Analysis of self-adjoint operators

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Certificate of Examination

This is to certify that the dissertation titled **Analysis of Self Adjoint operators** submitted by **Hitesh Gakhar** (Reg. No. MS09060) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr Lingaraj Sahu at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly. This thesis is a bonafide record of study done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Lingaraj Sahu
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Abstract

In this thesis, we try to analyze self adjoint operators on a Hilbert space $\mathcal{H}$. This thesis talks about the spectrum, the spectral decomposition and the perturbation of self adjoint operators. The need to study perturbation comes from the setting of Quantum mechanics. If we consider the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$, then the elements of $\mathcal{H}$ are the states of the system. Each observable is represented by a self adjoint linear operator acting on the state space. Each eigenstate of an observable corresponds to an eigenvector of the operator, and the associated eigenvalue corresponds to the value of the observable in that eigenstate. If the operator's spectrum is discrete, the expectation of observables can attain only those discrete eigenvalues. We denote the Hamiltonian by

$$H = -\Delta + V$$

where Δ is the Laplacian and V is the potential operator. In the later part of the thesis, we start the theory perturbation in different instances. First we see that the essential spectrum of a bounded operator is invariant under perturbation by a compact operator. Then we see that a small relatively bounded symmetric operator when added to a self adjoint operator gives us a self adjoint operator. Towards the end, we study a special case of rank one perturbations of self adjoint operator. The key result says that the absolutely continuous part of the spectrum stays invariant.

Contents

Chapter 1

Bounded operators on a Hilbert space

In this chapter, we look at the introductory theory to Hilbert spaces, bounded operators on a Hilbert space, their spectra and the spectral decomposition of a normal operator.
A crucial result in this chapter says that if we have a spectral measure or a resolution of identity for a bounded operator T, then we can make sense $f(T)$ for a bounded function f on $\sigma(T)$. For more detailed theory, see [1].

1.1 Bounded operators

Definition 1.1. *Let $\mathcal{H}$ be a complex vector space. A map from $\mathcal{H} \times \mathcal{H}$ to $\mathbb{C}$ is called an inner product if the following hold:*
(a) $(x, y) = \overline{(y, x)}$
(b) $(x + z, y) = (x, y) + (z, y)$
(c) $(ax, y) = a(x, y)$
(d) $(x, x) \geq 0$ *with* $(x, x) = 0$ *if and only if* $x = 0$
for $x, y \in \mathcal{H}, a \in \mathbb{C}$. $\mathcal{H}$ equipped with such a map is called an inner product space. It is called a Hilbert space if the space is complete with respect to the norm

$$\| x \| = (x, x)^{1/2}.$$

The following are a few examples of Hilbert spaces, taken from [3].
Examples:

(a) **Euclidean Space $\mathbb{C}^n$:** The space $\mathbb{C}^n$ is a Hilbert space with inner product defined by

$$(x, y) = x_1\overline{y_1} + x_2\overline{y_2} + ... + x_n\overline{y_n}$$

where $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ and $\overline{y_i}$ denotes the complex conjugate of y_i.

(b) **Space $\mathcal{L}^2[a, b]$:** Here $a, b \in \mathbb{R}$. $\mathcal{L}^2[a, b]$ is the set of all square integrable functions with the inner product defined by

$$(x, y) = \int_a^b x(t)y(t)dt.$$

Definition 1.2. *A Banach algebra is a complete normed vector space* $\mathbf{V}$ *with a unit element* e*(such that* $xe = ex = x$ *for* $x \in \mathbf{V}$*) over the field* $\mathbb{C}$ *in which multiplication is defined in a way such that it is associative and it satisfies*

$$(x + y)z = xz + yz,$$

$$x(y + z) = xy + xz,$$

$$\alpha(xy) = (\alpha x)y = x(\alpha y),$$

and the multiplicative inequality

$$\| xy \| \leq \| x \| \| y \|$$

for $x, y, z \in \mathbf{V}$ *and* $\alpha \in \mathbb{C}$.

Definition 1.3. *Let* $\mathcal{B}(\mathcal{H})$ *be the set of all bounded linear operators on* $\mathcal{H}$. *We define a norm on* $\mathcal{B}(\mathcal{H})$ *as:*

$$\| T \| = \sup\{\| Tx \| : x \in \mathcal{H}, \| x \| \leq 1\}$$

.

It is easy to see that $\mathcal{B}(\mathcal{H})$ forms a Banach Algebra.

Theorem 1.1. *Let* $\mathcal{H}$ *be a Hilbert space and* f *be a bounded-sesquilinear(i.e. linear in the first variable and conjugate linear in the second) functional*

$$f : \mathcal{H} \times \mathcal{H} \to \mathbb{C}.$$

Then there exists a a unique $S \in \mathcal{B}(\mathcal{H})$ that satisfies

$$f(x, y) = (x, Sy) \quad (x, y \in \mathcal{H})$$

Also, $\| S \| = \| f \|$.

Idea of the proof: Since f is bounded, the map

$$x \to f(x, y)$$

is a bounded linear functional on $\mathcal{H}$. Thus by Riesz representation theorem, there exists a unique $Sy \in H$ such that

$$f : \mathcal{H} \times \mathcal{H} \to \mathbb{C}.$$

Definition 1.4. *Let $\mathcal{H}$ be a Hilbert space and T be a bounded operator. The spectrum of T, denoted by $\sigma(T)$ is the the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is not invertible.*

Definition 1.5. *The spectral radius is defined as*

$$\gamma(T) = \sup\{| \lambda | : \lambda \in \sigma(T)\}$$

Remark:
(a) $\sigma(T)$ is compact and non empty.
(b) $\gamma(T)$ satisfies

$$\gamma(T) = \lim_{n \to \infty} \| T^n \|^{1/n}$$

Classification of spectra: How do we find values in $\sigma(T)$? If $\lambda I - T$ is not bounded below(or not injective in particular), then it must not be invertible. Hence, $\lambda \in \sigma(T)$. We define few types of spectral values:

Definition 1.6. *If for $T \in \mathcal{B}(\mathcal{H})$, $\lambda I - T$ is not injective, or for some x, $(\lambda I - T)x = 0$, then we say that λ is an eigenvalue. The set of all eigenvalues is called the eigenspectrum, or the point spectrum $\sigma_p(T)$.*

Examples: We take the simplest example in $\mathbb{R}^n$. Take T to be an operator that acts on the basis $\{v_1, v_2, ..., v_n\}$ in the following way:

$$T(v_i) = \lambda_i v_i$$

for $\lambda_i \in \mathbb{R}$. We see that $(\lambda_i I - T)v_i = 0$. Hence the collection of λ_is is the point spectrum of T.

Definition 1.7. *If for $T \in \mathcal{B}(\mathcal{H})$, $T - \lambda I$ is not bounded below, then we can find a sequence $\{x_n\}$ in H such that $\| x_n \| = 1 \forall n$ and $\| T(x_n) - \lambda x_n \| \to 0$ as $n \to \infty$. Such a k is called an approximate eigenvalue and the collection of all approximate eigenvalues is called the approximate eigenspectrum, denoted by $\sigma_a(T)$.*

Examples: Consider the bilateral shift T on $l^2(\mathbb{R})$ defined by

$$T(..., a_{-1}, \hat{a}_0, a_1, ...) = (..., \hat{a_{-1}}, a_0, a_1...)$$

where ˆ defines the zero-th position. It is easy to see that T doesn't have an eigenvalue. However, but for $| \lambda | = 1$, every λ is an approximate eigenvalue. Let

$$x_n = 1/\sqrt{n}(..., 0, 1, \lambda_{-1}, \lambda_{-2}, ..., \lambda_{1-n}, 0, ...).$$

Then $\| x_n \| = 1$ and

$$\| Tx_n - \lambda x_n \| = \sqrt{2/n} \to 0.$$

Definition 1.8. *An operator can be bounded below(and hence, injective) but may not be surjective. If $T - \lambda I$ is injective, but does not have a dense range, then $\lambda \in \sigma_r(T)$, the residual spectrum of T.*

Examples: The unilateral right shift on $l^2(\mathbb{N})$ given by

$$T(a_1, a_2, ...) = (0, a_1, a_2, ...)$$

is an example. This shift operator is an isometry, therefore bounded below by 1. But it is not invertible as it is not surjective(nor has dense range).

Definition 1.9. *Now, if for some λ, $T - \lambda I$ is injective, has a dense range, but is not surjective, then set of such λ is said to be continuous spectrum, denoted by $\sigma_c(T)$.*

1.2 Adjoint

Definition 1.10. *Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$. Then the adjoint T^* of T is the operator*

$$T^* : \mathcal{H} \to \mathcal{H}$$

such that $(Tx, y) = (x, T^*y)$ for $x, y \in \mathcal{H}$.

Remark: In the previous theorem, let $f(x, y) = (Tx, y)$ for $T \in \mathcal{B}(\mathcal{H})$. Then the unique S we get is the adjoint of T. Also, we see that $\| T^* \|=\| S \|=\| T \|$.
The following is an example of an adjoint, taken from [3].
Examples: Let $\mathcal{H}$ be a Hilbert space and $\{u_1, u_2, .., u_n\}$ be an orthonormal basis for $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, we can write the matrix

$$[A] = (\alpha_{i,j})$$

where $\alpha_{i,j} = (A(u_j), u_i)$. Since $A^* \in \mathcal{B}(\mathcal{H})$ and

$$(A^*(u_j), u_i) = (u_j, A(u_i)) = \overline{(A(u_i), u_j)}$$

for $i, j = 1, 2, ..n$. It follows that $[A^*] = (\overline{k_{j,i}})$ with respect to the same basis.

Proposition 1.1. *The map $T \to T^*$ is an involution on $\mathcal{B}(\mathcal{H})$, i.e. the following properties hold*

$$(T + S) = T^* + S^*$$

$$(\alpha(T))^* = \bar{\alpha}T^*$$

$$(ST)^* = T^*S^*$$

$$T^{**} = T$$

The proofs of the above properties are trivial. We just have to play around with the properties of the inner product. Now we define different types of operators.

Definition 1.11. *An operator $T \in \mathcal{B}(\mathcal{H})$ is called*
(a) a normal operator if $TT^* = T^*T$.
(b) a self adjoint operator if $T^* = T$.
(c) a unitary operator if $TT^* = I = T^*T$.
(d) a projection if $T^2 = T$.
(e) a compact operator if $\{Tx_n\}$ has a Cauchy subsequence for any bounded sequence $\{x_n\}$.

Having defined the operators, the following two theorems talk about the spectrum, its properties and the relations between various components of it.

Theorem 1.2. *Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$. Then*
(a) $k \in \sigma(T)$ iff $\bar{k} \in \sigma(T^*)$.
(b) $\sigma_p(T) \subset \sigma_a(T)$ and $\sigma(T) = \sigma_a(T) \cup \{k : \bar{k} \in \sigma_p(T^*)\}$.
(c) $\sigma_c(T) = \sigma_a(T) \setminus (\sigma_r(T) \cup \sigma_p(T))$.

Theorem 1.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator(i.e. $TT^* = T^*T$). Then*
(a) If $k \in \sigma_p(T)$, then $\bar{k} \in \sigma_p(T^*)$. Also, if for $x \in \mathcal{H}$, $Tx = \lambda x$, then $T^*x = \bar{\lambda}x$.
(b) If x_1 and x_2 are eigenvectors corresponding to distinct eigenvalues, then $(x_1, x_2) = 0$.
(c) $\sigma(T) = \sigma_a(T)$.
(d)T is self adjoint iff $\sigma(T) \subset \mathbb{R}$.
(e)T is unitary iff $\sigma(T) \subset \mathbb{S}^1$.

1.3 Gelfand theory

Definition 1.12. *Let A be a Banach algebra with an involution $x \to x^*$ which satisfies*

$$\| xx^* \| = \| x \|^2 .$$

Then A is called a C^* algebra.

Since for $T \in \mathcal{B}(\mathcal{H})$,

$$\| Tx \|^2 = (Tx, Tx) = (T^*Tx, x) \leq \| T^*T \| \| x \|^2$$

for all $x \in \mathcal{H}$, we get that $\| T \|^2 \leq \| T^*T \|$. Now since $\| T^* \| = \| T \|$, we get that

$$\| T^*T \| \leq \| T^* \| \| T \| = \| T \|^2 .$$

Hence, $\| T^*T \| = \| T \|^2 \quad \forall \;\; T \in \mathcal{B}(\mathcal{H})$, and we see that $\mathcal{B}(\mathcal{H})$ is a C^* algebra.

Definition 1.13. *Let A be a commutative Banach algebra. Let Δ be the set of all complex homomorphisms on A. The Gelfand transform $\hat{x}$ of $x \in A$ is a function $\hat{x} : \Delta \to \mathbb{C}$ defined as*

$$\hat{x}(h) = h(x) \qquad (h \in \Delta).$$

Let $\hat{A}$ be the set of all $\hat{x}; x \in A$. We define the Gelfand topology of Δ to be the weak topology induced by $\hat{A}$. Note that this topology makes every $\hat{x}$ continuous and is the weakest with this property.
Remark: $\hat{A} \subset C(\Delta)$.

Proposition 1.2. *Let A be a commutative Banach algebra and let Δ be the set of complex homomorphisms of A. Then*
(a)Every maximal ideal of A is the $\ker(h)$ *of some $h \in \Delta$.*
(b)If $h \in \Delta$, the $\ker(h)$ *is a maximal ideal of A.*

Remark: Note that Δ, with the Gelfand topology is called the maximal ideal space of A.

Lemma 1.1. *Let A be a commutative Banach algebra and let Δ be the set of all complex homomorphisms of A. Then $\lambda \in \sigma(x) \Longleftrightarrow h(x) = \lambda$ for some $h \in \Delta$.*

Theorem 1.4. *Let Δ be the maximal ideal space of A. Then*
(a) Δ is a compact Hausdorff space.
(b) The Gelfand transform is a homomorphism of A onto a subalgebra $\hat{A}$ of $C(\Delta)$, whose kernel is radA(i.e. the intersection of all maximal ideals of A).
(c) $R(\hat{x}) = \sigma(x)$. Hence $\| \hat{x} \|_\infty = \gamma(x) \leq \| x \|$.

Proof: We will prove (b) and (c) only. Let $y, x \in A, \alpha \in \mathbb{C}, h \in \Delta$. Then

$$\hat{\alpha x}(h) = h(\alpha x) = \alpha h(x) = \alpha \hat{x}(h)$$

$$x \hat{+} y(h) = h(x + y) = h(x) + h(y) = (\hat{x} + \hat{y})(h)$$

and

$$(xy)^(h) = h(xy) = h(x)h(y) = \hat{x}(h)\hat{y}(h) = (\hat{x}\hat{y})(h).$$

Hence, $x \to \hat{x}$ is a homomorphism whose null space consists of those $x \in A$ which satisfy $h(x) = 0 \ \ \forall \ h \in \Delta$. Now by previous proposition, null space of $x \to \hat{x}$ is the intersection of all maximal ideals of A, i.e. $\operatorname{rad}(A)$.
If $\lambda \in R(\hat{x})$, then $\lambda = h(x)$ for some $h \in \Delta$. By the above lemma, $\lambda \in \sigma(x)$.

Theorem 1.5. ***(Gelfand-Naimark)****: Suppose A be a commutative C^* algebra. Let maximal ideal space of A be Δ. The Gelfand transform is an isometric isomorphism of A onto $C(\Delta)$ satisfying for all $x \in A, h \in \Delta$,*

$$h(x^*) = \overline{h(x)}$$

or equivalently

$$\hat{(x^*)} = \bar{\hat{x}}.$$

Corollary: x is self adjoint if and only if $\hat{x}$ is a real valued function.

In the following theorem, we narrow down to a special case. Also, we talk about the inverse of the Gelfand transform.

Theorem 1.6. *If A is commutative C^*-algebra which contains an element x such that polynomials in x and x^* are dense in A, then*

$$\hat{\Psi f} = f \circ \hat{x}$$

defines an isometric isomorphism Ψ of $C(\sigma(x))$ onto A which satisfies

$$\Psi \bar{f} = (\Psi f)^*$$

for every $f \in C(\sigma(x))$. Moreover, if $f(\lambda) = \lambda$ on $\sigma(x)$, then $\Psi f = x$.

In the next section, we define the resolution of the identity, which later we use to define $f(T)$ for a bounded function f and $T \in \mathcal{B}(\mathcal{H})$.

1.4 Resolutions of Identity

Definition 1.14. *Let $\mathcal{M}$ be a $\sigma-$algebra in a set Ω. Let $\mathcal{H}$ be a Hilbert space. A resolution of identity E is a map from $\mathcal{M}$ to $\mathcal{B}(\mathcal{H})$ such that*
(a) $E(\phi) = 0$
(b) $E(\Omega) = I$
(c) $E(\omega)$ is a self adjoint projection for all ω
(d) $E(\omega' \cap \omega'') = E(\omega')E(\omega'')$

(e) If ω and ω' are disjoint, then $E(\omega \cup \omega') = E(\omega) + E(\omega')$
(f) For $x, y \in H$, we define a function $E_{x,y}$ by:

$$E_{x,y}(\omega) = (E(\omega)x, y)$$

We see that $E_{x,y}$ is a complex measure on $\mathcal{M}$.

Remarks:
(a) $E_{x,x}(\omega) = (E(\omega)x, x) = \| E(\omega)x \|^2$ since $E(\omega)$ is a self adjoint projection.
(b) Each $E_{x,x}$ is a positive measure on M with total variation

$$\| E_{x,x} \| = E_{x,x}(\Omega) = \| x \|^2$$

(c) Any two $E(\omega)$ commute.
(d) If ω and ω' are disjoint, then $\mathcal{R}(E(\omega)) \perp \mathcal{R}(E(\omega'))$
(e) E is finitely additive, but not countably additive in general.

Even though E is not countably additive, $E(.)x$ for every $x \in \mathcal{H}$. This happens because $(E(\omega)x, y)$ is a measure. In other words, $\omega \to E(\omega)x$ is a countably additive $\mathcal{H}$-valued measure on $\mathcal{M}$.

Examples:
(a)If T is a finite dimensional operator, then $T = \sum_{k=1}^{n} \lambda_k E_k$ where λ_k are the n eigenvalues.
(b)If T is a compact operator, then $T = \sum_{k=1}^{\infty} \lambda_k E_k$ where λ_k are the eigenvalues.

Proposition 1.3. *Suppose E is a resolution of the identity. If $\omega_n \in M$ and $E(\omega_n) = 0$ for $n = 1, 2, 3...$ and if $\omega = \cup_{n=1}^{\infty} \omega_n$, then $E(\omega) = 0$.*

The algebra $L^\infty(E)$: Let E be a resolution of identity on $\mathcal{M}$ and let f be a complex $\mathcal{M}$-measurable function on Ω. Since $\mathbb{C}$ is second countable space, there exists a countable collection $\{D_i\}$ of open discs which form a base for the topology of $\mathbb{C}$. Define V as

$$V = \cup D_j \qquad ; \; E(f^{-1}(D_j)) = 0$$

By the previous proposition, $E(f^{-1}(V)) = 0$.
The essential range of f is the compliment of V(by definition). If the essential range is bounded, we say that f is essentially bounded. The essential supremum is the largest absolute value of points in essential range. We denote the essential supremum as $\| f \|_\infty$.
Let B be the algebra of all bounded complex M-measurable functions on Ω, with the norm

$$\| f \| = sup\{| f(p) |: p \in \Omega\}$$

It is trivial to see that B is a Banach Algebra. Let N be an ideal defined by

$$N = \{f \in B : \| f \|_\infty = 0\}$$

We know by previous proposition that N is closed. Hence, we can define B/N, which we denote by $L^\infty(E)$.

Theorem 1.7. *If E is a resolution of the identity, then there exists an isometric* isomorphism Ψ from the $L^\infty(E)$ onto a normal subalgebra A of $\mathcal{B}(\mathcal{H})$, which is related to E by the formula*

$$(\Psi(f)x, y) = \int_\Omega f dE_{x,y} \qquad (x, y \in \mathcal{H}, f \in L^\infty(E))$$

This justifies the notation

$$\Psi(f) = \int_\Omega f dE$$

Moreover,

$$\| \Psi(f)x \|^2 = \int_\Omega | f |^2 \, dE_{x,x} \qquad (x \in \mathcal{H}, f \in L^\infty(E)).$$

By isometric isomorphism , we mean that Ψ is a one-one, linear, multiplicative and that*

$$\Psi(\bar{f}) = \Psi(f)^*.$$

The following is a proof taken from [1].
Proof: Let us start by proving the result for simple functions. Let $\{\omega_1, \omega_2, ..., \omega_n\}$ be a partition of Ω, with $\omega_i \in \mathcal{M}$ and let s be a simple function, such that $s = \alpha_i$ on ω_i.
Let us define $\Psi \in \mathcal{B}(\mathcal{H})$ as

$$\Psi(s) = \sum_{i=1}^{n} \alpha_i E(\omega_i)$$

We know that each $E(\omega_i)$ is self adjoint, hence

$$\Psi(s)^* = \sum_{i=1}^{n} \bar{\alpha}_i E(\omega_i 0) = \Psi(\bar{s})$$

If we have another partition, say $\{\omega_1^{'}, \omega_2^{'}, ..., \omega_n^{'}\}$, and another simple function $t = \beta_i$ on ω_i, then

$$\Psi(s)\Psi(t) = \sum_{i,j} \alpha_i \beta_j E(\omega_i) E(\omega_j^{'}) = \sum_{i,j} \alpha_i \beta_j E(\omega_i \cap \omega_j^{'})$$

We see that st is a simple function that equals $st = \alpha_i \beta_j$ on $\omega_i \cap \omega_j^{'}$, hence

$$\Psi(s)\Psi(t) = \Psi(st).$$

Similarly, we can show the linearity of Ψ. Now, if $x, y \in \mathcal{H}$,

$$(\Psi(s)x, y) = \sum_{i=1}^{n} \alpha_i (E(\omega_i)x, y) = \sum_{i=1}^{n} \alpha_i E_{x,y}(\omega_i) = \int_{\Omega} s \; dE_{x,y}$$

Since Ψ is multiplicative.

$$\| \Psi(s)x \|^2 = (\Psi(s)^* \Psi(s)x, x) = (\Psi(| s |^2)x, x) = \int_{\Omega} | s |^2 \; dE_{x,x}$$

so that

$$\| \Psi(s)x \| \leq \| s \|_{\infty} \| x \| .$$

Now, if $x \in R(E(\omega_j))$, then

$$\Psi(s)x = \alpha_j E(\omega_j)x = \alpha_j x$$

since $E(\omega_i)$ have orthogonal ranges. We can chose j so that $| \alpha_j | = \| s \|_{\infty}$. Hence,

$$\| \Psi(s) \| = \| s \|_{\infty} .$$

Now let us assume that $f \in L^{\infty}(E)$. Then there is a sequence of simple measurable functions s_k converging to f in the norm of $L^{\infty}(E)$. The operators $\Psi(s_k)$ form a Cauchy sequence in $\mathcal{B}(\mathcal{H})$ and hence converge in the norm to an operator that, that we denote by $\Psi(f)$. It is easy to see that $\Psi(f)$ does not depend on the choice of $\{s_k\}$.

We get

$$\| \Psi(f) \| = \| f \|_{\infty} .$$

Thus Ψ is an isometric* isomorphism from the $L^{\infty}(E)$ onto a normal subalgebra $A = \Psi(L^{\infty}(E))$ of $\mathcal{B}(\mathcal{H})$.

1.5 The spectral theorem

Definition 1.15. *A closed subalgebra A of $\mathcal{B}(\mathcal{H})$ is called a $^* -$ algebra if $I \in A$ and $T^* \in A$ whenever $T \in A$.*

Lemma 1.2. *Suppose $T \in A \subset \mathcal{B}(\mathcal{H})$. If $\sigma(T)$ doesn't separate $\mathbb{C}$, then $\sigma(T) = \sigma_A(T)$.*

Proposition 1.4. *Let $\mathcal{H}$ be a Hilbert space and T be a bounded operator. T has the same spectrum relative to all closed $^* -$ algebras in $\mathcal{B}(\mathcal{H})$ that contain T.*

Proof: Let A be a *-algebra in $\mathcal{B}(\mathcal{H})$ that contains T. Let us assume that T is invertible in $\mathcal{B}(\mathcal{H})$. Since TT^* is self adjoint, $\sigma(TT^*)$ is a compact subset of $\mathbb{R}$. Hence, it does not separate $\mathbb{C}$. Therefore, $\sigma(TT^*) = \sigma_A(TT^*)$ by the previous lemma. Since TT^* is invertible in $\mathcal{B}(\mathcal{H})$, it is invertible in A(because the spectrums are equal). Therefore, $(TT^*)^{-1} \in A$ and eventually $T^{-1} = T^*(TT^*)^{-1} \in A$.

Theorem 1.8. *If A is a closed normal subalgebra of $\mathcal{B}(\mathcal{H})$ containing the identity operator I and if Δ is the maximal ideal space of A, then the following assertions are true:*
(a) There exists a unique resolution E of the identity on the Borel subsets of Δ satisfying

$$T = \int_{\Delta} \hat{T} dE$$

where $T \in A$, where $\hat{T}$ is the Gelfand transform of T.
(b) The inverse of Gelfand transform(i.e. the map that takes $\hat{T}$ to T) extends to an isometric-isomorphism Φ of $L^{\infty}(E)$ onto a closed subalgebra B of $\mathcal{B}(\mathcal{H})$, $B \supset A$ given by*

$$\Phi f = \int_{\Delta} f dE.$$

Proof: We know that the first equation above means

$$(Tx, y) = \int_\Delta \hat{T} dE_{x,y}.$$

We know that $\mathcal{B}(\mathcal{H})$ is a C^* algebra. Since, A is normal, it is a commutative B^* algebra. By the Gelfand Naimark theorem, $T \to \hat{T}$ is an isometric *-isomorphism of A onto $C(\Delta)$.
To see a proof for uniqueness of E, we assume that an E exists and satisfies

$$(Tx, y) = \int_\Delta \hat{T} dE_{x,y}.$$

Since $\hat{T}$ ranges over all of $C(\Delta)$, the assumed regularity of complex Borel measures $E_{x,y}$ and the uniqueness assertion of Riesz representation theorem show that each $E_{x,y}$ is uniquely determined by the above equation. Now, since $(E(\omega)x, y) = E_{x,y}(\omega)$, each $E(\omega)$ is also uniquely determined.
Now we try to prove the existance of E. If $x, y \in \mathcal{H}$, Gelfand-Naimark theorem shows that $\hat{T} \to (Tx, y)$ is a bounded linear functional on $C(\Delta)$ of norm atmost $\| x \| \| y \|$, since $\| \hat{T} \|_\infty = \| T \|$. By Riesz representation theorem, we get unique Borel measures $\mu_{x,y}$ on Δ such that

$$(Tx, y) = \int_\Delta \hat{T} d\mu_{x,y}.$$

where $x, y \in \mathcal{H}, T \in A$. We know that LHS of above equation is a bounded sesquilinear functional on $\mathcal{H}$. Hence, so is RHS. Now, even if we replace our continuous $\hat{T}$ with an arbitary borel function f, the boundedness remains intact. To each f corresponds a $\Phi f \in B(H)$ such that

$$((\Phi f)x, y) = \int_\Delta f d\mu_{x,y}.$$

Comparing the above two equations, we see that $\Phi\hat{T} = T$. Hence, Φ is an extension of the inverse of Gelfand transform(since it is defined on Borel functions).
It is easy to see the linearity of Φ.
Gelfand Naimark theorem states that T is self adjoint iff $\hat{T}$ is real valued. For such a T,

$$\int_\Delta \hat{T} d\mu_{x,y} = (Tx, y) = (x, Ty) = \overline{(Ty, x)} = \overline{\int_\Delta \hat{T} d\mu_{y,x}}$$

and this implies that $\mu_{x,y} = \overline{\mu_{y,x}}$.
Hence,

$$((\Phi\bar{x})x, y) = \int_{\Delta} \bar{f} d\mu_{x,y} = \overline{\int_{\Delta} f d\mu_{y,x}} = \overline{((\Phi f)y, x)} = (x, (\Phi f)y)$$

for all $x, y \in \mathcal{H}$. Hence,

$$\Phi\bar{f} = (\Phi f)^*.$$

Now for $S, T \in A$, $(\hat{ST}) = \hat{S}\hat{T}$. Hence,

$$\int_{\Delta} \hat{S}\hat{T} d\mu_{x,y} = (STx, y) = \int_{\Delta} \hat{S} d\mu_{Tx,y}.$$

This holds for all $\hat{S} \in C(\Delta)$. Hence, we can replace $\hat{S}$ by any bounded Borel function f. Thus

$$\int_{\Delta} f\hat{T} d\mu_{x,y} = \int_{\Delta} f d\mu_{Tx,y} = ((\Phi f)Tx, y) = (Tx, z) = \int_{\Delta} \hat{T} d\mu_{x,z}$$

where $z = (\Phi f)^* y$. Now the first and last integrals remain equal if $\hat{T}$ is replaced by g, a borel function. We get

$$(\Phi(fg)x, y) = \int_{\Delta} fg d\mu_{x,y} = \int_{\Delta} g d\mu_{x,z} = ((\Phi g)x, z) = ((\Phi g)x, (\Phi f)^* y) = (\Phi(f)\Phi(g)x, y)$$

and hence, $\Phi(fg) = \Phi(f)\Phi(g)$.
Now we define E: If ω is a Borel subset of Δ, let χ_w be its characteristic function. Now define

$$E(\omega) = \Phi(\chi_\omega).$$

Since Φ is multiplicative,

$$E(\omega \cap \omega') = E(\omega)E(\omega').$$

If $\omega = \omega'$ above, we get that $E(\omega)$ is a projection.
When f is real, Φf is self adjoint. Hence each $E(\omega)$ is self adjoint. It is easy to see that $E(\phi) = \Phi(0) = 0$ and check the finite additivity of E. Now, $\forall x, y \in \mathcal{H}$,

$$E_{x,y}(\omega) = (E(\omega)x, y) = \int_{\Delta} \chi_\omega d\mu_{x,y} = \mu_{x,y}(\omega).$$

Hence (a) and (b) are proved.

Now, we narrow down to an operator.

Theorem 1.9. *If $T \in \mathcal{B}(\mathcal{H})$ is normal, then there exists a unique resolution of the identity E on the borel sets of $\sigma(T)$ satisfying*

$$T = \int_{\sigma(T)} \lambda dE(\lambda).$$

Proof: Let A be the smallest closed subalgebra of $\mathcal{B}(\mathcal{H})$ that contains I, T, T^*. Since T is normal, A is a normal subalgebra and previous theorem can be applied to A. We know from Gelfand theory that the maximal ideal space of A can be identified with $\sigma(T)$ in such a way that $\hat{T}(\lambda) = \lambda$ for every $\lambda \in \sigma(T)$. The existance of such a E follows from previous theorem.

Next we try to make sense of $f(T)$ for a given f.
f(T) for bounded f: If E is the spectral decomposition of a normal operator $T \in \mathcal{B}(\mathcal{H})$, and if f is a bounded Borel measurable function on $\sigma(T)$, it is customary to denote the operator

$$\Psi(f) = \int_{\sigma(T)} f dE$$

by $f(T)$.

In the next chapter, we will study about unbounded operators.

Chapter 2

Unbounded operators on a Hilbert space

In this chapter, we will study the theory of unbounded operators. The first thing we realize is that not all operators are defined on the whole Hilbert space.
In this chapter, like the previous one, we study tools to understand self adjoint operators. For a detailed theory, refer to [1].

2.1 Unbounded operators

Definition 2.1. *Let $\mathcal{H}$ be a Hilbert space. A linear operator $T : \mathcal{D}(T) \to \mathcal{H}$ is a map from a subspace $\mathcal{D}(T)$ of $\mathcal{H}$ to $\mathcal{H}$.*

Examples: Let $C([0,1])$ denote the space of continuous functions on the interval, and let$C_1([0,1])$ denote the space of continuously differentiable functions. Define the differentiation operator $\frac{d}{dx} : C_1([0,1]) \to C([0,1])$by

$$\frac{d}{dx}(f)(x) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon} \quad \forall x \in [0,1].$$

Since every differentiable function is continuous, $C_1([0,1]) \subset C([0,1])$.

If T is continuous , then T has a continuous extension to $\overline{\mathcal{D}(T)}$. Since $\overline{\mathcal{D}(T)}$ is complemented in $\mathcal{H}$, we can extend T to some member of $\mathcal{B}(\mathcal{H})$ over $\mathcal{H}$.

Definition 2.2. *Let $\mathcal{H}$ be Hilbert space. Then $\mathcal{H} \times \mathcal{H}$ is the space of all ordered pairs (x, y) where $x \in \mathcal{H}, y \in \mathcal{H}$. We can define an inner product on $\mathcal{H} \times \mathcal{H}$ by*

$$(\{a, b\}, \{c, d\}) = (a, c) + (b, d)$$

where (a, c) is the inner product in $\mathcal{H}$.

Remark: The norm on $\mathcal{H} \times \mathcal{H}$ is given by

$$\| \{a, b\} \|^2 = \| a \|^2 + \| b \|^2 .$$

Definition 2.3. *Let T be an operator on $\mathcal{H}$. Then the graph $\mathcal{G}(T)$ of T is the subspace of $\mathcal{H} \times \mathcal{H}$*

$$\{(x, Tx) : x \in \mathcal{D}(T)\}.$$

Definition 2.4. *Let S, T be two operators on $\mathcal{H}$. Then if $\mathcal{G}(T) \subset \mathcal{G}(S)$, we say that S is an extension of T.*

Definition 2.5. *If $\mathcal{G}(T)$ is a closed subspace of $\mathcal{H} \times \mathcal{H}$, then T is closed.*

Often it happens, that our T is not closed, but it may have a closed extension. The following notions of closability are taken from [3]

Definition 2.6. *If a linear operator T has an extension T_1 which is a closed linear operator, then we call T_1 a closed linear extension of T and T is closable.*

Definition 2.7. *A closed linear extension $\overline{T}$ of T is said to be minimal if every closed linear extension T_1 of T is a closed linear extension of $\overline{T}$. We call this minimal closed extension(if it exists) $\overline{T}$ as the closure of T.*

Remark: If $\overline{T}$ exists, it is unique.

Now we present an operator that is not closable from [8].

Example of Non Closable operator: Let $\{e_i\}$ be an orthonormal basis of an infinite-dimensional Hilbert space $\mathcal{H}$.Let us define a linear operator T as follows: $\mathcal{D}(T)$ is the set of all finite linear combinations of vectors $\{e_i\}$ and

$$Ae_k = ke_1.$$

We see that T is not closable.

2.2 Adjoint

Definition 2.8. *Let T be an operator in $\mathcal{H}$. The adjoint T^* of T is an operator on $\mathcal{H}$ such that*

$$(Tx, y) = (x, T^*y)$$

where $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(T^*) = \{y \in \mathcal{H} \mid x \to (Tx, y)$ is continuous $\}$.

Remark: If $y \in \mathcal{D}(T^*)$, then by Hahn-Banach theorem, the functional $x \to (Tx, y)$ can be extended to a continuous linear functional on $\mathcal{H}$. By Riesz representation theorem([2]), there exists T^*y such that

$$(Tx, y) = (x, T^*y).$$

It is easy to see that T^*y is uniquely defined iff $\mathcal{D}(T)$ is dense.
Some Trivial Properties: There are some basic properties that one should know regarding
(a) domains of sums and products

$$\mathcal{D}(S + T) = \mathcal{D}(S) \cap \mathcal{D}(T)$$

$$\mathcal{D}(ST) = \{x \in \mathcal{D}(T) : Tx \in \mathcal{D}(S)\}$$

(b) associativity

$$(R + S) + T = R + (S + T)$$

$$(RS)T = R(ST)$$

(c) distributivity

$$(R + S)T = RT + ST$$

$$T(R + S) \supset TR + TS$$

Note that in the last one, $(R + S)x$ may be in $\mathcal{D}(T)$, however one of Rx or Sx may not be. Hence, the inequality.

Proposition 2.1. *Let S, T and ST be operators on $\mathcal{H}$ with domains dense in $\mathcal{H}$. Then*

$$T^*S^* \subset (ST)^*$$

Now, if $S \in \mathcal{B}(\mathcal{H})$,

$$T^*S^* = (ST)^*$$

Let V be an operator on $\mathcal{H} \times \mathcal{H}$ given by

$$V\{a, b\} = \{-b, a\} \quad (a, b \in \mathcal{H}).$$

It is easy to see that $V^2 = -I$. Also, V is a unitary operator.
The next result relates the graph of T^* with the graph of T.

Proposition 2.2. *If T is densely defined in $\mathcal{H}$, then*

$$\mathcal{G}(T^*) = [V\mathcal{G}(T)]^{\perp}.$$

Theorem 2.1. *If T is an operator in H with dense $\mathcal{D}(T)$, then T^* is closed.*

Corollary: Self adjoint operators are closed.

Theorem 2.2. *If T is a closed operator defined densely in $\mathcal{H}$, then we can write $\mathcal{H} \times \mathcal{H}$ as a direct sum of two orthogonal subspaces in the following way:*

$$\mathcal{H} \times \mathcal{H} = V\mathcal{G}(T) \oplus \mathcal{G}(T^*).$$

The following is a result from [9].

Theorem 2.3. *Suppose that $T : \mathcal{H} \to \mathcal{H}$ is a densely defined operator on $\mathcal{H}$. Then*
(a) If T is closable, then $\overline{T}^* = T^*$.
(b) T is closable iff T^* is densely defined.
(c) If T is closable, then $\overline{T} = T^{**}$.

Proof: To prove (a) we use the operator V defined above as

$$V\{a, b\} = \{-b, a\}$$

for $a, b \in \mathcal{H}$. We see that if T is closable, then

$$\mathcal{G}(T^*) = (V\mathcal{G}(T))^{\perp} = (V\overline{\mathcal{G}(T)})^{\perp} = V\mathcal{G}(\overline{T})^{\perp} = \mathcal{G}(\overline{T^*})$$

by the previous theorems.
To prove (b), we take a sequence $\{x_n\}$ in $\mathcal{H}$ such that

$$x_n \to 0$$

$$Tx_n \to x$$

as $n \to \infty$. Let $y \in \mathcal{D}(T^*)$. Then

$$(Tx_n, y) = (x_n, T^*y).$$

Taking the limits as $n \to \infty$, we get

$$(x, y) = (0, T^*y) = 0.$$

Hence, if T^* is densely defined, we get that $x = 0$. Hence, T is closable.
To prove the other way, assume that T is closable and let $x \perp \mathcal{D}(T^*)$. By above argument,

$$(k, 0) \in V\mathcal{G}(T) = \mathcal{G}(T^*)^{\perp} = V\mathcal{G}(T)^{\perp\perp} = V\overline{\mathcal{G}(T)} = V\mathcal{G}(\overline{T}).$$

This implies that $(0, k) = \mathcal{G}(\overline{T})$. Since T is closable, $\overline{T}$ is well-defined and hence

$$k = \overline{T}0 = 0.$$

Therefore, $\mathcal{D}(T^*)$ is densely defined.
Now to prove (c), we assume that T is closable and T^* is densely defined. Then

$$\mathcal{G}(T^{**}) = (U\mathcal{G}(T^*))^{\perp}$$

where $U((y, x)) = (-x, y) = -V^{-1}(y, x)$. Therefore

$$\mathcal{G}(T^{**}) = U(V(\mathcal{G}(T)^{\perp})^{\perp} = \overline{\mathcal{G}(T)} = \mathcal{G}(\overline{T})$$

and hence, $\overline{T} = T^{**}$.

2.3 Symmetric operators

Definition 2.9. *An operator T in $\mathcal{H}$ is called symmetric if*

$$(Tx, y) = (x, Ty)$$

where $x, y \in \mathcal{D}(T)$.

Remarks:
If T is a dense symmetric operator, then

$$T \subset T^*$$

Definition 2.10. *If $T = T^*$, then T is said to be self adjoint.*

Remark: If T is densely defined and $(Tx, y) = (x, Sy)$, then $S \subset T^*$.

Theorem 2.4. *Let T be a densely defined operator in $\mathcal{H}$. Then if T is symmetric, $\overline{T}$ exists and is unique.*

Sketch of the proof: Let us first define the domain $M = \mathcal{D}(\overline{T})$ and then $\overline{T}$. We will prove that $\overline{T}$ is indeed the closure of T.
Let M be the set of all $x \in \mathcal{H}$ for which there is a sequence $\{x_n\} \in \mathcal{D}(T)$ and $y \in \mathcal{H}$ such that

$$x_n \to x$$

$$Tx_n \to y.$$

It is easy to see that M is a vector space and $\mathcal{D}(T) \subset M$. Now, we define the operator $\overline{T}$ on M as

$$y = \overline{T}x$$

for x and y given above.
The only thing that is left is to check $\overline{T}$ for all the properties by which the closure of a symmetric operator is defined.

Since, we have defined the symmetric operator, let us see an example of a symmetric operator which is not self adjoint.
Examples: In the complex space $\mathcal{L}^2[0, 1]$, let $A = id/dt$ be defined on the set D_A of

absolutely-continuous functions f on $[0,1]$ having a square-summable derivative and satisfying the condition $f(0) = f(1) = 0$. Then A is symmetric but not self-adjoint. However, if the condition on f is

$$f(0) = f(1)$$

, then A is a self adjoint.

Having seen that not all symmetric operators are self adjoint, the next section talks about when a symmetric operator is self adjoint.

2.4 When is a symmetric operator selfadjoint?

Theorem 2.5. *Suppose T is a symmetric operator densely defined in $\mathcal{H}$.*
(a) If $\mathcal{D}(T) = \mathcal{H}$, then T is self adjoint and $T \in \mathcal{B}(\mathcal{H})$.
(b) If T is self adjoint and one-one, then T^{-1} is self adjoint and $\mathcal{R}(T)$ is dense in $\mathcal{H}$.
(c) If $\mathcal{R}(T)$ is dense in $\mathcal{H}$, then T is a one-one map.
(d) If $\mathcal{R}(T) = \mathcal{H}$, then T is an self adjoint operator and $T^{-1} \in \mathcal{B}(\mathcal{H})$

Proof: To prove (a), we see that $T \subset T^*$ (since T is symmetric). If $\mathcal{D}(T) = \mathcal{H}$, that means that $\mathcal{D}(T^*) = \mathcal{H}$ and therefore, T is self adjoint. Hence, T is closed since adjoints are closed. Therefore, by closed graph theorem, T is continuous.
(b)Let $y \perp \mathcal{R}(T)$. Then the functional

$$x \to (Tx, y) = 0$$

is continuous in $\mathcal{D}(T)$. Hence, $y \in \mathcal{D}(T^*) = \mathcal{D}(T)$ and

$$(x, Ty) = (Tx, y) = 0 \ \ \forall \ x \in \mathcal{D}(T).$$

Hence, $Ty = 0$. Now it follows that $y = 0$, since T is one-one. We have proved that $\mathcal{R}(T)$ is dense in $\mathcal{H}$.
Now since $\mathcal{R}(T)$ is dense, T^{-1} is densely defined and $(T^{-1})^*$ exists. The relations

$$\mathcal{G}(T^{-1}) = V\mathcal{G}(-T)$$

and

$$V\mathcal{G}(T^{-1}) = \mathcal{G}(-T)$$

are easily verified:

$$(a,b) \in \mathcal{G}(T^{-1}) \Leftrightarrow (b,a) \in \mathcal{G}(T) \Leftrightarrow (b,-a) \in \mathcal{G}(-T) \Leftrightarrow (a,b) \in V\mathcal{G}(-T).$$

Now since T is self adjoint, T and $-T$ are closed. Hence T^{-1} is closed by above relations. Now we can orthogonally decompose T^{-1} and $-T$ as

$$\mathcal{H} \times \mathcal{H} = V\mathcal{G}(T^{-1}) \oplus \mathcal{G}((T^{-1})^*)$$

and

$$\mathcal{H} \times \mathcal{H} = V\mathcal{G}(-T) \oplus \mathcal{G}(-T) = \mathcal{G}(T^{-1}) \oplus V\mathcal{G}(T^{-1}).$$

Therefore

$$\mathcal{G}((T^{-1})^*) = [V\mathcal{G}(T^{-1})]^{\perp} = \mathcal{G}(T^{-1})$$

which shows that T^{-1} is self adjoint.

(c)Let $\mathcal{R}(T)$ is dense and let $Tx = 0$. Then

$$(x, Ty) = (Tx, y) = 0$$

for $y \in \mathcal{D}(T)$. Therefore $x \perp \mathcal{R}(T)$. Since $\mathcal{R}(T)$ is dense, $x = 0$.

(d)Assume that $\mathcal{R}(T) = \mathcal{H}$. Then T is one-one and $\mathcal{D}(T^{-1}) = \mathcal{H}$. Now, if $x, y \in \mathcal{H}$, then for some $w, z \in \mathcal{D}(T)$, we have $x = Tz$ and $y = Tw$, so that

$$(T^{-1}x, y) = (z, Tw) = (Tz, w) = (x, T^{-1}y)$$

making T^{-1} symmetric. By (a), T^{-1} is self adjoint and bounded and now it follows from (b) that $T = (T^{-1})^{-1}$ is self adjoint.

Theorem 2.6. *If T is an operator that is closed and densely defined in $\mathcal{H}$, then $\mathcal{D}(T^*)$ is dense in $\mathcal{H}$. Also $T^{**} = T$.*

Definition 2.11. *A symmetric operator T in $\mathcal{H}$ is said to be maximally symmetric if T has no proper symmetric extension, i.e. there exists no symmetric S such that $T \subset S$ and $T \neq S$.*

Theorem 2.7. *Self adjoint operators are maximally symmetric.*

2.5 Cayley transform

The Cayley tranform was originally described by Arthur Cayley as a map between skew symmetric matrices and special orthogonal matrices. Here we generalize to linear operators.

The Cayley Transform: The mapping

$$t \longrightarrow \frac{t-i}{t+i}$$

sets up a one-to-one map between $\mathbb{R}$ and the $S^1 \setminus \{1\}$. By functional calculus, we know that every self adjoint operator $T \in \mathcal{B}(\mathcal{H})$ is related(via a bijection) to a unitary operator(whose spectrum doesn't contain 1) as

$$U = (T - iI)(T + iI)^{-1}.$$

Now we can extend the relation $T \longleftrightarrow U$ to a one-one correspondence between symmetric operators and isometries.
Let T be a symmetric operator in $\mathcal{H}$. Then

$$\| Tx + ix \|^2 = \| x \|^2 + \| Tx \|^2 = \| Tx - ix \|^2 \qquad (x \in \mathcal{D}(T))$$

Hence, there is an isometry U such that $\mathcal{D}(U) = \mathcal{R}(T + iI)$ and $\mathcal{R}(U) = \mathcal{R}(T - iI)$ defined by

$$U(Tx + ix) = Tx - ix \quad (x \in \mathcal{D}(T))$$

Since $(T + iI)^{-1}$ maps $\mathcal{D}(U)$ onto $\mathcal{D}(T)$, U can be written as $U = (T - iI)(T + iI)^{-1}$.
If $T \in \mathcal{B}(\mathcal{H})$, then to see that U is unitary, we see that

$$(T - iI)(T + iI)^{-1}((T - iI)(T + iI)^{-1})^* = (T - iI)(T + iI)^{-1}((T + iI)^{-1})^*(T - iI)^*$$

$$= (T - iI)(T + iI)^{-1}(T - iI)^{-1}(T + iI) = I.$$

This U is called Cayley transform of T.

Lemma 2.1. *Suppose U is an operator in $\mathcal{H}$ which is an isometry: $\| Ux \| = \| x \|$ for every $x \in \mathcal{D}(U)$.*

(a) If $\mathcal{R}(I-U)$ is dense in $\mathcal{H}$, then $I-U$ is one-one.
(b) If any one of the three spaces $\mathcal{D}(U)$, $\mathcal{R}(U)$ and $\mathcal{G}(U)$ is closed, so are other two.

Theorem 2.8. *Suppose U is the Cayley transform of a symmetric operator T in $\mathcal{H}$. Then*
(a) U is closed iff T is closed.
(b) $\mathcal{R}(I-U) = \mathcal{D}(T)$, $I-U$ is one-one , and T can be reconstructed from U by the formula

$$T = i(I+U)(I-U)^{-1}$$

(c) U is unitary iff T is self adjoint.

Conversely, if V is an isometry in $\mathcal{H}$, and if $I-V$ is one-one, then V is the Cayley transform of a symmetric operator in $\mathcal{H}$.

2.6 Resolutions of the Identity

Let us recall resolutions of the identity from the last chapter.
Let $\mathcal{M}$ be a σ-algebra in a set Ω and let $\mathcal{H}$ be a Hilbert space. A resolution of identity on $\mathcal{M}$ is a mapping

$$E : \mathcal{M} \to \mathcal{B}(\mathcal{H})$$

such that
(a) $E(\phi) = 0$, $E(\Omega) = I$
(b) Each $E(\omega)$ is a self adjoint projection.
(c) $E(\omega' \cap \omega'') = E(\omega')E(\omega'')$
(d) If $\omega' \cap \omega'' = \phi$, then $E(\omega' \cup \omega'') = E(\omega') + E(\omega'')$.
(e) For every $x \in \mathcal{H}$ and $y \in \mathcal{H}$, the set function $E_{x,y}$ defined by

$$E_{x,y}(w) = (E(w)x, y)$$

is a complex measure on $\mathcal{M}$.

Theorem 2.9. *Let E be a resolution of identity on a set Ω.*
(a) To every measurable $f : \Omega \to \mathbb{C}$ corresponds a closed operator $\Psi(f)$ defined densely

in $\mathcal{H}$, with domain

$$\mathcal{D}(\Psi(f)) = D_f = \{x \in H : \int_\Omega \mid f \mid^2 dE_{x,x} < \infty\}$$

which is characterized by

$$(\Psi(f)x, y) = \int_\Omega f dE_{x,y} \qquad (x \in D_f, y \in \mathcal{H})$$

and satisfies

$$\| \Psi(f)x \|^2 = \int_\Omega \mid f \mid^2 dE_{x,x} \qquad (x \in D_f)$$

(b) The multiplication theorem holds in the following way: If f and g are measurable, then

$$\Psi(f)\Psi(g) \subset \Psi(fg)$$

and

$$\mathcal{D}(\Psi(f)\Psi(g)) = D_g \cap D_{fg}$$

Hence, $\Psi(f)\Psi(g) = \Psi(fg)$ if and only if $D_{fg} \subset D_g$.
(c) For every measurable $f : \Omega \to \mathbb{C}$,

$$\Psi(f)^* = \Psi(\bar{f})$$

and

$$\Psi(f)\Psi(f)^* = \Psi(\mid f \mid^2) = \Psi(f)^*\Psi(f)$$

The last part of the above theorem basically says that if f is a real valued function, then $\Psi(f)$ is a self-adjoint operator.

2.7 Spectral decomposition

Definition 2.12. *Let f be a complex valued measurable function. The essential range of f is the set*

$$\{z \in \mathbb{C} : \mu(\{x : |f(x) - w| < \epsilon\}) > 0 \ \ \forall\ \epsilon > 0\}.$$

Theorem 2.10. *Suppose E is a resolution of identity on Ω, $f : \Omega \to \mathbb{C}$ is measurable , and*

$$w_\alpha = \{p \in \Omega : f(p) = \alpha\} \qquad (\alpha \in (C))$$

(a) If α is in the essential range of f and $E(\omega_\alpha) \neq 0$, then $\Psi(f) - \alpha I$ is not one-one.
(b) If α is in the essential range of f but $E(\omega_\alpha) = 0$, then $\Psi(f) - \alpha I$ is a one-one mapping of D_f onto a dense proper subspace of $\mathcal{H}$, and there exist vectors $x_n \in \mathcal{H}$, with $\| x_n \| = 1$ such that

$$\lim_{n\to\infty} [\Psi(f)x_n - \alpha x_n] = 0$$

(c) $\sigma(\Psi(f))$ is the essential range of f.

Theorem 2.11. *Suppose*
(a) $\mathcal{M}$ and $\mathcal{M}'$ are σ-algebras in sets Ω and Ω'.
(b) $E : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ is a resolution of identity.
(c) $\phi : \Omega \to \Omega'$ has the property that $\phi^{-1}(\omega') \in \mathcal{M}$ for every $\omega' \in \mathcal{M}'$.
If $E'(\omega') = E(\phi^{-1}(\omega'))$, then $E' : \mathcal{M}' \to \mathcal{B}(\mathcal{H})$ is also a resolution of the identity, and

$$\int_{\Omega'} f dE'_{x,y} = \int_{\Omega} (f \circ \phi) dE_{x,y}$$

for every $\mathcal{M}'$-measurable $f : \Omega' \to \mathbb{C}$ for which either of these integrals exists.

Theorem 2.12. *To every self adjoint operator A in $\mathcal{H}$ corresponds a unique resolution E of the identity, on the Borel subsets of the real line such that*

$$(Ax, y) = \int_{-\infty}^{\infty} t dE_{x,y}(t) \qquad (x \in \mathcal{D}(A), y \in \mathcal{H})$$

Moreover, E is concentrated on $\sigma(A) \subset (-\infty, \infty)$,i.e. $E(\sigma(A)) = I$.

Proof: Let U be the Cayley transform of A, let $\Omega = S^1 \setminus \{1\}$. and let E' be spectral decomposition of U. Since $I - U$ is one-one, $0 = \mathcal{N}(I - U) = \mathcal{R}(E'(\{1\}))$, hence $E'(\{1\})$ and hence

$$(Ux, y) = \int_{\Omega)} \lambda dE'_{x,y}(\lambda) \qquad (x, y \in \mathcal{H})$$

Define

$$f(\lambda) = \frac{i(1 + \lambda)}{1 - \lambda} \qquad (\lambda \in \mathbb{C})$$

Define $\Psi(f)$ as earlier:

$$(\Psi(f)x, y) = \int_{\Omega)} f dE'_{x,y} \qquad (x \in D_f, y \in \mathcal{H})$$

Since f is real valued, $\Psi(f)$ is self adjoint and since $f(\lambda)(1-\lambda) = i(1+\lambda)$, the multiplication theorem gives

$$\Psi(f)(I-U) = i(I+U)$$

Thus, $\mathcal{R}(I-U) \subset \mathcal{D}(\Psi(f))$. Since U is the Cayley transform of A,

$$A(I-U) = i(I+U)$$

and $\mathcal{D}(A) = \mathcal{R}(I-U) \subset \mathcal{D}(\Psi(f))$. It can be seen that $\Psi(f)$ is a self adjoint extension of self adjoint operator A. Hence, $\Psi(f) = A$. Thus $(Ax, y) = \int_{\Omega)} f dE'_{x,y} \qquad (x \in \mathcal{D}(A), y \in \mathcal{H})$ By earlier result, $\sigma(A)$ is the essential range of f. Thus $\sigma(A) \subset (-\infty, \infty)$. Note that f is one-one in Ω. If we define

$$E(f(\omega)) = E'(\omega)$$

for every Borel set $\omega \subset \Omega$, we obtain the desired resolution E.

In the next chapter, we get to our study of perturbation by compact operators and relatively "small" symmetric operators.

Chapter 3

Perturbation of a self adjoint operator

In this chapter, we study two types of perturbations. Firstly, we see that perturbations of bounded operators by compact operators leave essential spectrum invariant([4]). In the later part of the chapter, we see that relatively bounded symmetric operators with relative bound less than 1 when added to a self adjoint operator leave the self adjointness preserved. A more detailed analysis can be found in [5].

Definition 3.1. *Let $\mathcal{H}$ be a Hilbert space. Then for $A \in \mathcal{B}(\mathcal{H})$, we define the left spectrum(or right spectrum) as $\{\lambda \in \mathbb{C} : A - \lambda I$ is not left invertible(or right invertible) $\}$.*

Proposition 3.1. *If $A \in \mathcal{B}(\mathcal{H})$, then the following are equivalent.*
(a) $\lambda \notin \sigma_a(A)$, i.e. $\inf\{\| (A - \lambda I)h \| : \| h \| = 1\} > 0$.
(b) $\mathcal{R}(A - \lambda I)$ is closed and $\dim \ker(A - \lambda I) = 0$.
(c) $\lambda \notin \sigma_l(A)$.
(d) $\bar{\lambda} \notin \sigma_r(A^*)$.
(e) $\mathcal{R}(A^* - \bar{\lambda} I) = \mathcal{H}$.

Proposition 3.2. *If N is a normal operator, then $\sigma(N) = \sigma_l(N) = \sigma_r(N)$. If λ is an isolated point of $\sigma(N)$, then $\lambda \in \sigma_p(N)$.*

Proof: To prove that $\sigma(N) = \sigma_l(N) = \sigma_r(N)$, it is sufficient to prove that for any $a \in \mathcal{A}-$ a $C*$-algebra

$$a \text{ is invertible} \Longleftrightarrow a \text{ is left invertible} \Longleftrightarrow a \text{ is right invertible.}$$

Assume that a is left invertible in $\mathcal{A}$. Hence there exists a $b \in \mathcal{A}$ such that $ba = 1$. Hence, for $x \in \mathcal{A}$,

$$\| x \| = \| bax \| \leq \| b \| \| ax \| .$$

This leads to

$$\| ax \| \geq \| b \|^{-1} \| x \| .$$

Since it is true for all $x \in \mathcal{A}$, it is true for $x \in C^*(a)$ in particular. Since a is normal, $C^*(a)$ is isomorphic to $C(\sigma(a))$ such that $a \longrightarrow z(z(w) = w)$. Hence, the above inequality becomes

$$\| zf \| \geq \| b \|^{-1} \| f \|$$

for every $f \in C(\sigma(a))$.
We need to show that $0 \notin \sigma(a)$. Let us assume on the contrary that $0 \in \sigma(a)$. Then, there is sequence $\{f_n\} \in C(\sigma(a))$ such that $0 \leq f_n \leq 1$, $f_n(0) = 1$ and

$$f_n(z) = 0; \quad z \in \sigma(a), | z | \geq 1/n.$$

Now, since $0 \in \sigma(a)$, $\| f_n \| = 1$. But $\| zf_n \| \leq 1/n$. This contradicts the inequality and so $0 \notin \sigma(a)$, i.e. a is invertible.
Now if we assume that a is invertible, we get that a is both left and right invertible.
Now let us assume that a is right invertible. Then a^* is left invertible. By the preceding argument, a^* is invertible and hence so is a.
Let λ be an isolated point of $\sigma(N)$ and $N = \int z dE(z)$, then $0 \neq E(\{\lambda\})H = \ker(N - \lambda)$ and we are done.

3.1 The essential spectrum

Definition 3.2. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the space of bounded operators on it. Let $\mathcal{B}_0(\mathcal{H})$ be the ideal of all compact operators on $\mathcal{H}$. The quotient $\mathcal{B}/\mathcal{B}_0$ is called the Calkin algebra.*

Definition 3.3. *Let $\mathcal{H}$ be a Hilbert space and $\pi : \mathcal{B} \to \mathcal{B}/\mathcal{B}_0$ be the natural map from $\mathcal{B}(\mathcal{H})$ into the Calkin algebra. For $A \in \mathcal{B}(\mathcal{H})$, the essential spectrum of A, is defined as*

$$\sigma_e(A) = \sigma(\pi(A)).$$

Similarily, we define the left and right essential spectrum as $\sigma_{le}(A) = \sigma_l(\pi(A))$ and $\sigma_{re}(A) = \sigma_r(\pi(A))$.

Proposition 3.3. *Let A be a bounded operator on $\mathcal{H}$.*
(a) $\sigma_e(A) = \sigma_{le}(A) \cup \sigma_{re}(A)$
(b) $\sigma_{le}(A) = \sigma_{re}(A^*)^*$
(c) $\sigma_{le}(A) \subseteq \sigma_l(A)$, $\sigma_{re}(A) \subseteq \sigma_r(A)$, $\sigma_e(A) \subseteq \sigma(A)$
(d) $\sigma_{le}(A), \sigma_{re}(A), \sigma_e(A)$ are compact in C.
(e)If $K \in \mathcal{B}_0(\mathcal{H})$, $\sigma_{le}(A+K) = \sigma_{le}(A), \sigma_{re}(A+K) = \sigma_{re}(A), \sigma_e(A+K) = \sigma_e(A)$

Proposition 3.4. *Let $A \in \mathcal{B}(\mathcal{H})$.*
(a) $\lambda \in \sigma_{le}(A)$ if and only if $\dim\ker(A-\lambda) = \infty$ or $\mathcal{R}(A-\lambda)$ is not closed.
(b) $\lambda \in \sigma_{re}(A)$ if and only if $\dim[\mathcal{R}(A-\lambda)]^{\perp} = \infty$ or $\mathcal{R}(A-\lambda)$ is not closed.

Proposition 3.5. *If $A \in \mathcal{B}(\mathcal{H})$, then*

$$\sigma_{ap}(A) = \sigma_{le}(A) \cup \{\lambda \in \sigma_p(A) : \dim\ker(A-\lambda) < \infty\}.$$

Proposition 3.6. *If N is a normal operator, and $\lambda \in \sigma(N)$, then $\mathcal{R}(N-\lambda)$ is closed iff λ is an isolated point.*

Proposition 3.7. *If N is normal, then*
(a) $\sigma_e(N) = \sigma_{le}(N) = \sigma_{re}(N)$
(b) $\sigma(N) \setminus \sigma_e(N) = \{\lambda \in \sigma(N) : \lambda$ is an isolated point in $\sigma(N)$ which is an eigenvalue with finite multiplicity $\}$.

Proof: To prove (a), we just have to apply Proposition 3.2 to Calkin algebra.
If λ is isolated the spectrum of N, then $\mathcal{R}(N-\lambda I)$ is closed by the previous result. So if $\dim\ker(N-\lambda)I < \infty$, then $\lambda \notin \sigma_{le}(N) = \sigma_{re}(N) = \sigma_e(N)$.
To prove the other side, we assume that $\lambda \in \sigma(N) \setminus \sigma_e(N)$, then $\mathcal{R}(N-\lambda I)$ is closed and $\dim\ker(N-\lambda I) < \infty$. By the previous result, λ is an isolated point of $\sigma(N)$.

Now this brings us to a decomposition of the spectrum $\sigma(N)$ as

$$\sigma(N) = \sigma_e(N) \cup \sigma_{disc}(N)$$

and that $\sigma_e(N)$ and $\sigma_{disc}(N)$ are disjoint, where $\sigma_{disc}(N)$ is the discrete spectrum of N and the complement of $\sigma_e(N)$. Formally, $\sigma_{disc}(N)$ is defined as

$$\sigma_{disc}(N) = \{\lambda \in \sigma(N) \mid \lambda \text{ is an eigenvalue of finite multiplicity } \}$$

The following remark has been taken from [6].

Remark 3.1. *$\sigma_{ess}(A)$ is always closed, whereas $\sigma_{disc}(A)$ may not be closed. $\sigma_{disc}(A)$ contains isolated eigenvalues of finite multiplicity.*

3.2 Perturbation of a self adjoint operator:

The stability of self-adjointness under the perturbation by a self adjoint operator is an important problem. A fundamental question that arises is how small a perturbation should be so that self adjointness stays preserved. We start off with a result that we will not prove.

Theorem 3.1. *Let T be a self adjoint operator. Then there is a $\delta > 0$ such that any symmetric closed operator S with $\hat{\delta}(S,T) < \delta$ is self adjoint, where $\hat{\delta}(S,T)$ is the gap between S and T defined by*

$$\hat{\delta}(S,T) = \max[(\sup_{\|u\|=1} dist(u,N)), (\sup_{\|v\|=1} dist(M,v))].$$

Even though the result is remarkable one, the definition of $\hat{delta}$ makes it complicated. Now, we try a different approach.

Definition 3.4. *An operator A is called T-bounded if $\mathcal{D}(A) \supset \mathcal{D}(T)$ and*

$$\| Au \| \leq a \| u \| + b \| Tu \| \qquad u \in \mathcal{D}(T)$$

or equivalently

$$\| Au \|^2 \leq a^{'2} \| u \|^2 + b^{'2} \| Tu \|^2 \qquad u \in \mathcal{D}(T).$$

Ofcourse, $a^{'}, b^{'}$ are different from a, b in general.

Definition 3.5. *The T-bound of A is the greatest lower bound of possible values of b or equivalently $b^{'}$.*

Theorem 3.2. *Let T be a self adjoint operator and A be a symmetric T-bounded operator with T-bound < 1, then $T + A$ is also self adjoint.*

Proof: We know that $T + A$ is symmetric and has the domain $\mathcal{D}(T)$. We may assume that the above equation holds with constants $a^{'}, b^{'}$ such that $a^{'} > 0, 0 < b^{'} < 1$. We can re-write the above equation as

$$\| Au \| \leq \| b^{'}T \mp ia^{'}u \|$$

for $u \in \mathcal{D}(T)$. Let $c^{'} = a^{'}/b^{'}$ and $(T \mp ic^{'})u = v$. We get

$$\| A(T \mp ic^{'})^{-1}v \| \leq b^{'} \| v \| .$$

Since T is self adjoint, v varies over all of H as u varies over $\mathcal{D}(T)$. Hence, we have

$$B^{\pm} = -A(T \mp ic^{'})^{-1} \in \mathcal{B}(\mathcal{H})$$

and $\| B^{\pm} \| \leq b^{'}$. Now, since by our assumption, $b^{'} < 1$, $(1 - B_{\pm})^{-1}$ exists and belongs to $\mathcal{B}(\mathcal{H})$. Hence, $1 - B_{\pm}$ maps $\mathcal{H}$ bijectively to $\mathcal{H}$. But we see that

$$T + A \mp ic^{'} = (1 - B_{\pm})(T \mp ic^{'})$$

and $\mathcal{R}(T \mp ic^{'}) = \mathcal{H}$ since T is self adjoint. Therefore $\mathcal{R}(T + A \mp ic^{'}) = \mathcal{H}$. Hence $T + A$ is self-adjoint.

Corollary: Let T be a self adjoint operator. If A is a symmetric bounded operator such that $\mathcal{D}(A) \supset \mathcal{D}(T)$, then $T + A$ is self-adjoint.

Definition 3.6. *Let T be a symmetric operator. If T^{**} is self adjoint, then we say that T is essentially self adjoint.*

Definition 3.7. *Let $\{u_n\} \in \mathcal{D}(\overline{T})$ be a sequence. It is called T-convergent to $u \in \mathcal{D}(\overline{T})$ if*

$$u_n \to u$$

and

$$Tu_n \to \overline{T}u.$$

Theorem 3.3. *Let T be an essentially self adjoint operator. If A is a symmetric T-bounded operator with T-bound < 1, then $T + A$ is essentially self adjoint and*

$(T \bar{+} A) = \bar{T} + \bar{A}$. *In particular, it is true when* $A \in \mathcal{B}(\mathcal{H})$ *is symmetric with* $\mathcal{D}(A) \supset \mathcal{D}(T)$.

Proof: We start by proving that $\overline{A}$ is $\overline{T}$-bounded, i.e.

$$D(\overline{A}) \supset D(\overline{T})$$

and

$$\| \overline{A}u \|^2 \leq a^{'2} \| u \|^2 + b^{'2} \| \overline{T}u \|^2$$

for $u \in D(\overline{T})$. For any $u \in D(\overline{T})$, there is a sequence $\{u_n\}$ which is T-convergent. Since A is T-bounded, it is easy to see that $\{u_n\}$ is A-bounded. Now if we replace u by u_n in the boundedness equation, and take the limit, we get the required equation. Since $\{u_n\}$ is both T-convergent and A-convergent, we get

$$(T + A)u_n \to (\overline{T} + \overline{A})u$$

so that $u \in D(\overline{T + A})$ and $\overline{T + A}u = (\overline{T} + \overline{A})u$. This shows that

$$\overline{T + A} \supset \overline{T} + \overline{A}.$$

Now, we apply the previous theorem to the pair $\overline{T}, \overline{A}$ that $\overline{T} + \overline{A}$ is selfadjoint(Note that here we use the assumption $b^{'} < 1$). Thus $\overline{T} + \overline{A}$ is closed extension of $T + A$ and therefore of $\overline{T + A}$. Hence, we get that $\overline{T} + \overline{A} = \overline{T + A}$.

The previous theorems are not symmetric with respect to T and $T + A(= S)$. So we have the following result:

Theorem: Let S,T be two symmetric operators such tha $\mathcal{D}(T) = \mathcal{D}(S) = D$ and

$$\| (S - T)u \| \leq a \| u \| + b(\| Tu \| + \| Su \|)$$

for $u \in D$, where a, b are non-negative constants with $b < 1$. Then S is essentially self adjoint iff T is.

Corollary: Let S,T be two operators satisfying the above properties, then S is self adjoint iff T is.

Lemma: Let $B \in \mathcal{B}(\mathcal{H})$ and $\| B \| \leq 1$. Then

$$Bu = u \Longleftrightarrow B^* u = u.$$

Such an operator is called a contraction.

Proof: Since $B^{**} = B$ and $\| B^* \| = \| B \| \leq 1$, it is sufficient to show that $Bu = u \rightarrow B^* u = u$. But we know that

$$\| B^* u - u \|^2 = \| B^* \|^2 + \| u \|^2 - 2Re(B^* u, u) \leq 2 \| u \|^2 - 2Re(u, Bu).$$

So if $Bu = u$, we get $B^* u = u$.

The following result is about the case of relative bound 1.

Theorem 3.4. *Let T be essentially self adjoint and let A be symmetric operator. If A is T-bounded and equation holds with $b' = 1$, then $T + A$ is essentially self adjoint.*

Proof: We start by assuming that T is self adjoint and defining $B_{\pm}$ as above. Since $b' < 1$, we get that $\| B_{\pm} \| \leq 1$ and $R(1 - B_{+})$ may not be H. But we will show that the range is dense in $\mathcal{H}$. Then by arguments used before, we will get that $R(T + A \mp ic')$ are dense in H and hence $T + A$ is essentially selfadjoint.
To see that $R(1 - B+)$ is dense in H, it suffices to show that a $v \in H$ orthogonal to this range must be zero. Now such a v would satisfy $B_{+}^* v = v$. According to the lemma above, $B_{+} v = v$, i.e.

$$A(T - ia')^{-1} v + v = 0.$$

Now set $u = (T - ia')^{-1} v \in \mathcal{D}(T)$. We get

$$(T + A - ia')u = 0.$$

But since $T + A$ is symmetric and $a' > 0$, this gives $u = 0$ and hence, $v = 0$. Hence, we have proved the theorem under the assumption that T is self adjoint.
Now we take the more general case of T being essentially self adjoint only. We proved the inclusion $\overline{T + A} \supset \overline{T} + \overline{A}$ without using the assumption that $b' < 1$. Now $\overline{T}$ is self

adjoint, $\mathcal{D}(\overline{A}) \supset \mathcal{D}(\overline{T})$ and

$$\| \overline{A}u \|^2 \leq a'^2 \| u \|^2 + \| \overline{T}u \|^2 .$$

Applying what was proved above to $\overline{A}, \overline{T}$, we see that $\overline{T} + \overline{A}$ is essentially selfadjoint. Since

$$\overline{T + A} \supset \overline{T} + \overline{A}$$

we see that closed symmetric operator $\overline{T + A}$ is an extension of an essentially self adjoint operator. Hence, $\overline{T + A}$ is self adjoint, i.e. $T + A$ is essentially selfadjoint.

In the next chapter, we will study a special case of rank one perturbations.

Chapter 4

Rank one perturbation of self adjoint operators

In this chapter, we study a special case of perturbations. We look at the perturbations of self adjoint operators by rank one operators. The key result says that the absolutely continuous part of the spectrum stays invariant. A detailed analysis of this can be found in [6].

4.1 Rank one perturbations

Let $\mathcal{H}$ be a Hilbert space and T be a self adjoint operator. Suppose ϕ is a normalized vector in $\mathcal{H}$. Let P_ϕ denote the orthogonal projection onto the subspace gennerated by ϕ. Now we look at the operator

$$T_\lambda = T + \lambda P_\phi$$

where $\lambda \in R$. These T_λ are the rank one perturbations of $\mathcal{H}$.

Proposition 4.1. *Let ψ be a unit vector in $\mathcal{H}$ and T_λ be as defined above. Then $\forall z \in \mathbb{C}^+$*

$$(\psi, (T_\lambda - z)^{-1}\phi) = \frac{(\psi, (T - z)^{-1}\phi)}{(\phi, (T - z)^{-1}\phi)} . \frac{1}{\lambda + (\phi, (T - z)^{-1}\phi)^{-1}}$$

Proof: Since $T_\lambda - T = \lambda P_\phi$, it can be rewritten as

$$T_\lambda - z - T + z = \lambda P_\phi$$

Taking left inverse of $T_\lambda - z$ and right inverse of $T - z$ on both sides, we get

$$(T - z)^{-1} - (T_\lambda - z)^{-1} = \lambda(T_\lambda - z)^{-1} P_\phi (T - z)^{-1}$$

or equivalently

$$(T_\lambda - z)^{-1} = (1 - \lambda(T_\lambda - z)^{-1} P_\phi)(T - z)^{-1}.$$

Now if $(\psi, (.)\phi)$ act on both sides and rearrange, we get the desired result.

Corollary: $(\phi, (T_\lambda - z)^{-1}\phi) = \frac{1}{\lambda + (\phi,(T-z)^{-1}\phi)^{-1}}$.

Definition 4.1. *A set $X \subset \mathcal{H}$ is called total in $\mathcal{H}$ if the set of linear combinations of elements of X is dense in $\mathcal{H}$.*

Definition 4.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a self adjoint operator. A vector $\phi \in \mathcal{H}$ is called cyclic if*

$$\{A^j \phi : 0 \leq j < \infty\}$$

is total in $\mathcal{H}$.

Theorem 4.1. *Consider a separable Hilbert space $\mathcal{H}$ and let T be a self adjoint operator on it. Let $\phi \in H$ be a unit vector. Now, for $\lambda \in \mathbb{R}$, assume that $T_\lambda \neq 0$. Then*
(a) If ϕ is cyclic for T, then it is cyclic for T_λ.
(b) Let $\mathcal{H}_\lambda$ and $\mathcal{H}_{\lambda'}$ be cyclic subspaces generated by T_λ and $T_{\lambda'}$ on ϕ. Then for $\lambda, \lambda' \in \mathbb{R}$,

$$\mathcal{H}_\lambda = \mathcal{H}_{\lambda'}.$$

Proof: We will prove the result for bounded T only. Since ϕ is cyclic for T, we can find by Gram Schmidt process, an orthonormal basis $\{\phi_n\}$ for $\mathcal{H}$, so that $\phi_0 = \phi$ and in this basis, T is tridiagonal. So there is no loss of generality in assuming that T is tridiagonal, i.e.

$$Tu_n = a_n u_{n+1} + b_n u_n + a_{n-1} u_{n-1}.$$

Then ϕ_0 is cyclic for T means that $(\phi_n, T\phi_{n+1}) \neq 0$ for any $n \geq 0$. Because if T is tridiagonal, $(\phi_k, T\phi_m) = 0$ if $\mid k - m \mid \geq 2$ and if $(\phi_n, T\phi_{n+1}) = 0$ for some n, then $a_n = 0$. Now by induction, we see that

$$(p(T)\phi, \phi_{n+1}) = 0$$

for any polynomial p of degree $> n$. This contradicts the assumption that ϕ is a cyclic vector for T. Now, we have, by definition of T_λ and ϕ, that $(\phi_0, T_\lambda \phi_0) = (\phi_0, T\phi_0) + \lambda$ and for any pair $(n, m) \neq (0, 0)$,

$$(\phi_n, T_\lambda \phi_m) = (\phi_n, T\phi_m) + \lambda(\phi_n, P_\phi \phi_m) = (\phi_n, T\phi_m).$$

This shows that ϕ_0 is cyclic for any T_λ.
Now if $\mathcal{H}_\prime$ is the cyclic subspace generated by T on ϕ, then the orthogonal complement $\mathcal{H}_\infty$ of $\mathcal{H}_\prime$ is left invariant by T and T_λ for any λ and on $\mathcal{H}_\infty$,

$$T = T_\lambda$$

since $\lambda P_\phi \mathcal{H}_\infty = \{\theta\}$. Thus we can write $T = B \oplus C$ amd $T_\lambda = B_\lambda + C$. Now an argument similar to the one used above would show that cyclic subspace generated by B_λ on ϕ agrees with $\mathcal{H}_\prime$ for any λ.

Definition 4.3. *Let μ be a measure on $\mathbb{R}$ satisfying the condition that*

$$\int_{\mathbb{R}} d\mu(x) \frac{1}{1 + x^2} < \infty.$$

Then the integral

$$\int_{\mathbb{R}} d\mu(x) \{ \frac{1}{x - z} - \frac{1}{1 + x^2} \}$$

defines the Borel transform F_μ of μ where $z \in \mathbb{C} \setminus \mathbb{R}$.

The above map is an analytic function in $\mathbb{C}^+ \cup \mathbb{C}^-$ and leaves both the components invariant.

Let E_λ be the resolution of identity related to the operator T_λ for $\lambda \neq 0$. Now we want to determine the behaviour of the spectral measures $E_{\lambda,\phi,\phi} = (\phi, E_\lambda(.)\phi)$ and associate it with T_λ and ϕ in terms of properties of measure $\mu_0 = (\phi, E_0(.)\phi)$. Therefore, consider the Borel transform

$$F_\lambda(z) = (\phi, (T_\lambda - z)^{-1} \phi) = \int_{\mathbb{R}} \frac{1}{x - z} d\mu_\lambda(x).$$

Now if we take all the ϕ with $\| \phi \|= 1$, then all μ_λ will be probability measures.

Since

$$(T_\lambda - z)^{-1} = (T - z)^{-1} - \lambda(T_\lambda - z)^{-1}P_\phi(T - z)^{-1},$$

we see that

$$F_\lambda(z) = \frac{F_0}{1 + \lambda F_0(z)}$$

$$Im(F_\lambda(z)) = \frac{Im(F_0)}{| 1 + \lambda F_0(z) |^2}.$$

Now let us define the following sets so we can relate μ_λ to μ_0:

$$S_{\lambda,0} = \{x \in \mathbb{R} : (DF_0)(x) < \infty, \lim_{\epsilon\to 0} F_0(x + i\epsilon) = -\lambda^{-1}\}$$

$$S_{\lambda,\infty} = \{x \in \mathbb{R} : (DF_0)(x) = \infty, \lim_{\epsilon\to 0} F_0(x + i\epsilon) = -\lambda^{-1}\}$$

$$L_\lambda = \{x \in \mathbb{R} : 0 < Im(\lim_{\epsilon\to 0} F_\lambda(x + i\epsilon)) < \infty\}$$

where $\lambda \neq 0$ and

$$DF_0(x) = \lim_{\epsilon\to 0} \int_{\mathbb{R}} \frac{1}{(x - y)^2 + \epsilon^2} d\mu_0(y).$$

We have a result regarding Lebesgue decomposition of μ_λ by Aronszajn and Donoghue, but we need to know about the decomposition of the spectrum before it.

4.2 Components of the spectrum

Now we want to study the decomposition of spectra by decomposing the Hilbert space $\mathcal{H}$.

Definition 4.4. *Let A be a self adjoint operator. Then we define*

$$\mathcal{H}_p(A) = \overline{span\{x \mid x \text{ is an eigenvector of } A\}}.$$

$\mathcal{H}_p(A)$ is the closure of set of all finite linear combinations of eigenvectors of A.

Lemma 4.1. *If* $\ker(A-\lambda_i)$ *denote the eigenspace corresponding to the eigenvalue* λ_i*, then*

$$\mathcal{H}_p(A) = \oplus_i \ker(A-\lambda_i).$$

It is easy to see that $\mathcal{H}_p(A)$ is a subspace of $\mathcal{H}$.

Definition 4.5. *Let* $\mathcal{H}$ *be a Hilbert space. Then we define*

$$\mathcal{H}_c(A) = (\mathcal{H}_p(A))^{\perp}.$$

Hence, $\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_c(A)$*. We call* $\mathcal{H}_c(A)$ *to be the continuous subspace of* (A)*.*

In the above context, $\mathcal{H}_p(A)$ is sometimes called the discontinuous subspace of A. The restriction of A to $\mathcal{D}(A) \cap \mathcal{H}_p(A)$ is denoted by A_p. $\mathcal{H}_p(A)$ is invariant under the action of A_p which is self adjoint in $\mathcal{H}_p(A)$.
Now we denote the restriction of A to $\mathcal{D}(A) \cap \mathcal{H}_c(A)$ by A_c. As earlier, A_c is self adjoint in $\mathcal{H}_c(A)$ and leaves it invariant.
Hence, we have a decomposition of A:

$$A = A_p \oplus A_c.$$

Now we define the continuous and the pure point spectrum of A.

Definition 4.6. *Let* A *be a self adjoint operator in* $\mathcal{H}$ *and let* $A = A_p \oplus A_c$ *as defined above. Then the continuous spectrum of* A *is defined as*

$$\sigma_c(A) = \sigma(A_c)$$

while the the pure point spectrum is defined as

$$\sigma_{pp}(A) = \sigma(A_p)$$

Remark 4.1. *If* $\sigma_p(A)$ *is the set of all eigenvalues of* A*, then*

$$\sigma_{pp}(A) = \overline{\sigma_p(A)}.$$

Moreover, the continuous subspace $\mathcal{H}_c(A)$ can be further decomposed into absolutely continuous and singularly continuous subspaces.

Definition 4.7. *Let m be the Lebesgue measure on $\mathbb{R}$. Then*

$$\mathcal{H}_{ac}(A) = \{x \in \mathcal{H}_c(A) \mid (x, E(\omega)x) = 0 \text{ if } m(\omega) = 0 \text{ for some Borel set } \omega\}$$

$$\mathcal{H}_{sc}(A) = \{x \in \mathcal{H}_c(A) \mid \text{there is a Borel set } \omega_x, m(\omega_x) = 0 \text{ but } E(\omega_x)x = x\}$$

$\mathcal{H}_{ac}(A)$ and $\mathcal{H}_{sc}(A)$ are subspaces of $\mathcal{H}_c(A)$.
Let us denote the restriction of A to $\mathcal{D}(A) \cap \mathcal{H}_{ac}(A)$ by A_{ac} and restriction of A to $\mathcal{D}(A) \cap \mathcal{H}_{sc}(A)$ by A_{sc}.

Now we define the absolutely continuous and singularly continuous spectrum by

$$\sigma_{ac}(A) = \sigma(A_{ac})$$

$$\sigma_{sc}(A) = \sigma(A_{sc})$$

Remark 4.2. *Now we have the following decompositions:*

$$\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_{sc}(A) \oplus \mathcal{H}_{ac}(A)$$

which can be re-written as

$$\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_c(A).$$

Now if we define, $\mathcal{H}_s(A) = (\mathcal{H}_{ac}(A))^{\perp}$ as called the singular subspace of A and like earlier, restriction of A to $\mathcal{D}(A) \cap \mathcal{H}_s(A)$ is denoted by A_s, and $\sigma_s(A) = \sigma(A_s)$ is called the singular spectrum of A, then we can rewrite above decomposition as

$$\mathcal{H} = \mathcal{H}_s(A) \oplus \mathcal{H}_{ac}(A).$$

Now we have a decomposition of the spectrum as

$$\sigma(A) = \sigma_{pp}(A) \cup \sigma_c(A)$$

or

$$\sigma(A) = \sigma_{pp}(A) \cup \sigma_{ac}(A) \cup \sigma_{sc}(A)$$

or

$$\sigma(A) = \sigma_{disc}(A) \cup \sigma_{ess}(A)$$

or

$$\sigma(A) = \sigma_{ac}(A) \cup \sigma_s(A).$$

Remark 4.3. *In general, $\sigma_{pp}, \sigma_{sc}, \sigma_{ac}$ are not disjoint.*

Now we come back to Lebesgue decomposition of μ_λ by Aronszajn and Donoghue.

Theorem 4.2. *Let T_λ and ϕ be as above. Then*
(a) The part $\mu_{\lambda,pp}$ is supported on the set $S_{\lambda,0}$. (b) The part $\mu_{\lambda,sc}$ is supported on the set $S_{\lambda,\infty}$.
(c) The part $\mu_{\lambda,ac}$ is supported on the set L_λ.

In the following theorem by Simon and Wolff, let F_0 denote the Borel transform of measure μ_0.

Theorem 4.3. ***Simon-Wolff:*** *Let T_λ and ϕ be as above. Consider the family of measures μ_λ for $\lambda \in \mathbb{R}$ and assume that for almost every λ, $\mu_\lambda([a,b]) \neq 0$. Then the following are equivalent:*
(a)For almost all λ, μ_λ is pure point in $[a,b]$.
(b)For almost every $x \in [a,b]$, $(DF_0)(x) < \infty$.

Now we come to the last result of the chapter by Barry Simon [7].

Theorem 4.4. ***Barry Simon*** *For $\lambda \neq 0$, the absolutely continuous parts of T_λ and T, i.e. $T_{\lambda,ac}$ and T_{ac} are unitarily equivalent.*

The theorem says that under perturbation by a rank one operator, the absolutely continuous part of the spectrum remains invariant.
Sketch of the proof: By definition,

$$T_\lambda = T + \lambda\phi.$$

If $f(x)dx$ and $g(x)dx$ be absolutely continuous positive measures on $\mathbb{R}$. These measures are equal iff $\{x \mid f(x) \neq 0\}$ and $\{x \mid g(x) \neq 0\}$ agree upto a null set.
Now it suffices to show that L_λ and L_0 agree upto sets of measure zero. Since

$$Im(F_\lambda(z)) = \frac{Im(F_0)}{\mid 1 + \lambda F_0(z) \mid^2},$$

we get

$$Im(F_\lambda(z)) > 0 \Leftrightarrow Im(F_0(z)) > 0.$$

Now L_λ and L_0 agree upto the sets where

$$\lim_{\epsilon \to 0} F(x + i\epsilon) = \infty$$

or

$$\lim_{\epsilon \to 0} F(x + i\epsilon) = -1/\lambda.$$

These sets are of measure zero and we are done.

Conclusion:

The theory of perturbation doesn't end here. It is trivial to see that the above result for invariance of absolutely continuous part of spectrum is also true for perturbation by finite rank operators. The proof is purely by induction. There are many more techniques in continuation to Borel transform to understand the spectra of a self adjoint operator. Fourier transform and Wavelet transform are just two of such techniques and will be a part of my future work.

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