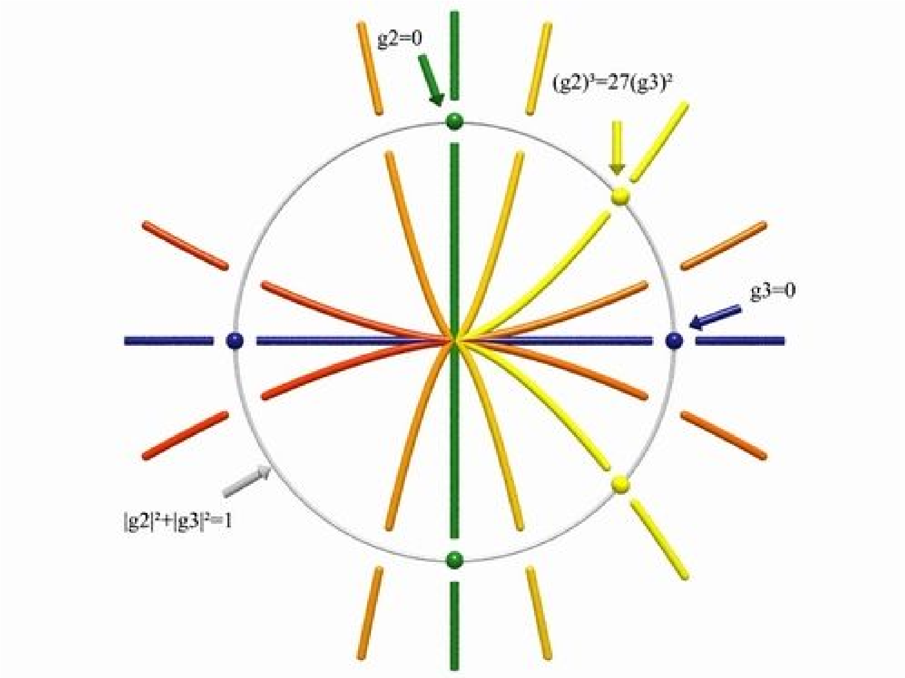

Figure 2.1: The space of lattices [10]

and by the definition of j-invariance α is forced to be 1 [4]. □

Thus we can say that the "space of all lattices " can be identified with collection of pairs (a, b) of complex numbers which satisfy $\triangle(a, b) \neq 0$ via the map

$$L = (\mathbf{E}_4(L), \mathbf{E}_6(L))$$

We already identified the space of lattices with the coset space $GL(2, \mathbb{R})/GL(2, \mathbb{Z})$. Thus we now have a representation of this space as

$$(\mathbb{C}^2 \backslash \{\triangle = 0\}) = \{(a, b) : \triangle(a, b) \neq 0\}$$

The picture above represents symbolically $\mathbb{C}^2$. The thing to note is that $\mathbb{C}^2$ here is $\mathbb{R}^4$, such that this is a four-dimensional picture! The horizontal blue axis corresponds to those lattices for which g_3=0. The vertical green axis corresponds to those lattices for which g_2=0. The yellow curve represents $\triangle = 0$, but again this is a one-dimensional curve over the complex numbers, and therefore a surface from the point of view of real numbers. In context with previous section one can look at flow as transformation explained below. The idea here is that given a lattice L of co-area 1, there is a natural way to shrink it in the y-direction and stretch it in the x-direction such that again the lattice is of co-area 1, this can be viewed as:

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 & \alpha x_2 \\ \alpha^{-1} y_1 & \alpha^{-1} y_2 \end{pmatrix}$$

Since $L =$ -L we may restrict our attention to positive α. In that case we can find a real number t so that $\alpha = \exp^t$. Considering t as a "time parameter" we can think of the transformation as giving a "flow" $L \mapsto \phi_t(L)$ in the space of lattices of co-area 1.

$$\phi_t = \begin{pmatrix} \exp^t & 0 \\ 0 & \exp^{-t} \end{pmatrix}$$

We can now use the identification given in the previous section to make this a flow on S^3 as follows. Given (a,b) in S^3, Let L(a,b) denote the associated lattice of co-area 1 so that

$$(a, b) = (\mathbf{E}_4(k.L(a, b)), \mathbf{E}_6(k.L(a, b)))$$

where $k = k(L(a, b))$.

At time t we send (a,b) to the image of $\phi_t(L(a, b))$under the identification

$$L \mapsto (\mathbf{E}_4(k(L)L), \mathbf{E}_6(k(L)L))$$

Thus the image $\phi_t(L(a, b))$ again is a lattice of area 1. The resulting flow on $\mathbf{S}^3$ is called the "modular" dynamical system on $\mathbf{S}^3$. We can see that there is another flow which is given by right multiplication i.e.

$$(\omega_1, \omega_2) \mapsto (\exp^t \omega_1, \exp^{-t} \omega_2)$$

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \mapsto \begin{pmatrix} \exp^t x_1 & \exp^{-t} x_2 \\ \exp^t y_1 & \exp^{-t} y_2 \end{pmatrix}$$

which is different from the flow we defined in $\mathbf{S}^3$.

2.5 Four dimensional object

What we have to do is here that the identification that we discovered earlier, we will use it to restrict our discussion to those lattices with co-area 1. Now we are about to discuss that whether can we view the above four dimensional object in a concrete way or not. It can be seen that given a lattice L, one can show that there exists a real number k (depending on L) so that $(\mathbf{E}_4(L, \mathbf{E}_6(L)$ lies on "the three dimensional sphere"

$$S^3 = \{(a, b) : |a|^2 + |b|^2 = 1\}$$

Note that when we replace L by kL in the formula, we get

$$(\mathbf{E}_4(kL)), \mathbf{E}_6(kL))) = (k^{-4}\mathbf{E}_4(L), k^{-6}\mathbf{E}_6(L))$$

Now we have to prove that there is a unique real kL for a given lattice L such that it lies on $\mathbf{S}^3$. Using the above replacement to satisfy this point in $\mathbf{S}^3$ we get

$$|k^{-4}E_4|^2 + |k^{-6}E_6|^2 = 1$$

$$|k^{-4} \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^4}|^2 + |k^{-6} \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^6}|^2 = 1$$

$$k^{-8}| \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^4}|^2 + k^{-12}| \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^6}|^2 = 1$$

$$k^{-8} \left(| \textstyle\sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^4}|^2 + k^{-4}| \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^6}|^2\right) = 1$$

put $k^{-8} = c^{-1}$, then the remaining expression is c. Thus

$$| \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^4}^2 + k^{-4} \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^6}|^2 = c$$

$$k^{-4} = \frac{c - \left|\sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^4}\right|^2}{\left|\sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^6}\right|^2}$$

Now put

$$\alpha = | \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^4}|^2$$

and

$$\beta = | \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^6}|^2$$

Then we have

$$\sqrt{c^{-1}} = \frac{c - \alpha}{\beta}$$

$$c^3 - 2\alpha c^2 + \alpha^2 c - \beta^2 = 0$$

Let s_1, s_2, s_3 be three roots of the above equation. Then we have

$$s_1 s_2 s_3 = \beta^2$$

$$s_1 + s_2 + s_3 = 2\alpha$$

$$s_1 s_2 + s_3 s_1 + s_2 s_3 = \alpha^2$$

Using the α we get the relation among the roots as

$$s_1^2 + s_2^2 + s_3^2 = 0$$

As α,β are non-zero, thus all the three roots cannot be zero and none of them can be zero.

For a lattice L of co-area 1,let *k(L)* denote this number and consider the map

$$L \mapsto (\mathbf{E}_4(kL), \mathbf{E}_6(kL))$$

This gives a map $SL(2,\mathbb{R})/SL(2,\mathbb{Z}) \mapsto S^3$. If two lattices L and $L^{'}$ have the same image then $k(L)L = k(L^{'})L^{'}$, so that $L^{'} = \frac{kL}{kL^{'}}$ L is the multiple of L by a real number. If L and $L^{'}$ both have co-area 1,we see that the real number must be $\pm$ 1, or we can say that $L = \pm L^{'} = L$. Now we can say that we have produced a representation of the space of lattices of co-area 1 as a subset of S^3:

$$SL(2,\mathbb{R})/SL(2,\mathbb{Z}) \cong (S^3 \backslash \{\triangle = 0\}) = \{(a,b) : |a|^2 + |b|^2 = 1 \ and \ \triangle(a,b) \ \neq \ 0\}$$

This means that if we have to study lattices up to rescaling we have to look at the complement in the unit sphere of the zero set of $\triangle$ i.e. $\triangle = 49E_6(L)^2$ - $20E_4(L)^3$. This unit sphere is 3-dimensional and it intersects the zero set in a one-dimensional object which turns to be a trefoil knot which is the simplest knot. This knot can be obtained by the stereographic projection from a point chosen on the sphere, which then has a projection to the tangent space opposite to the point chosen. Thus under this projection one can see that the set $\left\{\triangle = 0 \bigcap \mathbf{S}^3\right\}$ is a trefoil knot. Hence the space of lattices of co-area 1 is identified with the complement of a trefoil knot in the 3-sphere, which after deleting one point of a trefoil knot is the complement of the usual 3-space. Figure 2.5 is trefoil knot plotted using sage.

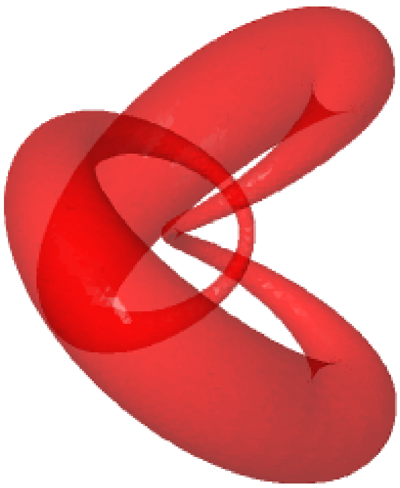

Figure 2.2: trefoil knot

Chapter 3

Apprehending Flows

3.1 Periodic orbits

In the quest of finding periodic orbits an example studied by Birman and Williams is the suspension flow on the complement of figure eight knot in S^1. But, remarkably Robert Ghirst showed that the every possible knot in S^1 arises as the periodic orbit of this flow, without exception. Birman and Williams analysed the flow associated with the famous Lorenz equations, in which they showed the family of knots arising as periodic orbits has special properties. The results were based on the fact that all periodic orbits of the Lorenz flow are described y a simple combinatorial construction, called the 'template'. In the important case of the hyperbolic flows Birman and Williams have proved that a template always exist but its explicit construction has been done in very specific cases. Even within the well-studied class of hyperbolic flows consisting of the geodesic flows on the unit tangent bundle of surfaces of the constant negative curvature, the first construction of a template for modular surfaces was achieved recently. Ghys established the extraordinary fact that the modular template coincides with the Lorenz template [13].

This work by Ghys has been inspiration for so many of us as it has raised questions such as to understand which properties of the periodic orbits of the modular surface hold for the orbits of the geodesic flows on other surfaces.

Ghys shows that the non-compact 3-dimensional homogeneous quotient space

$$Y = SL(2, \mathbb{R})/SL(2, \mathbb{Z})$$

is homeomorphic to the 3-sphere S^3 with the trefoil knot τ removed. Y carries a number of flows and corresponding non-vanishing vector fields and in particular the diagonal flow

$$\mathrm{G_t} \; for \; t \in \mathbb{R},$$

$$\mathrm{G_t}(ySL(2,\mathbb{Z})) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e{-}t/2 \end{pmatrix} ySL(2,\mathbb{Z})$$

This flow corresponds to the geodesic flow on the modular surface $X = \mathrm{H}/\Gamma$ with *PSL(2,ℤ)* and the primitive closed orbits of $\mathrm{G_t}$

3.2 Visualization

We have discussed so far about flows on modular surfaces. To give a gist of what Ghys and Jos Leys did, here are few visual descriptions of the knots on space of lattices. We now discuss the topological description of the periodic orbits of the modular flow. To understand in much simpler way consider a 2×2 matrix M with integral coefficients and determinant 1.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Clearly the matrix M preserves the standard square lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. Suppose that M is hyperbolic, i.e. |a + d| > 2, thus M is diagonalizable over real numbers. So it follows that there is 2×2 matrix Psuch that

$$\phi^t = PAP^{-1} = \pm \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

for some t. So if we define the lattice L to be the image of $\mathbb{Z}^2$ by P, one finds that L is fixed by ϕ^t. In this way, for each integral matrix with determinant 1, we find a fix point ϕ^t for some t, i.e. a periodic orbit for the modular flow. One should note that the periodic orbit of the flow has period t, which is the logarithm of the absolute value of an eigen value of M.

Hence, every hyperbolic matrix M defines a periodic orbit of the modular flow . Its is not difficult to see that if one replaces M by $\pm\, NMN^{-1}$, where N is some other integral matrix with determinant 1, one gets the same periodic orbit.

There is a natural bijection between the periodic orbits of the modular flow and the conjugacy classes of the hyperbolic integral matrices of determinant 1, up-to sign. Each of these periodic orbits is a closed curve in the space of lattices of area 1, hence

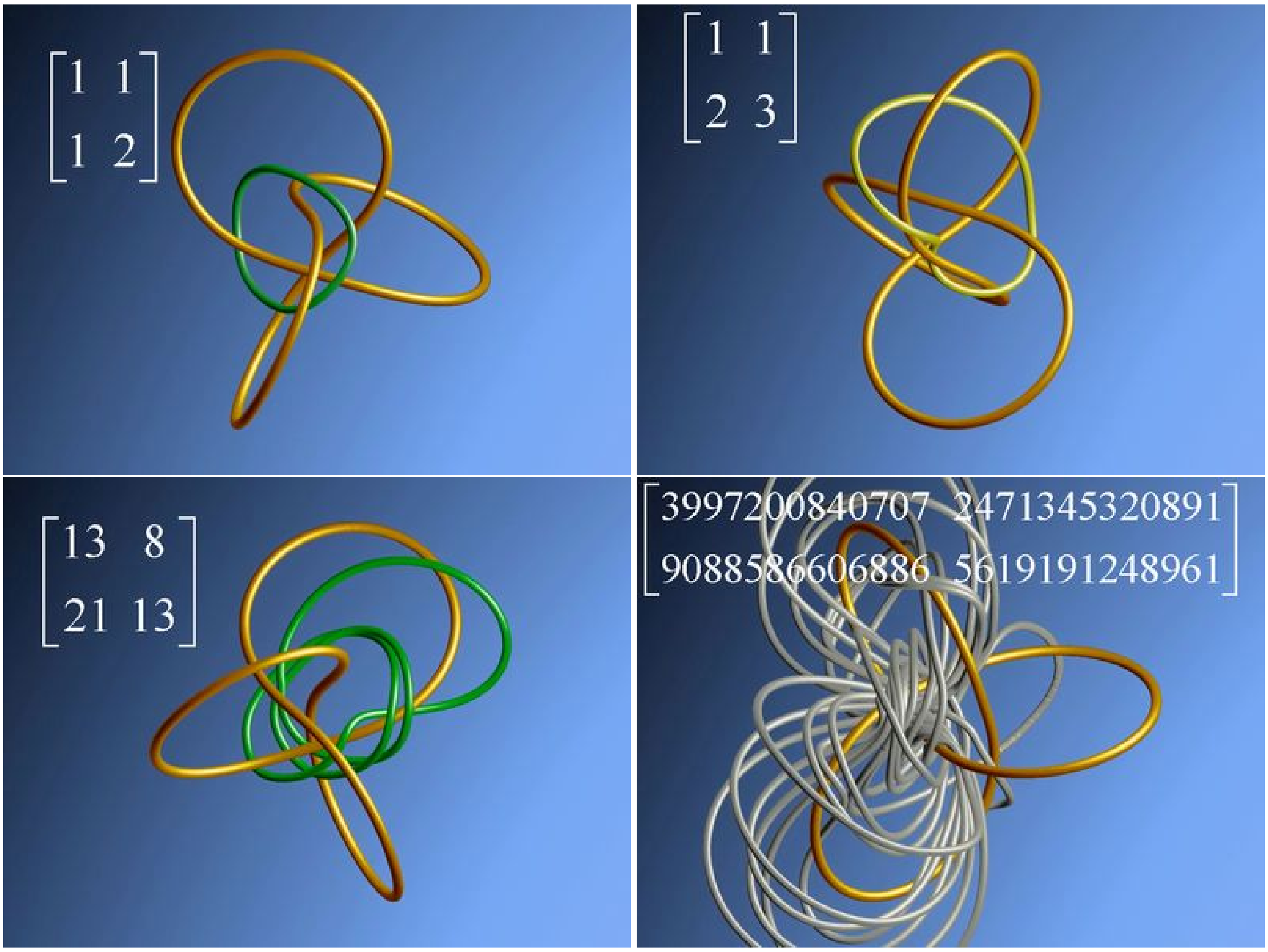

Figure 3.1: knots corresponding to hyperbolic matrices with trefoil knot in centre, [10].

defines the knot in the complement of the trefoil knot. In fig. 3.2 are the beautiful images produced by Jos Leys and Ghys using ultrafractal software, shows the knots corresponding to hyperbolic matrices with trefoil knot in centre (yellow).

So this was Ghys and Jos, where they discuss the knots corresponding to a single matrix. We wished to branch from them and hunt for all the knots appearing in S^3. How we do it?

3.3 Problem

The idea here is to detect cycle in S^3 with flow defined above using the Weierstrass functions, i.e. g_2,g_3 will define our lattice and with the flow defined we would allow this flow to traverse through whole S^3 and search for cycle and study the pattern of these cycles. The problem here is that the algorithm which we are trying to implement works for finite sets, but what we are dealing with, is not a finite set. The idea was to observe that, can we still find cycles ? .

Let S be any finite set, f be any function from S to itself, and x_0 be any element of S. For any $i > 0$, let $x_i = f(x_{i-1})$. Let μ be the smallest index such that the value $x\mu$ reappears infinitely often within the sequence of values x_i, and let λ (the loop length) be the smallest positive integer such that $x_\mu = x_{\lambda+\mu}$. The cycle detection problem is the task of finding λ and μ.

3.3.1 Brent's algorithm

Brent's algorithm works in linear time. Based on Floyd's Tortoise and the Hare algorithm, Brent's algorithm features a moving rabbit and a stationary turtle. We initialize both turtle and rabbit at the top of the list, searching for the smallest power of two 2^i that is larger than both λ and μ. For $i = 0$, 1, 2, . . ., the algorithm compares x^{2i-1} with each subsequent sequence value up to the next power of two, stopping when it finds a match. It has two advantages compared to the tortoise and hare algorithm: it finds the correct length λ of the cycle directly, rather than needing to search for it in a subsequent stage, and its steps involve only one evaluation of f.

The idea to use Brent's algorithm starting with two (obviously, corresponding our flow) f,f' as a function of time parameter t $\in \mathbb{R}$, such that one of the functions plays the role of turtle and other of the rabbit.

This is just the idea which we think can be useful if implemented, one can get much closer look at knots while studying the parameter λ which is length of the cycle and observe its pattern in S^3.

Implementing Algorithm: The following implementation is in reference to Sage (mathematical software):

Choose any pair of complex numbers (a, b) in S^3, such that if $a = x_1 + y_1 i$ and $b = x_2 + y_2 i$ then, a, b are such that its determinant is 1. For example, we can choose $a = 2 + i$ and $b = 1 + i$

```
def cal():   \\  to calculate  E4 and E6

 t = b/a \\ t is the argument (a complex number)
        \\ to compute q expansion of of E4 and E6

 g = eisenstein_series_qexp(4,10) \\ value of E4
 h = eisenstein_series_qexp(6,10) \\ value of E6
 g = g.substituting (q = 2pi*i*t )
 h = h.substituting (q = 2pi*i*t )

 if (49h^{2} - 20g^{3} != 0 ): \\ checking the condition
                              \\if discriminant is 0 or not.
    return g,h
 else:
    return 0
```

We now identify the lattices as points in S^3.

```
 \\ we must find a c such that the we are restricted to S^{3}.

 var = 'c,alpha,beta'
 eqn =  c^{3} -2alpha c^{2} + alpha^{2}c - beta^{2} = 0

 \\ solving to obtain c as discussed in the explanation.
 eqn.substitute (alpha = abs2(cal.g), beta =norm(cal.h )

 \\ calculating the norm of the complex numbers  E4and E6
 solve(eqn==0,c)
```

Thus using c we generate lattice on S^3

```
def latt():  \\ to generate lattice using E4 and E6 as basis
   u = random.randint(1, 1000)
   y = random.randint(1, 1000)
   S =  u*c^(1/8)*cal.g  + y*cal.h*c^(1/8) \\ lattice generated

def fu(): \\ this function defines our flow
 t=0.1 \\ step counter

 for t in range (0:100):
  v1=vector((exp(t), 0))
  v2=vector((0, exp(-t)))
  v3=vector((latt.s.re, latt.s.im ))
  \\ real and imaginary part of element of lattice generated
  v4=vector((latt.s.re, latt.s.im))
  \\ another vector from the lattice generated
  ma=matrix([v1,v2]); m; m.parent()
  \\ our flow
  na=matrix([v3,v4]); n; n.parent()
  \\ element on which flow acts.

  t=t+0.1
 return ma, na

 def main():
   tor = na
   har = ma*na
   while tor - har > 0.1:
     if pwr == la:
           tor = har
           pwr*= 2
           lam = 0

     har = ma*har
     la += 1

   mu = 0
```

```
tor = har = na

for e in range(la):
    har = ma*har

while tor - har > 0.1:
        tor = ma*tor
        har = ma*har
        mu += 1

return la, mu
```

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