Review of discrete Newtonian cosmology

by

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"I...a universe of atoms, an atom in the universe"

-Richard Feynman

Certificate of Examination

This is to certify that the dissertation titled Review of discrete Newtonian cosmology submitted by Mr.Imrankhan B. Mulani (Reg. No. MP12001) for the partial fulfilment of MS degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: July 31, 2015

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr.Jasjeet Bagla at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Imrankhan B. Mulani
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In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr.Jasjeet Bagla
(Supervisor)

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Contents

Abstract

Large scale properties of the universe can be explained by Einstein's general theory of relativity. Λ-CDM model incorporating general relativity predicts the accelerated expansion of universe and large scale distribution of galaxies. Structure formation equation obtained from this model are non linear and difficult to solve. Newtonian theory of gravity can be applied to situations after decoupling of matter and radiation. Newton himself tried to apply his theory to universe but failed. Reason for failure is that he considered static model with infinite stars. This result in divergences. The divergences can be avoided by assuming finite number of particles interacting only through gravitational attraction. This thesis is review of Newtonian cosmological theory of finite number of discrete particles interacting through gravitational attraction, also with Newtonian version of cosmological constant. Exact solution we get for Newtonian equations are homothetically expanding background with comoving positions constituting a central configuration. The scale factor satisfy Rauchaudhuri and Friedmann equations without making any fluid approximation. These solutions can be linearised to get perturbation theory for structure formation calculations.

Chapter 1

Introduction

Newtonian cosmological model contains universe made up of discrete particles interacting only via attractive gravitational force. This model do not require fluid dynamics or general relativity. The particles in this model can be identified with stars, galaxies, clusters, superclusters or even molecules. Cause any spherically symmetric isolated subsystem will move and gravitate like a point particle located at its 'centre'. The features of this model are :

1.1 No Fluid Model

The usual cosmological model assumes matter as continuous fluid distribution. Standard fluid properties are derived for particles that only undergoes short range interactions like collision. These short range interaction properties are not suitable to explain long range gravitational interaction between stars, galaxies or dark matter particles. Also if we want to identify galaxies as particle then the number of particles is much too small for good fluid approximation.

So this model assumes sets of gravitating particles embedded in vacuum and their interaction is long range gravitational attraction. Thus summation is used instead of integration in equations describing model.

1.2 No Divergences

Newtons assume infinite number of stars in his cosmological model and considered universe to be static. Failure to get cosmological solution was divergences associated with infinite number of particles. It leads to paradoxes like Olber's paradox and Bently's paradox. Discrete model assume finite number of particles and avoid these divergences and associated problems.

1.3 No Fourier analysis

In this model we try to get linear solutions to background model for structure formation. The basic gravitational interactions for structure formation are non linear and Fourier analysis would not work. For calculating actual dynamics we can work with actual distribution rather than its Fourier modes.

1.4 No periodic boundary conditions

Periodic boundary conditions restrict the nature of allowed solutions, also introduce artefacts. As periodic boundary conditions violate the conservation of angular momentum by breaking rotational symmetry, we consider open boundary condition.

1.5 Outcome

Outcome of this approach is that, if suitable discrete distribution of particles satisfying *central configuration* is given, we obtain an exact Newtonian version of standard FLRW models- a solution that expands homothetically and follow Raychaudhuri equations for pressure free matter. For large number of particles solutions are close to spatially homogeneous.

To get the FRLW like solution, initial distribution of particles must satisfy central configuration equation. Structure formation for this model can be obtained perturbing the dynamical equations of model. Also we can change background linearly and non linearly to see what happens using numerical integration methods. Perturbing the solutions gives us linearised Newtonian structure formation equations.We can also derive Demitriev-Zel'dovich equations, which is different approach to perturbation and in which we obtain equations governing the motion of point particles in the background cosmology. These equations are widely used in study of large scale structure in the universe.

Chapter 2

Discrete Newtonian cosmology

This chapter reviews dynamics of N point particles moving under the influence of gravity.

2.1 The basic equations

The gravitating masses follows Newton's laws of attraction. The equation of motion for interacting point particles, using inertial coordinates for discrete point particles at position $\mathbf{x}_a$ and with mass $m_a > 0$ is

$$m_a \frac{d^2\mathbf{x}_a}{dt^2} = \mathbf{F}_a + \sum_{b\neq a} \mathbf{F}_{ab} \tag{2.1}$$

where $\mathbf{F}_a$ are external forces, and $\mathbf{F}_{ab}$ are inter particle forces between particle a and b. a ranges over the value $1, 2, 3, ..., N$ if there are N particles. Gravitational force between any two particle is

$$\mathbf{F}_{ab} = -\frac{Gm_am_b}{|\mathbf{x}_a - \mathbf{x}_b|^3}(\mathbf{x}_a - \mathbf{x}_b) \tag{2.2}$$

where G is Newtons gravitational constant. Thus above equation become

$$m_a \frac{d^2\mathbf{x}_a}{dt^2} = -\sum_{b\neq a} Gm_am_b \frac{(\mathbf{x}_a - \mathbf{x}_b)}{|\mathbf{x}_a - \mathbf{x}_b|^3} + \mathbf{F}_a \tag{2.3}$$

We assume that universe consist of very large but finite number of particles and apply force law to all particles, so that there is no external force. We can also assume that according to symmetry considerations external forces vanish. then $\mathbf{F}_a = 0$ and we get for each a,

$$m_a \frac{d^2\mathbf{x}_a}{dt^2} = -\sum_{b\neq a} G m_a m_b \frac{(\mathbf{x}_a - \mathbf{x}_b)}{|\mathbf{x}_a - \mathbf{x}_b|^3} \tag{2.4}$$

and

$$\mathbf{F}_a^{(grav)} = -\frac{G m_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|^3}(\mathbf{x}_a - \mathbf{x}_b) \tag{2.5}$$

is total gravitational force exerted on a due to all other particles. We now drop the superscript and identify gravitational force by symbol $\mathbf{F}_a$

2.1.1 Potential energy

The gravitational force $\mathbf{F}_a$ acting on the a th particle can be represented as the derivative of a gravitational potential energy V_a acting on that particle. The potential energy $V_a(\mathbf{x}_c)$ for the gravitational force on the particle $\mathbf{x}_a$ is a function of the position $\mathbf{x}_a$ defined by

$$V_a(\mathbf{x}_a) := -\sum_{b\neq a} \frac{G m_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|} \tag{2.6}$$

This is the discrete version of the continuous definition of potential. we define

$$\mathbf{x}_{ba} := \mathbf{x}_b - \mathbf{x}_a, x_{ba} := |\mathbf{x}_{ba}| = ((\mathbf{x}_b - \mathbf{x}_a).(\mathbf{x}_b - \mathbf{x}_a))^{1/2}. \tag{2.7}$$

for $\mathbf{x}_a \neq \mathbf{x}_b$,

$$\frac{\partial}{\partial \mathbf{x}_a}(\frac{1}{x_{ba}}) = (\frac{1}{x_{ba}})^3(\mathbf{x}_b - \mathbf{x}_a) \tag{2.8}$$

$$\frac{\partial V_a(\mathbf{x}_a)}{\partial \mathbf{x}_a} = -\sum_{b\neq a} \mathbf{F}_{ab} = -\mathbf{F}_a \tag{2.9}$$

we can add any constant V_0 without affecting the result.

2.1.2 Symmetries

Equations of motion have following symmetries,

- time translation($t \to t + t_0$)

- spatial translation($\mathbf{x}_a \to \mathbf{x}_a + \mathbf{x}_0$)
- rotation about origin
- boost from one inertial frame to other

2.1.3 Conserved quantities

Central nature of the gravitational force guarantees conservation of momentum, angular momentum and energy. Mass of the isolated system of particles is also conserved:

$$dm_a/dt = 0 \tag{2.10}$$

It follows from the symmetries that total mass M, momentum$\mathbf{P}$, and angular momentum$\mathbf{L}$ about origin are conserved:

$$M = \sum_a m_a = M_0(constant) > 0, \mathbf{P} = \sum_a m_a \dot{\mathbf{x}}_a = \mathbf{P}_0(constant), \tag{2.11}$$

$$\mathbf{L} = \sum_a m_a(\mathbf{x}_a \times \dot{\mathbf{x}}_a) = \mathbf{L}_0(constant) \tag{2.12}$$

The conservation of momentum together with mass conservation implies that centre of mass moves with the constant velocity. Also energy $\mathcal{E}$ of system is conserved.

$$\mathcal{E} = T + V = \mathcal{E}_\prime(constant) \tag{2.13}$$

where T is kinetic energy and potential energy is V are given by

$$T(\dot{\mathbf{x}}_\mathbf{c}) := \frac{1}{2}\sum_a m_a(\dot{\mathbf{x}}_\mathbf{a})^2, V(\mathbf{x}_c) := \sum_a V_a = \sum_a \sum_{b \neq a} \frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|}. \tag{2.14}$$

The total gravitational potential energy, $V(\mathbf{x}_c)$ is homogeneous function of degree $K = -1$. T represents the total energy of motion of particle and V sum of potential energies of all the particles. These are just the numbers

2.1.4 Virial Relation

This result depends on the inverse square nature of the force law. Taking the dot product of $mathbfF_a$ given by (5) with $\mathbf{x}_a$, and sum over a to get

$$\sum_a \mathbf{x}_a.\mathbf{F}_a = -\sum_a \sum_{b\neq a} Gm_a m_b \frac{\mathbf{x}_a.(\mathbf{x}_a - \mathbf{x}_b)}{|\mathbf{x}_a - \mathbf{x}_b|^3} = \sum_a \sum_{b\neq a} \mathbf{x}_a.\partial_{\mathbf{x}_a}(\frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|}) \quad (2.15)$$

Euler's theorem on homogeneous function of degree k (that is function $f(V)$ such that $f(ax) = a^k f(x)$)says

$$x\partial f/\partial x = kf \quad (2.16)$$

in this case $f = \frac{1}{|\mathbf{x}_a - \mathbf{x}_b|}$ is of degree $k = -1$, so Euler's theorem says

$$\mathbf{x}_a.\partial \mathbf{x}_a f = -f \Rightarrow \mathbf{x}_a.\partial \mathbf{x}_a(\frac{1}{|\mathbf{x}_a - \mathbf{x}_b|}) = -\frac{1}{|\mathbf{x}_a - \mathbf{x}_b|}. \quad (2.17)$$

so that

$$\sum_a \sum_{b\neq a} \mathbf{x}_a.\partial_{\mathbf{x}_a}(\frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|}) = -\sum_a \sum_{b\neq a} (\frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|}) = V \quad (2.18)$$

from the equations above, we can calculate

$$\sum_a \mathbf{x}_a.\mathbf{F}_a = \sum_a m_a.\mathbf{x}_a.\frac{d^2\mathbf{x}_a}{dt^2} = \sum_a m_a(\frac{d}{dt}(\mathbf{x}_a.\frac{d\mathbf{x}_a}{dt}) - \frac{d\mathbf{x}_a}{dt}.\frac{d\mathbf{x}_a}{dt}) = \frac{d}{dt}\sum_a m_a \frac{1}{2}\frac{d}{dt}(\mathbf{x}_a.\mathbf{x}_a) - 2T \quad (2.19)$$

Using all the above equation we will get the scalar virial relation. In celestial mechanics following equation is called as Lagrange-Jacobi equation.

$$V = \frac{d^2 I}{dt^2} - 2T \quad (2.20)$$

where I is moment of inertia of the system. Taking time average$\langle\rangle$, If the average of the second derivative of the $I(t)$is zero we get the relation between kinetic and potential energy

$$\langle \frac{d^2 I}{dt^2} \rangle = 0 \Rightarrow \langle V \rangle = -2\langle T \rangle \quad (2.21)$$

2.2 Cosmological solutions

In this section we are going to discuss solutions of Newtons equations of motion which evolve by homotheties of the Euclidean space. Newton's equation of motion only allows homothetic solutions if the comoving position of the particle are constrained to form special configuration known as "central configuration". Central configuration extremize a certain function of position denoted by $\tilde{V}$. In case of large number of particles of equal mass maximizing $\tilde{V}$ form spherical and homogeneous ball.

2.2.1 Robertson-Walker like solutions

We assume the self similarity of the solutions. Let $S(t)$ be the homothetic factor, solutions are then given by

$$\mathbf{x}_a = S(t)\mathbf{r}_a, d\mathbf{r}_a/dt = 0 \tag{2.22}$$

where $\mathbf{r}_a$ are comoving coordinates of particle a. so velosity distance relation is given by

$$\mathbf{v}_a := \frac{d\mathbf{x}_a}{dt} = \dot{S(t)}\mathbf{r}_a = H(t)\mathbf{x}_a \tag{2.23}$$

where $H(t) := \frac{\dot{S(t)}}{S(t)}$. The gravitational law becomes

$$m_a\mathbf{r}_a\frac{d^2S(t)}{dt^2} = -\sum_{b\neq a} Gm_am_b\frac{(\mathbf{r}_a - \mathbf{r}_b)}{S^2(t)|\mathbf{r}_a - \mathbf{r}_b|^3} \tag{2.24}$$

we define

$$C(t) := S^2(t)\frac{d^2S(t)}{dt^2} \tag{2.25}$$

then equation becomes

$$C(t)m_a\mathbf{r}_a = -\sum_{b\neq a} Gm_am_b\frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3} \tag{2.26}$$

consistency requires that $C(t)$ is constant

$$\frac{\partial}{\partial dt}(C(t)m_a\mathbf{r}_a) = 0, \Rightarrow C(t) = Constant =: -G\tilde{M} \tag{2.27}$$

$\tilde{M}$ has units mass per unit volume.

2.2.2 Central configuration equation

By taking into account the above definitions gives us

$$\tilde{M} m_a \mathbf{r}_a = -\sum_{b \neq a} m_a m_b \frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3} \tag{2.28}$$

for all values of a. The above set of nonlinear time independent equations area known as central configuration equation. Small number of particles form polyhedra. For large number of particles there will be shell like structures in the solution. We will discuss more about the central configuration and how to find solutions starting from the random distribution of particles using numerical methods.

2.2.3 Time evolution equation

We have

$$-\frac{G\tilde{M}}{S^2(t)} = \frac{d^2 S(t)}{dt^2} \tag{2.29}$$

is Raychaudhuri equation. Multiplying by (dS/dt) on both sides gives

$$\frac{d}{dt}\left(\frac{G\tilde{M}}{S(t)}\right) = -\frac{G\tilde{M}}{S^2(t)}\frac{dS(t)}{dt} = \frac{d^2 S(t)}{dt^2}\frac{dS(t)}{dt} = \frac{1}{2}\frac{d}{dt}\left(\frac{dS(t)}{dt}\right)^2 \tag{2.30}$$

It can be integrated and gives Friedmann equation

$$\frac{G\tilde{M}}{S^3(t)} = \frac{1}{2}\left[\frac{\tilde{S(t)}}{S(t)}\right]^2 - \frac{E}{S^2(t)} \tag{2.31}$$

E is constant of integration. Thus we get the same result as in general relativity.

2.2.4 Virial Relation

For homothetic expansion moment of inertia becomes

$$I(t) = \frac{1}{2}\sum_a m_a x_a^2 = S^2(t)\sum_a m_a r_a^2 = S^2(t)\tilde{I}_0 \tag{2.32}$$

The virial relation become

$$V = 2\left(\frac{G\tilde{M}}{S(t)} + 2E\right)\tilde{I}_0 - 2T \tag{2.33}$$

In contrast to the earlier virial relation time average of the second derivative of the moment of inertia is not equal to zero here.

So conclusion of above discussion is "The Newtonian gravitational law of attraction for finite set of gravitating particles has an exact homothetic solution provided the time independent central configuration equation is satisfied for $a - 1toN$. The effect of gravitational attraction is to lead to a homothetic change in size the Raychaudhuri equation, with first integral the Friedmann equation."

Universe is expanding at present. So to incorporate we add cosmological constant term to the Newtonian equation and study their properties.

2.2.5 Cosmological constant

We add cosmological constant into the equations

$$m_a \frac{d^2\mathbf{x}_a}{dt^2} = -\sum_{b\neq a} G m_a m_b \frac{(\mathbf{x}_a - \mathbf{x}_b)}{|\mathbf{x}_a - \mathbf{x}_b|^3} + \frac{\Lambda m_a \mathbf{x}_a}{3} \tag{2.34}$$

putting homothetic factor and using separation of variable we get

$$m_a \mathbf{r}_a \frac{d^2 S(t)}{dt^2} = -\sum_{b\neq a} G m_a m_b \frac{S(t)(\mathbf{r}_a - \mathbf{r}_b)}{S^3(t)|\mathbf{r}_a - \mathbf{r}_b|^3} + \frac{\Lambda S(t) m_a \mathbf{r}_a}{3} \tag{2.35}$$

this gives the result

$$m_a \mathbf{r}_a S^2(t) \frac{d^2 S(t)}{dt^2} = -G\tilde{M} m_a \mathbf{r}_a + \frac{\Lambda S^3(t) m_a \mathbf{r}_a}{3} \tag{2.36}$$

Now the Raychaudhuri equation with cosmological constant become

$$\frac{1}{S(t)} \frac{d^2 S(t)}{dt^2} = -\frac{G\tilde{M}}{S^3(t)} + \frac{\Lambda}{3} \tag{2.37}$$

To integrate we multiply both sides by (dS/dt) to get

$$\frac{dS(t)}{dt} \frac{d^2 S(t)}{dt^2} = -\frac{G\tilde{M}}{S^3(t)} \frac{dS(t)}{dt} + \frac{\Lambda}{3} \frac{dS(t)}{dt} \tag{2.38}$$

which is

$$\frac{1}{2} \frac{d}{dt} (\frac{dS(t)}{dt})^2 = \frac{d}{dt} (\frac{G\tilde{M}}{S(t)}) + \frac{d}{dt} (\frac{\Lambda S^2(t)}{6}) \tag{2.39}$$

Integrating gives Friedmann equation

$$\frac{1}{2}[\frac{\tilde{S(t)}}{S(t)}]^2 = \frac{G\tilde{M}}{S^3(t)} + \frac{E}{S^2(t)} + \frac{\Lambda}{6} \tag{2.40}$$

where E is constant of integration.We can derive the standard Raychaudhuri and Friedmann equations for time dependant cosmology in exactly the same way for discrete Newtonian cosmology with $\Lambda \neq 0$ as for the case with $\Lambda = 0$ The central configuration equation required for homothetic solution is unchanged.

Chapter 3

Central Configuration equation

The central configuration equation

$$\tilde{M} m_a \mathbf{r}_a = -\sum_{b \neq a} m_a m_b \frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3} \tag{3.1}$$

is the initial value equation for discrete Newtonian cosmology, once it has been satisfied at an initial time, it will be satisfied for all times. Consider there are two forces acting on the particle in three spatial dimensions. One force is linear and other inverse square attraction. The central configuration for the system is then equilibrium between the two counteracting forces. We can think of central configuration in cosmological context as an equilibrium between gravitational attraction and fictitious auxiliary cosmological repulsion.

3.1 Properties of central configuration

3.1.1 Centre of mass

the centre of mass $\mathbf{r}_{CM}$ is given by

$$M\mathbf{r}_{CM} = \sum_a m_a \mathbf{r}_a = \sum_a \sum_{b \neq a} \frac{m_a m_b}{\tilde{M}} \frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3} = 0 \tag{3.2}$$

because sum is symmetric but summand antisymmetric. Thus the centre of mass of system lies at the origin. Total angular momentum and momentum of the system are zero. Vanishing angular and linear momentum are referred as 'relational'.

3.1.2 Effective Forces

One can represent the nature of central configuration in terms of effective forces and potentials. They are called effective because of their dependence on comoving distances $\mathbf{r}_a$ rather than actual distances $\mathbf{x}_a$ that occure in force equation. Starting with $m_a\mathbf{r}_a$, add and subtracting same term again gives

$$m_a\mathbf{r}_a = \frac{1}{M}m_a\mathbf{r}_a\sum_b m_b = \frac{1}{M}\sum_b m_a m_b(\mathbf{r}_a - \mathbf{r}_b) + \frac{m_a}{M}\sum_b m_b\mathbf{r}_b \tag{3.3}$$

using centre of mass equation we get

$$m_a\mathbf{r}_a == \frac{1}{M}\sum_{b\neq a} m_a m_b(\mathbf{r}_a - \mathbf{r}_b) \tag{3.4}$$

substituting into central configuration and multiplying by G we get

$$\sum_{b\neq a} m_a m_b(\mathbf{r}_a - \mathbf{r}_b)(\frac{G\tilde{M}}{M} - \frac{G}{r_{ab}^3}) = 0 \tag{3.5}$$

defining $r_{ab} = |\mathbf{r}_a - \mathbf{r}_b|$ and effective inter particle force

$$\tilde{\mathbf{F}}_{ab} := m_a m_b(\mathbf{r}_a - \mathbf{r}_b)(\frac{G\tilde{M}}{M} - \frac{G}{r_{ab}^3}) \tag{3.6}$$

Thus

$$\sum_{b\neq a}\tilde{\mathbf{F}}_{ab} = 0 \tag{3.7}$$

We can write $\tilde{\mathbf{F}}_{ab}$ as

$$\tilde{\mathbf{F}}_{ab} = \mathbf{F}_{ab}^{(TD)} + \tilde{\mathbf{F}}_{ab}^{(1)} \tag{3.8}$$

where

$$\tilde{\mathbf{F}}_{ab}^{(1)} := -Gm_a m_b\frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3} \tag{3.9}$$

is reduced inter particle gravitational force which is related to proper distances than comoving distance.

$$\mathbf{F}_{ab}^{(TD)} = G(\frac{\tilde{M}}{M})m_a m_b(\mathbf{r}_a - \mathbf{r}_b) \tag{3.10}$$

is top down effective force exerted on the spatial distribution because of the conformal expansion. It is an effective repulsive force.It is not due to a cosmological constant but arises solely due to configuration of particles.

3.2 Potential functions

We can write central configuration equation as

$$\tilde{\mathbf{F}}_a := \tilde{\mathbf{F}}_a^{(1)} + \tilde{\mathbf{F}}_a^{(2)} = 0 \tag{3.11}$$

where $\tilde{\mathbf{F}}_a^{(1)}$ is given by

$$\tilde{\mathbf{F}}_a^{(1)} = \sum_{b\neq a} \tilde{\mathbf{F}}_{ab}^{(1)} = -\sum_{b\neq a} G m_a m_b \frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3} \tag{3.12}$$

and $\tilde{\mathbf{F}}_a^{(2)}$ is defined by

$$\tilde{\mathbf{F}}_a^{(2)} := G\tilde{M} m_a \mathbf{r}_a \tag{3.13}$$

Define associated energies as

$$\tilde{V}_a := \tilde{V}_{(-1)a} + \tilde{V}_{(2)a} \tag{3.14}$$

where the effective gravitational potential energy is

$$\tilde{V}_{(-1)a} := -\sum_{b\neq a} \frac{G m_a m_b}{|\mathbf{r}_{ab}|} \tag{3.15}$$

which is homogeneous of degree $k = -1$, and the effective repulsion potential energy is

$$\tilde{V}_{(2)a} := -\frac{1}{2} G\tilde{M} m_a \mathbf{r}_a . \mathbf{r}_a \tag{3.16}$$

which is homogeneous of degree $k = 2$. from these definitions we can write

$$\tilde{\mathbf{F}}_a^{(1)} = -\frac{\partial \tilde{V}_{(-1)a}}{\partial \mathbf{r}_a}, \tilde{\mathbf{F}}_a^{(2)} = -\frac{\partial \tilde{V}_{(2)a}}{\partial \mathbf{r}_a} \tag{3.17}$$

Solution to central configuration equation are critical points of $\tilde{V}_a$:

$$\tilde{\mathbf{F}} = 0, \frac{\partial \tilde{V}_a}{\partial \mathbf{r}_a} = \frac{\partial \tilde{V}_{(-1)a}}{\partial \mathbf{r}_a} + \frac{\partial \tilde{V}_{(2)a}}{\partial \mathbf{r}_a} = 0 \tag{3.18}$$

Critical points of the function $\tilde{V}$ are in one-one correspondence with central configurations. There is at least one global and no global minima.

3.3 Numerical solutions

The conclusions of previous sections is that Newtonian cosmological solution exist for certain arrangement of particles known as central configuration. Particles satisfying these equations are special form of homothetically expanding solution. Given random distribution of particle we have to find extrema of the potential function $\tilde{V}$. Extrema of $\tilde{V}$ are central configuration.

Finding extrema of multivariable function like $\tilde{V}$ analytically is daunting task. We have to use numerical methods for optimizing multivariable functions. There are various methods used to find extremum of functions e.g. gradient descent method, simulated annealing method etc.

Chapter 4

Perturbations

In this chapter we are going to discuss perturbed form of the homothetic solution.

4.1 General case

General form of equation of motion is

$$m_a \ddot{\mathbf{x}}_a = -\frac{\partial V(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N)}{\partial \mathbf{x}_a} \tag{4.1}$$

where $V(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N)$ is mutual gravitational potential energy of N particles.

4.1.1 Potential form

Lets consider that background solution is given by $\bar{\mathbf{x}_a}$ and linear perturbation $\delta\mathbf{y}_a$ about this solution, so that

$$\mathbf{x}_a = \bar{\mathbf{x}_a} + \delta\mathbf{y}_a, |\bar{\mathbf{x}_a}| \gg |\delta\mathbf{y}_a| \tag{4.2}$$

Taylor expanding and neglecting the higher order terms in $\delta\mathbf{y}_a$ yields

$$m_a[\frac{d^2(\bar{\mathbf{x}_a})}{dt^2} + \frac{d^2(\delta\mathbf{y}_a)}{dt^2}] = m_a[\frac{d^2(\bar{\mathbf{x}_a} + \delta\mathbf{y}_a)}{dt^2}] \tag{4.3}$$

$$= -[\frac{\partial V(\bar{\mathbf{x}_a} + \delta\mathbf{y}_a)}{\partial \mathbf{x}_a}] \tag{4.4}$$

$$= -[\frac{\partial V(\bar{\mathbf{x}_a})}{\partial \bar{\mathbf{x}_a}} + \frac{\partial^2 V(\bar{\mathbf{x}_a})}{\partial \bar{\mathbf{x}_a} \partial \bar{\mathbf{x}_b}}.\partial \bar{\mathbf{x}_b}] \tag{4.5}$$

cancelling the background term the perturbation equation is

$$m_a \delta \ddot{\mathbf{y}_a} = -\sum_{b \neq a} \frac{\partial^2 V}{\partial \bar{\mathbf{x}_a} \partial \bar{\mathbf{x}_a}} (\bar{\mathbf{x}_1}, \bar{\mathbf{x}_2}, ..., \bar{\mathbf{x}_N}).\delta \mathbf{y}_a \tag{4.6}$$

The symmetric operator acting on $\delta \mathbf{y}_a$ is in fact minus the Hessian $\mathbf{E}_{ab}$ of V. Hessian is given by

$$\mathbf{E}_{ab} = -\frac{\partial^2 V}{\partial \bar{\mathbf{x}_a} \partial \bar{\mathbf{x}_a}} \tag{4.7}$$

The above equation is a linear ordinary differential equation for the perturbation $\delta \mathbf{y}_a(t)$ whose coefficients depends on background solution $\bar{\mathbf{x}_a(t)}$. Coefficients depends on time.

4.1.2 Force form

Force between the particles at $\mathbf{x}_a$ and $\mathbf{x}_b$ can be written as

$$\mathbf{F}_{ab} = -\frac{G m_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|^3} (\mathbf{x}_a - \mathbf{x}_b) \tag{4.8}$$

setting $\mathbf{x}_{ba} := \mathbf{x}_b - \mathbf{x}_a$, $\delta \mathbf{y}_{ba} := \delta \mathbf{y}_b - \delta \mathbf{y}_a$, $x_{ba} := |\mathbf{x}_b - \mathbf{x}_a| = ((\mathbf{x}_b - \mathbf{x}_a).(\mathbf{x}_b - \mathbf{x}_a))^{1/2}$ gives

$$\mathbf{F}_{ab}(\bar{\mathbf{x}_a} + \delta \mathbf{y}_b) = -\frac{G m_a m_b}{|(\bar{\mathbf{x}_b} - \bar{\mathbf{x}_a}) + \delta \mathbf{y}_{ba}|^3} ((\bar{\mathbf{x}_b} - \bar{\mathbf{x}_a}) + \delta \mathbf{y}_{ba}) \tag{4.9}$$

$$= -\frac{G m_a m_b}{|\bar{\mathbf{x}_b} - \bar{\mathbf{x}_a}|^3} (\bar{\mathbf{x}_{ab}} + \delta \mathbf{y}_{ab}) - \frac{\partial}{\partial \mathbf{x}_b} [\frac{G m_a m_b}{|\bar{\mathbf{x}_b} - \bar{\mathbf{x}_a}|^3}].\delta \mathbf{y}_{ba} \bar{\mathbf{x}_{ab}} + \mathcal{O}(\delta \mathbf{y}_a)^2 \tag{4.10}$$

This gives

$$\delta \mathbf{F}_{ab} = \mathbf{F}_{ab}(\bar{\mathbf{x}_{ab}} + \delta \mathbf{y}_{ab}) - \mathbf{F}_{ab}(\bar{\mathbf{x}_a}) \tag{4.11}$$

$$= -\frac{G m_a m_b}{|\bar{\mathbf{x}_{ab}}|^3} (\delta \mathbf{y}_{ab}) - 3 \frac{G m_a m_b}{|\bar{\mathbf{x}_{ab}}|^4} (\frac{\partial}{\partial \mathbf{x}_a} (x_{ab}).\delta \mathbf{y}_{ab}) \bar{\mathbf{x}_{ab}} \tag{4.12}$$

$$= \frac{G m_a m_b}{|\bar{\mathbf{x}_{ab}}|^5} \{\delta \mathbf{y}_{ba} \bar{\mathbf{x}_{ba}}^2 - 3(\bar{\mathbf{x}_{ba}}.\delta \mathbf{y}_{ba}) \bar{\mathbf{x}_{ba}}\} \tag{4.13}$$

and so

$$m_a(\delta \ddot{\mathbf{y}_a}) = \sum_{b \neq a} \frac{G m_a m_b}{|\bar{\mathbf{x}_{ab}}|^5} \{\delta \mathbf{y}_{ba} \bar{\mathbf{x}_{ba}}^2 - 3(\bar{\mathbf{x}_{ba}}.\delta \mathbf{y}_{ba}) \bar{\mathbf{x}_{ba}}\} \tag{4.14}$$

This applies generically to perturbation around any background.

4.2 Cosmological case

Applying the above general formalism to homothetically expanding background solution. Thus we have

$$\bar{\mathbf{x}}_a = S(t)\bar{\mathbf{r}}_a = constant, \bar{\mathbf{r}}_{ab} := \bar{\mathbf{r}}_a - \bar{\mathbf{r}}_b = constant, \bar{r}_{ab} := |\bar{\mathbf{r}}_a - \bar{\mathbf{r}}_b| \tag{4.15}$$

comoving perturbation variables are defined as $\mathbf{S}_a$, $\mathbf{S}_{ab} := \mathbf{S}_a - \mathbf{S}_b$ and $\delta\mathbf{y}_a = S(t)\mathbf{S}_a$ then the equation for perturbation becomes

$$m_a \frac{d^2}{dt^2}(S(t)\mathbf{S}_a) = \sum_{b \neq a} \frac{G m_a m_b}{S^5(t)|\bar{\mathbf{r}}_{ab}|^5} S^3(t)\{(\mathbf{S}_a - \mathbf{S}_b)|\bar{\mathbf{r}}_{ab}|^2 - 3(\bar{\mathbf{r}}_{ab}.\mathbf{S}_{ab})\bar{\mathbf{r}}_{ab}\} \tag{4.16}$$

so the cosmological perturbation equation become

$$S^2(t) m_a \frac{d^2}{dt^2}(S(t)\mathbf{S}_a) = \sum_{b \neq a} \frac{G m_a m_b}{|\bar{\mathbf{r}}_{ab}|^5}\{(\mathbf{S}_a - \mathbf{S}_b)|\bar{\mathbf{r}}_{ab}|^2 - 3(\bar{\mathbf{r}}_{ab}.\mathbf{S}_{ab})\bar{\mathbf{r}}_{ab}\} \tag{4.17}$$

this is second order ordinary differential equation for perturbation whose coefficients depend upon the background scale factor and background time independent central configuration, whose homothetic expansion we are perturbing about.

4.2.1 Asymptotic solution

The growth of perturbation is given by

$$m_a \frac{d^2}{dt^2}(S(t)\mathbf{S}_a) = \frac{1}{S^2(t)} \sum_{b \neq a} \frac{G m_a m_b}{|\bar{\mathbf{r}}_{ab}|^5}\{(\mathbf{S}_a - \mathbf{S}_b)|\bar{\mathbf{r}}_{ab}|^2 - 3(\bar{\mathbf{r}}_{ab}.\mathbf{S}_{ab})\bar{\mathbf{r}}_{ab}\} \tag{4.18}$$

as $S \to \infty$ Right hand side goes to zero. Thus at later times

$$S\mathbf{S}_a = \mathbf{w}_a t + \mathbf{q}_a \tag{4.19}$$

where $\mathbf{w}_a$, $\mathbf{q}_a$ are constant vectors. also $S \propto t^{2/3}$

$$\mathbf{S}_a = \mathbf{w}_a t^{1/3} + \frac{\mathbf{q}_a}{t^{2/3}} \tag{4.20}$$

First term grows only algebraically while second term decays.

4.3 Perturbation with cosmological constant

We are going to study perturbations in Newtonian equation of cosmology containing cosmological constant term. Newton's force law with cosmological constant term can be written as

$$m_a \frac{d^2\mathbf{x}_a}{dt^2} = -\sum_{b\neq a} G m_a m_b \frac{(\mathbf{x}_a - \mathbf{x}_b)}{|\mathbf{x}_a - \mathbf{x}_b|^3} + \frac{\Lambda m_a \mathbf{x}_a}{3} \tag{4.21}$$

4.3.1 Perturbation

Background solution is given by $\bar{\mathbf{x}_a}$ and linear perturbation $\delta\mathbf{y}_a$ about this solution. So the equations become

$$m_a[\frac{d^2(\bar{\mathbf{x}_a} + \delta\mathbf{y}_a)}{dt^2}] = m_a[\frac{d^2(\bar{\mathbf{x}_a})}{dt^2} + \frac{d^2(\delta\mathbf{y}_a)}{dt^2}] + \frac{\Lambda m_a(\bar{\mathbf{x}_a} + \delta\mathbf{y}_a)}{3} \tag{4.22}$$

$$= -[\frac{\partial V(\bar{\mathbf{x}_a} + \delta\mathbf{y}_a)}{\partial \mathbf{x}_a}] \tag{4.23}$$

$$= -[\frac{\partial V(\bar{\mathbf{x}_a})}{\partial \bar{\mathbf{x}_a}} + \frac{\partial^2 V(\bar{\mathbf{x}_a})}{\partial \bar{\mathbf{x}_a} \partial \bar{\mathbf{x}_b}} . \partial \bar{\mathbf{x}_b}] \tag{4.24}$$

cancelling the background terms the perturbation equation is

$$m_a \delta\ddot{\mathbf{y}}_a = -\sum_{b\neq a} \frac{\partial^2 V}{\partial \bar{\mathbf{x}_a} \partial \bar{\mathbf{x}_a}} (\bar{\mathbf{x}_1}, \bar{\mathbf{x}_2}, ..., \bar{\mathbf{x}_N}).\delta\mathbf{y}_a + \frac{m_a \Lambda \delta\mathbf{y}_a}{3} \tag{4.25}$$

4.3.2 Force form

Given the force form

$$\mathbf{F}_{ab} = -\frac{G m_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|^3} (\mathbf{x}_a - \mathbf{x}_b) + \frac{\Lambda m_a \mathbf{x}_a}{3} \tag{4.26}$$

proceeding as shown in the general case subsection we get the perturbation equation as follows

$$m_a(\delta\ddot{\mathbf{y}}_a) = \sum_{b\neq a} \frac{G m_a m_b}{|\bar{\mathbf{x}_{ab}}|^5} \{\delta\mathbf{y}_{ba} \bar{\mathbf{x}_{ba}}^2 - 3(\bar{\mathbf{x}_{ba}}.\delta\mathbf{y}_{ba})\bar{\mathbf{x}_{ba}}\} + \frac{m_a \Lambda \delta\mathbf{y}_a}{3} \tag{4.27}$$

4.3.3 Background cosmology

As before we put homothetic factor and separation of variable

$$m_a \mathbf{r}_a \frac{d^2 S(t)}{dt^2} = -\sum_{b\neq a} G m_a m_b \frac{S(t)(\mathbf{r}_a - \mathbf{r}_b)}{S^3(t)|\mathbf{r}_a - \mathbf{r}_b|^3} + \frac{\Lambda S(t) m_a \mathbf{r}_a}{3} \tag{4.28}$$

This implies Raychsudhuri equation with cosmological constant

$$\frac{1}{S(t)}\frac{d^2 S(t)}{dt^2} = -\frac{G\tilde{M}}{S^3(t)} + \frac{\Lambda}{3}(4.29)$$

Integrate to get Friedmann equation

$$\frac{1}{2}[\frac{\tilde{S(t)}}{S(t)}]^2 = \frac{G\tilde{M}}{S^3(t)} + \frac{E}{S^2(t)} + \frac{\Lambda}{6} \tag{4.30}$$

at late times

$$\frac{1}{2}[\frac{\tilde{S(t)}}{S(t)}]^2 = \frac{\Lambda}{6} \tag{4.31}$$

if $\Lambda > 0$

$$S(t) = S_0 exp(\sqrt{\frac{\Lambda}{3}})(t - t_0), \tag{4.32}$$

is scale free solution. We can apply perturbations general formalism to homothetically expanding background solutions with cosmological constant

$$m_a \frac{d^2}{dt^2}(S(t)\mathbf{S}_a) = \sum_{b\neq a} \frac{G m_a m_b}{S^5(t)|\bar{\mathbf{r}_{ab}}|^5} S^3(t)\{(\mathbf{S}_a - \mathbf{S}_b)|\bar{\mathbf{r}_{ab}}|^2 - 3(\bar{\mathbf{r}_{ab}}.\mathbf{S}_{ab})\bar{\mathbf{r}_{ab}}\} + \frac{\Lambda m_a S(t)\mathbf{S}_a(t)}{3} \tag{4.33}$$

4.3.4 Asymptotic Solution

Rearranging above equation growth of perturbation is given by

$$\frac{d^2}{dt^2}(S\mathbf{S}_a) = \frac{1}{S^2(t)}\sum_{b\neq a} \frac{G m_b}{|\bar{\mathbf{r}_{ab}}|^5}\{(\mathbf{S}_a - \mathbf{S}_b)|\bar{\mathbf{r}_{ab}}|^2 - 3(\bar{\mathbf{r}_{ab}}.\mathbf{S}_{ab})\bar{\mathbf{r}_{ab}}\} + \frac{\Lambda S(t)\mathbf{S}_a(t)}{3} \tag{4.34}$$

The first term on the right hand side goes to zero as $S \to \infty$, Thus at late times

$$\frac{d^2}{dt^2}(S\mathbf{S}_a) = \frac{\Lambda S \mathbf{S}_a}{3} \tag{4.35}$$

Assuming $\lambda > 0$ implies

$$\mathbf{S}_a = \frac{\mathbf{S}_0 exp(\sqrt{\frac{\Lambda}{3}})(t - t_0)}{S(t)} \tag{4.36}$$

4.4 Demitriev-Zel'dovich equation

This is different approach of perturbation theory. In this approach motion of the subgroup of particles is described on background which remain unaffected. Thus moving subgroup particles not only interact with themselves but also with the background, but the background remain same. The equation we obtained for such subgroup of particles is called Demitriev-Zel'dovich equations and are time dependent. Lets start with equation of motion for large but finite number of particles.

$$m_a\ddot{\mathbf{x}}_a = -\sum_{b\neq a} \frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|^3}(\mathbf{x}_a - \mathbf{x}_b) \tag{4.37}$$

particles are divided into two classes with $a = i.j.k...$ and $a = I, J, K...$ the second set forms the cosmological background. The second set remain unaffected. by the first group of particles whose motion is however affected by both the background and mutual interaction. Equation of motion splits into two sets, for background model

$$m_I\ddot{\mathbf{x}}_I = \sum_{J\neq I} \frac{Gm_I m_J}{|\mathbf{x}_J - \mathbf{x}_I|^3}(\mathbf{x}_J - \mathbf{x}_I) \tag{4.38}$$

and for the subgroup

$$m_i\ddot{\mathbf{x}}_i = \sum_{j\neq i} \frac{Gm_i m_j}{|\mathbf{x}_j - \mathbf{x}_i|^3}(\mathbf{x}_j - \mathbf{x}_i) + \sum_{J} \frac{Gm_i m_J}{|\mathbf{x}_J - \mathbf{x}_i|^3}(\mathbf{x}_J - \mathbf{x}_i) \tag{4.39}$$

Now consider that background particles move isometrically: $\mathbf{x}_a = S(t)\mathbf{r}_a$ by the above arguments there must be central configuration and $S(t)$ obeys the Friedmann equation, then motion of the subgroup is given by

$$m_i\ddot{\mathbf{x}}_i = \sum_{j\neq i} \frac{Gm_i m_j}{|\mathbf{x}_j - \mathbf{x}_i|^3}(\mathbf{x}_j - \mathbf{x}_i) + \sum_{J} \frac{Gm_i m_J S(t)}{|S(t)(\mathbf{r}_J - \mathbf{r}_i)|^3}(\mathbf{r}_J - \mathbf{r}_i) \tag{4.40}$$

by replacing the absolute position of particles by conformally scaled $\mathbf{x}_i = S(t)\mathbf{r}_i$ and obtain

$$m_i(S(t)\ddot{\mathbf{r}}_i+2\dot{S}(t)\dot{\mathbf{r}}_i+\ddot{S}(t)\mathbf{r}_i) = \frac{1}{S^2(t)}\sum_{j\neq i}\frac{Gm_im_j}{|\mathbf{r}_j-\mathbf{r}_i|^3}(\mathbf{r}_j-\mathbf{r}_i)+\frac{1}{S^2(t)}\sum_{J}\frac{Gm_im_J}{|\mathbf{r}_J-\mathbf{r}_i|^3}(\mathbf{r}_J-\mathbf{r}_i) \tag{4.41}$$

second term on the right hand side is force F_i exerted on the ith particle on the background particle. The numerical calculations provides good evidence that central configuration is to a very good approximation statistically spherically symmetric and homogeneous. It follows that force exerted by the background is radial.

$$\frac{1}{S^2(t)}\sum_{J}\frac{Gm_im_J}{|\mathbf{r}_J-\mathbf{r}_i|^3}(\mathbf{r}_J-\mathbf{r}_i) = -G\tilde{M}m_i\mathbf{r}_i \tag{4.42}$$

where $S^2\ddot{S} = -G\tilde{M}$. then force term on right side and third term on the left side cancels. we are left with

$$m_i(S(t)\ddot{\mathbf{r}}_i+2\dot{S}(t)\dot{\mathbf{r}}_i) = \frac{1}{S^2(t)}\sum_{j\neq i}\frac{Gm_im_j}{|\mathbf{r}_j-\mathbf{r}_i|^3}(\mathbf{r}_j-\mathbf{r}_i) \tag{4.43}$$

that is

$$\frac{d(S^2(t)\dot{\mathbf{r}}_i)}{dt} = \frac{1}{S(t)}\sum_{j\neq i}\frac{Gm_j}{|\mathbf{r}_j-\mathbf{r}_i|^3}(\mathbf{r}_j-\mathbf{r}_i) \tag{4.44}$$

These are the Demitriev-Zel'dovich equations.

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