On Second Order Linear Homogeneous Differential Equations

Neeraj

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Certificate of Examination

This is to certify that the dissertation titled **"On Second Order Linear Homogeneous Differential Equations"** submitted by **Neeraj** (Reg. No. MS09089) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Amit Kulshrestha   Dr. Yashonidhi Pandey   Dr. Varadharaj R. Srinivasan (Supervisor)

Dated: April 25, 2014.

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Varadharaj R. Srinivasan at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions.

Neeraj

Dated: April 25, 2014.

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Varadharaj R. Srinivasan
(Supervisor)

Dated: April 25, 2014.

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Notation

I	Identity Matrix (in appropriate dimensions).
$\mathbb{C}$	Complex numbers.
$\mathbb{R}$	Real numbers.
$GL(2, \mathbb{C})$	Group of 2×2 invertible matrices with coefficients in $\mathbb{C}$.
$SL(2, \mathbb{C})$	Group of 2×2 invertible matrices with determinant 1.

Contents

Chapter 1

Introduction

In this thesis, we will address the problem of finding closed form solutions of a second order linear homogeneous differential equation. The content of this thesis is based on the paper by Jerald J. Kovacic[**?**]. In that paper, Kovacic develops an algorithm to determine whether or not a given second order linear homogeneous differential equation defined over $\mathbb{C}(x)$, the field of rational functions in one variable x defined over the field of complex numbers, admits two linearly independent closed form solutions. The algorithm is implemented successfully in computer algebra systems and presently available in MAPLE and MACSYMA.

The rest of the thesis is arranged as follows. In chapter 2, we provide basic definitions and terminologies from differential algebra and from the Galois theory of linear differential equations. Then, we reduce the problem of finding closed form solutions of second order homogeneous linear differential equations to the problem of finding such solutions for equations of the kind $y'' = ry$, where $r \in \mathbb{C}(x)$. The latter has the added advantage that its differential Galois group can be identified with an algebraic subgroup of $SL(2, \mathbb{C})$. In chapter 3 we prove the Lie-Kolchin Theorem and classify the algebraic subgroups (up to conjugation) of $SL(2, \mathbb{C})$ into 4 distinct classes. In chapter 4, we use the classification of the Galois group of the differential equation $y'' = ry$, where $r \in \mathbb{C}(x)$, and obtain conditions that the poles of r must satisfy. In Chapter 5, we study the algorithm in detail and in Chapter 6 we provide several examples to illustrate how the algorithm works. In Chapter 7, we study the proof of correctness of the algorithm.

Chapter 2

Liouvillian solutions

In this chapter, we define the notion of a closed form solution of a linear homogeneous differential equation. We then define the Galois group of a linear differential equation and show that for differential equations of the form $y'' = ry$ for $r \in \mathbb{C}(x)$, the Galois group can be identified with an algebraic subgroup of $SL(2, \mathbb{C})$.

Definition 2.1. *A* ***differential field*** *F is a field with a map $' : F \longrightarrow F$ such that following conditions are satisfied :*

$$(x+y)' = x' + y'.$$
$$(xy)' = xy' + x'y \quad \forall x, y \in F.$$

The map $'$ is usually called a derivation.

The subfield $C = \{x \in F : x' = 0\}$ of F is called the field of constants of F.

A field extenstion E of a field F is said to be a differential field extension if the derivation map of E restricted to F is the same as the derivation map of F.

A differential field extension E of $\mathbb{C}(x)$ (equipped with the derivation $\frac{d}{dx}$) is called ***Liouvillian differential field*** *if there exist a tower of differential* *fields*

$$\mathbb{C}(x) = F_0 \subset F_1 \subset \cdots \subset F_n = E$$

such that for each $i = 1, 2, \cdots n$
either $F_i = F_{i-1}(\alpha)$ *where* $\dfrac{\alpha'}{\alpha} \in F_{i-1}$
or $F_i = F_{i-1}(\alpha)$ *where* $\alpha' \in F_{i-1}$
or F_i *is finite algebraic over* F_{i-1}.

Consider a linear differential polynomial $L(Y) = Y^{(n)} + a_{n-1}Y^{(n-1)} + \cdots + a_0 Y$, *where* $a_i \in \mathbb{C}(x)$ *and* $Y^{(i)}$ *denote the* i^{th} *derivative of* Y. *We say that* $L(Y) = 0$ *admits a Liouvillian (or a closed form) solution if there is a Liouvillian extension* E *of* $\mathbb{C}(x)$ *and an element* $z \in E$ *such that* $L(z) = 0$.

Lemma 2.2. *If one solution of the differential equation*

$$z'' + az' + bz = 0 \text{ where } a, b \in \mathbb{C}(x)$$

is Liouvillian then every solution of this differential equation is also Liouvillian .

Proof. Let η be a solution of $z'' + az' + bz = 0$ where $a, b \in \mathbb{C}(x)$. One can easily see that $\zeta = \eta \int \frac{1}{\eta^2} e^{-\int a dx}$ is a solution of above differential equation. Also ζ and η are linearly independent and ζ is Liouvillian if and only if η is Liouvillian. This completes the proof. □

Lemma 2.3. *The differential equation* $z'' + az' + bz = 0$ *where* $a, b \in \mathbb{C}(x)$ *can be reduced to*

$$y'' = ry, \ r \in \mathbb{C}(x)$$

without changing the Liouvillian nature of solutions.

Proof. Use the substitution $y = e^{\frac{1}{2}\int a} z$ and write $r = -b + \frac{1}{4}a^2 + \frac{1}{2}a'$. □

While solving the second order linear homogeneous differential equations we first reduce it using **??** to $y'' = ry$ where $r \in \mathbb{C}(x)$. If $r \in \mathbb{C}$ the solutions are easy to find and they are Liouvillian. So we only need to consider the case when $r \in \mathbb{C}(x) \setminus \mathbb{C}$.

Notation: From now on The DE means the differential equation

$$y'' = ry, r \in \mathbb{C}(x) \setminus \mathbb{C}.$$

Definition 2.4.

A set of two linearly independent solutions of a second order linear differential equation is called a fundamental system of solutions of the given differential equation.

*Let η, ζ be a fundamental system of solutions of the DE. A differential automorphism σ of $F = \mathbb{C}(x)(\eta, \zeta, \eta', \zeta')$ is a field automorphism of F such that $\sigma(a') = \sigma(a)'$ $\forall a \in F$. The group of all differential automorphisms of F that leave $\mathbb{C}(x)$ invariant is called the **Galois group** of F over $\mathbb{C}(x)$ (denoted by $Gal(F)$).*

Lemma 2.5. *$Gal(F)$ is isomorphic to a subgroup of $SL(2, \mathbb{C})$.*

Proof. Let η, ζ be a fundamental system of solutions of the DE. We note that, for $\sigma \in Gal(F)$, $\sigma(\eta)$ and $\sigma(\zeta)$ are two linearly independent solutions of the DE. Now, we can write $\sigma(\eta) = a_\sigma \eta + c_\sigma \zeta$ and $\sigma(\zeta) = b_\sigma \eta + d_\sigma \zeta$ for some $a_\sigma, b_\sigma, c_\sigma d_\sigma \in \mathbb{C}$. The map $\varphi : Gal\,(F) \longrightarrow GL\,(2, \mathbb{C})$ defined as $\varphi(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$ can be readily seen to be an injective group homomorphism. Moreover, W' (the derivation of Wronskian $W = \eta\zeta' - \eta'\zeta$) can be easily seen to be 0. Thus W is a constant. Also $W \neq 0$. As a result,W is kept fixed by $Gal(F)$. Hence $W = \sigma W$ $\forall \sigma \in Gal(F)$.

$$W = \sigma W = (a_\sigma d_\sigma - b_\sigma c_\sigma)W = det(c(\sigma))W$$

Hence $det(c_\sigma) = 1$. Therefore, $Gal(F) \subset SL(2, \mathbb{C})$. □

From now on, we shall be considering any subgroup of $SL(2, \mathbb{C})$ which is isomorphic to $Gal(F)$ the same as the Galois group of F over $\mathbb{C}(x)$.

Definition 2.6. *An algebraic subgroup of a group G, is a subgroup of G which is closed in Zariski topology.*

Theorem 2.7. *$Gal(F)$ is an algebraic subgroup of SL(2,$\mathbb{C}$).*

Proof. Let Y, Z, Y_1, Z_1 be indeterminates over $\mathbb{C}(x)$ and η, ζ be a fundamental system of solutions of the DE. Now, consider the substitution homomorphism:

$$\phi : \mathbb{C}[x, Y, Z, Y_1, Z_1] \longrightarrow \mathbb{C}[x, \eta, \zeta, \eta', \zeta']$$

Let $P = ker(\phi)$. Since $\mathbb{C}[x, Y, Z, Y_1, Z_1]/P \cong \mathbb{C}[x, \eta, \zeta, \eta', \zeta']$, we obtain that P is a prime ideal.
For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$, we define a map

$$\psi_A : \mathbb{C}[x, Y, Z, Y_1, Z_1] \longrightarrow \mathbb{C}[x, Y, Z, Y_1, Z_1]$$

defined by $\begin{pmatrix} Y & Z \\ Y_1 & Z_1 \end{pmatrix} \longmapsto \begin{pmatrix} Y & Z \\ Y_1 & Z_1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It is easy to prove that ψ_A is an isomorphism of rings. Now,one can easily prove $A \in Gal(F)$ if and only if $\psi_A(P) \subset P$. Define the following two maps :

$$\psi : \mathbb{C}[x, Y, Z, Y_1, Z_1] \longrightarrow \mathbb{C}[x, Y, Z, Y_1, Z_1, X_1, X_2, X_3, X_4]$$

defined by $\begin{pmatrix} Y & Z \\ Y_1 & Z_1 \end{pmatrix} \longmapsto \begin{pmatrix} Y & Z \\ Y_1 & Z_1 \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$ and

$$\rho : \mathbb{C}[x, Y, Z, Y_1, Z_1] \otimes \mathbb{C}[X_1, X_2, X_3, X_4] \longrightarrow \mathbb{C}[x, Y, Z, Y_1, Z_1, X_1, X_2, X_3, X_4]$$

where ρ is the natural isomorphism. Let $\mathcal{B}_P = \{g_\alpha : \alpha \in J\}$ be a basis of P over $\mathbb{C}$. We may extend $\mathcal{B}_P$ to a basis $\mathcal{B} = \{g_\alpha : \alpha \in J \text{ or } \alpha \in I\}$ of $\mathbb{C}[x, Y, Z, Y_1, Z_1]$ over $\mathbb{C}$ for some indexing sets I and J. Clearly, $\mathcal{B}$ is a basis of $\mathbb{C}[x, Y, Z, Y_1, Z_1, X_1, X_2, X_3, X_4]$ over $\mathbb{C}[X_1, X_2, X_3, X_4]$. And thus, $\rho^{-1}(\mathcal{B}) = \{\rho^{-1}(g_\alpha) : \alpha \in J \text{ or } \alpha \in I\}$ is a $\mathbb{C}[X_1, X_2, X_3, X_4]$ basis of $\mathbb{C}[x, Y, Z, Y_1, Z_1] \otimes \mathbb{C}[X_1, X_2, X_3, X_4]$. For $\alpha \in I \cup J$, we clearly have $\rho^{-1}(g_\alpha) = g_\alpha \otimes 1$. Also, it can be easily seen that $\psi_A = Eval_A \circ \psi$.
Let $g \in P$. Say $\psi(g) = \rho(\sum_{\alpha \in J \cup I} g_\alpha \otimes f_\alpha)$.
By definition of ρ,we get $\psi(g) = \sum_{\alpha \in J \cup I} g_\alpha f_\alpha$.
Since $\psi_A = Eval_A \circ \psi$ we have,$\psi_A(g) = \sum_{\alpha \in J \cup I} g_\alpha f_\alpha(a, b, c, d)$. But as $g \in P$,$\psi_A(g) \in$

P. Thus we have $g_\alpha f_\alpha(a,b,c,d) = 0$ for $\alpha \in I$ Hence $f_\alpha(a,b,c,d) = 0$ for $\alpha \in I$.
Now,we assume, $f_\alpha(a,b,c,d) = 0$ for $\alpha \in I$. Let $g \in P$ such that $\psi(g) = \sum_{\alpha \in J \cup I} g_\alpha f_\alpha$. Thus $\psi_A(g) \in P$. Hence $A \in Gal(F)$.
Now, we have a collection of polynomials $\{f_\alpha(x_1, x_2, x_3, x_4)\}$ such that $f_\alpha(a,b,c,d) = 0$ if and only if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gal(F)$. Hence, $Gal(F)$ is an algebraic subgroup of $\mathrm{SL}(2,\mathbb{C})$.

□

Chapter 3

Algebraic subgroups of $SL(2, \mathbb{C})$

3.1 Z Spaces

In this chapter we shall classify the algebraic subgroups of $SL(2, \mathbb{C})$. Most of the theorems in this chapter can be found in Chapter 5 of [?].

Let F be a field and V be an n-dimensional vector space over F. We define an algebraic manifold in V as the set of all common zeros of a collection of polynomials in n indeterminates.

Lemma 3.1. *An algebraic manifold M in V (an n-dimensional vector space over F) is the set of zeros of an ideal (i.e. the set of all common zeros of elements of the ideal) in $F[x_1, x_2, \cdots, x_n]$.*

Proof. Let $\{v_1, v_2, \cdots, v_n\}$ be a basis of V over F.

Let S be a set such that

$$M = \left\{ \sum_{i=1}^{n} a_i v_i : a_i \in F, f(a_1, a_2, \cdots, a_n) = 0 \ \forall f \in S \right\}.$$

Consider, $I = \langle S \rangle$ (the ideal generated by S in the ring $F[x_1, x_2, \cdots, x_n]$).

Write,

$$N = \left\{ \sum_{i=1}^{n} a_i v_i : a_i \in F, f(a_1, a_2, \cdots, a_n) = 0 \;\forall f \in I \right\}.$$

As $S \subset I$, we have $N \subset M$.
Let $v = \sum_{i=1}^{n} a_i v_i \in M$. Thus we have, $f(a_1, a_2, \cdots, a_n) = 0 \;\forall f \in S$ which implies, $f(a_1, a_2, \cdots, a_n) = 0 \;\forall f \in I$ (because S generates I). Therefore,$v \in N$. Hence,$M = N$ □

Definition 3.2. *1. A chain $U_1 \subset U_2 \subset \cdots \subset U_n \subset U_{n+1} \cdots$ of sets satisfies ascending chain condition if there exists $m \in \mathbb{N}$ such that $U_m = U_{m+j} \;\forall j \geq 1$.*

2. A ring R is said to be Noetherian if every chain of proper ideals satisfies ascending chain condition.

Theorem 3.3. *Let R be a Noetherian commutative ring. Then the polynomial ring, $R[x]$ is also Noetherian.*

Proof. The proof can be found in N. Jacobson[**?**]. □

Corollary 3.4. *$F[x_1, x_2, \cdots, x_n]$ is Noetherian.*

Proof. We prove it by induction on n.
For $n = 1$, since F is a field, it does not have any non-zero proper ideal. Thus F is Noetherian. And by Theorem **??** we get, $F[x_1]$ is Noetherian.
Now, if $F[x_1, x_2, \cdots, x_{n-1}]$ be Noetherian, then by Theorem **??**, $F[x_1, x_2, \cdots, x_n]$ is Noetherian. □

Remark:

1. Algebraic manifolds in V satisfy the descending chain condition.
2. Union of finite number of algebraic manifolds in V is a algebraic manifold in V (because finite union of ideals is an ideal.).
3. Intersection of arbitrary number of algebraic manifolds in V is a algebraic manifold in V (because arbitrary intersection of ideals is an ideal.).

4. We have a topology on V in which closed sets are algebraic manifolds. This topology is called as the Zariski topology on V.

Definition 3.5. *A T_1-space is a topological space in which singletons are closed sets.*

Lemma 3.6. *The Zariski topology on a vector space V over F is T_1.*

Proof. Let $\{v_1, v_2, \cdots, v_n\}$ be a basis of V over F. For $v = \sum_{i=1}^{n} a_i v_i \in V, a_i \in F$, define $f_i(x_1, x_2, \cdots, x_n) = x_i - a_i$ for $1 \leq i \leq n$. The only zero of the collection $\{f_1, f_2, \cdots, f_n\}$ is v. Thus, $\{v\}$ is an algebraic manifold. And hence singletons are closed in Zariski topology. Therefore this topology is T_1.

□

Definition 3.7. *A Z-space is a T_1-space which satisfies the ascending chain condition on open sets.*

Lemma 3.8. *1. Every subspace of a Z-space is a Z-space.*

2. If a T_1-space is a continuous image of a Z-space,it is itself a Z-space.

3. A Hausdorff Z-space is finite.

Proof. 1. Let A be a Z-space with topology τ (a collection of closed sets in A) and B be a subspace of A with subspace topology say τ' i.e. for any $C' \in \tau'$, there exists $C \in \tau$ such that $C \cap B = C'$.
Since A is a Z-space, we have $\{x\} \in \tau \ \forall x \in A$.
Now, for a fixed $x \in B$, $\{x\} \in \tau$ which implies $\{x\} \in \tau'$. Therefore τ' is T_1.
Now, let $C_1' \supset C_2' \cdots \supset C_n' \supset C_{n+1}' \cdots$ be a chain of closed sets in B. Thus there exists $C \in \tau$ such that $C \cap B = C'$. And therefore $C_1 \supset C_1 \cap C_2 \cdots \supset C_n \cap C_{n-1} \cdots \cap C_1 \supset C_{n+1} \cap C_n \cdots \cap C_1 \cdots$ is a descending chain of closed sets in A. Thus we have there exists m such that $C_m \cap C_{m-1} \cdots \cap C_1 = C_{m+j} \cap C_{m+j-1} \cdots \cap C_1 \ \forall j \geq 1$. And hence there exists m such that $(C_m \cap C_{m-1} \cdots \cap C_1) \cap B = (C_{m+j} \cap C_{m+j-1} \cdots \cap C_1) \cap B \ \forall j \geq 1$. Therefore, there exists m such that $(C_m \cap B) \cap (C_{m-1} \cap B) \cdots \cap (C_1 \cap B) = (C_{m+j} \cap B) \cap (C_{m+j-1} \cap B) \cdots \cap (C_1 \cap B) \ \forall j \geq 1$. As a result there exists m such that $C_m' \cap C_{m-1}' \cdots \cap$

$C'_1 = C'_{m+j} \cap C'_{m+j-1} \cdots \cap C'_1$ $\forall j \geq 1$. Hence there exists m such that $C'_m = C'_{m+j}$ $\forall j \geq 1$.

2. Let A be a Z-space and $\phi : A \to B$ be onto a continuous map where B is a T_1 space. Let $C'_1 \supset C'_2 \cdots \supset C'_n \supset C'_{n+1} \cdots$ be a chain of closed sets in B. Since, ϕ is continuous, $\phi^{-1}(C'_i)$ is a closed set in A $\forall i$. Also, $\phi^{-1}(C'_1) \supset \phi^{-1}(C'_2) \cdots \supset \phi^{-1}(C'_n) \supset \phi^{-1}(C'_{n+1})$ is a chain of closed sets in A. Thus, there exists m such that $\phi^{-1}(C'_m) = \phi^{-1}(C'_{m+j})$ $\forall j \geq 1$ and as ϕ is onto,$\phi(\phi^{-1}(C'_i)) = C'_i$. Therefore, there exists m such that $C'_m = C'_{m+j}$ $\forall j \geq 1$. And hence, B is a T_1-space.

3. Let X be a Hausdorff Z-space which is not finite. Define $V_i = \cap_{j \neq i} U_{i,j}$ where $U_{i,j}$ is a neighborhood of $x_i \in X$ which is disjoint from the neighborhood $U_{j,i}$ of $x_j \in X$. Now, we shall prove $V_i \cap V_j = \emptyset fori \neq j$ by using contradiction. Let $i \geq j$. And $V_i \cap V_j \neq \emptyset$ which implies $x \in V_i$ and $x \in V_j$
   But $U_{i,j} \cap U_{j,i} = \emptyset$ (By definition). Since,$i \geq j$, $V_i \subset U_{i,j}$ and $V_j \subset U_{j,i}$. we have $V_i \cap V_j = \emptyset$. Now, consider the ascending chain $V_1 \subset V_1 \cup V_2 \cdots \subset V_1 \cup V_2 \cup \cdots \cup V_n \subset \cdots$ of open sets in X which is not stationary.
   This is a contradiction to X being a Z-space. So,X is not infinite.
   Hence, a Hausdorff Z-space is finite.

□

Lemma 3.9. *A Z-space is the union of a finite number of disjoint open and closed connected sets.*

Proof. First we consider the following construction for a Z-space X:

1. Define, $Y_1 = X$.

2. If Y_i is disconnected open and closed subspace of X , then Y_i is a disconnected Z-space. Thus we can write, $Y_i = A \cup B$ with A and B being disjoint open and closed sets. Now, define $Y_{2i} = A$ and $Y_{2i+1} = B$.

3. If Y_i is connected open and closed set then define $Y_{2i} = Y_i$ and $Y_{2i+1} = \emptyset$.

Continue the above construction as long as the collection $L_i = \{Y_j : 2^i \leq j < 2^{i+1}\}$ contains only connected or empty sets for some i.
Now we claim that $\cup_{Y \in L_i} Y = X \ \forall i$ and $Y_{i_1} \cap Y_j = \emptyset$ for $Y_{i_1} \neq Y_j \in L_i$. We shall prove it by using induction on i.
For $i = 1$, $Y_1 = X$.
Let $\cup_{Y \in L_i} Y = X \ \forall i$ and $Y_{i_1} \cap Y_j = \emptyset for Y_{i_1} \neq Y_j \in L_i$ for $i < n$. Clearly, from the step 2 and step 3 of the construction as we go from L_{n-1} to L_n the union doesn't change.
Therefore, $\cup_{Y \in L_{n-1}} Y = \cup_{Y \in L_n} Y = X$. Also, $Y_{i_1} \cap Y_j = \emptyset$ for $Y_{i_1} \neq Y_j, Y_{i_1}, Y_j \in L_n$ (because this happens for elements of L_{n-1} and the construction just breaks elements of L_{n-1} into disjoint elements of L_n). Thus, if the construction stops at L_n then we get the collection $L_n = \{Y_j : 2^j \leq j < 2^{i+1}, Y_j \neq \emptyset\}$ satisfying the condition that L_n is finite set and $\cup_{Y \in L_n} Y = X$ and $y \in L_n$ is both open and closed connected set. Therefore, only case left to complete the proof of lemma is the case when there exists $Y \in L_n$ such that Y is not connected $\forall n$. Define $X_n = Y$ such that $Y \in L_n$ is disconnected.
Now we claim that for $i < j$ either $X_i \supset X_j$ or $X_i \cap X_j = \emptyset$. By definition of X_i , there exists i_1, i_2 such that $X_i = \ Y_{i_1}$ and $X_j = Y_{i_2}$ and $i_1 < i_2$.
Let $X_i \not\supseteq X_j$ and $X_i \cap X_j \neq \emptyset$. It implies that there exists $x \in X_i \cap X_j = Y_{i_1} \cap Y_{i_2}$. Therefore, Y_{i_1} and Y_{i_2} does not belong to same L_k for some k.
And by construction, $Y_{i_1} \supset Y_{i_2}$ which implies $X_i \supset X_j$. And this is a contradiction.

Now, consider the ascending chain of open and closed sets

$$X_1 \subset X_1 \cup X_2 \subset \cdots \subset X_1 \cup X_2 \cdots \cup X_n \subset \cdots .$$

By ascending chain condition on open sets, there exists n such that $\cup_{i=1}^n X_i = \cup_{i=1}^m X_i \ \forall \, m > n$. It implies, $X_{n+j} \subset \cup_{i=1}^n \ \forall j \geq 1$. And thus, there exists j such that X_j contains infinitely many $X_i's$ for $i > n$. Also $X_j \subsetneq X$.
Define $W_0 = X$ and $W_1 = X_j$ and for given W_i we can get $W_{i+1} \subsetneq W_i$. Clearly, we

have the chain

$$W_0 \supsetneqq W_1 \subsetneqq W_2 \cdots \subsetneqq W_n \subsetneqq W_{n+1} \cdots$$

which is not stationary. Hence our assumption was wrong. Thus there exists $Y \in L_i$ such that Y is not connected for some i and $\cup_{Y \in L_i} Y = X$ which Y's are open and closed connected sets.

□

Lemma 3.10. *Let V and W be m-dimensional and n-dimensional vector spaces over F with Zariski topology. Let $r_1, r_2, \cdots, r_n$ be rational functions in m-variables say $x_1, x_2, \cdots, x_m$. Let S be the set where the denominators of $r_1, r_2, \cdots, r_n$ vanish and $T = V - S$. Then the mapping:*

$$\phi : T \longrightarrow W$$

$$(x_1, x_2, \cdots, x_m) \longmapsto (y_1, y_2, \cdots, y_n)$$

such that $y_i = r_i(x_1, x_2, \cdots, x_m)$ is continuous.

Proof. Let A be a closed set in W. Then there exists $S = \{g_j : j \in J\}$ for some indexing set J such that

$$A = \{(y_1, y_2, \cdots, y_n) : g_j(y_1, y_2, \cdots, y_n) = 0 \ \forall j \in J\}$$

One can easily see that

$$\phi^{-1}(A) = \{(x_1, x_2, \cdots, x_m) : g_j(r_1, r_2, \cdots, r_n)(x_1, x_2, \cdots, x_m) = 0 \ \forall j \in J\}$$

which is a closed set in T. Thus ϕ is continuous. □

Definition 3.11. *A group G is a* $\mathbf{T_1}$ ***group*** *if it is a T_1 space such that $L_x : G \longrightarrow G$ mapping $y \longmapsto xy$, $R_x : G \longrightarrow G$ mapping $y \longmapsto yx$ and $Inv : G \longrightarrow G$ mapping $y \longmapsto y^{-1}$ are continuous maps. A* ***Z-group*** *is a T_1-group which is also a Z-space.*

Lemma 3.12. *$GL_n(F)$ is a T_1 group.*

Proof. $GL_n(F) \subset M_n(F)$ is a group under matrix multiplication and has the subspace topology due to the Zariski topology on $M_n(F)$ over F which makes it a Z-space.
Consider the map, $Inv : GL_n(F) \longrightarrow GL_n(F)$ mapping $A \in GL_n(F)$ to A^{-1}.
We define, $r_{i,j}(A) = \dfrac{1}{detA}C_{ij}$, $1 \leq i, j \leq n$ where C_{ij} denotes the cofactor of $(i, j)^{th}$ element of A.
By lemma **??**, Inv is continuous.
Now, consider the map $L_A : GL_n(F) \longrightarrow GL_n(F)$ mapping $B \in GL_n(F)$ to AB for $A \in GL_n(F)$. We define, $r_{i,j}(B) = \sum_{k=1}^{n} a_{ik}b_{kj}$ where a_{ij}, b_{ij} mean the ij^{th} entry of A and B respectively. Clearly, by lemma **??** L_A is a continuous map. Similarly, R_A can be shown to be a continuous map as well. Hence, $GL_n(F)$ is a T_1 group. □

Definition 3.13. *The component of the identity G_0 in a group G which also has a topology is the maximal closed connected subset of G that contains the identity of the group G.*

Lemma 3.14. *The component of the identity in a T_1-group is a closed normal group.*

Proof. Let G be a T_1-group and G_0 be the component of the identity in G. As, $Inv : G \longrightarrow G$ is continuous, the set $G_0^{-1} = \{x^{-1} : x \in G_0\}$ being the continuous image of a connected set, is connected. Note that $1 \in G_0^{-1}$. Now as G_0 is maximal connected set, $G_0^{-1} \subset G_0$. Since, L_g for $g \in G_0$ is continuous,gG_0 (being continuous image of a connected set) is connected $\forall\ g \in G_0$. Also $g \in G_0 \cap gG_0$ implies that $gG_0 \subset G_0\ \forall g \in G_0$. Hence G_0 is a group. Now since $x^{-1}G_0x = R_x(L_{x^{-1}}(G_0))$ for $g \in G$, we have $1 \in x^{-1}G_0x$. Now it follows that $x^{-1}G_0x$ is connected and therefore $x^{-1}G_0x \subset G_0$. Hence, G_0 is a closed normal subgroup of G. □

Lemma 3.15. *The component of the identity in a Z-group is a closed normal subgroup of finite index.*

Proof. Let G be a Z-group and G_0 be the component of the identity in G and suppose that $[G : G_0]$ is not finite. By lemma **??**, G can be broken into finitely many open and closed connected sets. We shall prove that G_0 is equal to one of them. Clearly,

G_0 is contained in exactly one of the open and closed connected sets say A. Now, as G_0 is maximal connected set, $A \subset G_0$. Hence, $A = G_0$.
Now, as $[G : G_0]$ is not finite, we have infinitely many cosets $G_0, x_1G_0, \cdots$ such that $x_i \in G$ are the representatives. Also each x_iG_0(being continuous image of a connected set) is open and closed connected set, which contradicts the lemma **??**. Therefore $[G : G_0] < \infty$. □

3.2 C-group

Definition 3.16. *A C-group G is a T_1 group in which the conjugation map $\phi_x : G \longrightarrow G$ mapping $a \in G$ to $a^{-1}xa$ for fixed $x \in G$.*

e.g. $GL_n(F) \subset M_n(F)$ under the subspace topology.

Lemma 3.17. *Let G be a C-group whose component of the identity G_0 has a finite index k. Then any finite conjugacy class of G has at-most k elements.*

Proof. Suppose there exists $x \in G$ such that $m = |C_x| > k$ and $m < \infty$ where $C_x = \{a^{-1}xa : a \in G\}$
As the mapping $\phi_x : a \mapsto a^{-1}xa$ is continuous and inverse image of a closed set is closed and singletons are closed in T_1 topology, the inverse image of each conjugate must be closed. Also, as $|C_x| = m < \infty$, each conjugate is open as well. (In the subspace topology on $Image(\phi_x)$). As, singletons are connected in G,$\phi_x(a^{-1})$ is also connected. But $\phi_x(a^{-1}) = \phi_x^{-1}(a^{-1}xa)$ and therefore, the inverse image of each conjugate is open and closed connected set which results in a decomposition of G into more than k open and closed connected sets. Therefore, there exists a coset of G_0 which contains more than one open and closed set which contradicts the connectedness of that coset. Hence any finite conjugacy class of G has at most k elements.

□

Lemma 3.18. *In a connected C-group, any non-central element has an infinite conjugacy class.*

Proof. Let G be a connected C-group. As G is connected, the component of the identity in G is G. By lemma **??**, any finite conjugacy class has at most 1 element, but any conjugacy class of a non central element contains at least 2 elements. So any conjugacy class of a non-central element has to be infinite. □

Lemma 3.19. *Arbitrary union of connected sets each having a fixed point in common with a fixed connected set in a T_1 topology is connected.*

Proof. Let $X = \cup_{j \in J} X_j$ where J is (fixed) indexing set. Let Y be the fixed connected set such that $Y \cap X_j \supset \{x\}$ $\forall j \in J$. Let X be not connected. It implies $X = A \cup B$ where A and B are non-empty disjoint open sets. As,$x \in X$ and $A \cap B = \emptyset$,we may assume that $x \in A$. Since, $B \neq \emptyset$, there exists $y \in B$ and therefore $y \in X_k$ for some k. But, as X_k is connected either $X_k \subset A$ or $X_k \subset B$ which is a contradiction to the fact that $x \in A \cap X_k$ and $y \in B \cap X_k$. Therefore X is connected.

□

Theorem 3.20. *If G is a connected C-group, then the commutator subgroup G' is also connected.*

Proof. Define D_k as the set of all products of k-commutators in G. Clearly, $\{1\} = D_0 \subset D_1 \subset \cdots$ and $\bigcup_{i=1}^{\infty} D_i = G'$. We shall prove that D_k is connected (by using induction on k).
Clearly, D_0 is connected.
Let D_{i-1} is connected for $i < n$. Consider the map

$$\phi_{b_1,b_2,\cdots,b_n,a_2,\cdots,a_n} : a_1 \mapsto a_1^{-1} b_1 - 1 a_1 b_1 a_2 b - 2 b_2 - 1 a_2 b_2 \cdots a_n^{-1} b_n - 1 a_n b_n$$

for fixed $b_1, b_2, \cdots, b_n, a_2, \cdots a_n \in G$.
Clearly, $\phi_{b_1,b_2,\cdots,b_n,a_2,\cdots,a_n}$ is continuous (being just the composition of right multiplication and conjugation). Therefore, the $Image(\phi_{b_1,b_2,\cdots,b_n,a_2,\cdots,a_n})$ is connected. Also, the image has a point in common with D_{k-1}(when $a_1 = b_1$).
But $D_k = \bigcup_{b_1,b_2,\cdots,b_n,a_2,\cdots,a_n \in G} \phi_{b_1,b_2,\cdots,b_n,a_2,\cdots,a_n}(G)$.
Therefore, by lemma **??** D_k is connected. Hence $G' = \bigcup_{i=1}^{\infty} D_i$ is connected as well. □

Lemma 3.21. *Let G be a C-group, H a closed subgroup of G. Suppose that the component of the identity of H is solvable. Suppose either H is of finite index in G or H is normal and G/H is abelian. Then the component of the identity in G is solvable.*

Proof. We prove it for the two cases separately.

Case 1: Let $[G : H]$ be finite. Say G_0 and H_0 are the components on the identity in G and H resp. Clearly, $H_0 \subset G_0$. Since $[G : H]$ is finite and H is closed, every coset of H is both open and closed. Hence G_0 is contained in exactly one of the cosets (because G_0 is connected.) But $G_0 \cap H \neq \emptyset$ and therefore, $G_0 = H_0$. Hence, the component of the identity in G is solvable.

Case 2: Let H be normal in G and G/H be abelian. Let G_0 and H_0 be the components on the identity in G and H respectively. As H is normal in G,$(ab)H = (aH)(bH)$ for $a, b \in G$ Therefore, $(ab)H = (ba)H$ i.e. $a^{-1}b^{-1}ab \in H$ i.e. $G^{'} \subset H$ which implies $G_0^{'} \subset H$. Now we claim that G_0 is a C-group. G_0 is a closed normal subgroup of G (By lemma **??**). Clearly, G_0 is a T_1 space. The map, $Inv_G : G \longrightarrow G$ mapping $x \mapsto x^{-1}$ is continuous. Now consider the map $Inv_{G_0} : G_0 \longrightarrow G_0$ mapping $x \mapsto x^{-1}$ and let Y be a open set in G_0. Then there exists $Y^{'} \subset G$ such that $Y^{'}$ is open in G and $Y^{'} \cap G_0 = Y$. Define $Y^{'^{-1}} = \{x^{-1} : x \in Y^{'}\}$. Clearly, $Y^{'^{-1}} = Inv_G^{-1}(Y^{'})$ is open in G and $Inv_{G_0}^{-1}(Y) = Y^{'^{-1}} \cap G_0$ which is open in G_0. Therefore, the map $Inv_{G_0} : G_0 \longrightarrow G_0$ is continuous.

Now, the map $\phi_{G,x} : G \longrightarrow G$ mapping $a \mapsto a^{-1}xa$ is continuous(for fixed $x \in G$). For $x \in G_0$, $x^{-1}G_0x \subset G_0$. Consider the map $\phi_{G_0,x} : G_0 \longrightarrow G_0$ mapping $a \mapsto a^{-1}xa$ for $x \in G_0$. Let Y be a open set in G_0. Then there exists $Y^{'} \subset G$ such that $Y^{'}$ is open in G and $Y^{'} \cap G_0 = Y$. Now,$\phi_{G_0,x}^{-1}(Y) = \phi_{G,x}^{-1}(Y^{'}) \cap G_0$. and therefore, $\phi_{G_0,x}$ is continuous.

Note that the map $\phi_{a_2} : G \longrightarrow G$ mapping $a_1 \mapsto a_1a_2$ is continuous for $a_2 \in G$. Consider the map, $\psi_{a_2} : G_0 \longrightarrow G_0$ mapping $a_1 \mapsto a_1a_2$ is continuous for $a_2 \in G_0$. Let Y be a open set in G_0. Then there exists $Y^{'} \subset G$ such that $Y^{'}$ is open in G and $Y^{'} \cap G_0 = Y$

$$\begin{aligned}
\psi^{-1}(Y) &= \{a_1 \in G_0 : a_1 a_2 \in G_0\} \\
&= \{a_1 \in G : a_1 a_2 \in Y\} \cap G_0 \\
&= \left\{a_1 \in G : a_1 a_2 \in Y^{'} \cap G_0\right\} \cap G_0 \\
&= \left\{a_1 \in G : a_1 a_2 \in Y^{'}\right\} \cap G_0 \cap \{a_1 \in G : a_1 a_2 \in G_0\} \\
&= Y'' \cap G_0
\end{aligned}$$

where $Y^{''}$ is open in G.

Therefore, ψ_{a_2} is continuous for $a_2 \in G_0$. Similarly the map, $\psi_{a_1} : G_0 \longrightarrow G_0$ mapping $a_2 \mapsto a_1 a_2$ is continuous for $a_1 \in G_0$ is continuous. Hence, G_0 is a C-group.

Now,by theorem **??**, $G_0^{'}$ is connected. Thus, $G_0^{'} \subset H_0$(because $1 \in G_0{'}$). Since, H_0 is solvable, $G_0^{'}$ has to be solvable as well. Hence G_0 is solvable.

□

Lemma 3.22. *In a C-group G, the normalizer of a closed subgroup is closed.*

Proof. Let S be a closed subgroup of G. As G is a C-group, for fixed $s \in S$ the map $\phi_s : G \longrightarrow G$ mapping $a \longmapsto a^{-1}sa$ is continuous. Now $\phi_s^{-1}(S) = \{a \in G | a^{-1}sa \in S\}$ is closed since S is closed for all $s \in S$. The set $S_1 = \cap_{s \in S} \phi_s^{-1}(S) = \{a \in G | a^{-1}Sa \subset S\}$ is intersection of closed sets and thus closed. Also, the set $S_2 = (Inv_G)^{-}1 \cap_{s \in S} \phi_s^{-1}(S) = \{a \in G | aSa^{-1} \subset S\}$ is also closed. Hence the normalizer of S being just the intersection of S_1 and S_2 is closed. □

Lemma 3.23. *Any commutative set of $n \times n$ invertible matrices over an algebraically closed field can be put in simultaneous triangular form.*

Proof. Let G be a commutative set of $n \times n$ invertible matrices over an algebraically closed field F. We shall prove the lemma by using induction on n.

For $n = 1$, G is already in simultaneous triangular form. So let us assume that for any m such that $2 \leq m < n$ any commutative set of $m \times m$ invertible matrices over F can be put in simultaneous triangular form. Choose $A_1 \in G$ and let $V = F^n$. As A_1 is invertible, $det(A_1) \neq 0$. So, there exists a non-zero eigenvalue say c with eigenvector $\alpha \in V = F^m$. Now, for any $B \in G$, $\alpha B A_1 = \alpha A_1 B = c\alpha B$. Thus, the set $W = \{\alpha \in V | \alpha A_1 = c\alpha\}$ is invariant under right multiplication by elements of G. Now, choose a basis of W say $\{v_1, v_2, \cdots, v_r\}$ and extend it to a basis say $\{v_1, v_2, \cdots, v_r, v_{r+1}, \cdots, v_n\}$ of V. With respect to this basis any matrix $A_i \in G$ is of form $\begin{bmatrix} B_i & 0 \\ * & C_i \end{bmatrix}$ where B_i is a $r \times r$ matrix with $r < n$. Consider the sets $G_1 = \{B_i | A_i \in G\}$ and $G_2 = \{C_i | A_i \in G\}$ of $r \times r$ and $(n-r) \times (n-r)$ matrices respectively. Now, we shall prove that these two sets are commutative. For $A_i, A_j \in G$, we have $A_i A_j = A_j A_i$.

Therefore, $\begin{bmatrix} B_i & 0 \\ * & C_i \end{bmatrix} \begin{bmatrix} B_j & 0 \\ * & C_j \end{bmatrix} = \begin{bmatrix} B_j & 0 \\ * & C_j \end{bmatrix} \begin{bmatrix} B_i & 0 \\ * & C_i \end{bmatrix}$.

It clearly implies that $\begin{bmatrix} B_i B_j & 0 \\ * & C_i C_j \end{bmatrix} = \begin{bmatrix} B_j B_i & 0 \\ * & C_j C_i \end{bmatrix}$. Thus $B_i B_j = B_j B_i$ and $C_i C_j = C_j C_i$. Hence G_1 and G_2 are both simultaneously triangulizable. If conjugation by M and N respectively triangulizes G_1 and G_2, then conjugation by $\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$ triangulizes G. Hence, G can be put in simultaneous triangular form. Hence the proof is complete. □

Theorem 3.24. *(**Lie Kolchin Theorem**) Any solvable connected (in the Zariski topology) multiplicative group of non singular matrices over an algebraically closed field can be put in simultaneous triangular form.*

Proof. Let G be a solvable multiplicative group of non singular matrices over an algebraically closed field F. Also let G is connected in Zariski topology. First we assume that G is reducible. We shall use induction on size of matrices in G. For a faithful homomorphism $\rho : G \longrightarrow GL(V)$, the vector space V admits a non-trivial invariant subspace say W. Now, choose a basis of W say $\{v_i : i \in \mathbb{N}, i \leq r\}$. Then extend it to a basis of V say $\{v_i : i \in \mathbb{N}, i \leq n\}$. We now write $A \in G$ in the chosen

basis of V. As W is invariant under G, A has the form $\begin{pmatrix} B & 0 \\ * & C \end{pmatrix}$, where $*$ indicates the unimportant matrix , B is a $r \times r$ matrix and C is a $(n-r) \times (n-r)$ matrix. Now we consider the sets $G_1 = \{B : A \in G\}$ and $G_2 = \{C : A \in G\}$ and the maps $\phi : G \longrightarrow G_1$ mapping $A \mapsto B$ and $\psi : G \longrightarrow G_2$ mapping $A \mapsto C$. Clearly, ϕ and ψ are well defined maps. It is easy to check that G_1 and G_2 are groups. Now,for $A_1 = \begin{pmatrix} B_1 & 0 \\ E_1 & C_1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} B_2 & 0 \\ E_2 & C_2 \end{pmatrix}$,we have, $\phi(A_1A_2) = B_1B_2 = \phi(A_1)\phi(A_2)$ and $\psi(A_1A_2) = C_1C_2 = \psi(A_1)\psi(A_2)$. Thus, ϕ and ψ are group homomorphisms.
Now,consider the r^2 functions in n^2 variables as $s_{ij} = a_{ij}$ for $1 \leq i, j \leq r$ where $A = (a_{ij})$. By lemma **??**, ϕ is a continuous function. Similarly, for the choice, $s_{ij} = a_{ij}$ for $r \leq i, j \leq n$, ψ is a continuous map. Thus, G_1 and G_2 are connected solvable matrix groups. Hence by induction hypothesis, G_1 and G_2 can be simultaneously put in triangular form. If conjugation by M and N respectively triangularizes G_1 and G_2, then conjugation by $\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$ triangularizes G. Therefore we may assume G to be a irreducible solvable multiplicative group of non singular matrices over an algebraically closed field F. Also let G be connected in the Zariski topology. We complete the proof by using the induction on the length of the derived series say n. If $n = 1$ then G is abelian. Hence by Lemma **??**, G can be put in simultaneous triangular form. Now, let the hypothesis be true for groups whose length of commutator series is less than n. Let length of commutator series of G be n. Then the length of commutator series of $G^{'}$ is $n-1$. And $G^{'}$ is a irreducible solvable multiplicative group of non singular matrices over an algebraically closed field F. Also by theorem **??**, $G^{'}$ is connected in Zariski topology. Hence $G^{'}$ can be put in simultaneous triangular form. Let V be a faithful representation of G. As G is irreducible V does not have any non-trivial invariant subspace.
Consider $W = \left\{\alpha \in V | \alpha T = c(T)\alpha \ \forall T \in G^{'}\right\}$. Now, we shall prove W is non-empty and invariant under the action of G. As $G^{'}$ can be put in simultaneous triangular form, there exists $\alpha (\neq 0) \in W$.Now, for $T \in G^{'}$ and $S \in G, STS^{-1} \in G^{'}$. For $\alpha \in W$, $\alpha STS^{-1} = c(STS^{-1})\alpha$ which implies $\alpha ST = C(STS^{-1})\alpha S$ i.e. $\alpha S \in W$.

Thus, W is invariant under G. As G is irreducible, $W = V$. Thus, all matrices in $G^{'}$ are diagonal.

Now, we shall prove that $G^{'} \subset Centre(G)$. For $T \in G^{'}$, $STS^{-1} \in G^{'}$ $\forall S \in G$.But $det(xI - T) = det(S^{-1}(xI - T)S) = det(xI - STS^{-1})$. Thus the set of eigenvalues of T and the set of eigenvalues of STS^{-1} are same. And thus, any $T \in G^{'}$ has only finitely conjugates. Hence, by lemma **??**,$G^{'} \subset Centre(G)$.

Now we shall prove that every matrix in $G^{'}$ is a scalar matrix. So let us assume that there exists $T \in G^{'}$which is not scalar. As T is invertible, $det(T) \neq 0$. Thus all eigenvalues of T are non-zero. Let c be an eigenvalue of T. Consider the subspace $W_1 = \{\alpha \in V | \alpha T = c\alpha\}$ of V. Now, as $TS = ST$ $\forall S \in G$, we have $\alpha ST = \alpha TS = c\alpha S$ which implies $\alpha S \in W$. Therefore, W is invariant under G. Thus $W = V$. So, we have $T = cI$, which is a contradiction to the fact that T is not scalar. Hence, all matrices in $G^{'}$ are scalar.

Now, for $T \in G^{'}$, $det(T) = 1$ and $T = cI$. Thus, c is a n^{th} root of unity. As, there are only finitely many n^{th} roots of unity, we get $|G^{'}| < \infty$. As $|G^{'}| \geq 2$ implies that $G^{'}$ is disconnected, we have that $G^{'} = I$. Thus G is a abelian group. And by lemma **??**, G can be put in simultaneous triangular form. Hence, any irreducible solvable connected (in the Zariski topology) multiplicative group of non singular matrices over an algebraically closed field can be put in simultaneous triangular form. □

Remark: The connectedness (in Zariski topology) can not be dropped from the hypothesis of theorem **??** beacuse of the following example:

$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}$ with discrete topology and usual multiplication. G is a solvable matrix group over the field $\mathbb{C}$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ upper triangularises G. Thus, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is upper triangular. Therefore $2cd = 0$. And $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is upper triangular implies that $c^2 = d^2$. Thus we get, $c = d = 0$ making A is non invertible. Hence G can not be simultaneously triangularised.

Corollary 3.25. *Any solvable matrix group over an algebraically closed field has a normal subgroup of finite index which admits simultaneous triangular form.*

Proof. Let G be a solvable matrix group over an algebraically closed field F. Let G_0 be the component of the identity of G. Then clearly, G_0 is a connected solvable matrix group which is of finite index in G. And by theorem **??**, G_0 admits simultaneous triangular form. $\square$

Theorem 3.26. *Let G be a subgroup of $SL(2, F)$, where F is an algebraically closed field. Let G be closed in Zariski topology and the component of the identity of G be solvable. Then exactly one of the following holds:*

1. *G can be put in simultaneous triangular form.*

2. *Case 1 does not hold and G is conjugate to a subgroup of*

$$D^{\dagger} = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} : c \in \mathbb{C}, c \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} : c \in \mathbb{C}, c \neq 0 \right\}.$$

3. *Cases 1 and 2 do not hold and G is finite.*

4. *$G = SL(2, F)$.*

Proof. Let $G \neq SL(2, F)$ and let G_0 be the component of the identity in G. By theorem **??**, G_0 can be put in simultaneous triangular form. Now, we assume that G_0 can be put in simultaneous diagonal form. Without loss of generality we can assume

$$G_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in X \subset F \right\}$$

for some set X. As G_0 is closed in G, G_0 satisfies a collection of polynomials in a finite number of variables, say m. Since we can write $G_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}$ such that $a \in X\ (\subset F)$ satisfies a collection of polynomials, either G_0 is finite or $X = F$.

Let W be the set of characteristic vectors of G_0 invariant under G where the action is left multiplication. Clearly, $v_1 = (1, 0)^t, v_2 = (0, 1)^t \in W$ and these the only characteristic vectors in $V = F^2$. Also, W is invariant in G. So only two cases are possible, either $G.v_1 = \alpha v_1$ and $G.v_2 = \beta v_2$ or there exists $g \in G$ such that $g.v_1 = \alpha v_2$ and $g.v_2 = \beta v_1$. A simple computation shows that if $G.v_1 = \alpha v_1$ and $G.v_2 = \beta v_2$ then $G = G_0$ and if there exists $g \in G$ such that $g.v_1 = \alpha v_2$ and $g.v_2 = \beta v_1$ then $[G : G_0] = 2$ and G is a subgroup of $D^\dagger$.
Now, let G_0 does not admit simultaneous diagonal form. But since G_0 admits simultaneous triangular form, G_0 has exactly one characteristic vector say α. The set $W = \{c\alpha : \alpha \in F\}$ is invariant under G. On extending the set $\{\alpha\}$ to a basis of $V = F^2$ we get G in simultaneous triangular form. This completes the proof. $\square$

Chapter 4

The Three Cases

For the DE $y'' = ry$, this chapter will give the conditions on r imposed by the galois group of the DE. The main reference for this chapter is Kovacic [**?**].

Lemma 4.1. *For a fundamental system of solutions η, ζ of the DE, if the field $\mathbb{C}(x)(\eta, \eta', \zeta, \zeta')$ is contained in a Liouvillian field then the component of the identity of the Galois group of $\mathbb{C}(x)(\eta, \eta', \zeta, \zeta')$ over $\mathbb{C}(x)$ is solvable.*

A proof of above lemma can be found in Galois theory of linear differential equations[**?**].

Theorem 4.2. *Exactly one of the following cases about the solutions of the DE holds in the respective cases of Theorem* **??** *:*

Case 1. There is a solution $e^{\int \omega}$ where $\omega \in \mathbb{C}(x)$.

Case 2. There is a solution $e^{\int \omega}$ with ω algebraic of degree 2 and case 1 does not hold.

Case 3. All solutions are algebraic over $\mathbb{C}(x)$, and both cases 1 and 2 do not hold.

Case 4. There are no Liouvillian solutions.

Proof. Let η, ζ be a fundamental system of solutions of the DE. Let G be the Galois group relative to the fundamental system η, ζ.

Case 1: By Case 1 of Theorem **??**, $G = Gal(F)$ can be put in simultaneous triangular form. So we can assume that G is upper triangular. Thus for $\sigma \in G$, we have $\sigma(\eta) = a_\sigma \eta$ where $a_\sigma \in \mathbb{C}$. So for $\omega = \frac{\eta'}{\eta}$, we have $\sigma(\omega) = \omega$, $\forall \sigma \in G$. Hence, $\omega \in \mathbb{C}(x)$. And by a simple computation one can see that $e^{\int \omega}$ is a solution of the DE.

Case 2: By Case 2 of Theorem **??**, $G = Gal(F)$ is conjugate to a subgroup of $D^\dagger$ and Case 1 does not hold. So, we can assume that G is a subgroup of $D^\dagger$. Thus for $\sigma \in G$ either $\sigma(\eta) = a_\sigma \eta$ and $\sigma(\zeta) = a_\sigma^{-1}\zeta$ or $\sigma(\eta) = -a_\sigma^{-1}\zeta$ and $\sigma(\zeta) = a_\sigma \eta$ where $a_\sigma \in \mathbb{C}$. Hence for $\omega = \dfrac{\eta'}{\eta}$ and $\phi = \dfrac{\zeta'}{\zeta}$ we have, either $\sigma(\omega) = \phi$ and $\sigma(\phi) = \omega$ or $\sigma(\omega) = \omega$ and $\sigma(\phi) = \phi$.
If $\forall \sigma \in G$, we have $\sigma(\omega) = \omega$ and $\sigma(\phi) = \phi$ then G is diagonal and hence triangular. This contradicts the fact that Case 1 does not hold. So, there exists $\sigma \in G$ such that $\sigma(\omega) = \phi$ and $\sigma(\phi) = \omega$. Thus $\omega\phi$ and $\omega + \phi$ are kept invariant by G. Hence, ω is quadratic over $\mathbb{C}(x)$. Now, a easy check that $e^{\int \omega}$ is a solution of the DE , completes the proof of this case.

Case 3: By Case 3 of Theorem **??**, $G = Gal(F)$ is a finite group and cases 1 and 2 do not hold. Let $G = \{\sigma_1, \sigma_2, \cdots \sigma_n\}$. One can easily check that elementary symmetric functions of $\sigma_1\eta, \sigma_2\eta, \cdots, \sigma_n\eta$ are invariant under G. Hence η is algebraic over $\mathbb{C}$. Similarly, ζ is algebraic over $\mathbb{C}(x)$. Also as, η and ζ are linearly independent. Every solution of the DE is algebraic.

Case 4: By Case 4 of Theorem **??**, $G = Gal(F) = SL(2, \mathbb{C})$. Let one solution of the DE is Liovillian then by lemma **??**, every solution of the DE is Liouvillian. Hence, $\mathbb{C}(x)(\eta, \eta', \zeta, \zeta')$ is contained in a Liouvillian field and thus the component of the identity of the Galois group of $\mathbb{C}(x)(\eta, \eta', \zeta, \zeta')$ over $\mathbb{C}(x)$ is solvable. But the component of the identity of the $SL(2, \mathbb{C})$ is $SL(2, \mathbb{C})$ which is not solvable. Hence the contradiction proves the statement for Case 4.

□

Definition 4.3. *Since $r \in \mathbb{C}(x)$, the zeros of the denominator of r in $\mathbb{C}$ are called the poles of r. Order of r at ∞ is defined as the order of ∞ as a zero of r.*

Theorem 4.4. *The following conditions are necessary for the respective cases of theorem* **??** *:*

Case 1. *Every pole of r must have even order or else have order 1. The order of r at ∞ must be even or else be greater than 2.*

Case2. *r must have at least one pole that either has odd order greater than 2 or else has order 2.*

Case 3. *The order of a pole of r cannot exceed 2 and the order of r at ∞ must be at least 2. If the partial fraction expansion of r is*

$$r = \sum_i \frac{\alpha_i}{(x - c_i)^2} + \sum_j \frac{\beta_j}{x - d_j},$$

then $\sqrt{1 + 4\alpha_i} \in \mathbb{Q}$, for each i, $\sum_j \beta_j = 0$, and if $\gamma = \sum_i \alpha_i + \sum_j \beta_j d_j$, then $\sqrt{1 + 4\gamma} \in \mathbb{Q}$.

Proof.
By Case 1 of theorem **??**, we know that there is a solution $e^{\int \omega}$ of the DE where $\omega \in \mathbb{C}(x)$. Thus ω satisfies the equation $\omega' + \omega^2 = r$. This equation is famous by the name of "Ricatti equation". As $\omega, r \in \mathbb{C}(x)$,they have a Lorentz series expansion around any $c \in \mathbb{C}$. To simplify the notation we take $c = 0$. Let

$$\text{Case 1: } \omega = bx^{\mu} + \cdots ; b \neq 0, \mu \in \mathbb{Z}$$
$$r = \alpha x^{\nu} + \cdots ; \alpha \neq 0, \nu \in \mathbb{Z}$$

(where dots represent the terms having higher degree terms.) On substituting the Lorentz series expansion of r and ω in Ricatti equation we get, $min(\mu - 1, 2\mu) = \nu$. We only need to prove that if $\nu < -3$ then ν is even. But, if $\nu < -3$ then clearly $min(\mu - 1, 2\mu) = 2\mu$. And thus $\nu = 2\mu$.
Now we consider the Lorentz series expansion of r and ω around ∞. Let

$$\omega = bx^{\mu} + \cdots ; b \neq 0, \mu \in \mathbb{Z}$$
$$r = \alpha x^{\nu} + \cdots ; \alpha \neq 0, \nu \in \mathbb{Z}$$

(where dots represent the terms having lower degree terms.) On substituting the Lorentz series expansion of r and ω in Ricatti equation we get, $max(\mu - 1, 2\mu) = \nu$. We only need to prove that if $\nu \geq -1$ then ν is even. But, if $\nu \geq -1$ then clearly $max(\mu - 1, 2\mu) = 2\mu$. And thus $\nu = 2\mu$. This completes the proof for Case 1.

Case 2: Let η, ζ be a fundamental system of solutions of the DE. Let G be the Galois group relative to the fundamental system η, ζ. By Case 2 of theorem **??** G is conjugate to a subgroup of

$$D^{\dagger} = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} : c \in \mathbb{C}, c \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} : c \in \mathbb{C}, c \neq 0 \right\}.$$

which is not triangulizable.
So for any $\sigma \in G$, $\sigma(\eta) = a_\sigma \eta$ and $\sigma(\zeta) = a_\sigma^{-1}\zeta$ or $\sigma(\eta) = -a_\sigma^{-1}\zeta$ and $\sigma(\zeta) = a_\sigma \eta$ where $a_\sigma \in \mathbb{C}$. Also $\forall \sigma \in G$,if we have $\sigma(\eta) = a_\sigma \eta$ and $\sigma(\zeta) = a_\sigma^{-1}\zeta$ then G is diagonal which is not possible. So, there exists$\sigma \in G$ such that $\sigma(\eta) = -a_\sigma^{-1}\zeta$ and $\sigma(\zeta) = a_\sigma \eta$. Thus, $\forall \sigma(\eta^2\zeta^2) = \eta^2\zeta^2$ and $\sigma(\eta\zeta) \neq \eta\zeta$. Hence, $\eta^2\zeta^2 \in \mathbb{C}(x)$ and $\eta\zeta \notin \mathbb{C}(x)$. Thus without loss of generality we can write,

$$\eta^2\zeta^2 = x^e \prod_i (x - c_i)^{e_i} \text{ where } c_i \in \mathbb{C} \text{ and } e, e_i \in \mathbb{Z}.$$

We define, $\theta = \frac{(\eta^2\zeta^2)'}{2\eta^2\zeta^2}$. By some calculation we get, $\theta = \frac{1}{2}ex^{-1} + \cdots$ (where the dots represent the terms having non negative exponent of x). Also by some calculations one can prove the following relation:

$$\theta'' + 3\theta'\theta + \theta^3 = 4r\theta + 2r'$$

Since $r \in \mathbb{C}(x)$, we can write $r = \alpha x^\nu + \cdots$ where $\alpha \neq 0, \nu \in \mathbb{Z}$. We substitute the Lorentz series expansion of r at 0 in the above relation. If $\nu > -2$ we get, $e - \frac{6}{8}e^2 + \frac{1}{8}e^3 = 0$ i.e. $e = 0$ or 2 or 4 which is a contradiction to the fact that e is odd. If $\nu \leq -2$ then, $e + \nu = 0$ which implies ν is odd. This completes the proof of Case 2.

Case 3: By Case 3 of theorem **??**, every solution of the DE is algebraic over $\mathbb{C}(x)$. Let η be a solution of the DE. Thus, η is algebraic over $\mathbb{C}$. Since the algebraic closure of field of Lorentz series over $\mathbb{C}$ is the field of Puiseaux series over $\mathbb{C}$, η has a Puiseaux series expansion around any point in $\mathbb{C} \cup \{\infty\}$. We can write Puiseaux series of η around 0 (say $\eta = ax^{\mu} + \cdots$ with $\mu \in \mathbb{Q}$), and Lorentz series of r around 0 (say $r = \alpha x^{\nu} + \cdots$ with $\nu \in \mathbb{Z}$.) Now, from the differential equation $y'' = ry$ we have

$$\mu(\mu - 1)ax^{\mu-2} + \cdots = \alpha a x^{\nu+\mu} + \cdots .$$

Thus $\nu \geq -2$, i.e. r has no pole of order greater than 2 . If $\nu = -2$ then the terms shown must cancel so $\mu(\mu - 1) = \alpha$, which implies that $\sqrt{1 + 4\alpha_i} \in \mathbb{Q}$. We can also write Puiseaux series of η around ∞ (say $\eta = ax^{\mu} + \cdots$ with $\mu \in \mathbb{Q}$), and Lorentz series of r around ∞ (say $r = \alpha x^{\nu} + \cdots$ with $\nu \in \mathbb{Z}$.) Now, from the DE $y'' = ry$ we have

$$\mu(\mu - 1)ax^{\mu-2} + \cdots = \alpha a x^{\nu+\mu} + \cdots .$$

Thus $\nu \leq -2$, i.e. r has the form

$$r = \sum_i \frac{\alpha_i}{(x - c_i)^2} + \sum_j \frac{\beta_j}{x - d_j}$$

$$r = (\sum_j \beta_j)x^{-1} + \gamma x^{-2} + \cdots \text{ with } \gamma = \sum_i \alpha_i + \sum_j \beta_j d_j$$

Using the DE,we get $\sum_j \beta_j = 0$ and $\mu(\mu - 1) = \gamma$. Thus, $\sqrt{1 + 4\gamma} \in \mathbb{Q}$. This completes the proof of the theorem.

□

Examples:

1. Airey's equation

$$y'' = xy$$

has no Liouvillian solution. This happens because necessary conditions fail for cases 1,2 and 3.

2. In Bessel's equation

$$y'' = \frac{4(n^2 - x^2) - 1}{4x^2} y \text{ where } n \in \mathbb{C}$$

only cases $1, 2$ and 4 are possible.

Chapter 5

Algorithm

This chapter shall explain the algorithm given by Kovacic[**?**] for solving second order linear homogeneous differential equations of type $y'' = ry$.

Algorithm for case 1:

Our aim is to find a solution of the differential equation which is of the form $\eta = Pe^{\int \omega}$, where $P \in \mathbb{C}[x]$ and $\omega \in \mathbb{C}(x)$. We first find out candidates for the partial fraction expansion of ω (using the poles of r). Then we search for P satisfying a known equation.

Let Γ be the set of poles of r.

Step 1. For each $c \in \Gamma \cup \{\infty\}$ we define $[\sqrt{r}]_c \in \mathbb{C}[x]$ and α_c^+, $\alpha_c^- \in \mathbb{C}$ depending on the order of pole of r at c as:

(c_1) If $c \in \Gamma$ is a pole of r of order 1, then we define

$$[\sqrt{r}]_c = 0, \qquad \alpha_c^+ = \alpha_c^- = 1.$$

(c_2) If $c \in \Gamma$ is a pole of r of order 2 , then we define

$$[\sqrt{r}]_c = 0; \qquad \alpha_c^{\pm} = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4b}.$$

where b is the coefficient of $1/(x-c)^2$ in the partial fraction expansion of r.

(c_3) If $c \in \Gamma$ is a pole of r of order $2\nu \geq 4$, then

$$[\sqrt{r}]_c = \frac{a}{(x-c)^\nu} + \cdots + \frac{d}{(x-c)^2}$$

is a part of the Laurent series expansion of $\sqrt{r}$ at c. One can choose any sign of $[\sqrt{r}]_c$ as this doesn't affect our results. We define,

$$\alpha_c^\pm = \frac{1}{2}\left(\pm\frac{b}{a} + \nu\right).$$

where b is the coefficient of $\frac{1}{(x-c)^{\nu+1}}$ in $r - [\sqrt{r}]_c^2$.
(∞_1) If the order of r at ∞ is > 2, then we define

$$[\sqrt{r}]_\infty = 0, \qquad \alpha_\infty^+ = 0, \qquad \alpha_\infty^- = 1.$$

(∞_2) If the order of r at ∞ is 2, then we define,

$$[\sqrt{r}]_\infty = 0$$

We define,

$$\alpha_\infty^\pm = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4b}.$$

where b is the coefficient of $1/x^2$ in the Laurent series expansion of r at ∞.
(∞_3) If the order of r at ∞ is $-2\nu \leq 0$, then

$$[\sqrt{r}]_\infty = ax^\nu + \cdots + d$$

is a part of the Laurent series expansion of $\sqrt{r}$ at ∞. One can choose any sign of $[\sqrt{r}]_c$ as this doesn't affect our results. Then

$$\alpha_\infty^\pm = \frac{1}{2}\left(\pm\frac{b}{a} - \nu\right).$$

where b is the coefficient of $x^{\nu-1}$ in $r - [\sqrt{r}]_\infty^2$.

Remark: In both (c_3) and (∞_3) the required terms can be found by using undetermined coefficients.

Step 2. For each family $s = (s(c), s(\infty))$ ($c \in \Gamma$), where $s(c)$ and $s(\infty)$ are either $+$ or $-$, let

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}.$$

If d is a non-negative integer, then

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

is a candidate for ω. If d is not a non-negative integer, then the family s has to be discarded.

Step 3. For each family from step 2 which hasn't been discarded, we search for a monic polynomial P of degree d with

$$P'' + 2\omega P' + (\omega' + \omega^2 - r)P = 0.$$

If such polynomial is found then $Pe^{\int \omega}$ is a solution of the DE. If not, then Case 1 cannot hold.

Algorithm for case 2

Let Γ be the set of poles of r.

Step 1. For each $c \in \Gamma \cup \{\infty\}$ we define E_c as:

(c_1) If $c \in \Gamma$ is a pole of r of order 1, then

$$E_c = \{4\}.$$

(c_2) If $c \in \Gamma$ is a pole of r of order 2, and if b is the coefficient of $1/(x-c)^2$ in the partial fraction expansion of r, then

$$E_c = \{2 + k\sqrt{1+4b}\} \cap \mathbb{Z}, \qquad k = 0, \pm 2.$$

(c_3) If $c \in \Gamma$ is a pole of r of order $\nu > 2$, then

$$E_c = \{\nu\}.$$

(∞_1) If r has order > 2 at ∞, then

$$E_c = \{0, 2, 4\}.$$

(∞_2) If r has order 2 at ∞, and b is the coefficient of x^{-2} in the Laurent series expansion of r at ∞, then

$$E_\infty = \{2 + k\sqrt{1+4b}\} \cap \mathbb{Z}, \qquad k = 0, \pm 2.$$

(∞_3) If r has order $\nu < 2$ at ∞, then

$$E_c = \{\nu\}.$$

Step 2. Consider the families $s = (e(c), e(\infty))$ $(c \in \Gamma)$, where $e(c) \in E_c$, and at least one of the coordinates is odd. Let

$$d = \frac{1}{2}\left(e_\infty - \sum_{c \in \Gamma} e_c\right).$$

If d is a non-negative integer, retain the family otherwise discard it.
Step 3. For each family retained from step 2 form the rational function

$$\theta = \tfrac{1}{2} \sum_{c \in \Gamma} \frac{e_c}{x - c},$$

and search for a monic polynomial P of degree d such that

$$P''' + 3\theta P'' + (3\theta^2 + 3\theta' - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P = 0.$$

If such a polynomial is found then define $\phi = \theta + P'/P$ and let ω be a solution of

$$\omega^2 + \phi\omega + (\tfrac{1}{2}\phi' + \tfrac{1}{2}\phi^2 - r) = 0.$$

Then $\eta = e^{\int \omega}$ is a solution of the differential equation.

If no such polynomial exists, then case 2 cannot hold.

Algorithm for case 3:

We first apply algorithm for $n = 4$. If no solution is found then we apply for $n = 6$. If no solution is found for $n = 4, 6$ then we apply this algorithm for $n = 12$. Here n is the degree of minimal polynomial of ω over $\mathbb{C}$ where ω satisfies the Ricatti equation.

Let Γ be the set of poles of r.

(c_1) If $c \in \Gamma$ is a pole of r of order 1, then

$$E_c = \{12\}.$$

(c_2) If $c \in \Gamma$ is a pole of r of order 2, and if b is the coefficient of $1/(x-c)^2$ in the partial fraction expansion of r, then

$$E_c = \{6 + \frac{12k}{n}\sqrt{1+4b}\} \cap \mathbb{Z}, \qquad k = 0, \pm 1, \pm 2, \cdots, \pm \frac{n}{2}.$$

(∞) If the Laurent series for r at ∞ is $r = bx^{-2} + \cdots$, where b may be 0, then

$$E_\infty = \{6 + \frac{12k}{n}\sqrt{1+4b}\} \cap \mathbb{Z}, \qquad k = 0, \pm 1, \pm 2, \cdots, \pm \frac{n}{2}.$$

Step 2. Consider families $s = (e(c), e(\infty))$ $(c \in \Gamma)$, where $e(c) \in E_c$. Let

$$d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right).$$

If d is a non-negative integer, retain the family otherwise discard it.

Step 3. For each family retained from step 2 form the rational function

$$\theta = \frac{n}{12}\left(\sum_{c\in\Gamma} \frac{e_c}{x-c}\right),$$

and the polynomial

$$S = \prod_{c \in \Gamma} (x - c).$$

Now search for a monic polynomial P of degree d which satisfies a certain differential equation that we write recursively as

$$\begin{aligned} P_n &= -P \\ P_{i-1} &= -SP_i' + ((n-i)S' - S\theta)P_i - (n-i)(i+1)S^2 r P_{i+1} \\ P_{-1} &= 0 \end{aligned}$$

We use the second formula to compute $P_{n-1}, P_{n-2} \cdots P_{-1}$ and compare the result with last equation.

If such a monic polynomial is found then for a solution ω of

$$\sum_{i=0}^{n} \frac{S^i P}{(n-i)!} \omega^i = 0.$$

we get $\eta = e^{\int \omega}$ as a solution of the differential equation.

If no such monic polynomial exists, then case 3 cannot hold.

Chapter 6

Examples

Here we consider some differential equations and try to solve them using the algorithm mentioned in the previous chapter. For a differential equation of type $y'' + ay' + by = 0$, we first reduce it using lemma **??**. Then we apply the algorithm from the previous chapter. Consider the linear differential equation $y'' = ry$ where

$$\begin{aligned} r &= \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4} \\ &= x^2 - 2x + 3 + \frac{1}{x} + \frac{7}{4x^2} - \frac{5}{x^3} + \frac{1}{x^4}. \end{aligned}$$

The only pole of r is at 0 of order 4. By theorem **??**, cases 2 and 3 are not possible so we only need to apply the algorithm of case 1. Thus by using (c_3) we have

$$[\sqrt{r}]_0 = \frac{1}{x^2}.$$

Also $b = -5$ and therefore

$$\alpha_0^+ = -3/2 \qquad \alpha_0^- = 7/2.$$

Thus only four families need to be considered in step 2.

$$\begin{aligned}
&s(0) = +, \quad s(\infty) = +, \quad d = \tfrac{1}{2} - (-\tfrac{3}{2}) = 2 \\
&s(0) = +, \quad s(\infty) = -, \quad d = -\tfrac{3}{2} - (-\tfrac{3}{2}) = 0 \\
&s(0) = -, \quad s(\infty) = +, \quad d = \tfrac{1}{2} - \tfrac{7}{2} = -3 \\
&s(0) = -, \quad s(\infty) = -, \quad d = -\tfrac{3}{2} - \tfrac{7}{2} = -5
\end{aligned}$$

In step 2 we consider the first two of these only and get:

$$\begin{aligned}
d = 2, \quad \omega &= +[\sqrt{r}]_0 + \frac{\alpha_0^+}{x} + [\sqrt{r}]_\infty = \frac{1}{x^2} - \frac{3}{2x} + x - 1 \\
d = 0, \quad \omega &= +[\sqrt{r}]_0 + \frac{\alpha_0^+}{x} - [\sqrt{r}]_\infty = \frac{1}{x^2} - \frac{3}{2x} - x + 1.
\end{aligned}$$

Now, we first search for a monic polynomial P of degree 2 satisfying

$$P'' + 2\omega P' + (\omega' + \omega^2 - r)P = 0.$$

After some calculation, we get

$$P = x^2 - 1,$$

so

$$\begin{aligned}
\eta = Pe^{\int \omega} &= (x^2 - 1)e^{\int (1/x^2 - 3/(2x) + x - 1)} \\
&= x^{-3/2}(x^2 - 1)e^{-1/x + x^2/2 - x}.
\end{aligned}$$

6.1 Airey's and Webers's equations

Airey's equation is $y'' = xy$. Since there are no poles ($\Gamma = \emptyset$), and the order at ∞ is -1, by theorem **??**, cases 1 ,2 and 3 are not possible. So this equation has no Liouvillian solution. Thus by theorem **??**, we obtain that the Galois group of this DE is $SL\,(2, \mathbb{C})$.

More generally, we consider the case where r is a polynomial of degree two:

$$y'' = \left((Ax+B)^2 + C\right) y.$$

There are no poles and the order at ∞ is -2, clearly cases 2 and 3 of theorem **??** are not possible. Hence we need to follow the algorithm for case 1. We find that

$$\begin{aligned} [\sqrt{r}]_\infty &= Ax + B \\ \alpha_\infty^{\pm} &= \tfrac{1}{2}\left(\pm(C/A) - 1\right) \\ d &= \alpha_\infty^{+} \quad \text{or} \quad \alpha_\infty^{-}. \end{aligned}$$

If C/A is not an odd integer then d cannot be an integer so case 1 cannot hold. So this linear differential equation has no Liouvillian solutions. If C/A is an odd integer then one can complete steps 2 and 3 and find a solution. So by theorem **??**, we obtain that the Galois group of this DE can be put in simultaneous triangular form if C/A is an odd integer and is $SL\,(2,\mathbb{C})$ if C/A is not an odd integer.

A special case of this is Weber's equation

$$y'' = (\tfrac{1}{4}x^2 - \tfrac{1}{2} - n)y, \quad n \in \mathbb{C}.$$

Here $A = -1/2, B = 0, C = -1/2 - n$. Thus $C/A = 2n + 1$ is an odd integer if and only if n is an integer. Hence, Weber's equation has Liouvillian solutions iff n is an integer.

6.2 Bessel's equation

Consider the Bessel's equation

$$y'' = \left(\frac{4n^2 - 1}{4x^2} - 1\right) y$$

where $n \in \mathbb{C}$. The only pole is at $c = 0$ of order 2. Clearly, case 3 of theorem **??** can not hold. So we need to apply algorithm for cases 1 and 2. First we apply algorithm for case 1. Thus

$$[\sqrt{r}]_0 = 0, \quad b = \frac{4n^2 - 1}{4}, \quad \alpha_0^{\pm} = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1 + 4b} = \tfrac{1}{2} \pm n.$$

At ∞ the order is 0 and $[\sqrt{r}]_\infty = i (= \sqrt{-1})$. Also $b = 0$ and $\alpha_\infty^{\pm} = 0$.

There are four families to consider.

$$\begin{aligned} s(0) &= +, \quad s(\infty) = +, \quad d = \tfrac{1}{2} - n \\ s(0) &= +, \quad s(\infty) = -, \quad d = -\tfrac{3}{2} - n \\ s(0) &= -, \quad s(\infty) = +, \quad d = \tfrac{1}{2} - n \\ s(0) &= -, \quad s(\infty) = -, \quad d = -\tfrac{3}{2} - n \end{aligned}$$

If n is not half an odd integer then d cannot be a non-negative integer and case 1 cannot hold.

If n is half an odd integer then case 1 of the algorithm can be carried out and one can find a Liouvillian solution which comes out to be

$$\eta = \left(\sum_{j=0}^{m} \frac{1}{(-2i)^{m-j}} \frac{(2m - j)!}{j!(m - j)!} x^j \right) e^{\int -\frac{m}{x} + i}.$$

Now, as Case 2 is also possible we apply the algorithm for Case 2 when n is not half an odd integer. As order of pole of r at 0 is 2, we get

$$E_0 = \{2, 2 + 4n, 2 - 4n\}$$

If $4n \notin \mathbb{Z}$ then we only need to consider

$$e_0 = 2 \qquad e_\infty = 0 \qquad d = -1 \notin \mathbb{N} \cup \{0\}$$

Thus here Case 2 can not hold.

If $4n \in \mathbb{Z}$, then we need to consider

$$e_0 = 2 + 4n \qquad e_\infty = 0 \qquad d = -1 - 2n$$
$$e_0 = 2 - 4n \qquad e_\infty = 0 \qquad d = -1 + 2n$$

If $d \in \mathbb{N} \cup \{0\}$, then n is half an integer. Hence n has to be half an even integer. Thus $n \in \mathbb{Z}$ which implies that e_0 and e_∞ are even. Thus Case 2 can not happen. Thus by theorem **??**, we obtain that the Galois group of this DE can be put in simultaneous triangular form when n is half an odd integer and is $SL(@, \mathbb{C})$ when n is not half an odd integer.

6.3 Example for Case 2

Consider the differential equation $y'' = ry$ where $r = \frac{1}{x} - \frac{3}{16x^2}$. Clearly, Cases 1 and 3 of theorem **??** do not hold. Thus we only need to apply the algorithm for Case 2.

As order of pole of r at 0 is 2, we get $E_0 = \{1, 2, 3\}$ As order of r at ∞ is 1, we get $E_\infty = \{1\}$ We only need to consider the cases,

$$e_0 = 1 \qquad e_\infty = 1 \qquad d = 0 \in \mathbb{N} \cup \{0\}$$
$$e_0 = 2 \qquad e_\infty = 1 \qquad d = -\frac{1}{2} \notin \mathbb{N} \cup \{0\}$$
$$e_0 = 3 \qquad e_\infty = 1 \qquad d = -1 \notin \mathbb{N} \cup \{0\}$$

Thus we only need to consider the first case here. Thus we get

$$\theta = \frac{1}{2x}$$

By the algorithm we need to find a monic polynomial P of degree 0 satisfying the equation for Case 2. Clearly, $P = 1$ satisfies iff

$$\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r' = 0$$

Thus $\phi = \theta = \frac{1}{2x}$. And hence $\omega = \frac{1}{4x} \pm \frac{1}{\sqrt{x}}$. As a result, we have $\eta = e^{\int \omega}$ as solutions of the differential equation. By theorem **??**, we obtain that the Galois group of this DE can not be put in triangular form and is conjugate to a subgroup of

$$D^\dagger = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} : c \in \mathbb{C}, c \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} : c \in \mathbb{C}, c \neq 0 \right\}.$$

6.4 Example for Case 3

Consider the differential equation $y'' = ry$ where $r = -\frac{3}{16x^2} - \frac{2}{9(x-1)^2} + \frac{3}{16x(x-1)}$. All cases of theorem **??** are possible. So we consider Case 1 and get

$$\alpha_0^+ = \frac{3}{4} \qquad \alpha_0^- = \frac{1}{4}$$
$$\alpha_1^+ = \frac{2}{3} \qquad \alpha_1^- = \frac{1}{3}$$
$$\alpha_\infty^+ = \frac{2}{3} \qquad \alpha_\infty^- = \frac{1}{3}$$

and $d = \alpha_\infty^\pm - \alpha_0^\pm - \alpha_1^\pm$ can never be a non-negative integer. Thus Case 1 fails. By applying the algorithm for Case 2, we get

$$E_0 = \{2, 3, 1\} \qquad E_1 = \{2\} \qquad E_\infty = \{2\}$$

In this case too, $d = e_\infty - e_0 - e_1$ can never be a non-negative integer. Thus Case 2 fails.
Now we apply the algorithm for Case 3. We get,

$$E_0 = \{3, 4, 5, 6, 7, 8, 9\} \qquad E_1 = \{4, 5, 6, 7, 8\}$$

Now we go to step 2 and calculate d using the relation $d = e_\infty - e_0 - e_1$. Only the following possibilities remain

$e_\infty = 7$	$e_0 = 3$	$e_1 = 4$	$d = 0$
$e_\infty = 8$	$e_0 = 3$	$e_1 = 4$	$d = 1$
$e_\infty = 8$	$e_0 = 3$	$e_1 = 5$	$d = 0$
$e_\infty = 8$	$e_0 = 4$	$e_1 = 4$	$d = 0$

Now we consider the first possibility, and get,

$$\theta = \frac{3}{x} + \frac{4}{x-1} \qquad S = x^2 - x$$

Now we need to check whether $P = 1$ satisfies $(\#)_{12}$. One can easily write a SAGE program to compute P_i where $i = 12, 11, \cdots - 1$. And in the end one shall get $P_{-1} = 0$. Therefore $\eta = e^{\int \omega}$ is a solution of the differential equation where ω satisfies $\sum_{i=0}^{12} \frac{(x^2-x)^i P_i}{(12-i)!}$. By using some program to factorize a polynomial we obtain that $\sum_{i=0}^{12} \frac{(x^2-x)^i P_i}{(12-i)!}$ is the cube of a polynomial. So by some results (which shall be proven later in Chapter 7), we obtain that the Galois group of this DE is the tetrahedral group.

Chapter 7

Proofs

This chapter contains the proof of correctness of the algorithm given by Kovacic [?]. We complete the proof case-wise in different sections of this chapter.

7.1 Case 1

By theorem **??**, the differential equation has a solution $\eta = e^{\int \theta}$ where $\theta \in \mathbb{C}(x)$. Since η satisfies $y'' = ry$, θ satisfies the Ricatti equation,

$$\theta' + \theta^2 = r$$

For $c \in \mathbb{C}$, we write

$$\theta = [\theta]_c + \frac{\alpha}{x-c} + \bar{\theta}_c$$

where $[\theta]_c$ is the component of partial fraction expansion of θ at c and $\bar{\theta}_c \in \mathbb{C}[x]$. For simplicity of notation, we assume $c = 0$ and drop the subscript 0. Thus,

$$\theta = [\theta] + \frac{\alpha}{x} + \bar{\theta}$$

We assume that necessary conditions for case 1 in theorem **??** hold. Now we shall prove that the algorithm for the case 1 is correct using the same steps as in the algorithm. Our aim is to find out α and $[\theta]$.

$(c1)$ Let the order of pole of r at 0 be 1. Therefore we can write,

$$r = \frac{*}{x} + \cdots .$$

On substituting in the Ricatti equation we get,

$$-\frac{\nu a_\nu}{x^{\nu+1}} + \cdots + \frac{a_\nu^2}{x^{2\nu}} + \cdots = \frac{*}{x} + \cdots .$$

If $\nu > 1$,then $2\nu > \nu + 1$ and since $a_\nu^2 \neq 0$, we get a contradiction to the above equality. Thus, $\nu \leq 1$. Hence $[\theta] = 0$. Now,we substitute $\theta = \frac{\alpha}{x} + \bar{\theta}$ in the Ricatti equation and get,

$$-\frac{\alpha}{x^2} + \bar{\theta}' + \frac{\alpha^2}{x^2} + \frac{2\alpha\bar{\theta}}{x} + \bar{\theta}^2 = \frac{*}{x} + \cdots .$$

It clearly implies, $\alpha^2 - \alpha = 0$. If $\alpha = 0$ then 0 is not a pole of left hand side of above equation and a pole of r, which is not possible. So, $\alpha = 1$. Hence,we get

$$\theta = \frac{1}{x} + \bar{\theta}.$$

$(c2)$ Let the order of pole of r at 0 be 2.Therefore we can write,

$$r = \frac{b}{x^2} + \frac{*}{x} + \cdots .$$

One can follow the similar reasoning as in $(c1)$ and get $\bar{\theta} = 0$ and $\alpha^2 - \alpha = b$. Thus we get,

$$\theta = \frac{\alpha^{s(0)}}{x} + \bar{\theta} \text{ where } \alpha^{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4b} \text{ and } s(0) = + \text{ or } - .$$

$(c3)$ Let the order of pole of r at 0 be $2\mu \geq 4$. In the proof of Case 1 of theorem **??**, we proved $\mu = \nu$. Then,

$$[\sqrt{r}] = \frac{a}{(x)^\nu} + \cdots + \frac{*}{(x)^2}$$

is a part of the Laurent series expansion of $\sqrt{r}$ at 0. Let $\bar{r} = \sqrt{r} - [\sqrt{r}]$. Using $r = 2\bar{r}[\sqrt{r}] + \bar{r}^2 + [\sqrt{r}]^2$, Ricatti equation and $\theta = [\theta] + \frac{\alpha}{x} + \bar{\theta}$ we get,

$$\left([\theta] - [\sqrt{r}]\right)\left([\theta] + [\sqrt{r}]\right) = -[\theta]' + \frac{\alpha}{x^2} - \bar{\theta}' - \frac{2\alpha}{x}[\theta] - 2\bar{\theta}[\theta] - \frac{\alpha^2}{x^2} - \frac{2\alpha}{x}\bar{\theta} - \bar{\theta}^2 + 2\bar{r}[\sqrt{r}] + \bar{r}^2$$

Clearly, the coefficients of $\frac{1}{x^i}$ for $i = \nu + 2, \nu + 3 \cdots, 2\nu$ are zero on the right hand side. Also, since at-least one of the factors of left hand side involves $\frac{1}{x^\nu}$, if the other factor were non-zero then there exists a highest $i < \nu$ such that the coefficient of $\frac{1}{x^i}$ is non-zero in the other factor which implies that the coefficient of $\frac{1}{x^{\nu+i}}$ in the product is non-zero which is not possible, so left hand side must be 0. Hence, $[\theta] = \pm[\sqrt{r}]$.

Let b be the coefficient of $\frac{1}{x^{\nu+1}}$ in $r - [\sqrt{r}]^2$. Now, the coefficient of $\frac{1}{x^{\nu+1}}$ on the right hand side of the main equation in this case is $\pm\nu a \mp 2\alpha a + b$. Hence $\alpha^{\pm} = \frac{1}{2}\left(\pm\frac{b}{a} + \nu\right)$. ($c4$) If the order of pole of r at 0 is 0 then by ($c1$), we have $[\theta] = 0$ and $-\alpha + \alpha^2 = 0$. Thus, the component of partial fraction expansion of r is either 0 or $\frac{1}{x}$.

Until now we have proven,

$$\theta = \sum_{c \in \Gamma}\left(s(c)\,[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + \sum_{i=1}^{d}\frac{1}{x - d_i} + R$$

where $s(c) = +$ or $-$ and $R \in \mathbb{C}[x]$.

Now we consider the Laurent series at ∞. Let

$$\theta = R + \frac{\alpha_\infty}{x} + \cdots$$

($\infty 1$) Let the order of r at ∞ be $\nu > 2$. By Ricatti equation, we get $R = 0$ and $\alpha_\infty = 0$ or 1.

($\infty 2$) Let the order of r at ∞ be 2. Therefore we can write,

$$r = \frac{b}{x^2} + \frac{*}{x^3} + \cdots .$$

One can follow the similar reasoning as in ($c1$) and get $R = 0$ and $\alpha_\infty^2 - \alpha_\infty = b$.
($\infty 3$) Let the order of r at ∞ be $-2\nu < 2$. By using the similar arguments as in ($c3$) we get, $R = \pm[\sqrt{r}]_\infty$ and $\alpha_\infty^\pm = \frac{1}{2}\left(\pm\frac{b}{a} - \nu\right)$ where a is the leading coefficient of $[\sqrt{r}]_\infty$ and b is the coefficient of $\frac{1}{x^{\nu-1}}$ in $r - [\sqrt{r}]_\infty^2$.
Now we know,

$$\theta = \sum_{c\in\Gamma}\left(s\left(c\right)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + \sum_{i=1}^{d}\frac{1}{x-d_i} + s\left(\infty\right)[\sqrt{r}]_\infty$$

Now, the coefficient of $\frac{1}{x}$ in Laurent series expansion of θ around ∞ is $\alpha_\infty^{s(\infty)}$. Thus, $\alpha_\infty^{s(\infty)} = d + \sum_{c\in\Gamma}\alpha_c^{s(c)}$. As $d \in \mathbb{N}\cup 0$, we have $\alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma}\alpha_c^{s(c)} \in \mathbb{N}\cup 0$.
Now, let $\omega = \sum_{c\in\Gamma}\left(s\left(c\right)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s\left(\infty\right)[\sqrt{r}]_\infty$ and $P = \prod_{i=1}^{d}\left(x - d_i\right)$.Thus, $\theta = \omega + \frac{P'}{P}$. As θ satisfies Ricatti equation we get,

$$P'' + 2\omega P' + \left(\omega' + \omega^2 - r\right)P$$

Now one can easily verify that if P is a solution of above equation then θ satisfies the Ricatti equation and $\eta = e^{\int\theta}$ satisfies the differential equation.

7.2 Case 2

Let $G \subset D^\dagger$ be the differential Galois group of the differential equation and η, ζ be a fundamental system of solutions of corresponding to G. As proven in theorem **??**

$\eta^2\zeta^2 \in \mathbb{C}(x)$ and $\eta\zeta \notin \mathbb{C}(x)$. Let Γ be the set of poles of r. We can write,

$$\eta^2\zeta^2 = g\prod_{c\in\Gamma}(x-c)^{e_c}\prod_{i=1}^{m}(x-d_i)^{f_i}$$

where $e_i, f_i \in \mathbb{Z}, g \in \mathbb{C}$. Now, let $\Phi = \frac{(\eta\zeta)'}{\eta\zeta} = \frac{1}{2}\frac{(\eta^2\zeta^2)'}{\eta^2\zeta^2} = \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} + \frac{1}{2}\sum_{i=1}^{m}\frac{f_i}{x-d_i}$. Since, $\Phi = \frac{\eta'}{\eta} + \frac{\zeta'}{\zeta}$ and η, ζ are solutions of the differential equation one can easily prove,

$$\Phi'' + 3\Phi\Phi' + \Phi^3 = 4r\Phi + 2r' \tag{7.1}$$

For simplification of notation, we shall assume $c = 0$.

(c_1) Let order of r at 0 be 1. Also let the Laurent series expansion of r and Φ around 0 be as follows,

$$r = \alpha x^{-1} + \cdots (\alpha \neq 0)$$

$$\Phi = \frac{1}{2}ex^{-1} + f + \cdots (e \in \mathbb{Z}, f \in \mathbb{C})$$

On substuting in **??** and comparing coefficients of x^{-3} and x^{-2} on both sides we get, $e - \frac{3}{4}e^2 + \frac{1}{8}e^3 = 0$ and $-\frac{3}{2}ef + \frac{3}{4}e^2 f = 2\alpha e - \alpha$. On using the fact that $\alpha \neq 0$, we get $e = 4$.

(c_2) Let the order of pole of r at 0 be 2. Also let the Laurent series expansion of r and Φ around 0 be as follows,

$$r = bx^{-2} + \cdots (b \neq 0)$$

$$\Phi = \frac{1}{2}ex^{-1} + f + \cdots (e \in \mathbb{Z}, f \in \mathbb{C})$$

On substuting in **??** and comparing coefficients of x^{-3} on both sides we get,

$$e - \frac{3}{4}e^2 + \frac{1}{8}e^3 = 2eb - 4b$$

Thus $e = 2, 2 \pm 2\sqrt{1+4b}$ and also since $e \in \mathbb{Z}$, any non-integral value of e must be discarded.

(c_3) Let the order of pole of r at 0 be $\nu > 2$. Also let the Laurent series expansion of r and Φ around 0 be as follows,

$$r = x^{-\nu} + \cdots (b \neq 0)$$

$$\Phi = \frac{1}{2}ex^{-1} + f + \cdots (e \in \mathbb{Z}, f \in \mathbb{C})$$

On substuting in **??** and comparing coefficients of $x^{-\nu-1}$ on both sides we get,$2\alpha e - 2\alpha\nu = 0$. Hence $e = \nu$.

Now to determine f_i we use the same calculation as in (c_1) (replacing α by 0 as d_i is not a pole of r) and get $f_i = 0, 2$ or 4. Thus we can write,

$$\eta^2 = \prod_{c\in\Gamma} (x-c)^{e_c} P^2$$

where $e_c \in E_c$ and $P \in \mathbb{C}[x]$. Now we set $\theta = \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c}$. Thus $\Phi = \theta + \frac{P'}{P}$.
Now, we need to find degree d of P. Let the Laurent series expansion of Φ around ∞ be,

$$\Phi = \frac{1}{2}e_\infty x^{-1} + f + \cdots (e \in \mathbb{Z}, f \in \mathbb{C})$$

By using **??**, we get, $e_\infty = \sum_{c\in\Gamma} e_c + 2d$.

(∞_1) Let order of r at ∞ be 1. By the same steps as in (c_1) we get, $e_\infty = 0, 2$ or 4.
(∞_2) Let order of r at ∞ be 2. Also let the Laurent series expansion of r around ∞ be as follows,

$$r = bx^{-2} + \frac{*}{x^{-3}} \cdots (b \neq 0)$$

By the same steps as in (c_2) we get, $e_\infty = 2, 2 \pm 2\sqrt{1+4b}$. and also since $e_\infty \in \mathbb{Z}$, any non-integral value of e_∞ must be discarded. (∞_3) Let order of r at ∞ be $\nu < 2$. As in (c_3), we get, $e_\infty = \nu$.
Also since $\eta\zeta \notin \mathbb{C}(x)$, at-least one of the e_c must be odd.

By using **??** and $\Phi = \theta + \dfrac{P'}{P}$ we get,

$$P''' + 3\theta P'' + (3\theta^2 + 3\theta' - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P = 0.$$

As above equation is linear homogeneous, it has a monic polynomial solution if and only if it has a polynomial solution. Now, suppose ω is a root of

$$\omega^2 + \phi\omega + (\tfrac{1}{2}\phi' + \tfrac{1}{2}\phi^2 - r) = 0. \tag{7.2}$$

Then we only need to prove $\eta = e^{\int \omega}$ is a solution of the differential equation.
On differentiating **??**, we get,

$$(2\omega - \Phi)\,\omega' = \Phi'\omega - \frac{1}{2}\Phi'' - \Phi\Phi' + r'$$

If $2\omega - \Phi = 0$ then $\omega' + \omega^2 - r = 0$ (from **??**). Thus $\eta = e^{\int \omega}$ is a solution of the differential equation where $\omega = \dfrac{\Phi}{2} \in \mathbb{C}(x)$ which clearly happens in Case 1. Thus, $2\omega - \Phi \neq 0$.
By using **??** and **??** we get,

$$2\,(2\omega - \Phi)\left(\omega' + \omega^2 - r\right) = -\Phi'' - 3\Phi\Phi' - \Phi^3 + 4r\Phi + 2r' = 0$$

Thus, $\omega' + \omega^2 - r = 0$ and hence, $\eta = e^{\int \omega}$ is a solution of the differential equation. this completes the proof of correctness of algorithm for Case 2.

7.3 Case 3

7.3.1 Finite subgroups of $SL(2, \mathbb{C})$

Theorem 7.1. *For any finite subgroup G of $SL_2(\mathbb{C})$,either*

1. *G is conjugate to a subgroup of $D^\dagger = D \cup \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} D$ where D is the group of all diagonal matrices in $SL_2(\mathbb{C})$, or*

2. G has order 24 (the "tetrahedral" case), or

3. G has order 48 (the "octahedral" case), or

4. G has order 120 (the "icosahedral" case).

Proof. Let case 1 does not happen. Let H be the set of all scalar matrices in G. Thus $|H| = 1$ or 2. Choose $x \in G\backslash H$. We denote the centralizer of x in G by $Z(x)$ and normalizer of $Z(x)$ in G by $N(x)$. Since order of x is finite, x has to be diagonalisable. Also a simple computation shows that any non-scalar diagonal matrix in $SL_2(\mathbb{C})$ has D as its centralizer in $SL_2(\mathbb{C})$. Thus $Z(x) = G \cap g^{-1}Dg$ for some $g \in SL_2(\mathbb{C})$. Let $y \in G$ be such that $y = h^{-1}dh$ for some $d \in D$. Then $y \in Z(x)$ implies that $Z(y) = G \cap h^{-1}Dh$. Thus, $Z(x) = Z(y)$ iff $y \in Z(x)$. Also,$Z(g^{-1}xg) = g^{-1}Z(x)g$.
For $x, y, g, h \in G$ either $Z(g^{-1}xg) \cap Z(h^{-1}yh) = H$ or $Z(g^{-1}xg) = Z(h^{-1}yh)$. (Hint:Let $x' = g^{-1}xg$ and $y' = h^{-1}yh$.
Clearly, $Z(x') \cap Z(y') \supset H$. Let $a \in Z(x') \cap Z(y')) - H$. On computation, we get $x'y' = y'x'$ i.e. $y' \in Z(x')$. Thus $Z(x') = Z(y')$.) From above fact,in the latter case we get $y \in Z(hg^{-1}xgh^{-1})$. Also, $Z(g^{-1}xg) = Z(h^{-1}yh)$ iff $gN(x) = hN(x)$. Also for $k \in G - H$, the order of k has to be greater than 2. Thus for $y \in G$ there exists $x_i \in G - H$ and $g \in G$ such that $y \in g^{-1}Z(x_i)g$. Hence we can break G as

$$G = H \cup \left(\bigcup_{i=1}^{s} \bigcup_{gN(x_j) \in \frac{G}{N(x_j)}} (gZ(x_i)g^{-1} - H) \right)$$

where $s \in \mathbb{N}$ and $x_1, x_2, \cdots, x_s \in G - H$.

A simple computation shows that the only matrices in $SL_2(\mathbb{C})$ which conjugate a diagonal matrix to a diagonal matrix are the elements of $D^\dagger$. Thus $N(x_i) = G \cap g^{-1}D^\dagger g$ for some $g \in SL_2(\mathbb{C})$ Also, $[N(x_i) : Z(x_i)] = 1$ or 2.
Let $M = [G : H]$ and $e_i = [Z(x_i) : H]$. By the representation of G as a disjoint union, we get

$$M|H| = |H| + \sum_{i=1}^{s} [G : N(x_i)] \left(e_i|H| - |H|\right)$$

$$M = 1 + \sum_{i=1}^{s} \frac{M}{[N(x_i) : Z(x_i)]\, e_i} (e_i - 1)$$

$$\frac{1}{M} = 1 + \sum_{i=1}^{s} \frac{1}{[N(x_i) : Z(x_i)]} \left(\frac{1}{e_i} - 1 \right)$$

Clearly $s \neq 0$ since $G \neq H$. Since $x_i \in G - H$, we have order(x)¿2 which in turn implies that $e_i \geq 2$.

If $s = 1$ then

$$\frac{1}{M} \geq \frac{1}{[N\,(x_i) : Z\,(x_i)]\, e_1} = \frac{1}{|\frac{N(x_1)}{H}|}$$

Thus $G = N\,(x_1)$. This is a contradiction to the fact that Case 1 does not happen. Now we have,

$$0 < \frac{1}{M} \leq \frac{1}{2} \sum_{i=1}^{s} \frac{1}{[N(x_i) : Z(x_i)]}$$

Thus,

$$\textstyle\sum_{i=1}^{s} \frac{1}{[N(x_i) : Z(x_i)]} < 2$$

Now as, $[N(x_i) : Z(x_i)] = 1$ or 2, only following three cases are possible:

$s = 2$ $\qquad [N(x_1) : Z(x_1)] = 2, [N(x_2) : Z(x_2)] = 2$

$s = 2$ $\qquad [N(x_1) : Z(x_1)] = 1, [N(x_2) : Z(x_2)] = 2$

$s = 3$ $\qquad [N(x_1) : Z(x_1)] = 2, [N(x_2) : Z(x_2)] = 2, [N(x_3) : Z(x_3)] = 2.$

As $[N(x_2) : Z(x_2)] = 2$ (for all above cases), G contains a matrix whose square is $-I$. So $|H| = 2$. Now, if $s \geq 2$ then $M > 2e_i$ (because $M = 2e_i$ implies $N(x_i) = G$ which is a contradiction to the fact that G is not conjugate to a subgroup of $D^{\dagger}$). So by first solution we get the equation:

$$\frac{1}{M} = \frac{1}{2e_1} + \frac{1}{2e_2}$$

which has no solution as existence of any solution would contradict $M > 2e_1$. Now, by the second solution, we get the equation:

$$\frac{1}{M} = \frac{1}{e_1} + \frac{1}{2e_2} - \frac{1}{2}$$

whose only solution is $e_1 = 3, e_2 = 2, M = 12$ (because $e_1 \geq 3$) . By the third solution, we get the equation:

$$\frac{2}{M} = \frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} - 1$$

Without loss of generality, we can assume $e_1 \leq e_2 \leq e_3$. As

$$\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} - 1 > 0$$

we get, $e_1 < 3$, so $e_1 = 2$ and

$$\frac{2}{M} = \frac{1}{e_1} + \frac{1}{e_2} - \frac{1}{2}$$

Also $e_2 = 3$ Thus the solutions are:
$e_1 = 2, \quad e_2 = 3, \quad e_3 = 3, \quad M = 12, \quad |G| = 24$
$e_1 = 2, \quad e_2 = 3, \quad e_3 = 4, \quad M = 24, \quad |G| = 48$
$e_1 = 2, \quad e_2 = 3, \quad e_3 = 5, \quad M = 60, \quad |G| = 120$
This completes the proof. □

Lemma 7.2. *Let G be a finite subgroup of $SL(2, \mathbb{C})$ such that G is not conjugate to a subgroup of $D^\dagger$. Let $H = \{1, -1\}$. Then G/H has no cyclic subgroup which is normal in G/H .*

Proof. Let xH be a normal cyclic subgroup of G/H. Thus clearly, the sub-group of G generated by x and $-x$ say K is diagonalizable. Since K has to be normal in G and thus $G = N(x)$ and hence G must be a conjugate of a subgroup of $D^\dagger$. The contradiction completes the proof. □

Theorem 7.3. *Let G be a subgroup of $SL(2, \mathbb{C})$ of order 24 such that G is not conjugate to a subgroup of $D^\dagger$. Let $H = \{1, -1\}$. Then $G/H \simeq A_4$. Moreover, G is conjugate* *to a group generated by*

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad , \quad \phi \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

where $\phi = 2\xi - 1$ *and* ξ *is a primitive* 6^{th} *root of unity.*

Proof. Since $|G/H| = 12$ and G/H has no normal cyclic subgroups, G/H must have exactly 4 sylow-3-subgroups. Also, G/H acts on the set of sylow-3-subgroups (say X) by conjugation. Thus we get a homomorphism $\Phi : G/H \longrightarrow S_4$. Let K_i be the set of elements of G/H which fixes the i^{th} sylow-3-subgroup. Since G/H acts transitively on X, we get $|K_i| = |K_j|\ \forall i$ and thus $|Im\,(\Phi)\,|$ is divisible by 4. Therefore, the only possible values for $|ker\,(\Phi)\,|$ are 1 or 2 or 3. By using previous lemma, we get $|ker\,(\Phi) = 1$. Now by the composition of signature homomorphism of S_4 and Φ, we get a homomorphism from G/H to $\{1, -1\}$. Since G/H must not have a normal subgroup of order 6, one can easily prove that $signature \circ \Phi$ has trivial image. Hence, $G/H \simeq A_4$.

Let $\tau : G \longrightarrow A_4$ be a homomorphism such that $ker\,(\tau) = H$. Let $A \in \tau^{-1}\,(234)$. By appropriate conjugation one can make sure that A is a diagonal matrix. Therfore without loss of generality we can assume $A = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$. Since $\tau\,(A^3) = 1$, we have that $A^3 \in H$. Also $\tau\,(A)\,, \tau\,(A^2) \neq 1$. Thus we can assume ξ to be a primitive 6^{th} root of unity. Let $B \in \tau^{-1}\,(23)\,(41)$. A simple computation shows that $\tau\,(AB) \neq \tau\,(BA)$. Thus B can not be a diagonal matrix. Hence at-least one of the non-diagonal entries of B is non-zero. Let $B = (B_{ij})$. On conjugating G by $\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ where $c^2 = B_{21}$ and $d^2 = \sqrt{2}B_{12}$ we get that A remains unchanged and B becomes of the form $\begin{pmatrix} \phi & \psi \\ 2\psi & -\chi \end{pmatrix}$. As $\tau(B^2) = 1$, we have that $B^2 \in H$. By direct computation we get $\chi = \phi$. By observation we get $\tau\,(BA^2) = \tau\left((AB)^2\right)$. Thus $BA^2 = \pm\,(AB)^2$. By computation we get $\phi\,(\xi^2 - 1) = \pm\xi^4$ (using $\xi \neq 0$). Replacing B by $-B$ (if needed) we may assume $\phi\,(\xi^2 - 1) = \xi^4$. Thus $3\phi = 2\xi - 1$ (using $\xi^2 = \xi - 1$). By using det(B)= 1 we get $\phi^2 + 2\psi^2 = -1$. And hence $3\psi = \pm\,(2\xi - 1)$. We get $3\psi = (2\xi - 1) = 3\phi$ by conjugating by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (if needed). As (234) and $(23)(41)$ generate A_4, A and B generate G. This completes the proof. This finite subgroup of $SL(2, \mathbb{C})$ is called as the tetrahedral group. □

Theorem 7.4. *Let G be a subgroup of $SL(2, \mathbb{C})$ of order 48 such that G is not conjugate to a subgroup of $D^\dagger$. Let $H = \{1, -1\}$. Then $G/H \simeq S_4$. Moreover, G is conjugate to a group generated by*

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad , \quad \phi \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

where $\phi = \xi\left(\xi^2 + 1\right)$ and ξ is a primitive 8^{th} root of unity.

Proof. Since $|G/H| = 24$ and G/H has no normal cyclic subgroups, G/H must have exactly 4 sylow-3-subgroups. Also, G/H acts on the set of sylow-3-subgroups (say X) by conjugation. Thus we get a homomorphism $\Phi : G/H \longrightarrow S_4$. Let K_i be the set of elements of G/H which fixes the i^{th} sylow-3-subgroup. Since G/H acts transitively on X, we get $|K_i| = |K_j|\ \forall i$ and thus $|Im\,(\Phi)\,|$ is divisible by 4. Therefore, the only possible values for $|ker\,(\Phi)$ are 1, 2, 3 and 6 . By using previous lemma, we get $|ker\,(\Phi)\,| = 1$. Hence, $G/H \simeq S_4$.

Let $\tau : G \longrightarrow S_4$ be a homomorphism such that $ker\,(\tau) = H$. Let $A \in \tau^{-1}(1234)$. By appropriate conjugation one can make sure that A is a diagonal matrix. Therefore without loss of generality we can assume $A = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$. Since $\tau\left(A^4\right) = 1$, we have that $A^4 \in H$. Also $\tau(A), \tau\left(A^2\right), \tau\left(A^3\right) \neq 1$. Thus we can assume ξ to be a primitive 8^{th} root of unity. Let $B \in \tau^{-1}(12)$. A simple computation shows that $\tau(AB) \neq \tau(BA)$. Thus B can not be a diagonal matrix. Hence at-least one of the non-diagonal entries of B is non-zero. Let $B = (B_{ij})$. On conjugating G by $\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ where $c^2 = B_{21}$ and $d^2 = B_{12}$ we get that A remains unchanged and B becomes of the form $\begin{pmatrix} \phi & \psi \\ \psi & -\chi \end{pmatrix}$. As $\tau(B^2) = 1$, we have that $B^2 \in H$. By direct computation we get $\chi = \phi$. By observation we get $\tau(BA^3) = \tau\left((AB)^2\right)$. Thus $BA^2 = \pm(AB)^2$. By computation we get $\phi(\xi^2 - 1) = \pm\xi$ (using $\xi \neq 0$) or equivalently, $2\phi = \pm\xi(\xi^2 + 1)$. Replacing B by $-B$ (if needed) we may assume $2\phi = \xi(\xi^2 + 1)$. Thus $2\phi^2 = -1$. By using det(B)= 1 we get $-\phi^2 - \psi^2 = 1$. And hence $2\psi^2 = -1$. We get $\psi = \phi$ by conjugating by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (if needed). As (1234) and (12) generate S_4, A and B generate G. This completes the proof. This finite subgroup of $SL(2, \mathbb{C})$ is called as the octahedral group. $\square$

Theorem 7.5. *Let G be a subgroup of $SL(2, \mathbb{C})$ of order 120 such that G is not conjugate to a subgroup of $D^\dagger$. Let $H = \{1, -1\}$. Then $G/H \simeq A_5$. Moreover, G is conjugate to a group generated by*

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad , \quad \begin{pmatrix} \phi & \psi \\ \psi & -\phi \end{pmatrix}$$

where $5\phi = 3\xi^3 - \xi^2 + 4\xi - 2$ and $5\psi = 3\xi^3 + 3\xi^2 - 2\xi + 1$ and ξ is a primitive 10^{th} root of unity.

Proof. The proof that G/H is isomorphic to A_5 can be found in Burnside (1955,127,p. 161-2) [?].

Let $\tau : G \longrightarrow A_5$ be a homomorphism such that $ker(\tau) = H$. Let $A \in \tau^{-1}(12345)$. By appropriate conjugation one can make sure that A is a diagonal matrix. Therefore without loss of generality we can assume $A = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$. Since $\tau(A^5) = 1$, we have that $A^5 \in H$. Also $\tau(A), \tau(A^2), \tau(A^3), \tau(A^3) \neq 1$. Thus we can assume ξ to be a primitive 10^{th} root of unity. Let $B \in \tau^{-1}(12)(34)$. A simple computation shows that $\tau(AB) \neq \tau(BA)$. Thus B can not be a diagonal matrix. Hence at-least one of the non-diagonal entries of B is non-zero. Let $B = (B_{ij})$. On conjugating G by $\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ where $c^2 = B_{21}$ and $d^2 = B_{12}$ we get that A remains unchanged and B becomes of the form $\begin{pmatrix} \phi & \psi \\ \psi & -\chi \end{pmatrix}$. As $\tau(B^2) = 1$, we have that $B^2 \in H$. By direct computation we get $\chi = \phi$. By observation we get $\tau(A^4B) = \tau\left((BA)^2\right)$. Thus $A^4B = \pm(BA)^2$. By computation we get $\phi(\xi^3+1) = \pm\xi^4$ or equivalently, $5\phi = \pm(3\xi^3 - \xi^2 + 4\xi - 2)$. Replacing B by $-B$ (if needed) we may assume $5\phi = (3\xi^3 - \xi^2 + 4\xi - 2)$. By using det($B$)= 1 we get $5\psi = \pm(3\xi^3 + 3\xi^2 - 2\xi + 1)$. We get $\psi = (3\xi^3 + 3\xi^2 - 2\xi + 1)$ by conjugating by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (if needed). As (12345) and $(12)(34)$ generate A_5, and since the group generated by A and B contains H we get A and B generate G. This completes the proof. This finite subgroup of $SL(2, \mathbb{C})$ is called as the icosahedral group. □

Theorem 7.6. *Let G be the differential Galois group of the DE $y'' = ry$ and let η, ζ be a fundamental system of solutions of the DE with respect to G. If G is tetrahedral then $(\eta^4 + 8\eta\zeta^3)^3 \in \mathbb{C}(x)$. If G is octahedral then $(\eta^5\zeta - \eta\zeta^5)^2 \in \mathbb{C}(x)$. If G is icosahedral group then $(\eta^{11}\zeta - 11\eta^6\zeta^6 - \eta\zeta^{11}) \in \mathbb{C}(x)$.*

Proof. If G is the tetrahedral group then ξ is a primitive 6^{th} root of unity. Thus, $\xi^2 = \xi - 1$. Also, we have $3\phi = 2\xi - 1$. An easy computation shows that, $(\eta^4 + 8\eta\zeta^3)^3$ is kept fixed by $\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ and $\phi\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$. And thus $(\eta^4 + 8\eta\zeta^3)^3 \in \mathbb{C}(x)$.

If G is the octahedral group then ξ is a primitive 8^{th} root of unity. Thus, $\xi^4 = -1$. Also, we have $2\phi = \xi(\xi^2 + 1)$. An easy computation shows that, $(\eta^5\zeta - \eta\zeta^5)^2$ is kept fixed by $\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ and $\phi\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. And thus $(\eta^5\zeta - \eta\zeta^5)^2 \in \mathbb{C}(x)$.

If G is the icosahedral group then ξ is a primitive 10^{th} root of unity. Thus , $\xi^4 = \xi^3 - \xi^2 + \xi - 1$. Also, $5\phi^2 = \xi^3 - \xi^2 - 3$, $5\psi^2 = -\xi^3 + \xi^2 - 2$ and $5\phi\psi = 2\xi^3 - 2\xi^2 - 1 = 5(\phi^2 - \psi^2)$. An easy computation shows that, $(\eta^{11}\zeta - 11\eta^6\zeta^6 - \eta\zeta^{11})$ is kept fixed by $\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ and $\begin{pmatrix} \phi & \psi \\ \psi & -\phi \end{pmatrix}$. And thus $(\eta^{11}\zeta - 11\eta^6\zeta^6 - \eta\zeta^{11}) \in \mathbb{C}(x)$. □

7.3.2 Proof of correctness of algorithm

We need to prove that the algorithms for finding $4^{th}, 6^{th}$ or 12^{th} degree equations for ω are correct when the Galois group is tetrahedral, octahedral or icosahedral respectively. Also we need to prove that equations obtained for ω are irreducible in respective cases and finally that the algorithm for finding degree 12 equation for ω is also correct (in this case the equation need no be irreducible).

First we prove that the equations of degree 4,6 and 12 for ω in respective cases are irreducible.

Notation: Let G denote the Galois group of the differential equation $y'' = ry$.Since we are assuming case 3 of theorem **??** holds, G must be tetrahedral or octahedral or icosahedral group. Also let η,ζ be a fundamental system of solutions of the differential equation with respect to G. Set $\omega = \dfrac{\eta'}{\eta}$.

Theorem 7.7. *If η_1 satisfies the DE and $\omega_1 = \dfrac{\eta_1'}{\eta_1}$ then $deg_{\mathbb{C}(x)}\omega_1 \geq 4$ when G is tetrahedral group, ≥ 6 when G is octahedral group and ≥ 12 when G is icosahedral group. Moreover, $deg_{\mathbb{C}(x)}\omega = 4$ when G is tetrahedral group , 6 when G is octahedral group and 12 when G is icosahedral group.*

Proof. Let G_1 be the subgroup of G generated by $\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ where ξ is a root of unity. Clearly, ω is fixed by G_1. Thus degree of ω over $\mathbb{C}(x)$ must be less than $[G : G_1] = 4, 6, 12$ in tetrahedral , octahedral, icosahedral case respectively.

Let G be the tetrahedral group. Now, let G_1 be the subgroup of G which fixes η_1. Let η_1, ζ_1 be a fundamental system of solutions of the DE and XGX^{-1} be the Galois group with respect to η_1, ζ_1. We can assume XGX^{-1} to be triangular. Thus if $A \in XGX^{-1}$ then A has the form $\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}$ Since G_1 is a finite group, we get $d = 0$ and $c^m = 1$ where $m = |G_1|$. Thus XGX^{-1}, being a subgroup of a cyclic group, is cyclic. Thus G_1/H is isomorphic to a cyclic subgroup of A_4. Thus $|G_1/H| \leq 3$. Therefore $|G_1| \leq 6$. And hence, $deg_{\mathbb{C}(x)}\omega_1 = [G : G_1] \geq 4$. A similar argument in case when G is octahedral and icosahedral group, gives the required result in those cases. □

Definition 7.8. *Consider the following recursively defined differential equation,*
$a_n = -1$

$a_{i-1} = -a_i' - za_i - (n-i)(i+1)ra_{i+1} \qquad (i = n, \cdots, 1, 0)$
By a solution of this differential equation we mean a function z such that if $a_n, a_{n-1}, \cdots, a_0, a_{-1}$ are defined as above then $a_{-1} = 0$. We shall denote this equation by $(\#)_n$

Theorem 7.9. *Let z be a solution of $(\#)_n$ and let ω be any root of $y^n - \sum_{i=0}^{n-1} \frac{a_i}{(n-i)!} y^i = 0$. Then $\eta = e^{\int \omega}$ is a solution of the DE $y'' = ry$.*

Proof. Let $A = \sum_{i=0}^{n} \frac{a_i}{(n-i)!} y^i$ $(a_n = -1)$ where y is an indeterminate. One can easily prove $\frac{\partial^{k+1} A}{\partial y^{k+1}} (y^2 - r) = \frac{\partial^{k+1} A}{\partial y^k \partial x} + [(n-2k) y + z] \frac{\partial^k A}{\partial y^k} + k(n-k+1) \frac{\partial^{k-1} A}{\partial y^{k-1}}$ by using induction on k. Now, we assume $\omega' + \omega^2 - r \neq 0$.
Since, $A(\omega) = 0$ we have $\frac{\partial A}{\partial y}(\omega)\omega' + \frac{\partial A}{\partial x}(\omega) = 0$.
Thus $\frac{\partial A}{\partial y}(\omega)(\omega' + \omega^2 - r) = -\frac{\partial A}{\partial x}(\omega) + (n\omega + z) A(\omega) + \frac{\partial A}{\partial x}(\omega) = 0$
Hence we get, $\frac{\partial A}{\partial y}(\omega) = 0$. Also by assuming $\frac{\partial^{k-1} A}{\partial y^{k-1}}(\omega) = \frac{\partial^k A}{\partial y^k}(\omega) = 0$, one can easily prove $\frac{\partial^{k+1} A}{\partial y^{k+1}}(\omega) = 0$ (using the above equation).
But $\frac{\partial^n A}{\partial y^n}(\omega) = -n! \neq 0$ which is a contradiction. Thus, $\omega^2 + \omega' - r = 0$. And an easy computation shows that $\eta = e^{\int \omega}$ is a solution of the DE. □

Theorem 7.10. *1. For k= 4,6 suppose $(\#)_k$ has a solution $z \in \mathbb{C}(x)$. Then the polynomial $y^k - \sum_{i=0}^{k-1} \frac{a_i}{(k-i)!} y^i \in \mathbb{C}(x)[y]$ is irreducible over $\mathbb{C}(x)$.*

2. If $(\#)_{12}$ has a solution $z \in \mathbb{C}(x)$ such that $(\#)_4$ and $(\#)_6$ do not have solutions in $\mathbb{C}(x)$. Then the polynomial $y^{12} - \sum_{i=0}^{11} \frac{a_i}{(k-i)!} y^i \in \mathbb{C}(x)[y] \in \mathbb{C}(x)[y]$ is irreducible over $\mathbb{C}(x)$.

Proof. 1. By previous theorems, any root of polynomial $y^k - \sum_{i=0}^{k-1} \frac{a_i}{(k-i)!} y^i \in \mathbb{C}(x)[y]$ has to have degree 4 or 6 or 12 over $\mathbb{C}(x)$. Thus for $k = 4$ we are done. And for $k = 6$, if given polynomial is irreducible then at-least one root must have degree less than 4, which is not possible. So this polynomial must be irreducible over $\mathbb{C}(x)$.

2. We only need to prove that if $deg_{\mathbb{C}(x)}\omega = n$ then $(\#)_n$ has a solution $z \in \mathbb{C}(x)$. Let $A = \sum_{i=0}^{n} \dfrac{a_i}{(n-i)!} y^i \in \mathbb{C}(x)\,[y]$ $(a_n = -1)$ be the minimal polynomial for ω over $\mathbb{C}(x)$.
Consider $B = \dfrac{\partial A}{\partial y}\left(r - y^2\right) + \dfrac{\partial A}{\partial x} + (ny + z)\,A$ where $z = a_{n-1} \in \mathbb{C}(x)$. One can easily check that the coefficient of y^n and the coefficient of y^{n+1} in B are 0. Thus $deg_y B < n$.
But $B(\omega) = \dfrac{\partial A}{\partial y}(\omega)\left(r - \omega^2\right) + \dfrac{\partial A}{\partial x}(\omega) + (n\omega + z)\,A(\omega)$
$= \dfrac{d}{dx}\left(A(\omega)\right) + (n\omega + z)\,A(\omega) = 0$
Thus $B = 0$. The coefficient of y^i in B is

$$\frac{1}{(n-i)!}\left[(n-i)(i+1)\,r a_{i+1} + a_{i-1} + a_i' + z a_i\right] = 0$$

where $a_{-1} = 0$. And these are the same equations which were needed for z to be a solution of $(\#)_n$.

□

Notation: Let $l\delta b = \dfrac{b'}{b}$ denote the logarithmic derivative of b.

Theorem 7.11. *If F is any homogeneous polynomial of degree n in solutions of the DE. Then $z = l\delta F$ is a solution of $(\#)_n$.*

Proof. Let $F = \prod_{i=1}^{n} \eta_i$ where $\eta_1, \eta_2, \cdots, \eta_n$ are solutions of the DE. Let $\omega = \dfrac{\eta_i'}{\eta_i}$ and $\sigma_{m,k}$ be the k^{th} symmetric function of $\omega_1, \omega_2, \cdots, \omega_m$. Clearly, $\sigma_{m,k} = 0$ if either $k = 0$ or $k > m$. By using induction one can easily prove
$\sigma_{m,k}' = (m + 1 - k)\,r\sigma_{m,k-1} - \sigma_{m,1}\sigma m, k + (k+1)\,\sigma_{m,k+1}$.
Now, by using induction on i, we shall prove $a_i = (-1)^{n-i+1}(n-i)!\sigma_{n,n-i}$.
Clearly, $a_n = -1, a_{n-1} = z = l\delta F = \sum_{i=1}^{n} \omega_i = \sigma_{n,1}$.

By using $(\#)_n$, we get

$$\begin{aligned} a_{i-1} &= -a_i' + za_i - (n-i)(i+1)ra_{i+1}. \\ &= (-1)^{n-i}(n-i)!\sigma_{n,n-i}' + \sigma_{n,1}(-1)^{n-i}(n-i)!\sigma_{n,n-i} \\ &- (n-i)(i+1)r(-1)^{n-i}(n-i-1)!\sigma_{n,n-i-1}. \\ &= (-1)^{n-i}(n-i)!\left[\sigma_{n,n-i}' + \sigma_{n,1}\sigma_{n,n-i} + (i+1)r\sigma_{n,n-i-1}\right]. \\ &= (-1)^{n-i}(n-i+1)!\sigma_{n,n-i+1}. \end{aligned}$$

Hence, $a_{-1} = 0$.
Thus if F_1 and F_2 are functions such that $l\delta F_1$ and $l\delta F_2$ are solutions of $(\#)_n$ then it is sufficient to prove that $l\delta(c_1F_1 + c_2F_2)$ is a solution of $(\#)_n$ for any $c_1, c_2 \in \mathbb{C}$. Let a_i^1, a_i^2, a_i^3 be the sequences obtained from $(\#)_n$ for $z = l\delta F_1, l\delta F_2, l\delta(c_1F_1 + c_2F_2)$ respectively.
We shall prove $(c_1F_1 + c_2F_2)(a_i^3) = c_1F_1a_i^1 + c_2F_2a_i^2$. For $i = n$, it is clear. Now, let $i \leq n$ and this is true for all $j \geq i$ such that $j < n$.

$$\begin{aligned} (c_1F_1 + c_2F_2)\left(a_i^3\right) &= (c_1F_1 + c_2F_2)\left[-\left(a_i^3\right)' - l\delta(c_1F_1 + c_2F_2)a_i^3\right] \\ &- (c_1F_1 + c_2F_2)\left[(n-i)(i+1)ra_{i+1}^3\right]. \\ &= -\left[(c_1F_1 + c_2F_2)a_i^3\right]' - (n-i)(i+1)r(c_1F_1 + c_2F_2)a_{i+1}^3. \\ &= -\left[\left(c_1F_1a_i^1 + c_2F_2a_i^2\right)\right]' - (n-i)(i+1)r\left(c_1F_1a_{i+1}^1 + c_2F_2a_{i+1}^2\right). \\ &= c_1F_1a_{i-1}^1 + c_2F_2a_{i-1}^2. \end{aligned}$$

Thus $(c_1F_1 + c_2F_2)a_{-1}^3 = c_1F_1a_{-1}^1 + c_2F_2a_{-1}^2 = 0$ which completes the proof.

□

Theorem 7.12. *1. If G is the tetrahedral group then $(\#)_4$ has a solution $z = l\delta u$ where $u^3 \in \mathbb{C}(x)$.*

2. If G is the octahedral group then $(\#)_6$ has a solution $z = l\delta u$ where $u^2 \in \mathbb{C}(x)$.

3. *If G is either the tetrahedral or the octahedral or the icosahedral group then $(\#)_{12}$ has a solution $z = l\delta u$ where $u \in \mathbb{C}(x)$.*

Proof. For 1. we may take $u = \eta^4 + 8\eta\zeta^3$
For 2. we may take $u = \eta^5\zeta - \eta\zeta^5$.
For 3. we may take $u = \left(\eta^4 + 8\eta\zeta^3\right)^3, \left(\eta^5\zeta - \eta\zeta^5\right)^2$ or $\eta^{11}\zeta - 11\eta^6\zeta^6 - \eta\zeta^{11}$ □

We write $u^{\frac{12}{n}} = \prod_{c\in\mathbb{C}} (x-c)^{e_c} \in \mathbb{C}(x)$ where $n = 4, 6 or 12$ and $e_c \in \mathbb{Z}$. Now, we shall determine the possibilities for e_c. For simplicity of notation, we will assume $c = 0$.

$$z = l\delta u = \frac{n}{12} l\delta\left(u^{\frac{12}{n}}\right)$$

We write the Laurent series of r and z at 0 as follows:

$$r = \alpha x^{-2} + \beta x^{-1} + \cdots (\alpha, \beta \in \mathbb{C}, \text{ possibly } 0)$$

$$z = \frac{n}{12} e x^{-1} + \cdots (e = e_0 \in \mathbb{Z}, \text{ possibly } 0)$$

Theorem 7.13. *If $\alpha = 0, \beta \neq 0$ then $e = 12$.*

Proof. Write $z = \frac{n}{12} e x^{-1} + f + \cdots$
We shall treat e and f as indeterminates. Then

$$a_i = A_i x^{i-n} + B_i x^{i-n+1} + C_i f x^{i-n+1} + \cdots$$

where A_i, B_i, C_i are polynomials in e with coefficients in $\mathbb{C}$. Using $(\#)_n$ we get,

$$A_n = -1 \qquad B_n = C_n = 0$$
$$A_{i-1} = \left(n - i - \frac{n}{12} e\right) A_i$$
$$B_{i-1} = \left(n - i - 1 - \frac{n}{12} e\right) B_i - (n-i)(i+1)\beta A_{i+1}$$
$$C_{i-1} = \left(n - i - 1 - \frac{n}{12} e\right) C_i - A_i \text{ (for } i = n, \cdots, 1, 0).$$

One can easily verify that the solution of these equations is :

$$A_i = -\prod_{j=0}^{n-i-1}\left(j - \frac{n}{12}e\right).$$

$$B_i = \beta\sum_{j=0}^{n-i-2}(j+1)(n-j)\prod_{k=0}^{n-i-2}\left(k - \frac{n}{12}e\right).$$

$$C_i = (n-i)\prod_{j=0}^{n-i-2}\left(j - \frac{n}{12}e\right) \qquad (i = n, \cdots, 1, 0).$$

Since,

$$0 = a_{-1} = A_{-1}x^{-n-1} + B_{-1}x^{-n}C_{-1}fx^{-n} + \cdots .$$

$$0 = A_{-1} = -\prod_{j=0}^{n}\left(j - \frac{n}{12}e\right).$$

$$\text{and } 0 = B_{-1} + C_{-1}f = \beta\sum_{j=0}^{n-1}(j+1)(n-j)\prod_{k=0,k\neq j}^{n-1}\left(k - \frac{n}{12}e\right)$$

$$+ (n+1)\prod_{j=0}^{n-1}\left(j - \frac{n}{12}e\right)f.$$

We get, $e = \frac{12}{n}l$ for some $l = 0, 1, \cdots, n$.
Now suppose $l \neq n$, then by above equation we get,

$$B_{-1} = \beta(l+1)(n-l)\prod_{k=0,k\neq l}^{n-1}\left(k - \frac{n}{12}e\right)$$

which implies $\beta = 0$. This is a contradiction, hence $l = n$. Therefore, $e = 12$. □

Now we consider the possibility that $\alpha \neq 0$. As before we write $a_i = A_i x^{i-n} + \cdots$.

Lemma 7.14. *$A_i \in \mathbb{Q}[\alpha][e]$ such that $deg_e A_i = n - i$ and leading coefficient of A_i is $-\left(-\frac{n}{12}\right)^{n-i}$.*

Proof. By using $(\#)_n$ we get,

$$A_n = -1$$
$$A_{i-1} = \left(n - i - \frac{n}{12}e\right) A_i - (n-i)(i+1)\,\alpha A_{i+1}$$

Rest of the proof is just an easy calculation. □

If $\alpha \neq \frac{1}{4}$, then the DE has Puiseaux series solutions of the form

$$\eta_1 = x^{\mu_1} + \cdots \text{ where } \mu_1 = \frac{1}{2} + \frac{1}{2}\sqrt{1+4\alpha}$$
$$\eta_2 = x^{\mu_2} + \cdots \text{ where } \mu_2 = \frac{1}{2} - \frac{1}{2}\sqrt{1+4\alpha}$$

By theorem **??**, we know that $l\delta\left(\eta_1^i \eta_2^{n-i}\right)$ is a solution of $(\#)_n$ for every $i = 0, 1, \cdots, n$. Since

$$l\delta\left(\eta_1^i \eta_2^{n-i}\right) = (i\mu_1 + (n-i)\,\mu_2)\,x^{-1} + \cdots$$
$$= \left(\frac{n}{2} - \left(\frac{n}{2} - i\right)\sqrt{1+4\alpha}\right) x^{-1} + \cdots$$

and A_{-1} must vanish thus we have, $\frac{12}{n}e = \frac{n}{2} - \left(\frac{n}{2} - i\right)\sqrt{1+4\alpha}$ for $i = 0, 1, \cdots, n$.

Theorem 7.15. *1. If G is tetrahedral group then*

$$e \in \left\{6 + k\sqrt{1+4\alpha} : k = 0, \pm 3, \pm 6\right\} \cap \mathbb{Z}.$$

2. If G is the octahedral group then $e \in \left\{6 + k\sqrt{1+4\alpha} : k = 0, \pm 2, \pm 4, \pm 6\right\} \cap \mathbb{Z}$.

3. If G is either the tetrahedral group or the octahedral group or the icosahedral group then $e \in \left\{6 + k\sqrt{1+4\alpha} : k = 0, \pm 1, \cdots, \pm 6\right\} \cap \mathbb{Z}$.

Proof. 1. In this case $n = 4$. If $\alpha \neq -\frac{1}{4}$, we use the lemma and remarks after that to get,

$$0 = A_{-1} = \prod_{i=0}^{4} \left(\frac{e}{3} - 2 + (2 - i) \sqrt{1 + 4\alpha} \right)$$

Thus, $e \in \left\{ 6 + k\sqrt{1 + 4\alpha} : k = 0, \pm 3, \pm 6 \right\} \cap \mathbb{Z}$. If $\alpha = -\frac{1}{4}$, we compute directly and obtain, $A_{-1} = \frac{1}{243} (e - 6)^5$.

2. In this case, $n = 6$. Similar to above case, if $\alpha \neq -\frac{1}{4}$, we obtain the result from the lemma and the remarks. And if $\alpha = -\frac{1}{4}$, a direct computation shows, $A_{-1} = \frac{1}{128} (e - 6)^7$. This completes the proof of this part.

3. In this case $n = 12$. Similar to the first case, if $\alpha \neq -\frac{1}{4}$, we obtain the result from the lemma and the remarks. And if $\alpha = -\frac{1}{4}$, a direct computation shows $A_{-1} = (e - 6)^{11}$. This completes the proof of theorem.

□

Now we consider the case when, $\alpha = \beta = 0$. Using the previous theorem we get, $\frac{ne}{12} \in \mathbb{Z}$.

Let Γ be the set of poles of r. We have proven that,

1. In tetrahedral case, $z = l\delta u$ is a solution of $(\#)_4$ where

$$u^3 = P^3 \prod c \in \Gamma \, (x - c)^{e_c}$$

with $P \in \mathbb{C}[x]$ and $e \in \left\{ 6 + k\sqrt{1 + 4\alpha} : k = 0, \pm 3, \pm 6 \right\} \cap \mathbb{Z}$.

2. In octahedral case, $z = l\delta u$ is a solution of $(\#)_6$ where

$$u^2 = P^2 \prod c \in \Gamma \, (x - c)^{e_c}$$

with $P \in \mathbb{C}[x]$ and $e \in \left\{ 6 + k\sqrt{1 + 4\alpha} : k = 0, \pm 2, \pm 4, \pm 6 \right\} \cap \mathbb{Z}$.

3. In tetrahedral or octahedral or icosahedral case, $z = l\delta u$ is a solution of $(\#)_{12}$ where

$$u = P \prod c \in \Gamma \left(x - c\right)^{e_c}$$

with $P \in \mathbb{C}[x]$ and $e \in \left\{6 + k\sqrt{1+4\alpha} : k = 0, \pm 1, \cdots, \pm 6\right\} \cap \mathbb{Z}$.
If $d = deg\ P$, then the Laurent series of r and z at ∞ are of the form:

$$z = \frac{n}{12}\left(\frac{12}{n}d + \sum_{c\in\Gamma} e_c\right) x^{-1} + \cdots$$

$$r = \gamma x^{-2} + \cdots$$

Let $e_\infty = \frac{12}{n}d + \sum_{c\in\Gamma} e_c$. By following the same steps done in previous theorem, we get that e_∞ also satisfies the same conditions as e_c. Also, $d = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \in \mathbb{N} \cup \{0\}$.
Now we shall show that the recursive relations in step 3 are identical with $(\#)_n$.
Let $\theta = \frac{n}{12}\sum_{c\in\Gamma} \frac{e_c}{x-c}$ and $S = \prod_{c\in\Gamma}(x-c)$. Thus $z = l\delta u = \frac{P'}{P} + \theta$. Also set $P_i = S^{n-i}Pa_i$.Using $(\#)_n$ we get,

$$\begin{aligned}
P_n &= -P \\
P_{i-1} &= S^{n-i+1}Pa_{i-1} \\
&= S^{n-i+1}P\left(-a_i' - za_i - (n-i)(i+1)\,ra_{i+1}\right) \\
&= -S\left(S^{n-i}Pa_i\right)' + (n-i)\,S^{n-i}S'Pa_i + S^{n-i+1}P'a_i \\
&\quad - S\left(P' + P\theta\right)S^{n-i}a_i - (n-i)(i+1)\,S^2 r\left(S^{n-i-1}Pa_{i+1}\right) \\
&= -SP' + (n - i - S\theta)\,P_i - (n-i)(i+1)\,S^2 r P_{i+1}
\end{aligned}$$

which is precisely the equation in the algorithm. Finally,

$$\omega^n = \sum_{i=0}^{n-1} \frac{a_i}{(n-i)!}\omega^i$$

can be rewritten as

$$0 = -S^n P \omega^n + \sum_{i=0}^{n-i} \frac{S^n P a_i}{(n-i)!} \omega^i = \sum_{i=0}^{n} \frac{S^i P_i}{(n-i)!} \omega^i$$

which completes the proof of correctness of the algorithm.

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