Conjugacy classes in Möbius groups

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A dissertation submitted for the partial fulfilment of BS-MS dual degree in Science

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Indian Institute of Science Education and Research Mohali

April 2014

Certificate of Examination

This is to certify that the dissertation titled **Conjugacy classes in Möbius groups** submitted by **Harjit Singh Sandhu** (Reg. No. MS09056) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Mahender Singh	Dr. Amit Kulshrestha	Dr. K. Gongopadhyay (Supervisor)

Dated: April 21, 2014

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Krishnendu Gongopadhyay at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Harjit Singh Sandhu
(Candidate)

Dated: April 21, 2014

In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Krishnendu Gongopadhyay
(Supervisor)

Acknowledgment

I would like to express my gratitude towards my supervisor Dr. Krishnendu Gongopadhyay for motivating me to carry out my thesis research in the area of ”Conjugacy Classes in Möbius Groups” and for providing me kind supervision and guidance in completing my dissertation. I am also grateful to Dr. Amit Kulshrestha and Dr. Mahender Singh for their valuable suggestions which helped me to improve the quality of my dissertation. I am also thankful to IISER Mohali for providing me the necessary infrastructure and facilities to accomplish my work. I would also like to thank Department of Science and Technology, Govt. of India, for providing me financial assistance in the form of INSPIRE fellowship.

I am also grateful to my family for supporting me and providing me mental strength to accomplish this work. I would like to thank my friends for helping me whenever needed specially Saksham for helping me with LaTeX and Rahul for helping me with mathematics.

Finally, I would like to acknowledge that the material presented in this thesis are based on other people's work. Every theorem in this thesis can be found elsewhere. At the beginning of every chapter, we have clearly mentioned the main texts that laid the foundation of the chapter. If I have made any contribution then it is the selection and presentation of the materials from different sources those are listed in the bibliography.

Harjit Singh Sandhu

List of Figures

Notation

$\mathbb{S}^n$	The conformal boundary of the hyperbolic space
$M(n)$	The group of conformal diffeomorphisms of $\mathbb{S}^n$
$M_o(n)$	The identity component of $M(n)$
$\mathbb{R}^n$	The n-dimensional real space
$\mathbb{H}^n$	The n-dimensional hyperbolic space
$O(1, n-1)$	The set of all Lorentzian $n \times n$ matrices
$SO(1, n-1)$	Subgroup of $O(1, n-1)$ with det. 1
$\mathfrak{J}(2:k)$	The space of all decomposition of $\mathbb{E}^{2k}$
$\mathcal{G}_k(n)$	The Grassmannian of k-dimensional vector subspaces of $\mathbb{R}^n$
$S_k(n)$	The Grassmannian of k-dimensional spheres in $\mathbb{S}^n$
$\mathcal{O}_k(n)$	The conjugacy classes of the regular k-rotations with no rotation angle π
$\mathcal{O}_k^-(n)$	The conjugacy classes of the regular k-rotations with rotation angle π
$\mathcal{A}_k(n)$	The collection of all k-dimensional affine subspaces of $\mathbb{R}^n$

Abstract

Let $\mathbb{H}^{n+1}$ denote the $n+1$-dimensional (real) hyperbolic space and let $\mathbb{S}^n$ denote the conformal boundary of the hyperbolic space. $M(n)$ denotes the group of conformal diffeomorphisms of $\mathbb{S}^n$ and $M_o(n)$ be defined as identity component which consists of all orientation preserving elements in $M(n)$. Conjugacy classes of isometrics in $M_o(n)$ depends on the conjugacy of T and T^{-1} in $M_o(n)$. For an element $T \in M(n)$, T and T^{-1} are conjugate in $M(n)$, but they may not be conjugate in $M_o(n)$. T is called real if T and T^{-1} are conjugate to each other in $M_0(n)$. Let T be an element in $M_o(n)$, so to T there is an associated element T_o in $SO(n+1)$. If the complex conjugate eigenvalues of T_o are given by $\{e^{i\theta_j}, e^{-i\theta_j}\}$, $0 < \theta j \leqslant \pi$, $j = 1, \cdots, k$, then $\theta_1, \cdots, \theta_k$ are called the rotation angles of T. T is called a *regular* element if the *rotation angles* of T are distinct from each-other. After classification of the real elements in $M_o(n)$ we have parametrized the conjugacy classes of regular elements in Mo(n). In the parametrization, when T is not conjugate to T^{-1}, then enlarge the group and consider the conjugacy class of T in $M(n)$. So each such conjugacy class can be induced with a fibration structure.

Contents

Chapter 1

Hyperbolic Geometry

The theory and proofs in this chapter are based on Ratcliffe, J.G.: Foundation of Hyperbolic Manifolds, Graduate Texts in Mathematics 149. Springer, Berlin (1994).

In the first half of the nineteenth century Hyperbolic geometry was created in order to prove the dependence of Euclid's fifth postulate on the first four. Around 300 B.C. Euclid wrote his famous elements. In thirteen volume work he brilliantly organized and presented the fundamental propositions of Greek geometry and number theory. Now in this chapter we will study the hyperbolic geometry defining a new inner product on $\mathbb{R}^n$ which we call the Lorentzian inner product, then we will proceed to the positive half of the sphere of unit imaginary radius in $\mathbb{R}^{n+1}$ which we call hyperbolic n-space $\mathbb{H}^n$.

1.1 Lorentzian n-Space

Let x, y be vectors in $\mathbb{R}^n$.The Lorentzian inner product of x, y is the real number given by

$$x \circ y = -x_1y_1 + x_2y_2 + \cdots + x_ny_n \tag{1.1}$$

Lorentzian n-space is the inner product space consisting of the vector space $\mathbb{R}^n$ with the Lorentzian inner product and is denoted by $\mathbb{R}^{1,n-1}$.

Let x be a vector in $\mathbb{R}^n$. The Lorentzian norm (length)of x is defined by the complex number

$$\|x\| = (x \circ x)^{1/2} \tag{1.2}$$

$\|x\|$ is either zero, positive, or positive imaginary. The absolute value of positive imaginary $\|x\|$ is denoted by $\||x\||$.

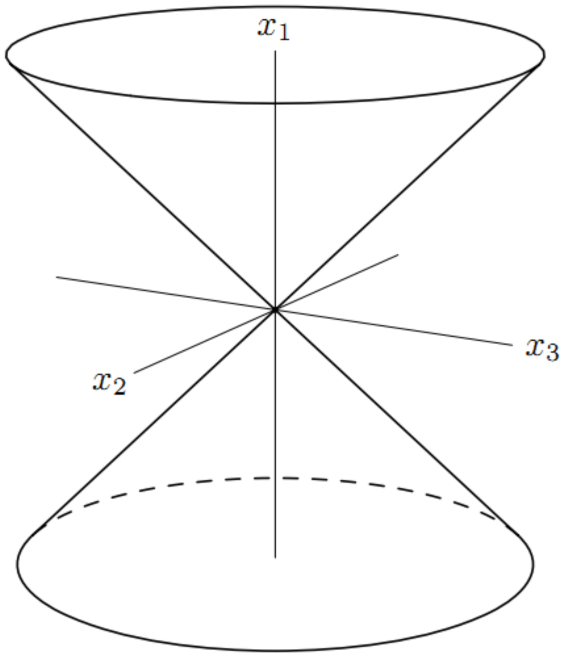

Figure 1.1: The light cone C^2 of $\mathbb{R}^{1,2}$

Image Courtesy: Ratcliffe, J.G.: Foundation of Hyperbolic Manifolds, Graduate Texts in Mathematics 149. Springer, Berlin p. 55 (1994).

We define a vector $\bar{x} \in \mathbb{R}^{n-1}$ by

$$\bar{x} = (x_2, x_3, \cdots, x_n) \tag{1.3}$$

Then

$$\|x\|^2 = -{x_1}^2 + |\bar{x}|^2 \tag{1.4}$$

If vectors $x, y \in \mathbb{R}^n$, then

$$x \circ y = -x_1 y_1 + \bar{x}\bar{y} \tag{1.5}$$

The hypercone C^{n-1} is defined by equation $|x_1| = |\bar{x}|$ which is set of all $x \in \mathbb{R}^n$ such that $\|x\| = 0$. The hypercone C^{n-1} is called the light cone of $\mathbb{R}^n$. If $\|x\| = 0$, then the vector x is said to be light-like. A light-like vector x in $\mathbb{R}^n$ is said to be positive (or negative) iff $x_1 > 0$ (or $x_1 < 0$).
If $\|x\| > 0$, then x is said to be space-like. So x is space-like iff $|x_1| < |\bar{x}|$. The exterior of C^{n-1} in $\mathbb{R}^n$ is the open subset of $\mathbb{R}^n$ consisting of all the space-like vectors.
If $\|x\|$ is imaginary, then x is said to be time-like. It can be easily seen that x is time-like iff $|x_1| > |\bar{x}|$. We say that time-like vector x is said to be positive (or negative) iff $x_1 > 0$ (or $x_1 < 0$). The interior of C^{n-1} in $\mathbb{R}^n$ is the open subset of $\mathbb{R}^n$ consisting of all the time-like vectors. A vector is said to be space-like if $\|x\| > 0$ then x is space-like iff $|x_1| < |\bar{x}|$.

Theorem 1. *Let x, y be nonzero nonspace-like vectors in $\mathbb{R}^n$ with the same parity. Then $x \circ y \leq 0$ with equality iff x, y are linearly dependent light-like vectors.*

Theorem 2. *If x, y are nonzero nonspace-like vectors in $\mathbb{R}^n$, with the same parity, and $t > 0$, then*

- *the vector tx has the same likeness and parity as x;*
- *the vector $x + y$ is nonspace-like with the same parity as x, y; moreover $x + y$ is light-like iff x, y are linearly dependent light-like vectors.*

A convex subset of $\mathbb{R}^n$ is the set of all positive (or negative) time-like vectors.

1.1.1 Lorentz Transformation

Definition 1.1. *A function $\phi : \mathbb{R} \to \mathbb{R}$ is a Lorentz transformation iff*

$$\phi(x) \circ \phi(y) = x \circ y \ \forall x, y \ in \ \mathbb{R}.$$

Let $\{v_1, v_2, \cdots, v_n\}$ is a basis of $\mathbb{R}^n$ then it called Lorentz orthonormal iff $v_1 \circ v_1 = -1$ and $v_i \circ v_j = \delta_{ij}$ otherwise. The standard basis $\{e_1, e_2, \cdots, e_n\}$ of $\mathbb{R}^n$ is Lorentz orthonormal.

Theorem 3. *A function $\phi : \mathbb{R}^n \in \mathbb{R}^n$ is a Lorentz transformation iff ϕ is linear and $\{\phi(e_1), \phi(e_2), \cdots, \phi(e_n)\}$ is a Lorentz orthonormal basis of $\mathbb{R}^n$.*

A real $n \times n$ matrix A called Lorentzian iff the associated linear transformation $A : \mathbb{R} \in \mathbb{R}$, given by $A(x) = Ax$, is Lorentzian. The set of all Lorentzian $n \times n$ matrices together with matrix multiplication forms a group $O(1, n-1)$, called the Lorentz group of $n \times n$ matrices. There exists an isomorphism between the groups $O(1, n-1)$ and the group of Lorentz transformations of $\mathbb{R}^n$.

Theorem 4. *Let A be a real $n \times n$ matrix, and let L be the $n \times n$ diagonal matrix defined by*

$$L = diag(-1, 1, \cdots, 1).$$

Then the following are equivalent:

- *The matrix A is Lorentzian.*
- *The equation $ALA^t = L$ is satisfied by the matrix A.*
- *The equation $A^tLA = L$ is satisfied by the matrix A.*
- *The columns of A creates a Lorentz orthonormal basis of $\mathbb{R}^n$.*
- *The rows of A creates a Lorentz orthonormal basis of $\mathbb{R}^n$.*

Suppose a Lorentzian matrix given by A. So $A^tLA = L$, clearly $(detA)^2 = 1$. Therefore $detA = \pm 1$. The set of all A in $O(1, n-1)$ with $detA = 1$ be $SO(1, n-1)$. So index two subgroup of $O(1, n-1)$ is $SO(1, n-1)$. This group $SO(1, n-1)$ is known as the special Lorentz group.
The set of positive and negative time-like vectors are the two connected components of all time-like vectors in $\mathbb{R}^n$. A Lorentzian matrix A is said to be positive (or negative) iff A transforms positive time-like vectors into positive (or negative) time-like vectors. For example, the matrix L is negative. So, a Lorentzian matrix is either negative or positive.
The set of all positive matrices in $O(1, n-1)$ is given by $PO(1, n-1)$. So $PO(1, n-1)$ is a subgroup of index two in $O(1, n-1)$. $PO(1, n-1)$, the group of positive matrices is called the positive Lorentz group. Similarly, the set of all positive matrices in $SO(1, n-1)$ is given by $PSO(1, n-1)$. So $PSO(1, n-1)$ is a index two subgroup of $SO(1, n-1)$. The group $PSO(1, n-1)$ is called the positive special Lorentz group.

Definition 1.2. *Two vectors x, y in $\mathbb{R}^n$ are Lorentz orthogonal iff $x \circ y = 0$.*

Theorem 5. *Let $x, y \neq 0$ Lorentz orthogonal vectors in $\mathbb{R}^n$. If x is time-like, then y is space-like.*

Definition 1.3. *Let V be a vector subspace of $\mathbb{R}^n$. Then V is said to be*

- *time-like iff V is time-like, or*
- *space-like iff every nonzero vector in V is space-like, or*
- *else light-like.*

Theorem 6. *For each dimension m, the natural action of $PO(1, n-1)$ n the set of m-dimensional time-like vector subspaces of $\mathbb{R}^n$ is transitive.*

Theorem 7. *Let x, y be positive (or negative) time-like vectors in $\mathbb{R}^n$. Then $x \circ y \leq \|x\|\|y\|$ with equality iff x, y are linearly dependent.*

1.1.2 The Time-Like Angle between Time-Like Vectors

Let x, y be positive (or negative) time-like vectors in $\mathbb{R}^n$. If a unique non-negative real number $\eta(x, y)$ is such that

$$x \circ y = \|x\|\|y\| \cosh \eta(x, y).$$

The $\eta(x, y)$ is called the Lorentzian time-like angle between x, y. Therefore, $\eta(x, y) = 0$ iff x, y are positive scalar multiples of each other.

1.2 Hyperbolic n-Space

A sphere of radius r in $\mathbb{R}^{n+1}$ is of constant curvature $1/r^2$ and hyperbolic n-space is of negative constant curvature, the duality between hyperbolic geometries and spherical geometries indicates that hyperbolic n-space should be a sphere of imaginary radius. Since imaginary lengths are possible in Lorentzian $(n+1)$-space, we will take model for hyperbolic n-space the sphere of unit imaginary radius

$$F^n = \{x \epsilon \mathbb{R}^{n+1} : \|x\|^2 = -1\}.$$

The set F^n is disconnected. The set F^n is a hyperboloid of two sheets given by the equation $x_1^2 - |\bar{x}|^2 = 1$. The subset of all x in F^n such that $x_1 > 0$ (or $x_1 < 0$) is called the positive (or negative) sheet of F^n. The hyperboloid model $\mathbb{H}^n$ of hyperbolic n-space is defined to be the positive sheet of F^n. Let x, y be vectors in $\mathbb{H}^n$ and the Lorentzian time-like angle between x, y is given by $\eta(x, y)$. The hyperbolic distance between x, y is defined to be the real number.

$$\eta(x, y) = d_H(x, y) \tag{1.6}$$

As $\|x\|\|y\|cosh\eta(x, y) = x \circ y$ the equation

$$coshd_H(x, y) = -x \circ y \tag{1.7}$$

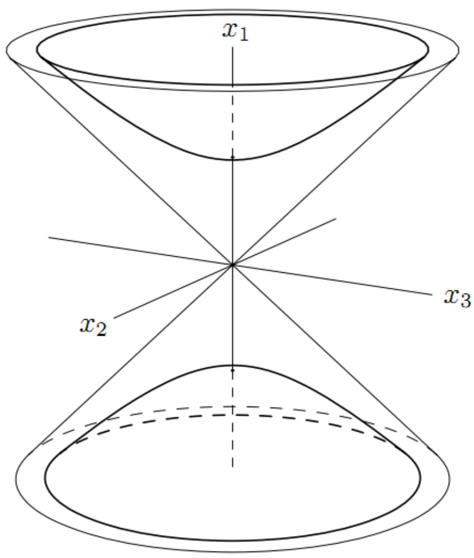

Figure 1.2: The Hyperboloid F^2 inside C^2

Image Courtesy: Ratcliffe, J.G.: Foundation of Hyperbolic Manifolds, Graduate Texts in Mathematics 149. Springer, Berlin p. 61 (1994).

1.2.1 Lorentzian Cross Products

Let the vectors $x, y \in \mathbb{R}^3$ and let $J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The Lorentzian cross product of x, y is given by

$$x \otimes y = J(x \times y) \tag{1.8}$$

Theorem 8. *If w, x, y, z are vectors in $\mathbb{R}^3$, then*

- $x \otimes y = -y \otimes x$
- $(x \otimes y) \circ z = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$
- $x \otimes (y \otimes z) = (x \circ y)z - (z \circ x)y$
- $(x \otimes y) \circ (z \otimes w) = \begin{vmatrix} x \circ w & x \circ z \\ y \circ w & y \circ z \end{vmatrix}$

Theorem 9. *If x, y are linearly independent, positive (negative), time-like vectors in $\mathbb{R}^3$, then $\|x \otimes y\| = \|x\|\|y\| sinh\eta(x, y)$ and $x \otimes y$ is space-like.*

Theorem 10. *If x, y are space-like vectors in $\mathbb{R}^3$, then*

- $|x \circ y| < \|x\|\|y\|$ *iff* $x \otimes y$ *is time-like,*
- $|x \circ y| = \|x\|\|y\|$ *iff* $x \otimes y$ *is light-like,*
- $|x \circ y| > \|x\|\|y\|$ *iff* $x \otimes y$ *is space-like*

Theorem 11. *The hyperbolic distance function d_H is a metric on H^n*

Theorem 12. *Every positive Lorentz transformation of $\mathbb{R}^{n+1}$ restricts to an isometry of $\mathbb{H}^n$, and every isometry of $\mathbb{H}^n$ extends to a unique positive Lorentz transformation of $\mathbb{R}^{n+1}$.*

1.2.2 Hyperbolic Geodesics

Definition 1.4. *A hyperbolic line of $\mathbb{H}^n$ is the intersection of $\mathbb{H}^n$ with a 2-dimensional time-like vector subspace of $\mathbb{R}^{n+1}$.*
Let x, y be two distinct points of $\mathbb{H}^n$. Then x, y span a 2-dimensional time-like subspace $V(x, y)$ of $\mathbb{R}^{n+1}$, and so

$$L(x, y) = \mathbb{H}^n \bigcap V(x, y)$$

is the unique hyperbolic line of $\mathbb{H}^n$ *containing both* x, y*.* $L(x, y)$ *is a branch of a hyperbola.*

Definition 1.5. *Three points* x, y, z *of* $\mathbb{H}^n$ *are hyperbolically collinear iff there is a hyperbolic line* L *of* $\mathbb{H}^n$ *containing points* x, y, z*.*

- If x, y, z are points of $\mathbb{H}^n$ and

$$\eta(x, y) + \eta(y, z) = \eta(x, z),$$

  then x, y, z are hyperbolically collinear.

Definition 1.6. *Two vectors* x, y *in* $\mathbb{R}^{n+1}$ *are Lorentz orthonormal iff* $\|x\|^2 = -1$ *and* $x \circ y = 0$ *and* $\|y\|^2 = 1$

Theorem 13. *Let* $\alpha : [a, b] \to \mathbb{H}^n$ *be a curve. Then the following are equivalent:*

- *The curve* α *is a geodesic arc.*
- *There are Lorentz orthonormal vectors* x, y *in* $\mathbb{R}^{n+1}$ *such that*

$$\alpha(t) = (cosh(t - a))x + (sinh(t - a))y.$$

- *The curve* α *satisfies the differential equation* $(\alpha'' - \alpha) = 0$*.*

Theorem 14. *A function* $\lambda : \mathbb{R} \to \mathbb{H}^n$ *is a geodesic line iff there are Lorentz orthonormal vectors* x, y *in* $\mathbb{R}^{n+1}$ *such that*

$$\lambda(t) = (cosht)x + (sinht)y.$$

- The geodesics of $\mathbb{H}^n$ are its hyperbolic lines.

1.3 The Hyperbolic Triangles

Define the angle between two hyperbolic lines in $\mathbb{H}^2$ as the interior angle between their tangents at the point of intersection. Let x, y, z be three hyperbolically non-collinear points in $\mathbb{H}^2$. Let $L(x, y)$ be the unique geodesic in $\mathbb{H}^2$ containing x, y. Let $H(x, y, z)$ be the closed half-space of $\mathbb{H}^2$ such that $L(x, y)$ is its boundary and z is in its interior. The hyperbolic triangle with vertices x, y, z is given by

$$T(x, y, z) = H(x, y, z) \bigcap H(y, z, x) \bigcap H(z, x, y)$$

Let $[x, y]$ be the segment of $L(x, y)$ joining x, y. The sides of $T(x, y, z)$ are denoted by $[x, y], [y, z], [z, x]$. Let a, b, c be the hyperbolic lengths of $[z, y], [z, x]$ and $[x, y]$ respectively.

Suppose $f : [0, a] \to \mathbb{H}^2, g : [0, b] \to \mathbb{H}^2$ and $h : [0, c] \to \mathbb{H}^2$ are the geodesic arcs from y to z, z to x, and x to y respectively. The angle α between the sides $[z, x]$ and $[x, y]$ of $T(x, y, z)$ is the interior angle between $h'(0)$ and $-g'(b)$, which is the interior angle between the tangents at the point of intersection of the sides. In the similar way as above, angles between the other pair of sides are obtained. Now we shall allow the vertices of a triangle to belong to the circle at infinity. If two geodesics intersect at the circle at infinity then angle between them is defined to be zero. If all the three vertices of a hyperbolic triangle lie on the circle at infinity, it is called an ideal triangle.

Ideal triangles in the hyperbolic plane

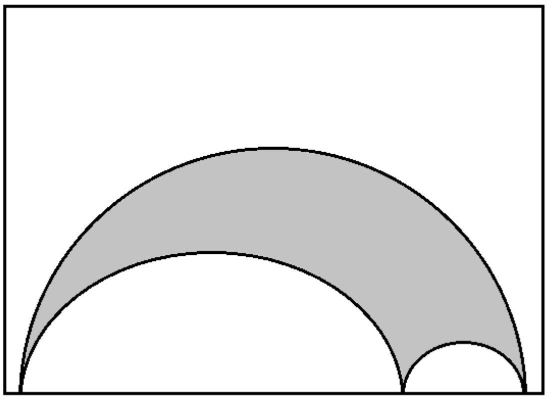

Figure 1.3: An ideal triangle with real vertices

The area of a set X in $\mathbb{H}^2$ is defined by

$$Area(X) = \int\int_X \frac{dxdy}{y^2} \tag{1.9}$$

The area in the unit-disk model D^2 is

$$\int\int_X \frac{2dxdy}{1 - x^2 - y^2} \tag{1.10}$$

The hyperbolic area is invariant under the isometries of $\mathbb{H}^2$.

Theorem 15. *Any ideal triangle in the hyperbolic space has area* $\mathbb{H}^2$.

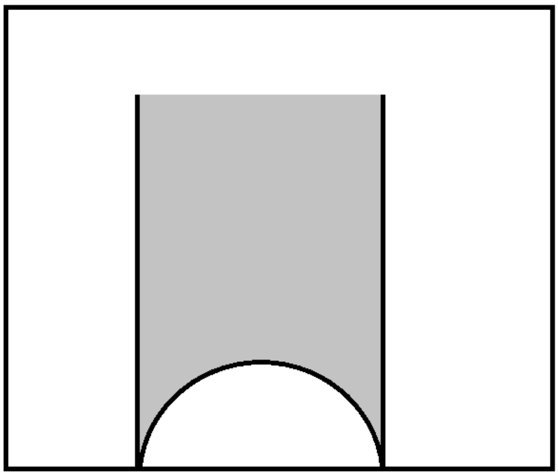

Figure 1.4: An ideal triangle with one vertex at infinity

Theorem 16 (Gauss Bonnet Theorem)**.** *Let Δ be a hyperbolic triangle with angles α, β, γ. Then*

$$Area(\Delta) = -(\alpha + \beta + \gamma)$$

Theorem 17. *Let α, β, γ be the angles of a hyperbolic triangle $T(x, y, z)$, then*
(1) $\eta(z \otimes x, x \otimes y) = \pi - \alpha$,
(2) $\eta(x \otimes y, y \otimes z) = \pi - \beta$
(3) $\eta(y \otimes z, z \otimes x) = \pi - \gamma$

Theorem 18. *Let x, y be space-like vectors in $\mathbb{R}^3$. If $x \otimes y$ is time-like, then*

$$|||x \otimes y||| = \|x\|\|y\| \sin \eta(x, y).$$

Proof Since $x \otimes y$ is time-like, the vector subspace of $\mathbb{R}^3$ spanned by the vectors x, y is space-like. We have

$$\begin{aligned}\|x \otimes y\|^2 &= (x \circ y)^2 - \|x\|^2\|y\|^2 \\ &= \|x\|^2\|y\|^2 \cos^2 \eta(x, y) - \|x\|^2\|y\|^2 \\ &= -\|x\|^2\|y\|^2 \sin^2 \eta(x, y).\end{aligned}$$

Theorem 19. *If α, β, γ are the angles of a hyperbolic triangle, then*

$$\alpha + \beta + \gamma < \pi.$$

Proof Let α, β, γ be the angles of $T(x, y, z)$. The vectors $x \otimes y$,$z \otimes y$, $z \otimes x$ are linearly independent. Let

$$u = \frac{x \otimes y}{\|x \otimes y\|},\ v = \frac{z \otimes y}{\|z \otimes y\|},\ w = \frac{z \otimes x}{\|z \otimes x\|}.$$

However

$$(x \otimes y) \otimes (z \otimes y) = ((x \otimes y) \circ z)y$$

and

$$(z \otimes y) \otimes (z \otimes x) = ((x \otimes y) \circ z)z$$

It is clear that both $u \otimes v$ and $v \otimes w$ are time-like vectors.

$$\begin{aligned}
&\cos(\eta(u, v) + \eta(v, w)) \\
&= \cos \eta(u, v) \cos \eta(v, w) - \sin \eta(u, v) \sin \eta(v, w) \\
&= (u \circ v)(v \circ w) + \|u \otimes v\| \|v \otimes u\| \\
&> (u \circ v)(v \circ w) + ((u \otimes v) \circ (v \otimes u)) \\
&= (u \circ v)(v \circ w) + ((u \circ w)(v \circ v) - (v \circ w)(u \circ v)) \\
&= u \circ w \\
&= \cos \eta(u, w).
\end{aligned}$$

Thus, either

$$\eta(u, w) > \eta(u, v) + \eta(v, w)$$

$$2\pi - \eta(u, w) < \eta(u, v) + \eta(v, w)$$

We have that $\eta(u, w) = \pi - \alpha, \eta(u, v) = \beta$, and $\eta(v, w) = \gamma$. Thus, either $\pi > \alpha + \beta + \gamma$ or $\pi + \alpha < \beta + \gamma$. Without loss of generality, assume that α is the largest angle. If $\pi + \alpha < \beta + \gamma$, so the contradiction

$$\pi + \alpha < \beta + \gamma < \pi + \alpha$$

$$\alpha + \beta + \gamma < \pi.$$

Theorem 20. *(Law of Sines) If α, β, γ are the angles of $T(x, y, z)$ and a, b, c are the lengths of the opposite sides of the hyperbolic triangle, then*

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}$$

Proof By taking norms of both sides of equations

$$(z \otimes x) \otimes (x \otimes y) = -((z \otimes x) \circ y)x,$$
$$(x \otimes y) \otimes (y \otimes z) = -((x \otimes y) \circ z)y,$$
$$(y \otimes z) \otimes (z \otimes x) = -((y \otimes z) \circ x)z,$$

It is clear that

$$\sinh b \sinh c \sin \alpha = |(x \otimes y) \circ z|,$$
$$\sinh c \sinh a \sin \beta = |(x \otimes y) \circ z|,$$
$$\sinh a \sinh b \sin \gamma = |(x \otimes y) \circ z|,$$

Theorem 21. *The First Law of Cosines) If α, β, γ are the angles of $T(x, y, z)$ and a, b, c are the lengths of the opposite sides of hyperbolic triangle, then*

$$\cos \gamma = \tfrac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}$$

Proof Since

$$(y \otimes z) \circ (x \otimes z) = \begin{pmatrix} y \circ z & y \circ x \\ z \circ z & z \circ x \end{pmatrix}$$

We have that

$$\sinh a \sinh b \cos \gamma = \cosh a \cosh b - \cosh c.$$

Theorem 22. *(The Second Law of Cosines) If α, β, γ are the angles of a hyperbolic triangle and a, b, c are the lengths of the opposite sides, then*

$$\cosh c = \tfrac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta}$$

Proof

$$x' = \tfrac{y \otimes z}{\|y \otimes z\|},\ y' = \tfrac{z \otimes x}{\|z \otimes x\|},\ z' = \tfrac{x \otimes y}{\|x \otimes y\|}$$

Then

$$x = \tfrac{y' \otimes z'}{|||y' \otimes z'|||},\ y = \tfrac{z' \otimes x'}{|||z' \otimes x'|||}$$

Now since

$$(y' \otimes z') \circ (z' \otimes x') = \begin{pmatrix} y' \circ x' & y' \circ z' \\ z' \circ x' & z' \circ z' \end{pmatrix}$$

we have

$$-\sin(\pi - \alpha)\sin(\pi - \beta)\cosh c = \cos(\pi - \gamma) - \cos(\pi - \alpha)\cos(\pi - \beta).$$

It is interesting to compare the hyperbolic sine law

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}$$

with the spherical sine law

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}$$

and the hyperbolic cosine laws

$$\cos \gamma = \frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b},$$

$$\cosh c = \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta}$$

with the spherical cosine laws

$$\cos \gamma = \frac{\cosh c - \cos a \cos b}{\sinh a \sinh b},$$

$$\cos c = \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta}$$

Recall that

$$\sin ia = i \sinh a \text{ and } \cos ia = \cosh a.$$

Hence, the hyperbolic trigonometry formulas can be obtained from their spherical counterparts by replacing a, b, c by ia, ib, ic, respectively.

1.3.1 Area of Hyperbolic Triangles

A sector of $\mathbb{H}^2$ is defined to be the intersection of two distinct, intersecting, nonopposite half-planes of $\mathbb{H}^2$. Any sector of $\mathbb{H}^2$ is congruent to a sector $S(\alpha)$ given by hyperbolic coordinates (η, θ) by the inequalities

$$-\alpha/2 \leq \theta \leq \alpha/2.$$

where α is the angle formed by the two sides of $S(\alpha)$ at its vertex e_1. Let $\beta = \alpha/2$. Then the geodesic rays that form the sides of $S(\alpha)$ are represented in parametric form by

$$(\cosh t)e_1 + (\sinh t)((\cos \beta)e_2 + (\sin \beta)e_3) \text{ for } t \geq 0$$

$$(\cosh t)e_1 + (\sinh t)((\cos\beta)e_2 - (\sin\beta)e_3) \text{ for } t \geq 0.$$

These geodesic rays are asymptotic to the 1-dimensional light-like vector subspaces spanned by the vectors $(1, \cos\beta, \sin\beta)$ and $(1, \cos\beta, -\sin\beta)$, respectively. These two light-like vectors span a 2-dimensional vector subspace V that intersects $\mathbb{H}^2$ in a hyperbolic line L. Suppose $T(\alpha)$ be the intersection of $S(\alpha)$ and the closed half-plane bounded by L and containing the vertex e_1.

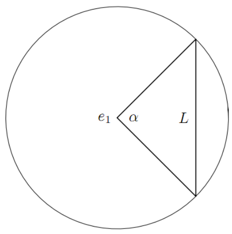

Figure 1.5: A generalized triangle with two ideal vertices

Image Courtesy: Ratcliffe, J.G.: Foundation of Hyperbolic Manifolds, Graduate Texts in Mathematics 149. Springer, Berlin p. 84 (1994).

The fascinating fact about $\mathbb{H}^2$ is that when viewed from the origin it looks like the projective disk model with the point e_1 at its center. It is clear that the two sides of the sector $S(\alpha)$ meet the hyperbolic line L at infinity. From this point of view, it is natural to regard $T(\alpha)$ as a hyperbolic triangle which has two ideal vertices at infinity. A generalized hyperbolic triangle in $\mathbb{H}^2$ can be defined in the similar way that we defined a hyperbolic triangle in $\mathbb{H}^2$ except that some of its vertices can be ideal. When observed from the origin, a generalized hyperbolic triangle in $\mathbb{H}^2$ is a Euclidean triangle in the projective disk model with its ideal vertices on the circle at infinity. The angle of a generalized hyperbolic triangle at an ideal vertex is defined to be zero. An infinite hyperbolic triangle is a generalized hyperbolic triangle with at least one ideal vertex and an ideal hyperbolic triangle is called an infinite hyperbolic triangle with three ideal vertices. Every infinite hyperbolic triangle with exactly two ideal vertices is congruent to $T(\alpha)$ for some angle α. Now we will find a parametric representation for the side L of $T(\alpha)$ in terms of hyperbolic coordinates (η, θ). The vector

$$(1, \cos\beta, \sin\beta) \times (1, \cos\beta, -\sin\beta) = (-2cos\beta\sin\beta, 2sin\beta, 0)$$

is normal with respect to the 2-dimensional vector subspace V whose intersection with $\mathbb{H}^2$ is L. Hence, the equation is satisfied

$$(\cos\beta)x_1 - x_2 = 0.$$

by the vectors in V. Now the points of $\mathbb{H}^2$ satisfy the system of equations

$$\begin{cases} x_1 = \cosh\eta, \\ x_2 = \sinh\eta\cos\theta, \\ x_3 = \sinh\eta\sin\theta \end{cases} \tag{1.11}$$

Thus, the points of L satisfy the equation

$$x_1 = \sec\beta\cos\theta\sqrt{x_1^2 - 1}.$$

While solving x_1, we see that

$$x_1 = \frac{\cos\theta}{\sqrt{\cos^2\theta - \cos^2\beta}}.$$

Hence

$$x_2 = \frac{\cos\theta\cos\beta}{\sqrt{\cos^2\theta - \cos^2\beta}}$$

and

$$x_3 = \frac{\sin\theta\cos\beta}{\sqrt{\cos^2\theta - \cos^2\beta}}.$$

Theorem 23. *Area* $T(\alpha) = \pi - \alpha$

Proof Let the polar angle parameterization of L be defined as

$$x(\theta) = (x_1(\theta), x_2(\theta), x_3(\theta)$$

So we have

$$AreaT(\alpha) = \int_{-\beta}^{\beta}\int_0^{\eta(e_1,x(\theta))} \sinh\eta d\eta d\theta$$

$$= \int_{-\beta}^{\beta}(\cosh\eta(e_1, x(\theta)) - 1)d\theta$$

$$= \int_{-\beta}^{\beta} x_1(\theta)d\theta - \alpha$$

and

$$\int_{-\beta}^{\beta} x_1(\theta)d\theta = \int_{-\beta}^{\beta} \frac{\cos\theta d\theta}{\sqrt{\cos^2\theta - \cos^2\beta}}$$

$$= \int_{-\beta}^{\beta} \frac{\cos\theta d\theta}{\sqrt{\sin^2\beta - \sin^2\theta}}$$

$$= \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}}, \text{ where } u = \sin\theta \sin\beta$$

$$= Arc\sin u|_{-1}^{1} = \pi.$$

Hence, we have that

$$\text{Area } T(\alpha) = \pi - \alpha.$$

Theorem 24. *The area of an ideal hyperbolic triangle is π.*

Proof Suppose T be any ideal hyperbolic triangle, take a point x in the interior of T. Then it is possible to subdivide T into three infinite hyperbolic triangles such that each of which has only finite vertex given by x. Define α, β, γ be the angles of the triangles at the vertex x as shown in the figure. Then we have

$$Area(T) = (\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) = \pi.$$

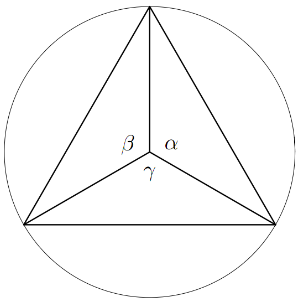

Figure 1.6: An ideal triangle subdivided into three infinite triangles

Image Courtesy: Ratcliffe, J.G.: Foundation of Hyperbolic Manifolds, Graduate Texts in Mathematics 149. Springer, Berlin p. 86 (1994).

Theorem 25. *If α, β, γ are the angles of a generalized hyperbolic triangle T, then*

$$Area(T) = \pi - (\alpha + \beta + \gamma).$$

Proof The formula holds if T has two or three ideal vertices. Let x and y be the only two vertices of T with angles α and β. On extending the finite side of T, from Figure, we observe that T is the difference of two infinite hyperbolic triangles T_x and T_y with only one finite vertex x, y, respectively. Therefore

$$Area(T) = Area(T_x) - Area(T_y) = (\pi - \alpha) - \beta.$$

Now let x, y, z be the three finite vertices of T with angles α, β, γ. On extending the sides of T, as in figure, we can have an ideal hyperbolic triangle T' which can be subdivided into four regions, one of which is T, and the others are infinite hyperbolic triangles T_x, T_y, T_z with only one finite vertex x, y, z, respectively. Hence, we have

$$Area(T') = Area(T) + Area(T_x) + Area(T_y) + Area(T_z).$$

Hence

$$\pi = Area(T) + (\alpha + \beta + \gamma).$$

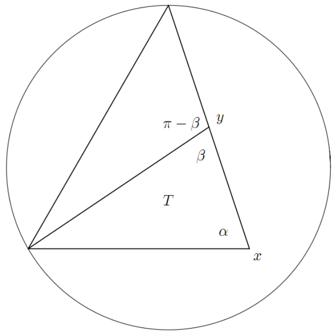

Figure 1.7: An infinite triangle T expressed as the difference of two triangles

Image Courtesy: Ratcliffe, J.G.: Foundation of Hyperbolic Manifolds, Graduate Texts in Mathematics 149. Springer, Berlin p. 87 (1994).

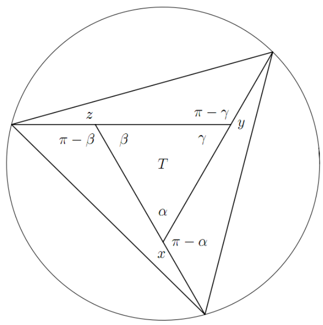

Figure 1.8: The ideal triangle found by extending the sides of $T(x, y, z)$

Image Courtesy: Ratcliffe, J.G.: Foundation of Hyperbolic Manifolds, Graduate Texts in Mathematics 149. Springer, Berlin p. 87 (1994).

Chapter 2

Inversive Geometry

The theory and proofs in this chapter are based on Ratcliffe, J.G.: Foundation of Hyperbolic Manifolds, Graduate Texts in Mathematics 149. Springer, Berlin (1994).

The group of transformations of Euclidean space E^n generated by reflections in hyperplanes and inversions in spheres. It turns out that this group is isomorphic to the group of isometries of hyperbolic space H^{n+1}.

2.1 Hyperplanes

Definition 2.1. *A hyperbolic m-plane of H^n is defined as the intersection of H^n with a $(m+1)$-dimensional time-like vector subspace of $\mathbb{R}^{n+1}$.*

Then a hyperbolic 1-plane of H^n is the hyperbolic line of H^n. A hyperbolic $(n-1)$-plane of H^n is known as a hyperplane of H^n.
Define x to be a space-like vector of $\mathbb{R}^{n+1}$. Then the Lorentzian complement of a vector subspace $\langle x \rangle$ spanned by x is an n-dimensional time-like vector subspace of $\mathbb{R}^{n+1}$. Hence a hyerplane H^n is given by $P = \langle x \rangle^L \cap H^n$. The hyperplane P is called the hyperplane of H^n Lorentz orthogonal to x.

Theorem 26. *Let x, y be linearly independent space-like vectors in $\mathbb{R}^{n+1}$. Then the following statements are equivalent:*

- *The vectors x, y satisfy the equation $|x \circ y| < \|x\|\|y\|$.*
- *The vector subspace V spanned by x, y is space-like.*

- *The hyperplanes P and Q of H^n Lorentz orthogonal to x, y, intersect.*

2.2 Reflections

Definition 2.2. *Let a be defined as a unit vector in E^n and t be a real number. Consider the hyperplane of E^n which is by*

$$P(a,t) = \{x \epsilon E^n : a \cdot x = t\}. \tag{2.1}$$

Every point x in $P(a,t)$ satisfies

$$a \cdot (x - ta) = 0. \tag{2.2}$$

Therefore $P(a,t)$ is a hyperplane of E^n with normal unit vector a passing from the point ta. Any hyperplane has exactly two representations $P(-a,-t)$ and $P(a,t)$. The reflection ρ of E^n in the plane $P(a,t)$ is given by the formula

$$\rho(x) = x + sa, \tag{2.3}$$

where s is a real scalar so that $x + \frac{1}{2}sa$ is in $P(a,t)$. This leads to a direct formula

$$\rho(x) = x + 2(t - a \cdot x)a. \tag{2.4}$$

Theorem 27. *If ρ is the reflection of E^n in the plane $P(a,t)$, then*

- *$\rho^2(x) = x$ for all x in E^n; and*
- *$\rho(x) = x$ iff x is in $P(a,t)$;*
- *ρ is an isometry.*

Theorem 28. *Every isometry of E^n is a composition of at most $n+1$ reflections in hyperplanes.*

Proof Let $\phi : E^n \to E^n$ be an isometry and define $v_0 = \phi(0)$. ρ_0 be the identity if $v_0 = 0$, or the reflection in the plane P $(v_0/|v_0|, |v_0|/2)$ otherwise. So $\rho_0(v_0) = 0$ and then $\rho_0\phi(0) = 0$. The map $\phi_0 = \rho_0\phi$ is an orthogonal transformation.
Let ϕ_{k-1} is defined as an orthogonal transformation of E^n which fixes $e_1, \cdots, e_{k-1}$.

Suppose $v_k = \phi_{k-1}(e_k) - ek$ and ρ_k be the identity if $v_k = 0$, or the reflection in the plane P $(v_k/|v_k|, 0)$ otherwise. So $\rho_k\phi_{k-1}$ fixes e_k. Also, for each $j = 1, \cdots, k-1$, we see that

$$\begin{aligned} v_k.e_j &= (\phi_{k-1}(e_k) - e_k).e_j \\ &= \phi_{k-1}(e_k).e_j \\ &= \phi_{k-1}(e_k).\phi_{k-1}(e_k) \\ &= e_k.e_j \\ &= 0. \end{aligned}$$

Thus e_j is in the plane P $(v_k/|v_k|, 0)$ and then is fixed by ρ_k. Hence, we observe that $\phi_k = \rho_k\phi_{k-1}$ fixes $e_1, \cdots, e_k$. So by induction we have that there are maps $\rho_0, \cdots, \rho_n$ such that each ρ_i is either the identity or a reflection and $\rho_n \cdots \rho_0\phi$ fixes $0, e_1, \cdots, e_n$. Thus $\rho_n \cdots \rho_0\phi$ is a identity and hence $\phi = \rho_0 \cdots \rho_n$.

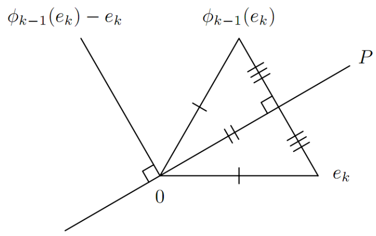

Figure 2.1: The reflection of the point $\phi_{k-1}(e_k)$ in the plane P

Image Courtesy: Ratcliffe, J.G.: Foundation of Hyperbolic Manifolds, Graduate Texts in Mathematics 149. Springer, Berlin p. 101 (1994).

2.3 Inversions

Let E^n be a euclidean space with a point a inside it and let r be a positive real number. A sphere of E^n with radius r centered at a is defined as

$$S(a, r) = \{x \epsilon E^n : |x - a| = r\}. \tag{2.5}$$

The reflection (or inversion) σ of E^n inside the sphere $S(a, r)$ is given by

$$\sigma(x) = a + s(x - a) \tag{2.6}$$

where s is a positive scalar such that

$$|\sigma(x) - a||x - a| = r^2. \tag{2.7}$$

This leads to direct formula

$$\sigma(x) = a + (\frac{r}{|x-a|})^2(x - a) \tag{2.8}$$

There is a fair geometric construction of the point $\sigma(x)$. Firstly we assume that x is inside the sphere, $S(a, r)$ then erect a chord of $S(a, r)$ passing through x which is perpendicular to the line joining x to a. Let u and v be the endpoints of the chord. So $\sigma(x)$ is the point x' of intersection of the lines tangent to $S(a, r)$ at the points u and v in the plane including a, u, and v. We can clearly see that the right triangles $T(a, x, v)$ and $T(a, v, x')$ are similar to each other.

$$\frac{|x' - a|}{r} = \frac{r}{|x - a|} \tag{2.9}$$

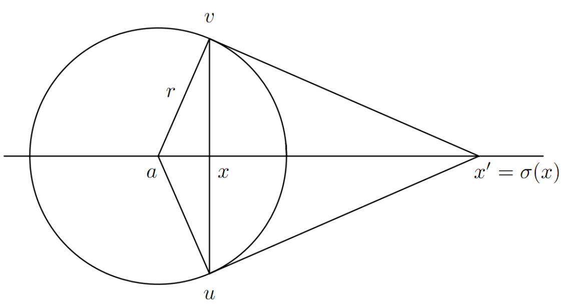

Figure 2.2: The construction of the reflection of a point x in a sphere $S(a, r)$

Image Courtesy: Ratcliffe, J.G.: Foundation of Hyperbolic Manifolds, Graduate Texts in Mathematics 149. Springer, Berlin p. 102 (1994).

Now let us assume that x is outside the sphere $S(a, r)$. Let y is the midpoint of the line segment $[a, x]$ and let C be the circle centered at y of radius $|x - y|$. Then the circle C intersects $S(a, r)$ at two points namely u, v and $\sigma(x)$ is the point x' of intersection of the line segments $[a, x]$ and $[u, v]$.

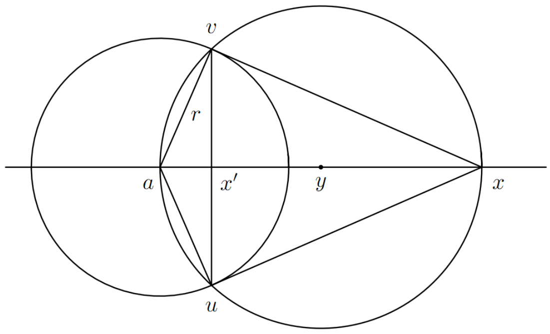

Figure 2.3: The construction of the reflection of a point x outside the sphere $S(a, r)$

Image Courtesy: Ratcliffe, J.G.: Foundation of Hyperbolic Manifolds, Graduate Texts in Mathematics 149. Springer, Berlin p. 103 (1994).

Theorem 29. *If σ is the reflection of E^n in the sphere $S(a, r)$, then*

- *$\sigma(x) = x$ iff x is in $S(a, r)$;*
- *$\sigma^2(x) = x$ for all $x \neq a$; and*
- *for all $x, y \neq a$,*

$$|\sigma(x) - \sigma(y)| = \frac{r^2|x-y|}{|x-a||y-a|}. \tag{2.10}$$

Proof (1) AS

$$|\sigma(x) - a||x - a| = r^2,$$

We observe that $\sigma(x) = x$ iff $|x - a| = r$

(2) We see that

$$\begin{aligned}\sigma^2(x) &= a + \left(\tfrac{r}{|\sigma(x)-a|}\right)^2 (\sigma(x) - a) \\ &= a + \left(\tfrac{|x-a|}{r}\right)^2 \left(\tfrac{r}{|x-a|}\right)^2 (x - a) \\ &= x.\end{aligned}$$

(3) We see that

$$\begin{aligned}|\sigma(x) - \sigma(y)| &= r^2 \left|\tfrac{(x-a)}{|x-a|^2} - \tfrac{(y-a)}{|y-a|^2}\right| \\ &= r^2 \left[\tfrac{1}{|x-a|^2} - \tfrac{2(x-a).(y-a)}{|x-a|^2|y-a|^2} + \tfrac{1}{|y-a|^2}\right]^{1/2} \\ &= r^2 \tfrac{|x-y|}{|y-a||x-a|}.\end{aligned}$$

2.4 Conformal Transformations

Let U be an open subset of E^n and let $\phi : U \to E^n$ be a C^1 function. Then ϕ is differentiable and has continuous partial derivatives. Let $\phi'(x)$ be the matrix $(\frac{\partial \phi_i}{\partial x_j}(x))$ of partial derivatives of ϕ. Then the function ϕ is said to be conformal iff there exists a function

$$\kappa : U \to \mathbb{R}_+,$$

called the scale factor of ϕ and $\kappa(x)^{-1}\phi'(x)$ is an orthogonal matrix for every x in U. The scale factor κ of a conformal function ϕ is uniquely determined by ϕ, as $[\kappa(x)]^n = |det\phi'(x)|$.

Theorem 30. *Let A be a real $n \times n$ matrix. Then there is a positive scalar k such that $k^{-1}A$ is an orthogonal matrix iff A preserves angles between nonzero vectors.*

Proof Suppose there exists a $k > 0$ such that $k^{-1}A$ is an orthogonal matrix. Then A is nonsingular matrix. Suppose x, y be nonzero vectors in E^n. Then Ax and Ay are nonzero, and A preserves angles, because

$$\cos\theta(Ax, Ay) = \frac{Ax.Ay}{|Ax||Ay|}$$

$$= \frac{k^{-1}Ax.k^{-1}Ay}{|k^{-1}Ax||k^{-1}Ay|}$$

$$= \frac{x.y}{|x||y|} = \cos\theta(x, y).$$

Conversely, assume that the matrix A preserves angles between nonzero vectors. Then A is a nonsingular matrix. Since $\theta(Ae_i, Ae_j) = \theta(e_i, e_j) = \pi/2$ for all $i \neq j$, the vectors $Ae_1, \cdots, Ae_n$ are orthogonal. Let B be an orthogonal matrix such that $Be_i = Ae_i/|Ae_i|$ for each i. Then $B^{-1}A$ also preserves angles and $B^{-1}Ae_i = c_ie_i$ where $c_i = |Ae_i|$. Hence, without loss of generality, we may assume that $Ae_i = c_ie_i$, with $c_i > 0$, for each $i = 1, \cdots, n$. As

$$\theta(A(e_i + e_j), Ae_j) = \theta(e_i + e_j, e_j)$$

for all $i \neq j$, we observe

$$\frac{(c_ie_i + c_je_j).c_je_j}{(c^22_i + c_j^2)^{1/2}c_j} = \frac{1}{\sqrt{2}}.$$

Hence $2c_j^2 = c_i^2 + c_j^2$ and so $c_i = c_j$ for all i and j. So, the common value of the c_i is a positive scalar k such that $k^{-1}A$ is orthogonal.

2.4.1 Angle Between Curves

Let $\alpha, \beta : [-b, b] \to E^n$ be a differential curve such that $\alpha(0) = \beta(0)$ and $\alpha'(0), \beta'(0)$ are both nonzero. The angle between α and β at 0 is defined to be the angle between $\alpha'(0)$ and $\beta'(0)$.

Theorem 31. *Let U be an open subset of E^n and let $\phi : U \to E^n$ be a C^1 function. Then ϕ is conformal iff ϕ preserves angles between differentiable curves in U.*

Proof Let us assume that the function ϕ is conformal. Then there is a function $\kappa : U \to \mathbb{R}_+$ such that $\kappa(x)^{-1}\phi'(x)$ is orthogonal to each x in U. Let $\alpha, \beta : [-b, b] \to U$ be differentiable curves such that $\alpha(0) = \beta(0)$ and $\alpha'(0)$, $\beta'(0)$ are both nonzero. We observe

$$\begin{aligned} &\theta((\phi\alpha)'(0), (\phi\beta)'(0)) \\ &= \theta(\phi'(\alpha(0))\alpha'(0), \phi'(\beta(0))\beta'(0)) \\ &= \theta(\alpha'(0), \beta'(0)). \end{aligned}$$

Hence, the angle between $\phi\alpha$ and $\phi\beta$ at 0 is the same as the angle between α and β at 0.

Theorem 32. *Every reflection of E^n in a hyperplane or sphere is conformal and reverses orientation.*

Proof Let ρ be the reflection of E^n in the plane $P(a, t)$. Then

$$\begin{aligned} \rho(x) &= x + 2(t - a.x)a, \\ \rho'(x) &= (\delta_{ij} - 2a_i a_j) = I - 2A \end{aligned}$$

where A is the matrix $(a_i a_j)$. Since $\rho'(x)$ is independent of t, without loss of generality we may assume that $t = 0$. Then ρ is an orthogonal transformation and

$$\rho(x) = (I - 2A)x.$$

Hence $I - 2A$ is an orthogonal matrix, and then ρ is conformal. There is an orthogonal transformation ϕ such that $\phi(a) = e_1$. So

$$\begin{aligned} \phi\rho\phi^{-1}(x) &= \phi(\phi^{-1}(x) - 2(a.\phi^{-1}(x))a) \\ &= x - 2(a.\phi^{-1}(x))e_1 \\ &= x - 2(\phi(a).x)e_1 \\ &= x - 2(e_1.x)e_1. \end{aligned}$$

Hence $\phi\rho\phi^{-1}$ is the reflection in $P(e_1, 0)$. Then by chain rule,

$$det(\phi\rho\phi^{-1})'(x) = det\rho'(x).$$

For the computation of determinant of $\rho'(x)$, we may assume that $a = e_1$. So

$$I - 2A = \begin{pmatrix} -1 & & & \\ & 1 & & 0 \\ & & \cdot & \\ 0 & & & 1 \end{pmatrix}$$

Hence det $\rho'(x) = -1$, and thus ρ reverses orientation.
Let σ_r be the reflection of E^n in the sphere $S(0, r)$. So

$$\sigma r(x) = \frac{r^2 x}{|x|^2}$$

and so

$$\sigma_r'(x) = r^2 \left(\frac{\delta_{ij}}{|x|^2} - \frac{2x_i x_j}{|x|^4} \right) = \frac{r^2}{|x|^2}(I - 2A),$$

where A is the matrix $(x_i x_j / |x|^2)$. As $I - 2A$ is orthogonal, and therefore σ_r is conformal; moreover σ_r reverses orientation, as

$$\begin{aligned} det\sigma_r'(x) &= \left(\frac{r}{|x|}\right)^{2n} det(I - 2A) \\ &= -\left(\frac{r}{|x|}\right)^{2n} < 0 \end{aligned}$$

Let σ be the reflection with respect to $S(a, r)$ and let τ be the translation by a. So $\tau'(x) = I$ and $\sigma = \tau\sigma_r\tau^{-1}$. Thus $\sigma'(x) = \sigma_r'(x - a)$. Hence σ is conformal and reverses orientation.

2.5 Sterographic Projection

Identify E^n with $E^n \times \{0\}$ in E^{n+1} . The stereographic projection π of E^n onto $S^n - \{e_{n+1}\}$ is given by projecting x in E^n towards (or away from) e_{n+1} unless it meets the sphere S^n in the distinct point $\pi(x)$ other than e_{n+1}. Since $\pi(x)$ is a point on the line which passes through x in the direction of $e_{n+1} - x$, there is a scalar s such that

$$\pi(x) = x + s(e_{n+1} - x). \qquad (2.11)$$

The condition $|\pi(x)|^2 = l$ leads to the value

$$s = \frac{|x|^2 - 1}{|x|^2 + 1} \tag{2.12}$$

and the direct formula

$$\pi(x) = (\frac{2x_1}{1+|x|^2}, \cdots, \frac{2x_n}{1+|x|^2}, \frac{|x|^2-1}{|x|^2-1}). \tag{2.13}$$

Let σ be the reflection of E^{n+1} in the sphere $S(e_{n+1}, \sqrt{2})$. So

$$\sigma(x) = e_{n+1} + \frac{2(x - e_{n+1})}{|x - e_{n+1}|^2} \tag{2.14}$$

If x is in E^n, then we have

$$\sigma(x) = e_{n+1} + \frac{2}{1+|x|^2}(x_1, \cdots, x_n, -1) = (\frac{2x_1}{1+|x|^2}, \cdots, \frac{2x_n}{1+|x|^2}, \frac{|x|^2-1}{|x|^2-1}) \tag{2.15}$$

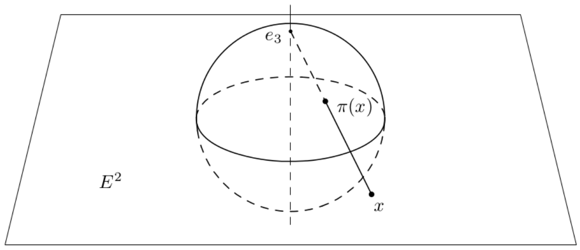

Figure 2.4: The stereographic projection π of E^2 into S^2

Image Courtesy: Ratcliffe, J.G.: Foundation of Hyperbolic Manifolds, Graduate Texts in Mathematics 149. Springer, Berlin p. 107 (1994).

Suppose ∞ be a point which is not in E^{n+1} and define $\hat{E}^n = E^n \cup \{\infty\}$. Now π is extended to a bijection $\hat{\pi} : \hat{E}^n \to S^n$ by setting $\hat{\pi}(\infty) = e_{n+1}$, and define a metric d on $\hat{E}^n$ by

$$d(x, y) = |\hat{\pi}(x) - \hat{\pi}(y)|. \tag{2.16}$$

The metric d is known as the chordal metric on E^n.

2.5.1 Cross Section

Let u, v, x, y be the points of $\hat{E}^n$ such that $x \neq y$ and $u \neq v$. The cross ratio of these points is defined be the real number

$$|(u, v, x, y)| = \frac{d(u,x)d(v,y)}{d(u,v)d(x,y)} \tag{2.17}$$

The cross ratio is defined as the continuous function of four variables, as the metric $d : \hat{E}^n \times \hat{E}^n \to \mathbb{R}$ is a continuous function.

Theorem 33. *If u, v, x, y are points of E^n such that $u \neq v$ and $x \neq y$, then*

$$[u, v, x, y] = \frac{|u-x||v-y|}{|u-v||x-y|} \tag{2.18}$$

$$[\infty, v, x, y] = \frac{|v-y|}{|x-y|} \tag{2.19}$$

$$[u, \infty, x, y] = \frac{|u-x|}{|x-y|} \tag{2.20}$$

$$[u, v, \infty, y] = \frac{|v-y|}{|u-v|} \tag{2.21}$$

$$[u, v, x, \infty] = \frac{|u-x|}{|u-v|} \tag{2.22}$$

Theorem 34. *If x, y are in E^n, then*

$$d(x, \infty) = \frac{2}{(1+|x|^2)^{\frac{1}{2}}} \tag{2.23}$$

$$d(x, y) = \frac{2|x-y|}{(1+|x|^2)^{\frac{1}{2}}(1+|y|^2)^{\frac{1}{2}}} \tag{2.24}$$

Proof (1) We see that

$$d(x, \infty) = |\hat{\pi}(x) - \hat{\pi}(\infty)|$$

$$= |\pi(x) - e_{n+1}|$$

$$\left|\left(\tfrac{2x_1}{1+|x|^2}, \cdots, \tfrac{2x_n}{1+|x|^2}, \tfrac{-2}{1+|x|^2}\right)\right|$$

$$= \tfrac{2}{(1+|x|^2)^{\frac{1}{2}}}.$$

(2) We have

$$d(x,y) = \frac{2|x-y|}{|x-e_{n+1}||y-e_{n+1}|}$$

$$= \frac{2|x-y|}{(1+|x|^2)^{\frac{1}{2}}(1+|y|^2)^{\frac{1}{2}}}.$$

Theorem 35. *Every reflection of $\hat{E}^n$ in an extended hyperplane is a homeomorphism.*

Proof Let ρ be the reflection of E^n in a hyperplane. Then ρ is continuous. As $\lim_{x\to\infty} \rho(x) = \infty$, we have that $\hat{\rho}$ is continuous at ∞. Thus $\hat{\rho}$ is a continuous function. As $\hat{\rho}$ is inverse of its own, it is a homeomorphism.
Let σ be defined as the reflection of E^n in the sphere $S(a,r)$. We extend σ to a map $\hat{\sigma} : \hat{E}^n \to \hat{E}^n$ by setting $\hat{\sigma}(a) = \infty$ and $\hat{\sigma}(x) = (a)$. Then $\hat{\sigma}(x) = (x)$ for all x in $S(a,r)$ and $\hat{\sigma}^2$ is the identity. The map $\hat{\sigma}$ is called the reflection of $\hat{\sigma}$ in the sphere $S(a,r)$.

Theorem 36. *Every reflection of $\hat{E}^n$ in a sphere of E^n is a homeomorphism.*

Proof Let σ be defined as the reflection of E^n in the sphere $S(a,r)$ and let $\hat{\sigma}$ be the extended reflection of $\hat{E}^n$. As $\hat{\sigma}^2$ is the identity, $\hat{\sigma}$ is a bijection with inverse $\hat{\sigma}$. The map $\hat{\sigma}$ is continuous, since σ is continuous, $\lim_{x\to\infty} \rho(x) = \infty$, and $\lim_{x\to\infty} \rho(x) = a$.Thus $\hat{\sigma}$ is a homeomorphism.

2.6 Möbius Transformations

A sphere Σ of $\hat{E}^n$ is defined to be either a Euclidean sphere $S(a,r)$ or an extended plane $\hat{P}(a,t) = P(a,t) \cup \infty$. $\hat{P}(a,t)$ is topologically a sphere.

Definition 2.3. *A Möbius transformation of $\hat{E}^n$ is a finite compostion of reflections of $\hat{E}^n$ in sphere.*

Let $M(\hat{E}^n)$ be defined as the set of all Möbius transformations of $\hat{E}^n$. Then clearly $M(\hat{E}^n)$ clearly forms a group under composition. Every isometry of E^n may be extended in a distinct way to a Möbius transformation of $\hat{E}^n$. Hence, we may regard the group of Euclidean isometries $I(E^n)$ as a subgroup of $M(\hat{E}^n)$.
Let k be a positive constant and let $\mu_k : \hat{E}^n \to \hat{E}^n$ be the function given by $\mu_k(x) = k$. Then μ_k is a Möbius transformation, since μ_k is the composite of the reflection in $S(0,1)$ followed by the reflection in $S(0,\sqrt{k})$. Since each similarity of E^n is the

composite of an isometry followed by μ_k for some k, every similarity of E^n extends in a distinct way to a Möbius transformation of $\hat{E}^n$. Hence, we can also consider the group of Euclidean similarities $S(E^n)$ as a subgroup of $M(\hat{E}^n)$.

Theorem 37. *If σ is the reflection of $\hat{E}^n$ in the sphere $S(a, r)$ and σ_1 is the reflection in $S(0, 1)$, and $\phi : \hat{E}^n \to \hat{E}^n$ is given by $\phi(x) = a + rx$, then $\sigma = \phi\sigma_1\phi^{-1}$.*

Proof We see that

$$\sigma(x) = a + \left(\tfrac{r}{|x-a|}\right)^2 (x - a)$$

$$= \phi\left(\tfrac{r(x-a)}{|x-a|^2}\right)$$

$$= \phi\sigma_1\left(\tfrac{(x-a)}{r}\right) = \phi\sigma_1\phi^{-1}(x)$$

Theorem 38. *A function $\phi : \hat{E}^n \to \hat{E}^n$ is a Möbius transformation iff it preserves cross ratios.*

Proof Let ϕ be a Möbius transformation. Since ϕ is a composition of the reflections, we can assume that ϕ is a reflection. A Euclidean similarity preserves cross ratios, and so $\phi(x) = x/|x|^2$. We observe that

$$|\phi(x) - \phi(y)| = \tfrac{|x-y|}{|x||y|}$$

We conclude that

$$[\phi(u), \phi(v), \phi(x), \phi(y)] = [u, v, x, y]$$

if u, v, x, y are all finite and nonzero. Then the remaining cases follow by continuity. Hence ϕ preserves cross ratios.

Theorem 39. *A Möbius transformation ϕ of $\hat{E}^n$ fixes ∞ iff ϕ is a similarity of E^n.*

2.6.1 The Isometric Sphere

Let ϕ be a Möbius transformation of $\hat{E}^n$ with $\phi(\infty) \neq \infty$. Suppose $a = \phi^{-1}(\infty)$ and σ be the reflection of $\hat{E}^n$ in the sphere $S(a, 1)$. So $\phi\sigma$ fixes ∞. Thus $\phi\sigma$ is a similarity of E^n. Therefore, there is a point b of E^n, a scalar $k > 0$, and an orthogonal transformation A of E^n such that

$$\phi(x) = b + kA\sigma(x). \tag{2.25}$$

And

$$|\phi(x) - \phi(y)| = \frac{k|x-y|}{|x-a||y-a|}. \tag{2.26}$$

Now assume that x, y are in $S(a, r)$. So $|\phi(x)-\phi(y)| = |x-y|$ iff $r = \sqrt{k}$. Hence ϕ acts as an isometry on the sphere $S(a, \sqrt{k})$, and $S(a, \sqrt{k})$ is distinct with this property among the spheres of E^n centered at the point a. $S(a, \sqrt{k})$ is known as the isometric sphere of ϕ.

Theorem 40. *Let ϕ be a Möbius transformation of $\hat{E}^n$ with $\phi(\infty) \neq \infty$. Then there is a unique reflection σ in a Euclidean sphere Σ and a unique Euclidean isometry ψ such that $\phi = \psi\sigma$. Moreover Σ is the isometric sphere of ϕ.*

2.6.2 Preservation of Spheres

The equation defining a sphere $S(a, r)$ or $\hat{P}(a, t)$ in $\hat{E}^n$ is given by

$$|x|^2 - 2a \cdot x + |a|^2 - r^2 = 0 \tag{2.27}$$

or

$$-2a \cdot x + 2t = 0, \tag{2.28}$$

respectively, and these can be given as

$$a_0|x|^2 - 2a \cdot x + a_{n+1} = 0 \text{ with } |a|^2 > a_0 a_{n+1}.$$

Conversely, any vector $(a_0, \cdots, a_{n+1})$ in $\mathbb{R}^{n+2}$ such that $|a|^2 > a_0 a_{n+1}$, where $a = (a_1, \cdots, a_n)$ determines a sphere Σ of $\hat{E}^n$ satisfying the equation

$$a_0|x|^2 - 2a \cdot x + a_{n+1} = 0.$$

If $a_0 \neq 0$, then we have

$$\Sigma = S\left(\tfrac{a}{a_0}, \tfrac{(|a|^2 - a_0 a_{n+1})^{1/2}}{|a_0|}\right)$$

If $a_0 = 0$,then

$$\Sigma = \hat{P}\left(\tfrac{a}{|a|}, \tfrac{a_{n+1}}{2|a|}\right)$$

The vector $(a_0, \cdots, a_{n+1})$ is known as coefficient vector for Σ, and it is distinctly determined by Σ up to multiplication by a nonzero scalar.

2.7 Möbius Transformations of Upper Half-Space

Definition 2.4. *A Möbius transformation of upper half-space U^n is a Möbius transformation of $\hat{E}^n$ that leaves U^n invariant.*

Let $M(U^n)$ be the set of all Möbius transformations of U^n. Then $M(U^n)$ is defined as a subgroup of $M(\hat{E}^n)$.
The group $M(U^n)$ of Möbius transformations of U^n is isomorphic to $M(\hat{E}^{n-1})$.

Theorem 41. *Every Möbius transformation of U^n is the composition of reflections of $\hat{E}^n$ in spheres orthogonal to $\hat{E}^{n-1}$.*

Two spheres Σ and Σ' of $\hat{E}^n$ are said to be orthogonal iff they intersect in E^n and at each point of intersection in E^n their normal lines are orthogonal.

Theorem 42. *Two spheres of $\hat{E}^n$ are orthogonal under the following conditions:*
(1) The spheres $\hat{P}(a,r)$ and $\hat{P}(b,s)$ are orthogonal iff a and b are orthogonal.
(2) The spheres $S(a,r)$ and $\hat{P}(b,s)$ are orthogonal iff a is in $P(b,s)$.
(3) The spheres $S(a,r)$ and $S(b,s)$ are orthogonal iff r and s satisfy the equation $|a-b|^2 = r^2 + s^2$.

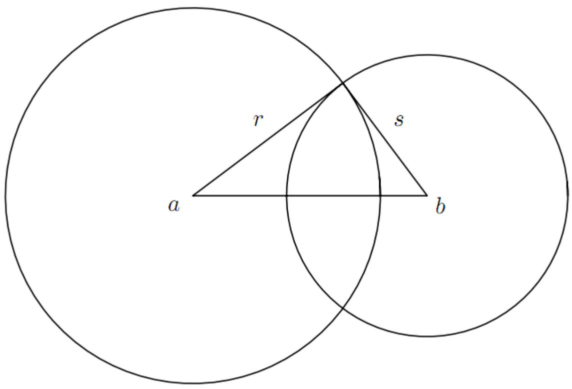

Figure 2.5: Orthogonal circles $S(a,r)$ and $S(b,s)$

Image Courtesy: Ratcliffe, J.G.: Foundation of Hyperbolic Manifolds, Graduate Texts in Mathematics 149. Springer, Berlin p. 118 (1994).

Remark: The two spheres Σ and Σ' of $\hat{E}^n$ are orthogonal iff they are orthogonal on a single point of intersection in E^n.

2.7.1 Möbius Transformations of the Unit n-Ball

Let σ be the reflection of $\hat{E}^n$ in the sphere $S(e_n, \sqrt{2})$. Then

$$\sigma(x) = e_n + \frac{2(x - e_n)}{|x - e_n|^2} \tag{2.29}$$

Therefore

$$|\sigma(x)|^2 = 1 + \frac{4e_n(x - e_n)}{|x - e_n|^2} + 4|x - e_n|^2 \tag{2.30}$$

Hence

$$|\sigma(x)|^2 = 1 + \frac{4x_n}{|x - e_n|^2}. \tag{2.31}$$

This suggests that σ maps lower half-space -U^n into the open unit n-ball

$$B^n = \{x \in E^n : |x| < 1\} \tag{2.32}$$

Since σ is a homeomorphism of $\hat{E}^n$, it maps every component of $\hat{E}^n - \hat{E}^{n-1}$ homeomorphically onto a component of $\hat{E}^n - S^{n-1}$. Hence σ maps -U^n homeomorphically onto B^n and vice versa. Let ρ be the reflection of $\hat{E}^n$ in $\hat{E}^{n-1}$ and we define $\eta = \sigma\rho$. Then η maps U^n homeomorphically onto B^n. The Möbius transformation η is known as the standard transformation from U^n to B^n.

Definition 2.5. *A Möbius transformation of S^n is a function $\phi : S^n \to S^n$ such that $\pi^{-1}\phi\pi$ is a Möbius transformation of $\hat{E}^n$, where $\pi : \hat{E}^n \to S^n$ is stereographic projection.*

Let $M(S^n)$ is given by the set of all Möbius transformations of S^n. Then $M(S^n)$ forms a group under composition. The mapping $\psi \to \pi\psi\pi^{-1}$ is an isomorphism from $M(\hat{E}^n)$ to $M(S^n)$.

Definition 2.6. *A Möbius transformation of the open unit ball B^n is a Möbius transformation of $\hat{E}^n$ which leaves B^n invariant.*

Theorem 43. *Let ϕ be a Möbius transformation of B^n. If $\phi(\infty) = \infty$, then ϕ is orthogonal. If $\phi(\infty) \neq \infty$, then the isometric sphere Σ of ϕ is orthogonal to S^{n-1} and $\phi = \psi\sigma$, where σ is given as the reflection in Σ and ψ is an orthogonal transformation.*

2.8 The Conformal Ball Model

By redefining the Lorentzian inner product on $\mathbb{R}^{n+1}$ to be

$$x \circ y = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}. \tag{2.33}$$

The Lorentz group of $\mathbb{R}^{n,1}$ is given by $O(n,1)$. Identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^{n+1}$. The stereographic projection ζ of the open unit ball B^n onto hyperbolic space H^n is given by projecting x in B^n away from $-e_{n+1}$ unless it meets H^n in the distinct point $\zeta(x)$. Since $\zeta(x)$ is on the line passing through x in the direction of $x + e_{n+1}$, there exists a scalar s such that

$$\zeta(x) = x + s(x + e_{n+1}).$$

The condition $\|\zeta(x)\|^2 = -1$ leads to the value

$$s = \tfrac{1+|x|^2}{1-|x|^2}$$

and the formula

$$\zeta(x) = \left(\frac{2x_1}{1-|x|^2}, \cdots, \frac{2x_n}{1-|x|^2}, \frac{1+|x|^2}{1-|x|^2} \right) \tag{2.34}$$

The map ζ is a bijection of B^n onto H^n. The inverse of ζ is defined by

$$\zeta(y)^{-1} = \left(\frac{y_1}{1+y_{n+1}}, \cdots, \frac{y_n}{1+y_{n+1}} \right) \tag{2.35}$$

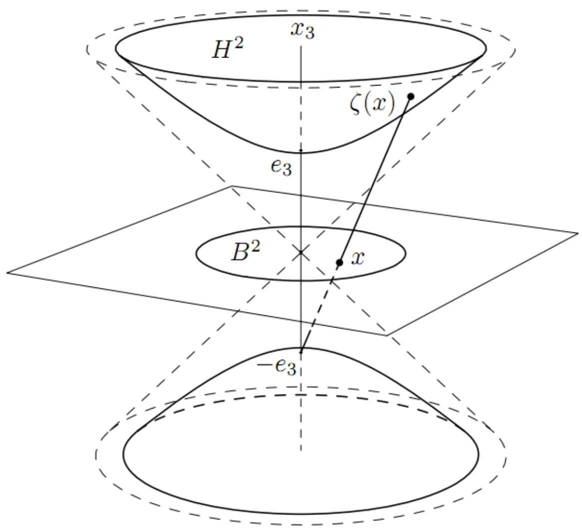

Figure 2.6: The stereographic projection ζ of B^2 onto H^2

Image Courtesy: Ratcliffe, J.G.: Foundation of Hyperbolic Manifolds, Graduate Texts in Mathematics 149. Springer, Berlin p. 122 (1994).

2.8.1 Hyperbolic Translation

Let $S(a,r)$ be defined as a sphere of E^n orthogonal to S^{n-1}. We have $r_2 = |a|^2 - 1$, and so a determines r. Suppose σ_a be the reflection in $S(a,r)$. Then σ_a leaves B^n invariant. Suppose ρ_a be the reflection in the hyperplane $a \cdot x = 0$. Then ρ_a also leaves B^n invariant, and thus the composite $\sigma_a\rho_a$ also leaves B^n invariant. Define

$$a^* = a/|a|^2.$$

It is obvious that

$$\sigma_a\rho_a(x) = \frac{(|a|^2-1)x+(|x|^2+2x\cdot a^*+1)a}{|x+a|^2}$$

$\sigma_a\rho_a(0) = a^*$

Let b be a nonzero point of B^n and let $a = b^*$. Then $|a| > 1$ and $a^* = b$, then $r = (|a|^2 - 1)^{1/2}$. Obviously $S(a,r)$ is orthogonal to S^{n-1}. Thus, define a Möbius transformation of B^n by

$$\tau_b = \sigma_{b^*}\rho_{b^*}.$$

Then

$$\tau_b(x) = \frac{(|b^*|^2-1)x+(|x|^2+2x\cdot b+1)b^*}{|x+b^*|^2}$$

As τ_b is the composite of two reflections in hyperplanes orthogonal to the line $(-b/|b|, b/|b|)$, the transformation τ_b acts as a translation along this line. We also define τ_0 to be the identity. So $\tau_b(0) = b$ for all b in B^n. The map τ_b is said to be the hyperbolic translation of B^n by b.

Theorem 44. *Every Möbius transformation of B^n restricts to an isometry of the conformal ball model B^n, and every isometry of B^n extends to a unique Möbius transformation of B^n.*

Theorem 45. *A subset S of B^n is a hyperbolic sphere of B^n iff S is a Euclidean sphere of E^n that is contained in B^n.*

2.9 The Upper Half-Space Model

Let η be defined as the standard transformation from upper half-space U^n to the open unit ball B^n. Then $\eta = \sigma\rho$, where ρ is the reflection of $\hat{E}^n$ in the hyperplane $\hat{E}^{n-1}$ and σ is the reflection of $\hat{E}^n$ in the sphere $S(e_n, \sqrt{2})$. We define a metric d_U on U^n which is given by

$$d_U(x, y) = d_B(\eta(x), \eta(y)).$$

The metric d_U is said to be the *Poincaré* metric on U^n. By definition, η is an isometry from U^n, with the metric d_U, to the conformal ball model B^n of hyperbolic n-space. The metric space consisting of U^n together with the metric d_U is called the upper half-space model of hyperbolic n-space.

Theorem 46. *Every Möbius transformation of U^n restricts to an isometry of the upper half-space model U^n, and every isometry of U^n extends to an unique Möbius transformation of U^n.*

2.10 Classification of Transformations in Unit Ball Model

Let ϕ be defined as a Möbius transformation of B^n.Then ϕ maps the closed ball $\bar{B}^n$ to itself. By the Brouwer fixed point theorem, we have that ϕ has a fixed point in $\bar{B}^n$. The transformation ϕ is called
(1) elliptic if ϕ fixes a point of B^n;
(2) parabolic if ϕ fixes no point of B^n and fixes a unique point of S^{n-1};
(3) hyperbolic if ϕ fixes no point of B^n and fixes two points of S^{n-1}.

Let F_ϕ be the set of all the fixed points of ϕ in $\bar{B}^n$,and let ψ be a Möbius transformation of B^n. So

$$F_{\psi\phi\psi^{-1}} = \psi(F_\phi) \tag{2.36}$$

Hence ϕ is elliptic, parabolic, or hyperbolic iff $\psi\phi\psi^{-1}$ is elliptic, parabolic, or hyperbolic, respectively. Therefore, being elliptic, parabolic, or hyperbolic depends only on the conjugacy class of ϕ in $M(B^n)$.

2.10.1 Elliptic Transformations

Theorem 47. *A Möbius transformation ϕ of B^n is elliptic iff ϕ is conjugate in $M(B^n)$ to an orthogonal transformation of E^n.*

2.11 Classification of Transformations in Upper-Half Space Model

Parabolic and hyperbolic transformations can be analyzed easily in the upper half-space model U^n of hyperbolic space. Elliptic, parabolic, and hyperbolic Möbius transformations of U^n are defined in the similar way as in the conformal ball model B^n. Let ϕ be a Möbius transformation of U^n. The transformation ϕ is said to be
(1) elliptic if ϕ fixes a point of U^n;
(2) parabolic if ϕ fixes no point of U^n and fixes a unique point of $\hat{E}^{n-1}$;
(3) hyperbolic if ϕ fixes no point of U^n and fixes two points of $\hat{E}^{n-1}$.

Remark: Being elliptic, parabolic, or hyperbolic depends only on the conjugacy class of ϕ in $M(U^n)$.

Theorem 48. *A Möbius transformation ϕ of U^n is parabolic iff ϕ is conjugate in $M(U^n)$ to a fixed point free isometry of E^{n-1}.*

Chapter 3

Conjugacy Classes in Möbius Groups

The theory and proofs in this chapter are influenced from Gongopadhyay, K., Conjugacy classes in Möbius groups, Geom Dedicata (2011) 151:245-258 Springer Science+ Business Media B.V. (2010).

3.1 Introduction

The $n+1$-dimensional hyperbolic space is denoted by $\mathbb{H}^{n+1}$ and the conformal boundary of the hyperbolic space is denoted by $\mathbb{S}^n$. $M(n)$ denotes the group of conformal diffeomorphisms of $\mathbb{S}^n$ and $M_o(n)$ be defined as identity component which consists of all orientation preserving elements in $M(n)$. Conjugacy classes of isometrics in $M_o(n)$ depends on the conjugacy of T and T^{-1} in $M_o(n)$. An element $T \in M(n)$, T and T^{-1} are conjugate in $M(n)$, but they may not be conjugate in $M_o(n)$. T is called real if T and T^{-1} are conjugate to each other in $M_0(n)$. Let T be an element in $M_o(n)$, so to T there is a related element T_o in $SO(n+1)$. If the complex conjugate eigenvalues of T_o are given by $\{e^{i\theta_j}, e^{-i\theta_j}\}$, $0 < \theta j \leqslant \pi$, $j = 1, \cdots, k$, then $\theta_1, \cdots, \theta_k$ are called the rotation angles of T. T is called a *regular* element if the *rotation angles* of T are different from each-other. After classification of the real elements in $M_o(n)$ we have parametrized the conjugacy classes of regular elements in $Mo(n)$. In the parametrization, when T is not conjugate to T^{-1}, then enlarge the group and consider the conjugacy class of T in $M(n)$. So each such conjugacy class can be induced with a fibration structure.

3.2 Real Elements in Möbius Groups

Definition 3.1. *An element g in a linear algebraic group G is known as real if it is conjugate in G to its own inverse. Every element in $M(n)$ is real.*

The $n+1$-dimensional hyperbolic space is denoted by $\mathbb{H}^{n+1}$ and the conformal boundary of the hyperbolic space is denoted by $\mathbb{S}^n$. $M(n)$ denotes the group of conformal diffeomorphisms of $\mathbb{S}^n$ and $M_o(n)$ be defined as identity component which are all orientation preserving elements in $M(n)$. The group of isometries of $\mathbb{H}^{n+1}$ is identified with the group $M(n)$. In the ball model and the upper-half space model of the hyperbolic space, we denote the isometry group and its identity component by $M(n)$ and $M_o(n)$, respectively.

Determine the conjugacy classes in $M(n)$ by the minimal polynomial and the characteristic polynomial of an isometry. In general, an element in $M(n)$ is conjugate to its inverse. Rather, this does not holds for elements in $M_o(n)$. Take an example of an unipotent isometry $T \in M_o(1)$ which is not conjugate to its inverse. It is clear by identifying $M_o(1)$ with $PSL(2,\mathbb{R})$. therefore a conjugacy class in $M(n)$ possibly breaks into conjugacy classes in $M_o(n)$. The inverse of an isometry T is not conjugate to itself in $M_o(n)$. So to investigate the conjugacy classes in $M_o(n)$ it is necessary to classify the elements which are conjugate in $M_o(n)$ to their inverse. The classification of such type of elements essentially helps in classifying the conjugacy classes of $M_o(n)$.

Let $\mathbb{V}$ be a real vector space of dimension $n+1$ provided a non-degenerate quadratic form $\mathcal{Q}$ with signature $(n,1)$, i.e. corresponding to a suitable coordinate system $\mathcal{Q}$ has the form $\mathcal{Q}(x) = x_0^2 + \cdots + x_{n-1}^2 - x_n^2$. The full group of isometries of $(V, \mathcal{Q})$ is denoted by $O(n,1)$ and $SO(n,1)$ subgroup of $O(n)$ with all isometries of det 1. It is easy to see that $SO(n,1)$ is a index 2 subgroup in $O(n,1)$. It has two components.

- $v \in \mathbb{V}$ time-like, if $\mathcal{Q}(v) < 0$,
  $v \in \mathbb{V}$ space-like, if $\mathcal{Q}(v) > 0$,
  $v \in \mathbb{V}$ light-like, if $\mathcal{Q}(v) = 0$.

- A subspace W is time-like, if $\mathcal{Q}|_{\mathbb{W}}$ is non-degenerate and indefinite,
  A subspace W is space-like, if $\mathcal{Q}|_{\mathbb{W}} > 0$,
  A subspace W is light-like, if $\mathcal{Q}|_{\mathbb{W}} = 0$.

There are two components of hyperboloid $\{v \in \mathbb{V} | \mathcal{Q}(v) = -1\}$. One component with the vector $e_n = (0, 0, \cdots, 0, 1)$ is known as hyperboloid or linear model of the hyperbolic space $\mathbb{H}^n$. The index 2 subgroup of $O(n,1)$ is the isometry group $I(\mathbb{H}^n)$, it preserves the hyperplane $\mathbb{H}^n$. The group $M(n)$ and $I(\mathbb{H}^{n+1})$ are identified with each other, by this similar identification $M_o(n) = SO_o(n+1,1)$. The identity component of the groups $M(n)$ and $SO(n+1,1)$ are same, the difference is due to their second components. In $M(n)$ the second component consists of the orientation-reversing isometries of hyperbolic $n+1$-space, $\mathbb{H}^{n+1}$ having det -1.

Theorem 49. *1. Every element in $SO_o(n,1)$ is real iff $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$.*

2. If $n \equiv 1 \pmod 4$, then an element T in $SO_o(n,1)$ is real iff it is either a hyperbolic isometry with at least one eigenvalue ± 1, or, it is not hyperbolic.

3. If $n \equiv 2 \pmod 4$,then an element T in $SO_o(n,1)$ is real iff one of the following holds.
(a) T is hyperbolic,
(b) T is a non-hyperbolic with at least one eigenvalue -1,
(c) T is non-hyperbolic, it has no eigenvalue -1, and there is at least one eigenvector to 1 which is space-like.

Proof Consider T is an elliptic element of $SO_o(n,1)$. So T fixes a time-like eigen vector v. The space-like orthogonal complement to the 1-dimensional subspace spanned by v is $\mathbb{W}$ and $dim\,(W)$ is n, then we write $V = \lambda v \oplus w$ for $\lambda \in \mathbb{R}$. So $T_o = T|_{\mathbb{W}}$ is an element in $SO(n)$. If $n \not\equiv 2 \pmod 4$, so there exists an orthogonal map $S_o : \mathbb{W} \to \mathbb{W}$ provided that determinant $S_o = 1$ and $S_o T_o S_o^{-1} = T_o^{-1}$. therefore there exists $S = \begin{pmatrix} S_o & 0 \\ 0 & 1 \end{pmatrix}$ in $SO_o(n,1)$ provided that $STS^{-1} = T^{-1}$. therefore T is real in $SO_o(n,1)$. Now consider $n \equiv 2 \pmod 4$, and consider T has no space-like eigenvalue ± 1. So in this case, any choice of S_o has determinant essentially -1. therefore it is not possible to choose any S as above. Thus T can not be real.
Consider T be a hyperbolic. So T has a real eigenvalue $r > 0$. As a result, $\mathbb{V}$ has an orthogonal decomposition $\mathbb{V} = \mathbb{V}_r \oplus \mathbb{W}$, such that $\mathbb{V}_r$ is a 2-dimensional orthogonally indecomposable time-like subspace and $\mathbb{W}$ is its space-like orthogonal complement of dimension $(n-1)$. Denote $T_r = T|_{\mathbb{V}_r}$, $T_o = T|_{\mathbb{W}}$. Since T is semi simple(i.e. a diagonalizable element of finite dimensional vector space V), and $\mathbb{V}_r$ is an eigen space of T, by considering any element S which conjugates T to T^{-1} must preserve $\mathbb{V}_r$. It is clear that any f in $I(\mathbb{H}^1)$ provided that $fT_rf^{-1} = T_r^{-1}$ must have det -1. Therefore T is real

in $SO_o(n,1)$ iff it is possible choose an S_o in $O(n-1)$ provided that $S_oT_oS_o^{-1} = T_o^{-1}$ and determinant $S_o = -1$ (because if T is real then T must be strongly real thus product of two involutions). This happens only when $n-1 \not\equiv 0 \pmod 4$, i.e. $n \not\equiv 1 \pmod 4$, or T_o has an eigenvalue ± 1.
Consider $n \equiv 1 \pmod 4$ and T has no space-like eigenvalue ± 1. As shown in above paragraph, any choice of S_o would have determinant essentially 1, and thus T can not be real in $SO_o(n,1)$.

Consider T be a parabolic. Implies that T has a time-like non-degenerate indecomposable sub-space $\mathbb{V}_1$ having dimension 3, and $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{W}$, such that $\mathbb{W}$ is the space-like $(n-2)$-dimensional orthogonal complement of $\mathbb{V}_1$. Minimal polynomial of $T|_{\mathbb{V}_1}$ is $(x-1)^3$. Suppose $T_1 = T|_{\mathbb{V}_1}, T_o = T|_{\mathbb{W}}$. Then $\mathbb{V}_1$ must be invariant under an isometry S with property that T is conjugate to T^{-1}. Therefore signature of $\mathcal{Q}|_{\mathbb{V}_1}$ is $(2,1)$, thus suppose T_1 as an unipotent isometry in $I(\mathbb{H}^2)$. It is clear that any isometry S_1 in $I(\mathbb{H}^2)$ which conjugates T_1 to T_1^{-1} with det -1. Thus T is real for an element S_o in $O(n-2)$ provided that $S_oT_oS_o^{-1} = T_o^{-1}$ and determinant $S_o = -1$ (by similar argument as in above paragraph). This only happens when $n-2 \equiv 0 \pmod 4$ i.e. $n \not\equiv 2 \pmod 4$, or S_o has an eigenvalue ± 1. It is clear that when $n \not\equiv 2 \pmod 4$ and T has no space-like eigenvalue ± 1, subsequently any element S which conjugates T to T^{-1} have determinant essentially -1, and thus it is not possible that T is real in this case.

Definition 3.2. *An element T in $SO_o(n,1)$ is called strongly real if it can be written as a product of two involutions in $SO_o(n,1)$.*

Theorem 50. *An element T in $SO_o(n,1)$ is strongly real iff it is real.*

Proof Consider T is an isometry of $\mathbb{H}^{n+1}$ and T is real. It is sufficient to construct an involution g in $SO_o(n,1)$ provided that $gTg^{-1} = T^{-1}$. The construction of g as done in the above theorem, and the decomposition of V as shown in the above proof.

3.3 Reality properties of conjugacy classes in $\mathbf{M}_o(n)$

Definition 3.3. *Let G be a group. An element g in G is called real if there exists h in G provided that $hgh^{-1} = g^{-1}$. An element g in G is an involution if $g^2 = 1$.*

Definition 3.4. *An element g in G is called strongly real if it can be written as a product of two involutions in G.*

A strongly real element in G is always real. Conversely, a real element $g \in G$ is strongly real iff there is a conjugating element in G which is an involution.

3.3.1 Reality in $SO(n)$

Suppose $\mathbb{V}$ be an n-dimensional vector space over $\mathbb{R}$ which has with a non-degenerate positive definite quadratic form q. The isometry group is denoted by $O(n)$ and let $SO(n)$ denote the subgroup $O(n)$ of index two contaning isometries with det 1. We can identify $\mathbb{V}$ with Euclidean space $\mathbb{E}^n$.

Theorem 51. *Let T be an element in $SO(n)$. Then T is strongly real in $SO(n)$ iff $n \not\equiv 2 \pmod 4$ or an orthogonal decomposition of $\mathbb{V}$ into orthogonally indecomposable T-invariant subspaces contains an odd dimensional summand.*

Proof Let $T \in SO(n)$ and has no eigenvalue ± 1. Then $\mathbb{V}$ has an orthogonal decomposition.

$$\mathbb{V} = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_k,$$

into two dimensional invariant subspaces. Each $\mathbb{V}_i$ with dimension 2, so in this case for each $i = 1, \cdots, k$, there is an involution f_i provided that $f_i T|_{\mathbb{V}_i} f_i^{-1} = T|_{\mathbb{V}}^{-1}$. Let $f = f_1 \oplus f_2 \oplus \cdots \oplus f_k$. Then f is an involution, and $fTf^{-1} = T^{-1}$, and determinant $f = (-1)^{\frac{n}{2}}$. therefore determinant $f = 1$ iff either of the conditions provided in the theorem are satisfied. If determinant $f = 1$, then $T = f.fT$, the product of two involutions.

Theorem 52. *Consider n is even and T be an element in $SO(n)$. Consider the minimal polynomial of T is a power of an irreducible quadratic polynomial over $\mathbb{R}$. Then T is real iff $n \not\equiv 2 \pmod 4$.*

Proof As the minimal polynomial of T is a irreducible power polynomial over $\mathbb{R}$, assume that the only eigenvalues over $\mathbb{C}$ are $\{e^{i\theta}, e^{-i\theta}\}$. Let $\chi_T(x) = (x_2 - 2\cos\theta x + 1)^m$ be the characteristic polynomial of T. So $n = 2m$. Let $S \in O(n)$ provided that $STS^{-1} = T^{-1}$. Let $\mathbb{V}_c = \mathbb{V} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification then identify T with $T \otimes_{\mathbb{R}} id$ and also view it as an operator on $\mathbb{V}_c$. Let $\mathbb{V}_c = \mathbb{V}_\theta + \mathbb{V}_{-\theta}$ be the decomposition into its eigenspaces. So S interchanges $\mathbb{V}_\theta$ and $\mathbb{V}_{-\theta}$. It is clear that determinant $S = (-1)^m$. thus S is an element of $SO(n)$ iff m is even.

Theorem 53. *Let T be an element in $SO(n)$. Consider 1 and -1 are not eigenvalues of T. Then T is real iff $n \not\equiv 2 \pmod 4$.*

Proof If ± 1 is not an eigenvalue of T, then the only possibility of n is being even. For T in $SO(n)$ there exists a decomposition of $\mathbb{V}$ into T-invariant subspaces

$$\mathbb{V} = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_k,$$

where for each $i = 1, 2, \cdots, k$, $\mathbb{V}_i \simeq \mathbb{R}[x]/(x^2 - 2cos\theta_i x + 1)^{m_i}$ for $m_i \geq 1$. Let S be an element in $O(n)$ provided that $STS^{-1} = T^{-1}$. Then S keeps each $\mathbb{V}_i$ invariant. Let $S_i = S|_{\mathbb{V}_i}, T_i = T|_{\mathbb{V}_i}$ are restrictions of S and T on $\mathbb{V}_i$. Then $S_i T_i S_i^{-1} = T_i^{-1}$. therefore det $S_i = (-1)^{m_i}$. therefore det $S = \Pi_{i=1}^{k} detS_i = (-1)^{\frac{n}{2}} n$. thus det $S = 1$ iff $\frac{n}{2} = 2m$. This completes the theorem.

Theorem 54. *Let T be an element in $SO(n)$. Then T is real in $SO(n)$ iff either $n \not\equiv 2 \pmod 4$ or T has an eigenvalue ± 1.*

Proof Consider $S \in O(n)$ be provided that $STS^{-1} = T^{-1}$. If T has no eigenvalue $1, -1$ and $n \not\equiv 2 \pmod 4$, then it can be clearly seen from the above theorem that det S can not be equal to 1. Thus T is not real.

Consider T has an eigenvalue ± 1. If -1 is an eigen value of T, then it should have an even multiplicity. Therefore an orthogonal decomposition of $\mathbb{V}$ into T-invariant subspaces

$$\mathbb{V} = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_k \oplus \Lambda_1 \oplus \Lambda_{-1},$$

where each $\mathbb{V}_i$ is T-invariant and even dimensional, $T|_{\Lambda_1} = I$, $T|_{\Lambda_{-1}} = -I$. Consider $\mathbb{W} = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_k$. Then dimension of $\mathbb{W}$ is even. By the reality of the orthogonal group, there exists an orthogonal map $S_w : \mathbb{W} \to \mathbb{W}$ provided that $S_w T_w S_w^{-1} = T_w^{-1}$ with det $S_w = I$ or -1. Since the maps I and $-I$ commutes with every element in the orthogonal group, after selecting such S_w, the maps $S_1 : \Lambda \to \Lambda$ can be selected accordingly provided that $STS^{-1} = T^{-1}$ and det $S = 1$, where $S = S_w \oplus S_1 \oplus S_{-1}$. thus T is real in $SO(n)$.

Theorem 55. *An element T in $SO(n)$ is real iff it is strongly real.*

Proof Let T is an element of $SO(n)$ provided that $n \not\equiv 2 \pmod 4$. Then by theorem (54) T is real and by theorem (51) T is strongly real, so clearly T is real iff T is strongly real for $n \not\equiv 2 \pmod 4$.

T has an eigen value of ± 1 iff T is only real by theorem (54). But T has an eigen value ± 1 iff orthogonal decompostion of $\mathbb{V}$ into orthogonally indecomposable T invariant subspaces contains an odd dimensional summand. therefore by theorem (51) T is strongly real.

Therefore T is real iff it is strongly real.

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