Counting manifolds upto homotopy and homeomorphy

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Certificate of Examination

This is to certify that the dissertation titled **Counting manifolds upto homotopy and homeomorphy** submitted by **Mr. Debi Prasad Kar** (Reg. No. MS09043) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Mahender Singh   Dr. Varadharaj R. Srinivasan   Dr. Alok Maharana
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Dated: April 25, 2014

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Alok Maharana at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

Debi Prasad
(Candidate)
Dated: April 25, 2014

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Alok Maharana
(Supervisor)

Acknowledgement

I would like to express my gratitude to my supervisor Dr. Alok Maharana. His guidance helped me in all the time of project and writing of this thesis.

I am grateful to phd student Mr. Soumya Dey for his generous support and encouragement.

I am also thankful to the faculty and the research students at IISER Mohali for maintaining research environment. A special thanks to the Library, IISER Mohali and JSTOR for providing access to the research articles. I would also like to appreciate my friends and family members for continuous support and encouragement.

Finally, I would like to acknowledge that the material presented in this thesis is completely based on the work of other mathematicians. At the beginning of every chapter, I have clearly mentioned all the books and papers that were used. My contribution in this thesis is of purely expository nature and in a close reading and presentation of the cited material. Primarily a book by Munkres and certain works of Burgess, Mather and Cheeger-Kister have been used and referred to.

Debi Prasad kar

Contents

Introduction

The aim of this thesis is to closely study two results in the theory of classification of manifolds attempting to count manifolds upto homotopy and homeomorphy and to give a detailed presentation of the classification of topological surfaces.

Two proofs of classification of surfaces is presented. One notices that the homeomorphism classes of topological surfaces is countable motivating the question of counting the number of homeomorphism classes of all manifolds.

This question is intercepted with a seemingly simpler question about homotopy classes of all topological manifolds possibly with boundary. Mather gives a pleasing answer to this question by proving that the homotopy classes of all compact manifolds possibly with boundary, are countable.

We then study the question of homeomorphism classes of all compact topological manifolds closely following the work of Cheeger and Kister who prove that even this collection is countable.

The plan of the thesis is as follows. First chapter gives a proof of classification theorem of surfaces assuming the difficult but classical theorem that all surfaces are triangulable. We follow the proof given in the book "Topology" by Munkres[7] very closely. Second chapter gives another proof of the classification theorem of surfaces by C.E. Burgess[1]. Third chapter explains Mather's theorem[6] on homotopy classification of all topological manifolds. The last chapter sketches the proof of a theorem of Cheeger and Kister[2] on homeomorphism classes of all topological manifolds.

All manifolds in this thesis are topological manifolds, possibly with boundary specifically mentioned, everywhere.

Chapter 1

Classification of surfaces - first proof

In this chapter we give a detailed classification of compact surfaces, without boundary, upto homeomorphism. We follow the proofs as they appear in Munkres[7].

1.1 Fundamental Groups of Surfaces

Definition 1.1.1. If the topology of a topological space X has a countable basis, then X is said to be ***second countable***.

For example the line $\mathbb{R}$ has a countable basis. The collection of all open intervals (a, b) with rational end points are countable basis for $\mathbb{R}$.

Definition 1.1.2. A **surface** is a compact connected metric space that is locally homeomorphic to the Euclidean plane $\mathbb{R}^2$.

For example sphere and torus are some surfaces.

Definition 1.1.3. Given a point c of $\mathbb{R}^2$, and given $a > 0$. Consider the circle of radius a in $\mathbb{R}^2$ with center at c. Given a finite sequence $\theta_0 < \theta_1 < \cdots < \theta_n$ of real number, where $n \geq 3$ and $\theta_n = \theta_0 + 2\pi$, consider the points $p_i = c + a(\cos\theta_i, \sin\theta_i)$, which lies on this circle. They are numbered in the counterclockwise order around the circle and $p_n = p_0$. The line through p_{i-1} and p_i splits the plane into two closed half-plane. Let H_i be the one that contains all the points p_k. Then the space

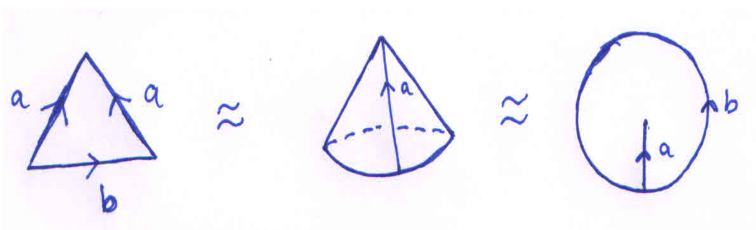
$$P = H_1 \cap \cdots \cap H_n$$

is called the **polygonal region** determined by the points p_i. The points p_i are called the **vertices** of P; the line segment joining p_{i-1} and p_i is called an edge of P; the union of the edges of P is denoted $\mathrm{Bd}(P)$; and $P \setminus Bd(P)$ is denoted $\mathrm{Int}P$.

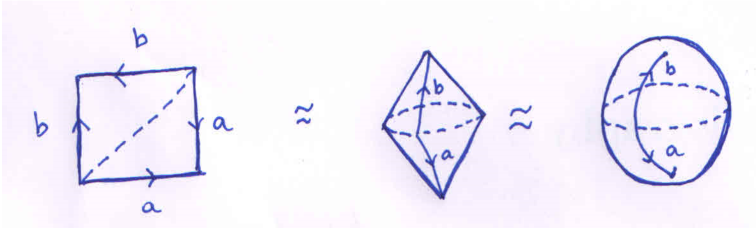
Definition 1.1.4. Given a line segment L in $\mathbb{R}^2$, an **orientation** of L is simply an ordering of its end points; the first, say a, is called the **initial point**, and the second, say b, is called the **final point**, of the orientated line segment. We often say that L is oriented **from** a **to** b and we picture the orientation by drawing an arrow on L that points from a towards b. If L' is another line segment, oriented from c to d, then the **positive linear map** of L onto L' is the homeomorphism h that carries the point $x = (1-s)a + sb$ of L to the point $h(x) = (1-s)c + sd$ of L'.

Definition 1.1.5. Let P be a polygonal region in the plane. A **labelling** of the edges of P is a map from the set of edges of P to a set S called the set of **labels**. Given an orientation of each edge of P, and given a labelling of the edges of P, we defined an equivalence relation on the points of P as follows. Each point of $\mathrm{Int}P$ is equivalent only to itself. Given any two edges of P that have the same label, let h be the positive linear map of one onto the other, and defined each point x of the first edge to be equivalent to the point $h(x)$ of the second edge. This relation generates relation on P. The quotient space X obtained from this equivalance relation is said to have been obtained by **pasting the edges of P together** according to the given orientations and labelling.

Example 1.1.6 Consider the orientations and labelling of the edges of the triangular region pictured below. The figure indicates how one can show that the resulting quotient space is homeomorphic to the unit ball.



The orientations and labelling of the edges of the square pictured in the figure given below give rise to a space that is homeomorphic to the sphere S^2.

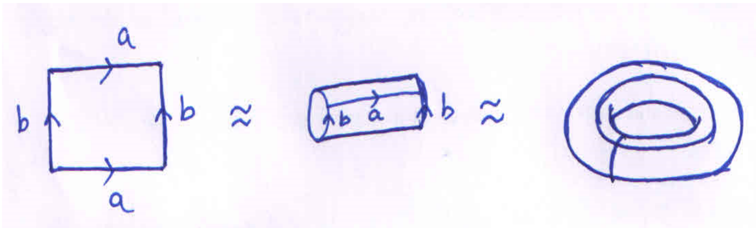


Definition 1.1.7. Let P be a polygonal region with succesive vertices $p_0, \cdots p_n$, where $p_0 = p_n$. Given orientation and a labelling of the edges of P, let $a_1, \cdots, a_m$ be the distinct labels that are assigned to the edges of P. For each k, let a_{i_k} be the label assigned to the edge $p_{k-1}p_k$, and let $\epsilon_k = +1$ or -1 according as the orientation assigned to this edge goes from p_{k-1} to p_k or the reverse. Then the number of edges of P, the orientations of the edges, and the labelling are completely specified by the symbol

$$\omega = (a_{i_1})^{\epsilon_1}(a_{i_2})^{\epsilon_2} \cdots (a_{i_n})^{\epsilon_n}$$

We call this symbol a **labelling scheme of length n** for the edges of P; it is simply a sequence of labels with exponents +1 or -1.

Example 1.1.8. Torus can be expressed as a quotient space of the unit square by means of the quotient map $p \times p : I \times I \to S^1 \times S^1$. This same quotient space can be specified by the square indicated below in the figure. It can be specified also by the scheme $aba^{-1}b^{-1}$.



Theorem 1.1.9. *Let X be the space obtained from a finite collection of polygonal region by pasting edges together according to some labelling scheme. Then X is a compact Hausdorff space.*

Proof. For simplicity, we can assume the case where X forms a single polygonal region. We can prove the general case in the similar manner.

X would be compact because quotient map is a continuous map and we know that A continuous function maps compact sets into compact sets.

To show that X is Hausdorff, it is sufficient to show that the quotient map π is a closed map. It can be seen that if $\pi : E \to X$ be a closed quotient map, and E is normal, then so is X. We also know that if a space is normal then it is Hausdorff too. $\square$

Theorem 1.1.10. *Let P be a polygonal region, and*

$$\omega = (a_{i_1})^{\epsilon_1} \cdots (a_{i_n})^{\epsilon_n}$$

be a labelling scheme for the edges of P. Let X be the resulting quotient space and let $\pi : P \to X$ be the quotient map. If π maps all the vertices of P to a single point x_0 of X, and if $a_1, \cdots, a_k$ are the distinct labels that appear in the labelling scheme, then $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on k generators $\alpha_1, \cdots, \alpha_k$ by the least normal subgroup containing the element

$$((a_{i_1})^{\epsilon_1} \cdots (a_{i_n})^{\epsilon_n}).$$

Example 1.1.11. In the case of torus the quotient map π maps all the vertices of the polygonal region to a single point of a torus. Let X denote the torus. Then $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on the generator α, β by the least normal subgroup containg the element $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$.

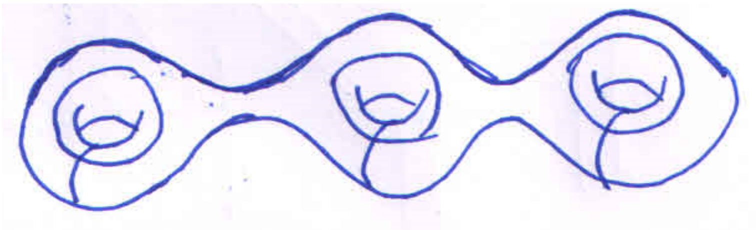
Definition 1.1.12. Consider the space obtained from a $4n$-sided polygonal region P by the means of the labelling scheme

$$(a_1 b_1 a_1^{-1} b_1^{-1})(a_2 b_2 a_2^{-1} b_2^{-1}) \cdots (a_n b_n a_n^{-1} b_n^{-1})$$

This space is called the **n-fold connected sum of tori**, or simply the n-fold torus, and is denoted $\mathbb{T} \# \cdots \# \mathbb{T}$.

Example 1.1.13. Below we can see a 2-fold torus. If we split the polygonal region P along the line c as indicated in the picture given below, then each of the piece will form a torus with a open disc removed.

Proceeding in a similar manner we see that the 3-fold torus $\mathbb{T}\#\mathbb{T}\#\mathbb{T}$ can be pictured as the surface below:



Theorem 1.1.14. *Let X denote the n-fold torus and $x_0 \in X$ a point. Then $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on the $2n$ generators $\alpha_1, \beta_1, \cdots, \alpha_n, \beta_n$ by the least normal subgroup containg the element*

$$[\alpha_1, \beta_1][\alpha_2, \beta_2] \cdots [\alpha_n, \beta_n],$$

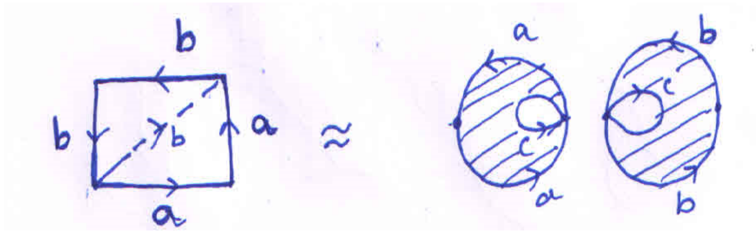
where $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$, as usual.

Definition 1.1.15. Let $m > 1$. Consider the space obtained from a $2m$-sided polygonal region P in the plane by means of the labelling scheme

$$(a_1 a_1)(a_2 a_2) \cdots (a_m a_m)$$

This space is called the **m-fold connected sum of projective planes** or simply the **m-fold projective plane** denoted $\mathbb{P}^2\# \cdots \#\mathbb{P}^2$.

Example 1.1.16. The 2-fold projective plane $\mathbb{P}^2\#\mathbb{P}^2$ is pictured below. This figure shows how this connected space can be obtained from two copies of projective plane by deleting one open disc from each of the projective plane and attaching the resulting space together along the boundaries of the deleted disc.



Theorem 1.1.17. *Let X denote the m-fold projective plane. The $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on m generators $\alpha_1, \cdots, \alpha_m$ by the least normal subgroup containing the element*

$$(\alpha_1)^2(\alpha_2)^2 \cdots (\alpha_m)^2$$

.

1.2 Homology of Surfaces

We saw in the last section how to compute fundamental groups of n-fold torus and m-fold projective plane. It is however not so easy to show that double torus is not homeomorphic to triple torus. This requires developing techniques to decide whether or not two groups are isomorphic. It turns out to be easier if we pass to the abelian group $\pi_1/[\pi_1, \pi_1]$, where $\pi_1 = \pi_1(X, x_0)$. This prove to be a good invariant to work with as explained in this section.

We know that if x is a path-connected space and α is a path in X from x_0 to x_1, then there is an isomorphism $\hat{\alpha}$ of fundamental group based at x_0 with the fundamental group based at x_1, but the isomorphism depends on the choice of the path α. In the case of $\pi_1/[\pi_1, \pi_1]$, the isomorphism of the *abelianized fundamental group* based at x_0 with the one based at x_1, induced by α, is in fact independent of the choice of path α.

We can verify this fact easily. Take two paths α and β from x_0 to x_1. The the path $g = \alpha \star \bar{\beta}$ induces the identity isomorphism of $\pi_1/[\pi_1, \pi_1]$ with itself.

Definition 1.2.1 If X is a path-connected space, let

$$H_1(X) = \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)].$$

We call $H_1(X)$ the **first homology group** of X. We omit the base point from the notation because there is a unique path-induced isomorphism between the abelianized fundamenatal groups based at two different points.

To compute $H_1(X)$ for the surfaces considered earlier, we need the following result:

Theorem 1.2.2. *Let F be a group, N be a normal subgroup of F and let $q : F \to F/N$ be the projection. The projection homomorphism*

$$p : F \to F/[F, F]$$

induces an isomorphism

$$\phi : q(F)/[q(F), q(F)] \to p(F)/p(N)$$

.

This theorem roughly states that if one divides F by N and then abelianizes the quotient, one gets the same result if one first abelianizes F and then divides by the image of N in this abelianization.

Corollary 1.2.3. *Let F be a free group with free generators $\alpha_1, \cdots, \alpha_n$, let N be the least normal subgroup of F containing the element x of F and let $G = F/N$. Let $p : F \to F/[F, F]$ be projection. Then $G/[G, G]$ is isomorphic to the quotient of $F/[F, F]$ which is free abelian with basis $p(\alpha_1), \cdots, p(\alpha_n)$, by the subgroup generated by $p(x)$.*

Theorem 1.2.4. *If X is the n-fold connected sum of tori, then $H_1(X)$ is a free abelian group of rank 2n.*

Proof. If we apply the Corollary 1.2.3 in the Theorem 1.1.14 we get that $H_1(X)$ is isomorphic to the quotient of the free abelian group F' on the 2n generator $\alpha_1, \beta_1, \cdots, \alpha_n, \beta_n$ by the least normal subgroup generated by the element $[\alpha_1, \beta_1]$, where $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. As F' is abelian $\alpha\beta\alpha^{-1}\beta^{-1}$ equals the identity element. Hence F' is a free abelian group of rank 2n. □

Definition 1.2.5. If G is a group, the set of **torsion elements** $T(G)$ of G is defined to be the set of elements g in G such that $g^n = e$ for some natural number n where e is the identity element of G. In the case that G is Abelian, $T(G)$ is a subgroup and is called the **torsion subgroup** of G.

Theorem 1.2.6. *If X is the m-fold connected sum of projective planes, then the torsion subgroup $T(X)$ of $H_1(X)$ has order 2 and $H_1(X)/T(X)$ is a free abelian group of rank $m-1$.*

Theorem 1.2.6. *Let $\mathbb{T}_n$ and $\mathbb{P}_m$ denote the n-fold connected sum of tori and the m-fold connected sum of projective planes, respectivily. Then the surfaces S^2; $\mathbb{T}_1, \mathbb{T}_2, \cdots$; $\mathbb{P}_1, \mathbb{P}_2, \cdots$ are non homeomorphic to each other.*

1.3 Cutting and Pasting

These technique show how to take a space X that is obtained by pasting together the edges of one or more polygonal regions according to some labelling scheme and to represent X by different collection of polygonal regions and a different labelling scheme.

Theorem 1.3.1. *Suppose X is the space obtained by pasting the edges of m polygonal region together according to the labelling scheme*

($\star$) $\qquad y_0 y_1, \omega_2, \cdots, \omega_m$

Let c be a label not appearing in this scheme. If both y_0 and y_1 have length at least two, then X can also be obtained by pasting the edges of $m+1$ polygonal regions together according to the scheme

($\star\star$) $\qquad y_0 c^{-1}, c y_1, \omega_2, \cdots, \omega_m$

Conversely, if X is the space obtained from $m+1$ polygonal regions by means of the scheme ($\star\star$), it can also be obtained from m polygonal regions by means of the scheme($\star$), providing that c does not appear in scheme ($\star$).

Elementary operartion on schemes

We now list a number of elemenatry operations that can be performed on a labelling scheme $\omega_1, \cdots, \omega_m$ without affecting the resulting quotient spaace X.

1. ***Cut***: One can replace the scheme $\omega_1 = y_0y_1$ by the scheme y_0c^{-1} and cy_1, provided c does not appear elsewhere in the total scheme and y_0 and y_1 have length at least two.

2. ***Paste***: One can replace the scheme y_0c^{-1} and cy_1 by the scheme y_0y_1, provided c does not appear elsewhere in the total scheme.

3. ***Relable***: One can replace all occurences of any given label by some other label that does not appear elsewhere in the scheme. Similarly, one can change the sign of the exponent of all occurences of a given label a; this amounts to reversing the orientations of all the edges labelled "a''. Neither of this alterations affects the pasting map.

4. ***Permute***: One can replace any one of the schemes ω_i by a cyclic permutation of ω_i. Specificially, if $\omega_i = y_0y_1$, we can replace ω_i by y_1y_0. This amount to renumbering the vertices of the polygonal region P_i so as to begin with different vertex; it does not affect the resulting space.

5. ***Flip***: one can replace the scheme

   $$\omega_i = (a_{i_1})^{\epsilon_1} \cdots (a_{i_n})^{\epsilon_n}$$

   by its formal inverse

   $$\omega_i^{-1} = (a_{i_1})^{-\epsilon_1} \cdots (a_{i_n})^{-\epsilon_n}$$

   This amounts simply to "flipping the polygonal region P_i over".

6. ***Cancel***: One can replace the scheme $\omega_i = y_0aa^{-1}y_1$ by the scheme y_0y_1, provided a does not appear elsewhere in the total scheme and both y_0 and y_1 have length at least two.

7. ***Uncancel***: This is the reverse of the operation Cancel. It replaces the scheme y_0y_1 by the scheme $y_0aa^{-1}y_1$, where a is a label that does not appear elsewhere in the total scheme.

Definition 1.3.2. We define two labelling schemes for the collections of polygonal regions to be **equivalent** if one can be obtained from the other by a sequence of elementary scheme operations. Since each elementary operation has as its inverse as another such operation, this is an equivalence relation.

Example 1.3.3. The Klein bottle K is the space obtained from the labelling scheme $aba^{-1}b$. We will show that K is homeomorphic to the 2-fold projective plane $\mathbb{P}^2\#\mathbb{P}^2$ by elementary operations.
$aba^{-1}b \to abc^{-1}$ and $ca^{-1}b$ by cutting
$\to c^{-1}ab$ and $b^{-1}ac^{-1}$ by permuting the first and flipping the second
$\to c^{-1}aac^{-1}$ by pasting
$\to aacc$ by permuting and relabelling

Example 1.3.4. Here we will show that connected sum of $\mathbb{T}\#\mathbb{P}^2$ is homeomorphic to $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$. From Example 1.3.3 it is clear that Klein bottle is homeomorphic to $\mathbb{P}^2\#\mathbb{P}^2$. So $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$ is homeomorphic to $K\#\mathbb{P}^2$. We will show that $K\#\mathbb{P}^2$ is homeomorphic to $\mathbb{T}\#\mathbb{P}^2$ by elementary operartions. Labellings scheme of $K\#\mathbb{P}^2$ is $abab^{-1}cc$. The following operation show how to obtain the labelling scheme of $\mathbb{T}\#\mathbb{P}^2$ from $K\#\mathbb{P}^2$.
$abab^{-1}cc \to cabab^{-1}c$ by permuting
$\to cabd^{-1}$ and $dab^{-1}c$ by cutting along d
$\to abd^{-1}c$ and $c^{-1}ba^{-1}d^{-1}$ by permuting the first and flipping the second
$\to abd^{-1}ba^{-1}d^{-1}$ by pasting
$\to a^{-1}d^{-1}abd^{-1}b$ by permuting
$\to a^{-1}d^{-1}abe$ and $e^{-1}d^{-1}b$ by cutting along e
$\to ea^{-1}d^{-1}ab$ and $b^{-1}de$ py permuting the first and by flipping the second
$\to ea^{-1}d^{-1}ade$ by pasting
$\to a^{-1}d^{-1}adee$ by permuting
Here $a^{-1}d^{-1}adee$ is a labelling scheme of $T\#P^2$.

1.4 The Classification Theorem for Surfaces

Definition 1.4.1. let ω be a proper labelling scheme for a single polygonal region. We say that ω is of **torus type** if each label in it appears once with exponent +1 and once with exponent -1. Otherwise, we say ω is of **projective type**.

Definition 1.4.2. Suppose $\omega_1, \cdots, \omega_k$ is a labelling scheme for the polygonal regions $P_1, \cdots, P_k$. If each label appears exactly twice in this scheme, we call it a **proper** labelling scheme.

Lemma 1.4.3. *Let ω be a proper scheme of the form*

$$\omega = (y_0)a(y_1)a(y_2),$$

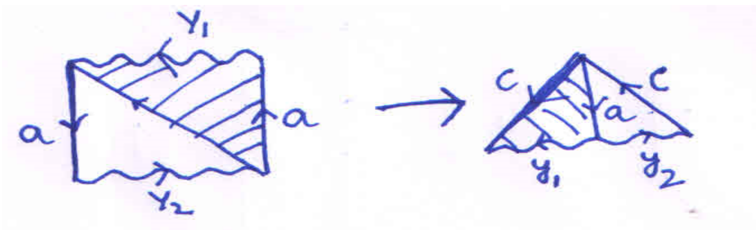
where some of the y_i may be empty. Then one has the equivalence

$$\omega \sim aa(y_0 y_1^{-1} y_2)$$

where y_1^{-1} denotes the formal inverse of y_1.

Proof. Step 1: If y_0 is empty, then we have to prove that

$$a(y_1)a(y_2) \sim aa(y_1^{-1} y_2)$$



Here if y_1 is empty then we can see it directly. If y_2 is empty then also it is clear by first fliping, then permuting and then relabelling. If both y_1 and y_2 are non empty then by elementary operations
$a y_1 a y_2 \to c^{-1} a y_1$ and $a y_2 c$ by cutting
$\to y_1^{-1} a^{-1} c$ and $a y_2 c$ by fliping the first one
$\to c y_1^{-1} a^{-1}$ and $a y_2 c$ by permuting the first one
$\to c y_1^{-1} y_2 c$ by pasting
$\to c c y_1^{-1} y_2$ by permuting

Step 2: Where y_0 is non empty. Then

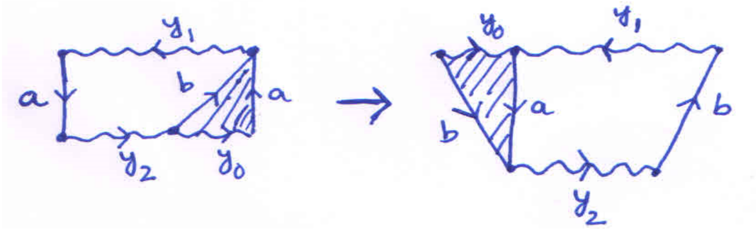
$$\omega = (y_0)a(y_1)a(y_2).$$

If both y_1 and y_2 are empty then permutation suffices. Otherwise we can apply cutting and pasting arguments to show that

$$\omega \sim b(y_2)b(y_1 y_0^{-1}).$$

It follows that

$\omega \sim bb(y_2^{-1}y_1y_0^{-1})$ by step 1
$\sim (y_0y_1^{-1}y_2)b^{-1}b^{-1}$ by flipping
$\sim aa(y_0y_1^{-1}y_2)$ by permuting and relabelling.



□

Corollary 1.4.4. *If ω is a scheme of projective type, then ω is equivalent to a scheme of the same length of the form*

$$(a_1a_2)(a_2a_2)\cdots(a_ka_k)\omega_1$$

where $k \geq 1$ and ω_1 is either empty or of torus type.

Proof. If $\omega = (y_0)a(y_1)(y_2)$, then by applying Lemma 1.4.3 ω is equivalent to a scheme of form $\omega_1 = aa\omega^1$, where $\omega^1 = y_0y_1^{-1}y_2$. Here ω and ω_1 have the same length. If ω^1 is torus type, we are finished. Otherwise, we can write ω_1 as

$$\omega_1 = aa(z_0)b(z_1)b(z_2) = (aaz_0)b(z_1)b(z_2).$$

Again applying the Lemma 1.4.3 we can see that ω_1 is equivalent to a scheme ω_{11}, where ω_{11} is of the form

$$\omega_{11} = bb(aaz_0z_1^{-1}z_2) = bbaa\omega_2, \text{ where } \omega_2 = z_0z_1^{-1}z_2.$$

Here both ω and ω_{11} have the same length. If ω_2 is torus type, then we are finished. Otherwise, we will proceed similarly. □

Lemma 1.4.5. *Let ω be a proper scheme of the form $\omega = \omega_0\omega_1$, where ω_1 is a scheme of torus type that does not contain two adjacent terms having the same label. Then ω is equivalent to a scheme of the form $\omega_0\omega_2$, where ω_2 has the same length as ω_1 and has the form*

$$\omega_2 = aba^{-1}b^{-1}\omega_3.$$

where ω_3 is of torus type or is empty.

Our final step of classification of surfaces is to show that a connected sum of projective planes and tori is equivalent to a connected sum of projective planes alone.

Lemma 1.4.6. *Let ω be a proper scheme of the form*

$$\omega = \omega_0(cc)(aba^{-1}b^{-1})\omega_1,$$

Then ω is equivalent to the scheme

$$\omega^1 = \omega_0(aabbcc)\omega_1.$$

Proof. We have already stated that for proper schemes we have

$(\star)$ $$(y_0)a(y_1)a(y_2) \sim aa(y_0y_1^{-1}y_2)$$

We proceed as follows:

$\omega \sim (cc)(aba^{-1}b^{-1}\omega_1\omega_0)$ by permuting

$= cc(ab)(ba)^{-1}(\omega_1\omega_0) \sim (ab)c(ba)c(\omega_1\omega_0)$ by $(\star)$ read backwards

$= (a)b(c)b(ac\omega_1\omega_0)$

$\sim bb(ac^{-1}ac\omega_1\omega_0)$ by $(\star)$

$= (bb)a(c)^{-1}a(c\omega_1\omega_0)$

$\sim aa(bbcc\omega_1\omega_0)$ by$(\star)$

$\sim \omega_0 aabbcc\omega_1$ by permuting. □

Now we come to the classification theorem for surfaces upto homeomorphism.

Theorem 1.4.7. (The Classification Theorem). *Let X be the quotient space obtained from a polygonal region in the plane by pasting its edges together in pairs. Then X is homeomorphic either to S^2, to the n-fold torus $\mathbb{T}_n$, or to the m-fold projective plane $\mathbb{P}_m$.*

Proof. Let ω be the labelling scheme by which one forms the space X from a polygonal region P. Then ω is the proper scheme of length at least 4. We will show that ω is equivalent to one of the following schemes:

1. $aa^{-1}bb^{-1}$,

2. $abab$,

3. $(a_1a_1)(a_2a_2)\cdots(a_ma_m)$ with $m \geq 2$,

4. $(a_1b_1a_1^{-1}b_1{-}1)(a_2b_2a_2^{-1}b_2{-}1)\cdots(a_nb_na_n^{-1}b_n{-}1)$ with $n \geq 1$

The first scheme denotes the space S^2, and the second scheme denotes the space P^2. The third scheme leads to the space $\mathbb{P}_m$ and fourth one to the space $\mathbb{T}_n$

Step 1: Let ω be a proper scheme of torus type. Then in step 1 we show that ω is either equivalent to sheme of type (1) or to a scheme of type (4). If ω has length four, then it can be written in one of the forms mentioned below.

$$aa^{-1}bb^{-1} \quad \textit{or} \quad aba^{-1}b^{-1}$$

The first one is of type (1) and the second one is of type (4).

We proceed by induction when ω has length greater than four. If ω contains two adjacent terms having the same label but opposite exponents, then we apply cancelling operation to reduce ω to a shorter scheme. If ω is equivalent to a shorter scheme of torus type, then we will apply the induction hypothesis. Otherwise, if ω contains no pairs of adjacent terms having the same label, then we will apply Lemma 1.4.5 by taking two cases: ω_0 is empty and ω_0 is non empty. If ω_0 is empty, ω is equivalent to a scheme having the same length as ω and the form

$$(aba^{-1}b^{-1})(cdc^{-1}d^{-1})\omega_4,$$

where ω_4 is empty or torus type. If ω_4 is empty, then we are finished. Otherwise, it will continue similarly by applying the lemma 1.4.5 again.

Step 2: Let ω be a proper scheme of projective type. In step 2 we show that ω is either equivalent to scheme of type (2) or to a scheme of type (3). If ω has length four, then by corollary 1.4.4 ω is equivalent to one of the schemes $aabb$ or $aab^{-1}b$. The first one is of type (3). By applying lemma 1.4.5 $aab^{-1}b$ can be written as $abab$, which is of type (2).

When ω has length greater than four, ω is equivalent to a scheme of the form

$$\omega^1 = (a_1a_1)\cdots(a_ka_k)\omega_1,$$

where $k \geq 1$ and ω_1 is of torus type or empty by the application of Corollary 1.4.4. If ω^1 is equivalent to a shorter scheme of projective type then induction hypothesis will be applied. Otherwise, by applying Lemma 1.4.5 ω^1 is equivalent to a scheme of the form

$$\omega^{11} = (a_1 a_1) \cdots (a_k a_k) a b a^{-1} b^{-1} \omega_2,$$

where ω_2 is empty or of torus type. Then we apply Lemma 1.4.6 to conclude that ω^{11} is equivalent to the scheme

$$(a_1 a_1) \cdots (a_k a_k) a a b b \omega_2.$$

We continue similarly. □

Chapter 2

Classification of surfaces - second proof

In this chapter we give another proof of classification of surfaces due to Burgess[1]. We closely follow the proof as it appears in [1].

For this chapter "surfaces" means compact connected 2-manifold without a boundary.

Theorem 2.1. *Every surface is homeomorphic with a space obtained by removing a finite number of disjoint disks from a 2-sphere and replacing each of them with a Mobius band or a punctured torus.*

To prove this theorem we need to model the given surface with a polygonal disk D whose edges are identical in pairs. The construction of such a polygonal disk depens upon the classical theorem that every surface can be triangulated.

Notation:

1. Int$D$ means the set of interior of D.
2. Bd(D)=$D \setminus IntD$

Definition 2.2. Let M_1 and M_2 be two disjoint surfaces and D_1 and D_2 be disks in M_1 and M_2, respectively. Let $M = (M_1 - Int\ D_1) \sqcup (M_2 - Int\ D_2)$, where $M_1 - Int\ D_1$ and $M_2 - Int\ D_2$ are identified on their boundaries, i.e., for some homeomorphism h

of Bd D_1 onto Bd D_2, M is the quotient space

$$[(M_1 - Int\ D_1) \sqcup (M_2 - Int\ D_2)]/(x, h(x))|x \in Bd\ D_1.$$

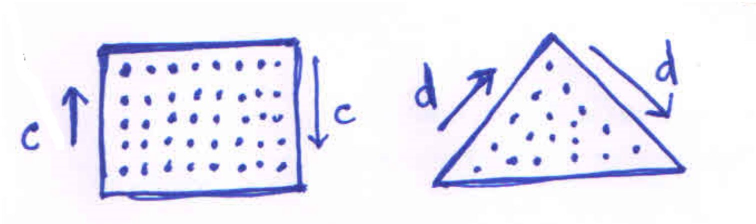
The surface M is called the **connected sum** of M_1 and M_2 and is denoted by $M_1 \# M_2$.

We know that $M_1 \# M_2$ is independent of the choice of D_1 and D_2 and that

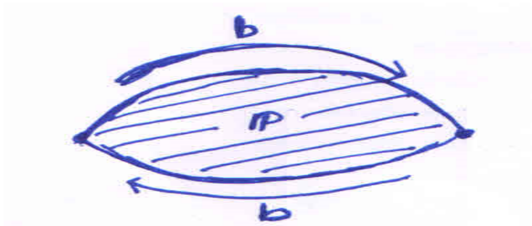
$$M_1 \#(M_2 \# M_3) = (M_1 \# M_2) \# M_3$$

Definition 2.3. A **2-sphere**, denoted by S^2, it is a space that is homeomorphic with the graph $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$.

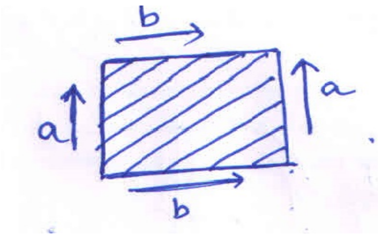
Definition 2.4. A **Mobius band** is a space obtained by identifying, or sewing, two opposite edges of a rectangular disk as indicated in Fig. below. Equivalently, a Mobius band is obtained by identifying two adjacent edges of a triangular disk as indicated in the figure below.



Definition 2.5. A **projective plane**, denoted by $\mathbb{P}^2$, it is a space obtained by sewing a Mobius band and a disk together on their boundaries. Equivalently, a projective plane is obtained by the identification indicated in the figure below.

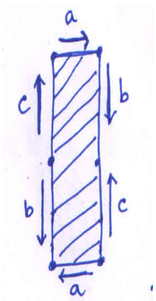


Definition 2.6. A **torus**, denoted by $\mathbb{T}$ is a space homeomorphic with the Cartesian product $S^1 \times S^1$, where S^1 denotes a circle. A torus is obtained by identifying the four edges of the square disk as indicated in the figure below.

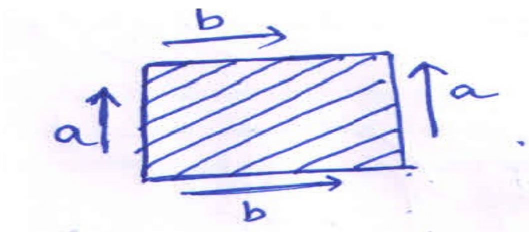


Notation: Let $D(n)$ denotes a disk with n edges on its boundary, where n is even, and $M(n)$ denotes a surface that is obtained by identifying, in pairs, the edges of D_n.

Definition 2.7. Two identified edges of a disk are called a twisted pair if the identification involves the same direction for the two edges around the boundary of D



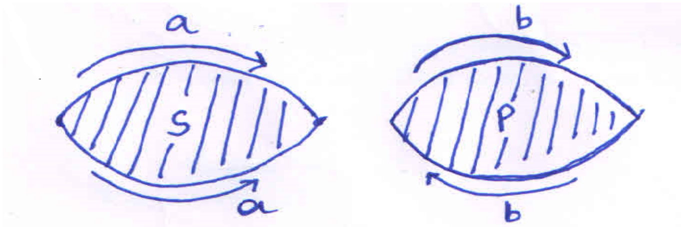
Definition 2.8. Two pairs of identified edges of disk D are called separated pairs if the two edges in one pair separate two in the other pair on the boundary of D.



The second step in proof of Theorem 2.1 is to give two lemmas that will be used in the inductive procedure in the fourth step.

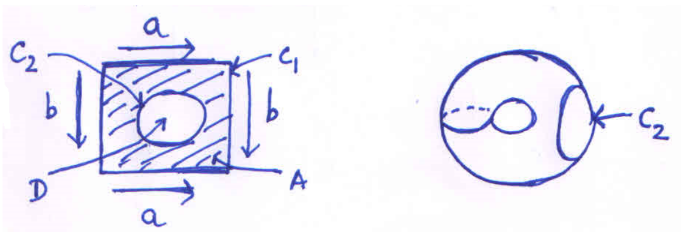
Lemma 2.9. *If $M(2)$ is a surface obtained by identifying the two edges of disk D(2), then M(2) is either a sphere or a projective plane.*

We can see this by the figure below:



Lemma 2.10. *If A is annulus with n edges (n even) on one component C_1 of its boundary, then any space obtained by identifying these n edges in pairs is homeomorphic with a punctured $M(n)$, i.e., there is a disk D in $M(n)$ such that the resulting space is homeomorphic with $M(n) \setminus IntD$.*

We can see that a disk D' with boundary C_1 can be obtained by identifying the boundary of a disk D with the other component C_2 of the boundary of an annulus A. So $D' = D \cup A$. We can identify the edges of D' to produce a surface $M(n)$ that contains D. We can see this for a special case where identification of edges of D' produces a torus:



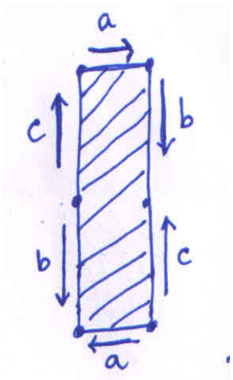
The third step in the proof is to notice that Theorem 2.1 can be re-stated as follow:

Theorem 2.11. *If M is a surface, different from a sphere, then $M = M_1 \# M_2 \# \cdots \# M_j$, where for each i, M_i is either a projective plane or a torus.*

The fourth step of the proof is to prove Theorem 2.11 by induction on the number of edges in the disk D obtained by the first step. Let D_n be a disk with edges identified to obtain $M(n)$. We assume by Lemma 2.9 that $n \geqslant 4$. The inductive argument is separated into four cases, with some overlap among them.

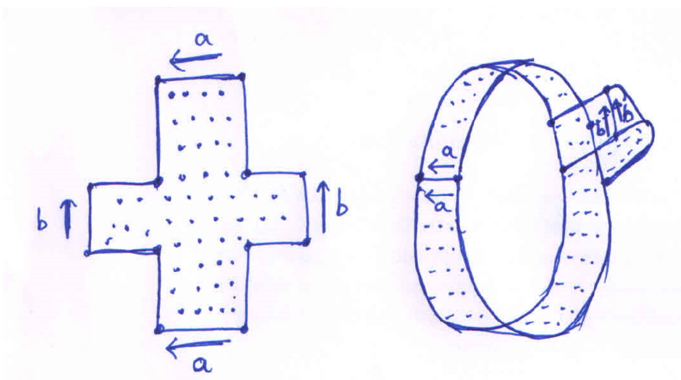
Case 1: If there is a twisted pair in the identification of the edges of $D(n)$, then we can identify the two edges in such a twisted pair to obtained a Mobius band B with $n-2$ edges remains in its boundary. In the figure given below we can see that there is a annulus A in B such that one component of B is the boundary of A. Let j be the other component of the boundary of A. Here we can see that $B-A$ is a mobius band with j its boundary. Idetifying the edges in the boundary of B and applying the Lemma 2.10, we can see that

$$M(n) = \mathbb{P}^2 \# M(n-2)$$

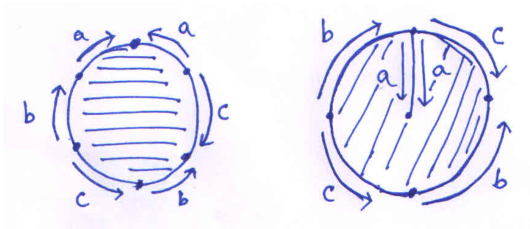


Case 2: If there are two separated pairs of edges of $D(n)$, that are nontwisted. If $n=4$, then we can directly conclude that $M(n)$ is torus. If $n \geq 6$,then by given figure we get a puntured torus results from identifying the edges in two separed pairs of edges that are nontwisted. Then by applying Lemma 2.10

$$M(n) = \mathbb{T} \# M(n-4)$$

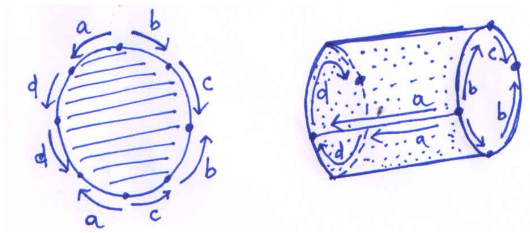


Case 3: If there is a twisted pair of adjacent edges in $D(n)$, then a disk $D(n-2)$ is obtained by identifying these two adjacent edges as in the given figure.



Case 4: If there is a nontwisted pair of nonadjacent edges of $D(n)$ that does not separate any other identified pair then an annulus A is obtained by identifying the edges in some such nontwisted pair as shown in the figure below. If $n \geq 6$, then there are two positive even integers p and q such that $p + q = n - 2$, where p donotes the number of edges in one component of the boundary of annulus A and q the number in the other component. Each edge in each of these two components must be identified with an edge in the same component . Then by Lemma 2.10,

$$M(n) = M(p)\#M(q) = M(n-2).$$



Remark. We have already shown in Example 1.3.4 that connected sum of three projective planes is topologically the same as the connected sum of a torus and a projective plane. This can be used to obtain the following more precise classification of surfaces.

Theorem 2.12. *Any surface different from a sphere is either a connected sum of a finite number of* $\mathbb{T}$ *or a connected sum of finite number of* $\mathbb{P}^2$.

Chapter 3

Homotopy types of manifolds

In this chapter proves that homotopy types of all manifolds with boundary, are countable in number. We closely follow Mather[6] and the proof there in. This theorem of Mather should be compared to the classification of surfaces which can also be used to show that there are countable homotopy classes of compact surfaces without boundary. Mather's achieves more in that manifolds with boundary are also admitted in his theorem.

Definition 3.1. A CW complex is a space X constructed in the following way:

1. Start with a discrete sets X^0, the 0-cells of X.

2. Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps

$$\Phi_\alpha : S^{n-1} \to X^{n-1}$$

   This means that X^n is the quotient space of $X^{n-1} \sqcup_\alpha D^n_\alpha$ under the identification $x \backsim \Phi_\alpha(x)$ for $X \in \partial D^n_\alpha$. The cell e^n_α is the homeomorphic image of $D^n_\alpha \setminus \partial D^n_\alpha$ under the quotient map.

3. $X = \cup_n X^n$ with the weak topology. A set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

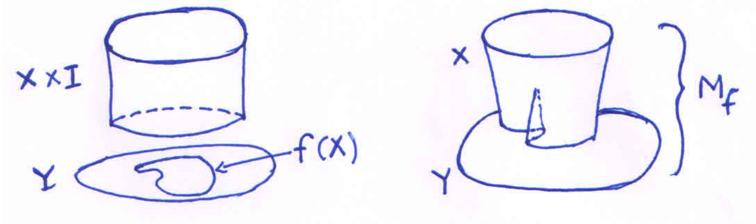
We can see that S^2 is a CW-complex. It consists of only two cells X^0 and X^2, constructed via map

$$\Phi_\alpha : S^1 \to X^0$$

Where S^1 is the boundary of a disk.

Definition 3.2. Let f: $X \longrightarrow Y$ and g: $Y \longrightarrow X$ be continuous maps. Suppose that the map $g \circ f$: $X \longrightarrow X$ is homotopic to the identity map of X and the map $f \circ g$: $Y \longrightarrow Y$ is homotopic to the identity map of Y. Then the maps f and g are called **homotopy equivalance** and each is said to be a homotopy inverse of the other. Two spaces that are homotopy equivalance are said to have the same **homotopy type**.

Definition 3.3. For a map $f : X \longrightarrow Y$, the **mapping cylinder** M_f is the quotient space of the disjoint union $(X \times I) \sqcup Y$ obtained by identifying each $(x,1) \in X \times I$ with $f(X) \subset Y$.



Remark. A mapping cylinder M_f deformation retracts to the subspace Y by sliding each point (x, t) along the segment $\{x\} \times I \subset M_f$ to the endpoint $f(x) \in Y$.

Definition 3.4. Let X and Y be CW-complexes. Then a map $f : X \longrightarrow Y$,satisfying $f(X^n) \subset Y^n$ for all n, is called a **cellular map**.

Definition 3.5. Let A be the subspace of X. We say that A is a **strong deformation retract** of X, if the identity map of X is homotopic to a map that carries all of X into A such that each point of X remains fixed during the homotopy. In the other words, there is a continuous map $H : X \times I \to X$ such that

1. $H(x, 0) = x$ for all $x \in X$.

2. $H(x, 1) \in A$ for all $x \in X$.

3. $H(a, t) = a$ for all $a \in A$.

Lemma 3.6. *Let A be a strong deformation retract of X; let $x_0 \in A$. Then the inclusion map*

$$j : (A, x_0) \to (X, x_0)$$

induces an isomorphism of fundamental groups.

Example 3.7. let B denote the z-axis in $\mathbb{R}^3$. Consider the space $\mathbb{R}^3 - B$. It has the punctured xy-plan $(\mathbb{R}^2 - 0) \times 0$ as a strong deformation retract. The map H defines by the equation

$$H(x, y, z, t) = (x, y, (1-t)z)$$

is a strong deformation retract, it gradually collapses each line parallel to the z-axis into the point where the line intersects the xy-plane.

Definition 3.8. Let A be a subspace of X. We say that A is a **deformation retract** of X, if there is a continuous map $H\colon X \times I \to X$ such that

1. $H(x, 0) = x$ for all $x \in X$.
2. $H(x, 1) \in A$ for all $x \in X$.
3. $H(a, 1) = a$ for all $a \in A$.

Definition 3.9. A space Y is said to be **dominated** by a space X if there are maps $Y \xrightarrow{i} X \xrightarrow{r} Y$ with $r \circ i \simeq 1_Y$. This makes the notion of a retract into something that depends only on the homotopy types of the spaces involved.

Definition 3.10. If there exists an open set U such that

$$A \subset U \subset X$$

and A is a retract of U then A is called a **neighborhood retract** of X.
A space X is an **absolute neighborhood retract** (or ANR) if for every normal space Y that embeds X as a closed subset, X is a neighborhood retract of Y. The n-sphere S^n is an absolute neighborhood retract.

Theorem 3.11. *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$.*

1. *The homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism.*
2. *Let $H = p_*(\pi_1(E, e_0))$. The lifting correspondence ϕ induces an injective map*

$$\phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$$

of the collection of right cosets of H into $p^{-1}(b_0)$, which is bijective if E is path connected.

We have assumed throughout this chapter that both E and B are locally path connected and path connected. So from the above theorem it is clear that p_* is injective, then

$$H_0 = p_*(\pi_1(E, e_0))$$

is a subgroup of $\pi_1(B, b_0)$ isomorphic to $\pi_1(E, e_0)$.

Theorem 3.12. *Let C be a topological space dominated by a finite CW-complex K. Then $C \times S^1$ has the homotopy type of a finite CW-complex.*

Proof. Here C is a topological space dominated by a finite CW-complex K. So by definition there exist maps

$$C \xrightarrow{i} K \xrightarrow{r} C$$

such that $r \circ i \simeq 1_C$. We know that mapping cylinder of the given map $K \xrightarrow{r} C$ is $M_r = (K \times I) \sqcup C$ identifying each $(k, 1) \in K \times I$ with $r(k) \in Y$. Then mapping cylinder M_r has a strong deformation retract to the subspace C. Clearly M_r and C have same homotopy type as C. Therefore it is sufficient to prove the result for M_r in place of c. The map $i : M_r \to K$ induces a map $f : M_r \to M_r$ and the image of f lies in K embeded in M_r and $f \simeq 1_{M_r}$. We may further suppose that $f|_K$ is cellular.

Define the mapping torus $T(f)$ of f by taking $M_r \times I$ and identifying $m \times 1$ with $f(m) \times 0$ for each $m \in M_r$. We know that $1 \simeq f$ which implies that $T(1) \simeq T(f)$. The space $T(1) = M_r \times S^1$, So $M_r \times S^1 \simeq T(f)$. Now define a homotopy $h_t : T(f) \to T(f)$ by

$$h_t(m \times s) = m \times (s + t) \ for \ s + t \leq 1$$

$$= f(m) \times (s + t - 1) \ for \ s + t \geq 1$$

This homotopy can be easily visualised just by pushing the mapping torus through an angle $2\pi t$. Here

$$h_0(m \times s) = m \times s$$

$$h_1(m \times S) = f(m) \times s$$

$$= m \times (s + 1)$$

Notice that $h_1(m \times s) \in T(f|_K)$ and $h_1(T(f|_K))$ is identity of $T(f|_K)$, so this homotopy is a weak deformation retract of $T(f)$ to $T(f|_K)$, naturally embedded in $T(f)$. We know that if a space is weak retraction of another space, then both spaces have same homotopy type. Therefore $T(f) \simeq T(f|_K)$. Since $T(f|_K)$ is a finite CW-complex, this prove the theorem.

□

Theorem 3.13. *The set of homotopy types of spaces dominated by finite CW-complex is countable.*

Proof. Let C be any topological space which is dominated by a finite CW-complex. Then by Theorem 3.12, $C \times S^1$ is homotopy equivalant to a finite CW-complex K. We need to prove that the set of homotopy types of finite CW-complex is countable.

We are already given that $C \times S^1 \simeq K$. Choose a particular homotopy equivalance $j : C \times S^1 \to K$ for each such space C(we suppose that all spaces have base points,which are preserved by maps but not by homotopies). Now we can see that $C \times \mathbb{R}$ is the covering space of $C \times S^1$ because $\pi_1(C \times \mathbb{R}) = \pi_1(C) \times \pi_1(\mathbb{R}) = \pi_1(C)$ is a subgroup of $\pi_1(C \times S^1)$. Here C is homotopy equivalant to $C \times R$ because C is a strong deformation retract of $C \times \mathbb{R}$. The map H defined by the equation

$$H(c, r, t) = (c, (1 - t)r)$$

is a strong deformation retraction. It follows that C is also homotopy equivalent to the covering space of K, which is determined by the subgroup $h_*\pi_1(C)$ of $\pi_1(K)$. But $\pi_1(K)$ is countable because K is a finite CW-complex and $\pi_1(C)$ is finitely generated. Hence there are only countable number of such subgroups. This proves the theorem.

□

Corollary 3.14. *The set of homotopy types of compact topological manifold is countable.*

Proof. By a theorem of Hanner[5], homotopy types of compact topological manifold are compact ANR and Compact ANR are dominated by a finite CW-complex. Therefore we can apply Theorem 3.13, which says that the set of homotopy types of spaces dominated by finite CW-complexes is countable. This prove the corollary. □

Chapter 4

Homeomorphism types of manifolds

We have seen in chapter 3 that there are countable homotopy classes of manifolds. One would like to ask for even more: what is the number of homeomorphism classes of manifolds. This chapter explains the answer which is a theorem of Cheeger-Kister[2]. We closely follow the proof as it appears in [2].

Notation.

1. B_r is a closed ball of radius r in $\mathbb{R}^n$.
2. M^n denotes a manifold of dimension n.

Theorem 4.1 *There are precisely a countable number of compact topological manifolds (boundary permitted), up to homeomorphism.*

Sketch of Proof: We will consider compact topological manifolds without boundary for simplicity.
Step 1: Suppose the theorem is false. Then there are an uncountable number of compact topological manifolds of some fixed dimension n, which are not homeomorphic to each other.
Step 2: Let $\{M^n_\alpha\}_{\alpha \in A}$ be uncountable number of compact topological manifolds of dimension n. Then for each M^n_α, we can find a collection of imbedding $h_{\alpha j} : B_2 \to M^n_\alpha$ $j = 1, 2, \cdots, k_\alpha$, such that $\{h_{\alpha j}(B_1)\}_{j=1}^{k_\alpha}$ covers M^n_α.
Step 3: By possibly choosing an uncounatble subcollection from $\{M^n_\alpha\}$, we can assume without loss of generality that $k_\alpha = k$ for all α. We can also assume that $h_{\alpha j}|B_1$

can be extended to an imbedding $h_{\alpha j}: B_{k+1} \to M^n_\alpha$ of B_{k+1} into M_α, $j = 1, 2, \cdots, k$, by reperametrizing.

Step 4: By Whitney's theorem, each M^n_α has an embedding in $\mathbb{R}^l$ for $l = 2n + 1$.

Step 5: If d is the metric in $\mathbb{R}^l$, define $\epsilon_{\alpha jm} = d(h_{\alpha j}(B_m), \overline{M_\alpha - h_{\alpha j}(B_{m+1})}\ )$ and let $\epsilon_\alpha = min_{j,m}\{\epsilon_{\alpha jm}\}$ for $\alpha \in A$ and $j = 1, 2, \cdots, k$,$m = 1, 2, \cdots, k + 1$. Clearly $\epsilon_\alpha > 0$ for $\alpha \in A$.

Step 6: By choosing a subcollection again we can assume there exist an $\epsilon > 0$ such that $\epsilon_\alpha > \epsilon$ for all α in A.

Step 7: Each M^n_α determines an imbedding $g_\alpha : B_{k+1} \to \mathbb{R}^{kl} = \mathbb{R}^l \times \mathbb{R}^1 \cdots \times \mathbb{R}^l$ by $g_\alpha(x) = (h_{\alpha_1}(x), \cdots h_{\alpha_k}(x))$.

Step 8: The set of all such imbeddings under the uniform metric:

$$d_u(g_\alpha, g_\beta) = max_{x \in B(k+1)} d(g_\alpha(x), g_\beta(x)),$$

is a separable metric space, hence some g_{α_0} is a limit point of a sequence of distinct imbedding $g_{\alpha_1}, g_{\alpha_2}, \cdots$.

Now we will show that M_{α_0} is homeomorphic to M_{α_i}, for i sufficient large. Which is a contradiction to our initial assumption. Furthermore, this homeomorphism can be taken to be arbitarily close to the identity as measured by the metric d.

Step 9: Let $V_j(m) = h_{\alpha_0 j}(B_m)$, $j = 1, 2, \cdots, k$, $m = 1, 2, \cdots, k + 1$. To simplify our notation we denote M_{α_i} by M', for fixed but arbitary large i, and we denote $h_{\alpha_i j}(B_m) \subset M'$ by $V'_j(m)$, $j = 1, 2, \cdots, k$, $m = 1, 2, \cdots, k + 1$. Now let

$$U_j(m) = \bigcup_{p=1}^{j} V_p(m) \ and \ U'_j(m) = \bigcup_{p=1}^{j} V'_p(m).$$

Note that $U_k(1) = M$ and $U'_k(1) = M'$ by normal test in step 2.

Step 10: Define $f_j : V_j(k+1) \to V'_j(k+1)$ as $h_{\alpha_i j} o h^{-1}_{\alpha_0 j}\ for j = 1, \cdots, k$ and note that each f_i can be taken as close to the identity as we like for M' sufficient far out in the sequence $M_{\alpha_1}, M_{\alpha_2}, \cdots$. We proceed to construct a homeomorphism from M to M' inductively on the sets $U_j(m)$.

Step 11: Suppose we can construct an imbedding $g_i : U_j(m) \to M'$ as close as we like to the identity by choosing M' sufficient far out in the sequence. We will show that we can construct an imbedding $g_{i+1} : U_{j+1}(m - 1) \to M'$ as close to the identity as we please. Hence, we will start by letting $g_1 = f_1$ and $m = k$, in $k-1$ steps we will have an imbedding $g_k : U_k(1) \to M'$, the desired homeomorphism.

Step 12: First we see that $g_j(U_j(m) \cap V_{j+1}(m)) \subset V'_{j+1}(m+1)$ if M' is choosen sufficient far out and g_i is close to the identity (relative to our previous ϵ). Then $f_{j+1}^{-1} g_j$ is defined on $U_j(m) \cap V_{j+1}(m)$ and close to the identity.

Step 13: Letting N be an open set in M with $U_j(m-1) \cap V_{j+1}(m-1) \subset N \subset U_j(m) \cap V_{j+1}(m)$. We can extent $f_{j+1}^{-1} g_j | N : N \cdots V_{j+1}(m)$ to an onto homeomorphism $h : V_{j+1}(m) \to V_{j+1}(m)$ close to the identity, using the theorem of [3].

Step 14: Now define $g_{i+1} : U_{j+1}(m-1) \to M'$ by

$$g_{j+1}(x) = \begin{cases} g_j(x) \ for \ x \ in \ U_j(m-1) \\ f_{j+1} \ h(x) \ for \ x \ in \ V_{j+1}(m-1). \end{cases}$$

By definition of h, g_{j+1} is well defined. Since g_{j+1} can be extended to N as well using either half of the definition, it is easily seen that g_{j+1} is local homeomorphism, since it is an imbedding on the two open sets $U_j(m-1) \cup (N \cap U_{j+1}(m-1))$ and $V_{j+1}(m-1) \cup (N \cap U_{j+1}(m-1))$.

Step 15: It would fail to be an imbedding only if $g_{j+1}(x) = g_{j+1}(y)$ for some x and y in $U_j(m-1) - N$ and $V_{j+1}(m-1) - N$ respectively, two compact disjoint sets with a positive distance between them.

Step 16: Since these two sets are independent of the choice of M, and the g_j we startwith, we choose M' sufficient far out in the sequence and g_j, and hence g_{j+1}, close enough to the identity so that g_{j+1} is 1-1. This completes the induction and the proof in the case of manifolds without boundary.

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