Quark Mass Matrices & Textures

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Ever tried. Ever failed. No matter. Try again. Fail again. Fail better.

- Samuel Beckett

Certificate of Examination

This is to certify that the dissertation titled "**Quark Mass Matrices & Textures**"submitted by Mr. Ashish Thakur (Reg No. MP12009) for the partial fulfillment of MS degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute.The committee finds the work done by the candidate satisfactory and recommends the report to be accepted.

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Dated: 07-08-2015

Declaration

This work presented in the dissertation has been carried out by me under the guidance of Prof. Manmohan Gupta (Panjab University) at the Indian Institute of Science Education, Mohali.
Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

Ashish Thakur
Dated: 07-08-2015

In my capacity as the supervisor of the candidate's project work, I certify the above statements by the candidate are true to the best of my knowledge.

Prof. Manmohan Gupta
(Supervisor)

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Abstract

Texture specific mass matrices provide a good example of “Bottom-Up”approach to deal with the fermion mass matrices and their implications for flavour physics. In the context of quarks, we have studied the implication of “Weak Basis”transformations and the naturalness condition. Interestingly, we find that the present data related to quark mixings and masses allow us to deduce almost a unique set of viable quark mass matrices.

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Chapter 1

The Standard Model & Fermion Mass Matrices

1.1 Introduction

Our present understanding of the fundamental particles and their interactions is neatly encapsulated in a theory know as "The Standard Model". The present form of the model surfaced in the late 1970s, almost after a two decade long endeavour. The Standard Model provides a remarkable insight into the fundamental structure of matter and their interactions. The model has successfully predicted a wide variety of phenomena which were later on confirmed by experiments with unprecedented precision. It has also explained almost all the experimental results.

The constituents of the SM can be broadly categorized as: matter forming and force carrying. The world around us is built of elementary particles. These elementary particles occur in two basic types called quarks and leptons. In The Standard Model both quarks and leptons come in six flavours (types). The quarks and leptons are responsible for the matter formation and they interact with each other through the exchange of force carriers known as the gauge bosons. The Standard Model incorporates the three out of the four fundamental forces, namely the weak force, the electromagnetic force and the strong force.

In mathematical parlance, the standard model is a quantum field theory based on the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ where $SU(3)$ is the gauge group of the strong interaction and $SU(2) \times U(1)$ is the gauge group of the electroweak interaction. The Standard Model, in spite of its impressive success has many unexplained features. The questions, such as "What is dark matter?", or "Why universe contains more matter than antimatter?", "Why are there exactly three generations of fermions with different mass scales?", don't find an answer within the standard model. It has also very little to say about the origin of electroweak symmetry breaking, smallness of neutrino masses and the origin of flavour mixing. The presence of a large number of free parameters in the SM also points towards its incompleteness. The free parameters include six quark mass masses, three mixing angles, three charged lepton masses, three gauge couplings, two parameters for Higgs potential, one CP violating phase in the quark sector, one strong CP parameter which add up to a total of nineteen parameters. The presence of an arbitrarily large number of parameters forces us to re-evaluate the status of the SM as a true fundamental theory. The suspicion is that the SM is merely an effective theory which has its origin in a more fundamental, yet unknown theory. It's important to highlight the fact that most of the free parameters reside in the fermionic sector also known as the Yukawa sector of the SM. Therefore, it's quite natural to assume that any new effort to understand the physics beyond the standard model should keep fermionic sector at its core. The phenomenological models attempting to reveal the mystery of fermion masses and mixings broadly fall into two categories, viz., "top-down"approach and "bottom-up"approach. In the top-down approach fermion masses are formulated using certain fundamental principles like grand unification, supersymmetry, horizontal symmetries, extra dimensions etc..

The bottom-up approach of understanding the flavour problem has progressed along three different directions. Firstly, on the lines of Fritzsch, the mass matrices are formulated in such a way that certain elements are assumed to be zero. The viability of mass matrices hence obtained are ensured by checking them against the low energy data obtained from experiments.

The other approach involves the freedom to make unitary transformations, referred to as the "Weak Basis (WB) Transformations"which only affect the mass matrices without changing the mixing matrices. WB transformations result in the reduction of free parameters of a general mass matrix.

The third approach put forward by Peccei and Wang relies on formulating "Natural Mass Matrices"wherein the elements of these matrices imitate the hierarchical structure of the CKM matrix.

The outline of the thesis is the following. In Chapter 1, we introduce the idea of fermion mass matrices and quark mass matrices. Chapter 2 discusses the current landscape of flavour mixing and efforts to understand that in the light of texture zeroes and weak basis transformations [1]. The idea of natural mass matrices have been discussed in Chapter 3 [2]. In Chapter 4, we explore the possibility of quark mass matrices which are in tune with the data [3].

1.2 Fermion Mass Matrices

Within the SM, the fermions are considered to be the elementary particles. The notion of elementary particles has kept on evolving with time. The advent of powerful accelerators have led us to probe deeper into the structure of matter and we are somewhat confident about our current classification of elementary particles. At the level of our current understanding, the elementary particles are quarks and leptons which fall into three distinct generations.

$$\text{Quarks: } \begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix},$$

$$\text{Leptons: } \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}$$

In the standard model of strong, weak and electromagnetic interactions the Brout-Englert-Higgs mechanism provides a consistent framework to generate masses for gauge bosons and fermions. The fermions acquire masses, after the spontaneous symmetry breaking of $SU(2) \times U(1)$ gauge group to $U(1)$, through the Yukawa couplings and the vacuum expectation value of the neutral Higgs field. The Lagrangian of the Yukawa sector of the standard model reads [4]:

$$\mathcal{L} = Y_d^{ij} \bar{Q}_L^i \phi D_R^j + Y_u^{ij} \bar{Q}_L^i \tilde{\phi} U_R^j + Y_e^{ij} \bar{L}_L^i \phi E_R^j + h.c. \tag{1.1}$$

where ϕ is the Higgs doublet under $SU(2)$ and $\tilde{\phi} = \iota \tau_2 \phi^\dagger$

Here, Y_u, Y_d, Y_e are 3×3 matrices with 36 real parameters each. After the SSB, the Higgs acquire a vacuum expectation value (VEV) v

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix}, \tilde{\phi} = \frac{1}{\sqrt{2}} \begin{pmatrix} v+h \\ 0 \end{pmatrix} \tag{1.2}$$

which leads to the introduction of undiagonalized 3×3 quark mass matrices (ignoring the lepton part for present purpose)

$$M_u^{ij} = \frac{v}{\sqrt{2}} Y_u^{ij} \tag{1.3}$$

$$M_d^{ij} = \frac{v}{\sqrt{2}} Y_d^{ij} \tag{1.4}$$

In the most general case, the above mass matrices contain 36 parameters (18 each) in total. To simply things, we invoke the polar decomposition theorem of matrix algebra; by which a general complex matrix can be written as a product of a hermitian and unitary matrix. In the SM, the unitary matrix can be absorbed by a rotation on right-handed quark fields. This makes all the mass matrices hermitian and brings down the number of free parameters from 36 to 18.

1.3 Quark Mass Matrices

The origin of quark mass matrices lies in the Higgs fermion couplings. These matrices, M_U and M_D are arbitrary. The total number of free parameters (36 in case of two 3×3 complex matrix) are greater than the number of observables. When the mass matrices are considered hermitian, the total number of free parameters reduces from 36 to 18. The matrices M_U and M_D have to produce six observables, i.e., six quark masses, three mixing angles and a CP violating phase.

In the general case mass terms are quadratic in terms of fermion fields. The quark mass terms, below the electroweak symmetry breaking, read

$$\bar{Q}_{U_L} M_U Q_{U_R} + \bar{Q}_{D_L} M_D Q_{D_R} \tag{1.5}$$

where $Q_{U_{L(R)}}$ and $Q_{D_{L(R)}}$ are left handed (right handed) quark fields for up sector (u, c, t) and down sector (d, s, b) respectively. The matrices M_U and M_D are for the up and down sector quarks respectively. The above equation has to be re-expressed in terms of physical quark fields to make any sense. This is achieved by diagonalizing the mass matrices via bi-unitary transformations.

$$V_{U_L}^{\dagger} M_U V_{U_R} = {M_U}^{diag} \equiv \text{diag}\,(m_u, m_c, m_t) \tag{1.6}$$

$$V_{D_L}^{\dagger} M_D V_{D_R} = {M_D}^{diag} \equiv \text{diag}\,(m_d, m_s, m_b) \tag{1.7}$$

where m_u, m_d, etc. are eigenvalues of the quark mass matrices which correspond to physical quark masses. The equation (1.5) can be re-written using Eqs. (1.6) and (1.7) as

$$\bar{Q}_{U_L} V_{U_L} M_U^{diag} V_{U_R}^{\dagger} Q_{U_R} + \bar{Q}_{D_L} V_{D_L} M_D^{diag} V_{U_D}^{\dagger} Q_{U_D} \tag{1.8}$$

which in terms of physical fields are

$$\bar{Q}_{U_L}^{phys} M_U^{diag} Q_{U_R}^{phys} + \bar{Q}_{D_L}^{phys} M_D^{diag} Q_{D_R}^{phys} \tag{1.9}$$

where $Q_{U_L}^{phys} = V_{U_L}^{\dagger} Q_{U_L}$ and $Q_{D_L}^{phys} = V_{D_L}^{\dagger} Q_{D_L}$ and so on. The mismatch in the diagonalization of up and down matrices leads to the definition of quark mixing matrix, known as the Cabibbo-Kobayashi-Maskawa (CKM) matrix, given by

$$V_{CKM} = V_{U_L}^{\dagger} V_{D_L} \tag{1.10}$$

The CKM matrix describes the weak interaction eigenstates (d', s', b') of the quarks in terms of their flavour eigenstates (d, s, b), e.g.,

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{\text{ud}} & V_{\text{us}} & V_{\text{ub}} \\ V_{\text{cd}} & V_{\text{cs}} & V_{\text{cb}} \\ V_{\text{td}} & V_{\text{ts}} & V_{\text{tb}} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{1.11}$$

The CKM matrix is a unitary matrix which describes the transition of one quark into another. A general $n \times n$ unitary matrix has n^2 parameters, $\frac{n(n-1)}{2}$ of these are the Eulers angles and remaining $\frac{n(n+1)}{2}$ are the phases. However, some of these phases can be rotated away. So, in a $n \times n$ we are left with only $\frac{(n-1)(n-2)}{2}$ measurable phases. Thus, in the case of three families of quarks, the mixing matrix is expressed in terms of three angles and one phase, the latter being responsible for CP violation.

The SM imposes the unitarity constraint on the quark mixing matrix. The unitarity of CKM matrix leads to nine relations, three being the normalization conditions and the rest six are non-diagonal relations which are defined in the follwing way

$$\sum_{\alpha=d,s,b} V_{i\alpha} V^*_{j\alpha} = \delta_{ij} \tag{1.12}$$

$$\sum_{i=u,c,t} V_{i\alpha} V^*_{i\beta} = \delta_{\alpha\beta} \tag{1.13}$$

where the Greek indices run over the down type quarks (d, s, b) and the Latin ones run over the up type quarks(u, c, t).

Chapter 2

Weak Basis Transformations

2.1 The Technology

Understanding fermion masses and mixings is one of the fundamental problems in high energy physics. In the absence of any compelling theoretical framework, the issues concerning fermion mixings and masses are understood with "Bottom Up" approaches. Texture specific mass matrices provide a good example of "Bottom Up" approach to have a viable description of fermion mixing and masses. The mass matrices in the Standard Model are completely arbitrary 3×3 complex matrices. However, they can be reduced to hermitian matrices without loss of generality. The reduction of the matrices to the hermitian form brings down the number of free parameters by half. However, the above prescription still leaves us with eighteen free parameters which are still in excess when compared to the number of observables, viz. six quark masses, three mixing angles and a CP violating phase. To account for this redundancy, we require some additional assumptions. In this context the concept of textures was introduced implicitly by Weinberg [5] and explicitly by Fritzsch [6], where in certain elements of the mass matrices are assumed to be highly suppressed or can be considered zero also. The zero elements of the mass matrices can be characterized as texture zeros defined in a particular manner.

A particular texture structure is said to be texture n zero, if it has n number of non-trivial zeros,for example, if the sum of the number of diagonal zeros and half the number of the symmetrically placed off diagonal zeros is n.

The Fritzsch's-like texture specific hermitian quark mass matrices have the following form.

$$M_U = \begin{pmatrix} 0 & A_U & 0 \\ A^*{}_U & D_U & B_U \\ 0 & B^*{}_U & C_U \end{pmatrix}, \; M_D = \begin{pmatrix} 0 & A_D & 0 \\ A^*{}_D & D_D & B_D \\ 0 & B^*{}_D & C_D \end{pmatrix} \tag{2.1}$$

Here, $A_i = |A_i| \exp^{\iota\alpha_i}$ and $B_i = |B_i| \exp^{\iota\beta_i}$ with $i = U, D$. Each of the above matrix is texture 2 zero type.

One particular facility available to achieve texture zeroes is of the Weak Basis Transformations. Branco et al [1] initiated the idea of WB transformations to introduce the texture zeroes compatible with the SM so as to lend predictability to the general mass matrices. Initially, texture zeroes were introduced as ansatz. However, efforts have been made to deduce these from symmetry considerations as well as from general considerations. In this chapter we would attempt the introduction of textures though general considerations.
In the SM one has the freedom to make a unitary transformation W on the quark fields e.g.,

$$q_L \to Uq_L,\ q_R \to Uq_R,\ q'_L \to Uq'_L,\ q'_R \to Uq'_R \tag{2.2}$$

under which gauge currents

$$\mathcal{L}_W = \frac{g}{\sqrt{2}}\overline{(u,c,t)}\gamma^\mu \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L W_\mu + hc \tag{2.3}$$

remain real and diagonal but the mass matrices transform as

$$M_u \to U^\dagger M_u U\ ,\ M_d \to U^\dagger M_d U \tag{2.4}$$

2.2 The (1,1) Weak Basis Zero

It is interesting to note that certain sets of zeroes in a texture specific mass matrices may be devoid of any physical significance. These zeroes can be obtained through appropriate WB transformations on arbitrary quark mass matrices. WB transformations only affect the mass matrices. The gauge currents remain real and diagonal under WB transformations. The quark mass matrices related by WB transformations display the same physical content.
In this section we present the results of Branco et al. [1]. We discuss the zeroes occurring at (1,1) position in up and down quark mass matrices. The most general transformation that leaves the mass matrices hermitian is:

$$M_u \longrightarrow M'_u = U^\dagger M_u U \tag{2.5a}$$

$$M_d \longrightarrow M'_d = U^\dagger M_d U \tag{2.5b}$$

where U is an arbitrary unitary matrix. In such a basis, we can always find a set of unitary matrices $\{U_u, U_d\}$ which can diagonalize the mass matrices such that

$$D'_u = U_u^\dagger M_u U_u \tag{2.6a}$$

$$D'_d = U_d^\dagger M_d U_d \tag{2.6b}$$

where $D_u \equiv$ diag (m_u, m_c, m_t) and $D_d \equiv$ diag (m_d, m_s, m_b). We choose to work in basis where M_u

is diagonal and M_d is hermitian, i.e.

$$M_u = D_u \quad (2.7a)$$
$$M_d = V D_d V^\dagger \quad (2.7b)$$

The matrix V is an arbitrary unitary matrix. Effecting a WB transformation with U, under which M_u and M_d transform as:

$$M_u \longrightarrow M'_u = U^\dagger D_u U, \quad (2.8a)$$
$$M_d \longrightarrow M'_d = U^\dagger V D_d V^\dagger U \quad (2.8b)$$

that $(M'_u)_{11} = (M'_d)_{11} = 0$. This requires the solution of the following system of equations.

$$m_u \mid U_{11} \mid^2 + m_c \mid U_{12} \mid^2 + m_t \mid U_{31} \mid^2 = 0 \quad (2.9a)$$
$$m_d \mid X_{11} \mid^2 + m_s \mid X_{12} \mid^2 + m_b \mid X_{31} \mid^2 = 0 \quad (2.9b)$$
$$\mid U_{11} \mid^2 + \mid U_{12} \mid^2 + \mid U_{13} \mid^2 = 1 \quad (2.9c)$$

where $X = V^\dagger U$ and thus:

$$|X_{i1}|^2 = |V_{1i}|^2|U_{11}|^2 + |V_{2i}|^2|U_{21}|^2 + |V_{3i}|^2|U_{31}|^2 + 2Re(V^*_{1i}U_{11}V_{2i}U^*_{21}) + 2Re(V^*_{1i}U_{11}V_{3i}U^*_{31}) + 2Re(V^*_{2i}U_{21}V_{3i}U^*_{31}), \quad \text{(i = 1, 2, 3)} \quad (2.10)$$

The system of Eqs. (2.9) has a real solution only if, at least one of the mass parameters m_u, m_c, m_t and one of the parameters m_d, m_s, m_b is negative. For the arbitrary mass matrices M_u and M_d, one has to find a unique U satisfying (2.9). It is not always possible to find analytic solutions for (Eqn 2.9). For the simple case, when $V = \mathbb{1}$, $X = U$ and we obtain the follwing solutions:

$$\mid U_{11} \mid^2 = \frac{m_c m_b - m_s m_t}{\Delta} \quad (2.11a)$$
$$\mid U_{21} \mid^2 = \frac{m_d m_t - m_u m_b}{\Delta} \quad (2.11b)$$
$$\mid U_{31} \mid^2 = \frac{m_u m_s - m_d m_c}{\Delta} \quad (2.11c)$$

where

$$\Delta = (m_t - m_u)(m_b - m_s) - (m_t - m_c)(m_b - m_d) \quad (2.12)$$

Next, if we choose V to be a realistic CKM matrix

$$V = \begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.13)$$

In this case, Eqs.(2.9) become

$$\begin{aligned} \mid X_{11} \mid^2 &= cos^2\theta \mid U_{11} \mid^2 + sin^2\theta \mid U_{21} \mid^2 - sin2\theta\, U_{11}U_{21} && (2.14a) \\ \mid X_{21} \mid^2 &= sin^2\theta \mid U_{11} \mid^2 + cos^2\theta \mid U_{21} \mid^2 + sin2\theta\, U_{11}U_{21} && (2.14b) \\ \mid X_{31} \mid^2 &= \mid U_{31} \mid^2 && (2.14c) \end{aligned}$$

Using unitarity, we can write

$$(m_u - m_t) \mid U_{11} \mid^2 + (m_c - m_t) \mid U_{21} \mid^2 + m_t = 0 \qquad (2.15a)$$

$$\begin{aligned} (m_d cos^2\theta + m_s sin^2\theta - m_b) \mid U_{11} \mid^2 + m_d sin^2\theta + m_s cos^2\theta - m_b) \mid U_{21} \mid^2 \\ + (m_s - m_d)\, sin2\theta\, U_{11}U_{21} + m_b = 0 \end{aligned} \qquad (2.15b)$$

Parametrizing the solutions as:

$$\sqrt{m_t - m_u}\, U_{11} = \sqrt{m_t}\, cos\phi \qquad (2.16a)$$

$$\sqrt{m_t - m_u}\, U_{21} = \sqrt{m_t}\, sin\phi \qquad (2.16b)$$

Denoting

$$\begin{aligned} a &= m_b - (m_b - m_d sin^2\theta - m_s cos^2\theta)\frac{m_t}{m_t - m_c} && (2.17a) \\ b &= (m_s - m_d)\frac{m_t sin2\theta}{\sqrt{(m_t - m_u)(m_t - m_c)}}, && (2.17b) \\ c &= m_b - (m_b - m_d cos^2\theta - m_s sin^2\theta)\frac{m_t}{m_t - m_u} && (2.17c) \end{aligned}$$

introducing $z \equiv \tan\phi$, the solution is given by the quadratic equation

$$az^2 + bz + c = 0 \qquad (2.18)$$

If $\theta = 0$ and $V = \mathbb{1}$, we recover the results of Eqs. (2.11).

2.3 The (One Three, Three One) Problem

In this section we present our attempts and partial results to obtain texture two zero matrices from the most general 3×3 unitary matrix using the recipe of weak basis transformations. Fritzch in his paper [7] had discussed the possibility of achieving the texture two form given below,

$$M_U = \begin{pmatrix} E_U & A_U & 0 \\ A_U^* & D_U & B_U \\ 0 & B_U^* & C_U \end{pmatrix},\ M_D = \begin{pmatrix} E_D & A_D & 0 \\ A_D^* & D_D & B_D \\ 0 & B_D^* & C_D \end{pmatrix} \qquad (2.19)$$

starting from the hermitian mass matrices,

$$M_q = \begin{pmatrix} E_q & A_q & F_q \\ A_q^* & D_q & B_q \\ F_q^* & B_q^* & C_q \end{pmatrix}, \ (q = U, D) \tag{2.20}$$

through a common unitary transformation. We tried to find out the exact form of the unitary matrix which accomplishes this task. We start by choosing a basis in which M_U is diagonal and M_D hermitian.

$$M_U = \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{pmatrix}, \ M_D = \begin{pmatrix} \mu_{11} & \mu_{12}e^{i\eta_{12}} & \mu_{13}e^{i\eta_{13}} \\ \mu_{12}e^{-i\eta_{12}} & \mu_{22} & \mu_{23}e^{i\eta_{23}} \\ \mu_{13}e^{-i\eta_{13}} & \mu_{23}e^{-i\eta_{23}} & \mu_{33} \end{pmatrix} \tag{2.21}$$

The unitary matrix for effecting the weak basis transformation is the following :

$$U = U_1 \begin{pmatrix} \cos\alpha\cos\gamma & \sin\alpha\cos\gamma & \sin\gamma e^{i(\alpha_3-\delta)} \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta\sin\gamma e^{i\delta} & \cos\alpha\cos\beta - \sin\alpha\sin\beta\sin\gamma e^{i\delta} & \sin\beta\cos\gamma \\ \sin\alpha\sin\beta - \cos\alpha\cos\beta\sin\gamma e^{i\delta} & -\cos\alpha\sin\beta - \sin\alpha\cos\beta\sin\gamma e^{i\delta} & \cos\beta\cos\gamma \end{pmatrix} U_2 \tag{2.22}$$

where U_1 and U_2 are given by

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i(\alpha_4-\alpha_3)} & 0 \\ 0 & 0 & e^{i(\alpha_5-\alpha_3)} \end{pmatrix}, \ U_2 = \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & e^{i\alpha_3} \end{pmatrix} \tag{2.23}$$

The result of the weak basis transformation on the matrices is the following.

$$M'_U = U^\dagger M_U U \tag{2.24a}$$

$$M'_D = U^\dagger M_D U \tag{2.24b}$$

Since, we are interested in only $(M'_U)_{13}$ and $(M'_D)_{13}$, we study the transformation of only those elements.

$$(M'_D)_{13} = {U^\dagger}_{1i}(M_D)_{ij}U_{j3} \tag{2.25}$$

, where i, j = 1, 2, 3 or

$$\begin{aligned} (M'_D)_{13} = & {U^\dagger}_{11}\{M_{11}U_{13} + M_{12}U_{23} + M_{13}U_{33}\}+ \\ & {U^\dagger}_{12}\{M_{21}U_{13} + M_{22}U_{23} + M_{23}U_{33}\}+ \\ & {U^\dagger}_{13}\{M_{31}U_{13} + M_{32}U_{23} + M_{33}U_{33}\} \end{aligned} \tag{2.26}$$

which translates into

$$
\begin{aligned}
(M'_D)_{13} = 0 = \mu_{11}\cos\alpha\cos\gamma\sin\gamma e^{i(\alpha_3-\alpha_1-\delta)} + \\
\mu_{22}\sin\beta\cos\gamma(\sin\alpha\cos\beta - \cos\alpha\sin\beta\sin\gamma e^{-i\delta})e^{i(\alpha_3-\alpha_1)} \\
+\mu_{33}\cos\beta\cos\gamma(\sin\alpha\sin\beta - \cos\alpha\cos\beta\sin\gamma e^{-i\delta})e^{i(\alpha_3-\alpha_1)} \\
+\mu_{12}[\cos\alpha\cos^2\gamma\sin\beta e^{i(\alpha_4-\alpha_1+\eta_{12})} + sin\gamma(\sin\alpha\cos\beta - \cos\alpha\sin\gamma e^{-i\delta})e^{i(2\alpha_3-\alpha_1-\alpha_4-\eta_{12}-\delta)}] \\
+\mu_{13}[\cos\alpha\cos\beta\cos^2\gamma e^{i(\alpha_4-\alpha_1+\eta_{13})} + \sin\gamma(\sin\alpha\sin\beta - cos\alpha\sin\beta\sin\gamma e^{-i\delta})e^{i(2\alpha_3-\alpha_5-\alpha_1-\eta_13-\delta)}] \\
+\mu_{23}[\cos\beta\cos\gamma(\sin\alpha\cos\beta - \cos\alpha\sin\beta\sin\gamma e^{-i\delta}) \\
e^{i(\alpha_5+\alpha_3-\alpha_4-\alpha_1+\eta_{23})} + \sin\beta\cos\gamma(\sin\alpha\sin\beta - \cos\alpha\cos\beta\sin\gamma e^{-i\delta})e^{i(\alpha_4+\alpha_3-\alpha_5-\alpha_1-\eta_{23})}]
\end{aligned}
\tag{2.27}
$$

Similarly, the other equation is:

$$
\begin{aligned}
(M'_U)_{13} = 0 = m_{11}\cos\alpha\cos\gamma\sin\gamma e^{i(\alpha_3-\alpha_1-\delta)} \\
+m_{22}\sin\beta\cos\gamma(\sin\alpha\cos\beta - \cos\alpha\sin\beta\sin\gamma e^{-i\delta})e^{i(\alpha_3-\alpha_1)} \\
+m_{33}\cos\beta\cos\gamma(\sin\alpha\sin\beta - \cos\alpha\cos\beta\sin\gamma e^{-i\delta})e^{i(\alpha_3-\alpha_1)}
\end{aligned}
\tag{2.28}
$$

Now, we have to simultaneously solve Eqs. (2.27 & 2.28). We make the following assumptions to simplify the above equations.

$$
\begin{aligned}
\alpha_3 &= \alpha_1 \\
\delta &= 0 \\
\alpha_4 - \alpha_3 + \eta_{12} &= 0 \\
\alpha_5 - \alpha_3 + \eta_{13} &= 0 \\
\alpha_5 - \alpha_4 + \eta_{23} &= 0
\end{aligned}
\tag{2.29}
$$

The assumptions of Eqn. (2.29), along with $\gamma = 0$ reduces Eqn. (2.28) to

$$m_{22}\sin\alpha\sin 2\beta + m_{33}\sin\alpha\sin 2\beta = 0 \tag{2.30}$$

$\implies$ either $\sin\alpha = 0$ or $\sin 2\beta(m_{22} + m_{33}) = 0$. If $\sin\alpha \neq 0$,then

$$\sin 2\beta(m_{22} + m_{33}) = 0 \tag{2.31}$$

which gives $\beta = 0, \frac{\pi}{2}$. $\gamma = 0$ and $\beta = 0$, reduces Eqn. (2.27) to

$$\mu_{13}\cos\alpha + \mu_{23}\sin\alpha = 0 \tag{2.32a}$$

$$\tan\alpha = \frac{-\mu_{13}}{\mu_{23}} \tag{2.32b}$$

whereas $\gamma = 0$ and $\beta = \frac{\pi}{2}$, reduces Eqn. (2.27) to

$$\tan\alpha = \frac{-\mu_{12}}{\mu_{23}} \tag{2.33}$$

On the other hand, if $\sin\alpha = 0 \implies \alpha = 0$

We obtain yet another solution with $\alpha = 0$ and $\gamma = 0$ which is

$$\tan\beta = \frac{-\mu_{12}}{\mu_{13}} \tag{2.34}$$

With $\gamma = 0$ $\beta = \frac{\pi}{2}$ and $\tan\alpha = \frac{-\mu_{12}}{\mu_{23}}$, the matrix U becomes

$$U = \begin{pmatrix} \frac{\mu_{23}}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & -\frac{\mu_{12}}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & 0 \\ 0 & 0 & 1 \\ -\frac{\mu_{12}}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & -\frac{\mu_{23}}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & 0 \end{pmatrix} \tag{2.35}$$

By virtue of Eqn. (2.24a), M'_U becomes

$$M'_U = \begin{pmatrix} \frac{m_{33}\mu_{12}^2}{\mu_{12}^2+\mu_{23}^2} + \frac{m_{11}\mu_{23}^2}{\mu_{12}^2+\mu_{23}^2} & \frac{m_{33}\mu_{12}\mu_{23}}{\mu_{12}^2+\mu_{23}^2} - \frac{m_{11}\mu_{12}\mu_{23}}{\mu_{12}^2+\mu_{23}^2} & 0 \\ \frac{m_{33}\mu_{12}\mu_{23}}{\mu_{12}^2+\mu_{23}^2} - \frac{m_{11}\mu_{12}\mu_{23}}{\mu_{12}^2+\mu_{23}^2} & \frac{m_{11}\mu_{12}^2}{\mu_{12}^2+\mu_{23}^2} + \frac{m_{33}\mu_{23}^2}{\mu_{12}^2+\mu_{23}^2} & 0 \\ 0 & 0 & m_{22} \end{pmatrix} \tag{2.36}$$

Similary, Eqn. (2.24b) leads to

$$M'_D = \begin{pmatrix} \frac{\mu_{23}\left(\frac{\mu_{11}\mu_{23}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}-\frac{\mu_{12}\mu_{13}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}\right)}{\sqrt{\mu_{12}^2+\mu_{23}^2}} - \frac{\mu_{12}\left(\frac{\mu_{13}\mu_{23}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}-\frac{\mu_{12}\mu_{33}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}\right)}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & -\frac{\mu_{12}\left(\frac{\mu_{11}\mu_{23}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}-\frac{\mu_{12}\mu_{13}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}\right)}{\sqrt{\mu_{12}^2+\mu_{23}^2}} - \frac{\mu_{23}\left(\frac{\mu_{13}\mu_{23}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}-\frac{\mu_{12}\mu_{33}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}\right)}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & 0 \\ \frac{\mu_{23}\left(-\frac{\mu_{11}\mu_{12}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}-\frac{\mu_{13}\mu_{23}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}\right)}{\sqrt{\mu_{12}^2+\mu_{23}^2}} - \frac{\mu_{12}\left(-\frac{\mu_{12}\mu_{13}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}-\frac{\mu_{23}\mu_{33}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}\right)}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & -\frac{\mu_{12}\left(-\frac{\mu_{11}\mu_{12}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}-\frac{\mu_{13}\mu_{23}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}\right)}{\sqrt{\mu_{12}^2+\mu_{23}^2}} - \frac{\mu_{23}\left(-\frac{\mu_{12}\mu_{13}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}-\frac{\mu_{23}\mu_{33}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}\right)}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & -\frac{\mu_{12}^2}{\sqrt{\mu_{12}^2+\mu_{23}^2}} - \frac{\mu_{23}^2}{\sqrt{\mu_{12}^2+\mu_{23}^2}} \\ 0 & -\frac{\mu_{12}^2}{\sqrt{\mu_{12}^2+\mu_{23}^2}} - \frac{\mu_{23}^2}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & \mu_{22} \end{pmatrix} \tag{2.37}$$

We notice that M_d has been put in the texture two zero form (Eqn. 2.19) though the weak basis transformation but the same form couldn't be achieved for M_u. We have additional zeroes on symmetrical positions (2,3) & (3,2). Efforts were made to get rid of these zeroes using another weak basis transformation but that couldn't be achieved without destroying zeroes at (1,3) & (3,1) position.

Chapter 3

Natural Mass Matrices

3.1 Preliminaries

The elements of the quark mixing matrix display a well defined hierarchy. Peccei and Wang used this hierarchy to reconstruct the quark mass matrices which are referred to as "natural mass matrices" [2]. The key idea is to manifest the hierarchical structure CKM matrix in the elements of the mass matrices by avoiding fine tuning. In this chapter, we review the construction of these natural mass matrices.

In its standard form the famous Cabibbo-Kobayashi-Maskawa matrix is

$$[CKM] = \begin{pmatrix} c_1c_3 & s_1c_3 & s_3e^{-i\delta} \\ -s_1c_2 - c_1s_2s_3e^{i\delta} & c_1c_2 - s_1s_2s_3e^{i\delta} & s_2c_3 \\ s_1s_2 - c_1c_2s_3e^{i\delta} & -c_1s_2 - s_1c_2s_3e^{i\delta} & c_2c_3 \end{pmatrix} \tag{3.1}$$

With the help of experimental hierarchy in the mixing angles, one can define $s_1 \equiv \sin\theta_1 \equiv \lambda \simeq 0.22,\ s_2 \equiv \sin\theta_2 \equiv A\lambda^2,\ s_3 \equiv \sin\theta_3 \equiv A\sigma\lambda^3$ with A, σ being of $O(1)$. The CKM matrix assumes the Wolfenstein form

$$[CKM] = \begin{pmatrix} 1 - \frac{\lambda^2}{2} - \frac{\lambda^4}{8} & \lambda & A\sigma\lambda^3e^{-i\delta} \\ -\lambda & 1 - \frac{\lambda^2}{2} - (\frac{A^2}{2} + \frac{1}{8})\lambda^4 & A\lambda^2 \\ A\lambda^3(1-\sigma e^{i\delta}) & -A\lambda^2 + A\frac{\lambda^4}{2} & 1 - \frac{A^2\lambda^4}{2} \end{pmatrix} \tag{3.2}$$

3.2 The Notion of Naturalness in Two Generation

In a general 2×2 hermitian mass matrix for the first two quark families, the phases can be rotated away. The matrix thus obtained is a real symmetric matrix. The orthogonal matrix O_u and O_d diagonalizing, the matrices for u quark and d quark, $M_u \equiv m_c\tilde{M}_u$, $M_d \equiv m_s\tilde{M}_d$ results in a Cabibbo quark mixing matrix C. We have

$$O_u^T \tilde{M}_u O_u = \tilde{M}_u^{diag} \equiv \begin{pmatrix} \xi_{uc}\lambda^4 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.3}$$

$$O_d^T \tilde{M}_d O_d = \tilde{M}_d^{diag} \equiv \begin{pmatrix} \xi_{ds}\lambda^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.4}$$

with

$$C = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -sin\theta_C & \cos\theta_C \end{pmatrix} \tag{3.5}$$

The matrices O_u and O_d has the same form as the matrix C. In Eqn.

(3.5), $\theta_C \; = \theta_d - \theta_u$.

Since $\sin\theta_C \equiv \lambda \ll 1$, it possible to have have both $\sin\theta_u \equiv \lambda \ll 1$ and $\sin\theta_d \equiv \lambda \ll 1$. This leads us to three possibilities for the angles θ_u and θ_d.

$$\begin{aligned} &(a)\ \sin\theta_d \equiv \lambda,\ \sin\theta_u \equiv \lambda \\ &(b)\ \sin\theta_d \equiv \lambda,\ \sin\theta_u \lesssim \lambda^2 \\ &(c)\ \sin\theta_d \lesssim \lambda^2,\ \ \sin\theta_u \equiv \lambda \end{aligned} \tag{3.6}$$

Evaluating $\tilde{M}_u$ and $\tilde{M}_d$, we obtain

$$\tilde{M}_u = O_u \tilde{M}_u^{diag} O_u^T \simeq \begin{pmatrix} \xi_{uc}\lambda^4 + \sin^2\theta_u & \sin\theta_u \\ sin\theta_u & 1 \end{pmatrix} \tag{3.7}$$

$$\tilde{M}_d = O_d \tilde{M}_d^{diag} O_d^T \simeq \begin{pmatrix} \xi_{ds}\lambda^2 + \sin^2\theta_u & \sin\theta_d \\ sin\theta_d & 1 \end{pmatrix} \tag{3.8}$$

The equations (3.6 a) & (3.6 c) require fine tuning of the matrix element $[\tilde{M}_u]_{11}$. Such a fine tuning seems unnatural. Consequently, the to generation mass matrices assume the following form corresponding to Eqn. (3.6 b).

$$\tilde{M}_u \simeq \begin{pmatrix} \alpha'_u\lambda^4 & \alpha_u\lambda^2 \\ \alpha_u\lambda^2 & 1 \end{pmatrix},\ \tilde{M}_d \simeq \begin{pmatrix} \alpha'_d\lambda^2 & \alpha_d\lambda \\ \alpha_d\lambda & 1 \end{pmatrix} \tag{3.9}$$

where

$\sin\theta_u = \alpha_u\lambda^2,\ \sin\theta_d = \alpha_d\lambda\,, {\alpha'_u}^2 - {\alpha_u}^2 = \xi_{uc},\ \ {\alpha'_d}^2 - {\alpha_d}^2 = \xi_{ds}$

3.3 Three Generation Extension

The three generation extension of mass matrices is based on a perturbative expansion in λ. We begin with the 3×3 hermitian mass matrices $M_u \equiv m_t(m_t)\tilde{M}_u$ and $M_d \equiv m_b(m_t)\tilde{M}_d$ which can be diagonalized by unitary matrices U and D.

$$\tilde{M}_u = U\tilde{M}_u^{diag}U^\dagger \tag{3.10}$$

$$\tilde{M}_d = D\tilde{M}_d^{diag}D^\dagger \tag{3.11}$$

$$[CKM] = U^\dagger D \tag{3.12}$$

If we change $N \to NU$ and $D \to ND$, CKM remains unchanged, where N is some arbitrary unitary matrix then

$$\tilde{M}_u = NU\tilde{M}_u^{diag}U^\dagger N^\dagger \tag{3.13}$$

$$\tilde{M}_d = ND\tilde{M}_d^{diag}D^\dagger N^\dagger \tag{3.14}$$

Using eqns. (3.10) & (3.11) and eqns. (3.13) & (3.14), we notice that $\tilde{M}_u$ and $\tilde{M}_d$ are unique up to a common unitary transformation $\tilde{M}_u \leftrightarrow N^\dagger\tilde{M}_u N$, $\tilde{M}_d \leftrightarrow N^\dagger\tilde{M}_d N$. The unitarity of N keeps $[CKM] \simeq 1$. This restricts our attention to small transformations, i.e., $U \simeq 1$, $D \simeq 1$.

If, in particular $N = \phi_L$, ϕ_L being a phase matrix, the changes in $\tilde{M}_u$ and $\tilde{M}_d$

$$\tilde{M}_u \to \phi_L\tilde{M}_u\phi_L^\dagger \tag{3.15a}$$

$$\tilde{M}_d \to \phi_L\tilde{M}_d\phi_L^\dagger \tag{3.15b}$$

can be absorbed by redefining the phases.

A change of in the construction of CKM $[CKM] \to \phi_u^\dagger[CKM]\phi_d \leftrightarrow D \to D\phi_d$, $U \to U\phi$ doesn't lead to any changes as $\phi_u\tilde{M}_u^{diag}\phi_u^\dagger = \tilde{M}_u^{diag}$ and $\phi_d\tilde{M}_d^{diag}\phi_d^\dagger = \tilde{M}_d^{diag}$ Thus we can write

$$U^\dagger D = [CKM] \equiv \phi_u^\dagger[CKM]\phi_d \tag{3.16}$$

with $D \equiv \phi_L^\dagger D_s\phi_d$ and $U \equiv \phi_L^\dagger U_s\phi_u$. The phase matrices ϕ_L& ϕ_d are chosen such that D_s has the same form as the CKM matrix. We also obtain $[CKM]_s = U_s^\dagger D_s$.

Now, we construct the mass matrices as

$$\tilde{M}_u = U_s\tilde{M}_u^\dagger U_s^\dagger \tag{3.17}$$

$$\tilde{M}_d = D_s\tilde{M}_d^\dagger D_s^\dagger \tag{3.18}$$

The CKM matrix (Eqn. 3.1) can be broken down into a product of following matrices

$$[CKM] = C_2 \Delta C_3 \Delta^\dagger C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & s_2 \\ 0 & -s_2 & c_2 \end{pmatrix} \Delta \begin{pmatrix} c_3 & 0 & s_3 \\ 0 & 1 & 0 \\ -s_3 & 0 & c_3 \end{pmatrix} \Delta^\dagger \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.19}$$

where

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \tag{3.20}$$

D also takes the CKM form by assumption

$$D = C_{2d} \Delta_d C_{3d} \Delta_d^\dagger C_{1d} \tag{3.21}$$

where matrices C_{id} have been defined in analogy with the Cabibbo quark mixing matrix C with new angles $\theta_{id} (i = 1, 2, 3)$. We define three more orthogonal matrices C_{iu} which satisfy the relation

$$C_{iu}^T C_{id} = C_i \tag{3.22}$$

where

$$\theta_i = \theta_{id} - \theta_{iu} \, , i = (1, 2, 3) \tag{3.23}$$

From Eqn. (3.16) we see that

$$U = D[CKM]^\dagger = \{C_{2u}\}\{C_2(\Delta_d C_{3u} \Delta_d^\dagger) C_2^\dagger\}\{C_2(\Delta_d C_3 \Delta_d^\dagger) C_{1u} (\Delta C_3^\dagger \Delta^\dagger) C_2^\dagger\} \tag{3.24}$$

The expansion in λ is given by

$$\theta_{1d} \equiv \sum_{n=1} \alpha_n \lambda^n, \; \theta_{2d} \equiv \sum_{n=2} \beta_n \lambda^n, \; \theta_{3d} \equiv \sum_{n=4} \gamma_n \lambda^n \tag{3.25}$$

The expansion of θ_{iu} is constrained by Eqn. (3.22) For naturalness we require

$$\begin{aligned} &(a)\; \theta_{1d} \sim \lambda, \; \theta_{1u} \lesssim \lambda^2 \\ &(b)\; \theta_{2u} \sim \theta_{2d} \sim \lambda^2 \; or \; \theta_{2u} \sim \lambda^2 \gg \theta_{2d} \; or \; \theta_{2d} \sim \lambda^2 \gg \theta_{2u} \\ &(c)\; \theta_{3u} \sim \theta_{3d} \sim \lambda^4 \; or \; \theta_{3u} \sim \lambda^4 \gg \theta_{3d} \; or \; \theta_{3d} \sim \lambda^4 \gg \theta_{3u} \end{aligned} \tag{3.26}$$

For a particular set of angles like $\theta_{1d} \sim \lambda$, $\theta_{1u} \sim \theta_{2d} \sim \lambda^2$, $\theta_{2u} \sim \theta_{3u} \sim \lambda^4, \theta_{3d} \sim \lambda^5$, we obtain the following mass pattern

$$\tilde{M}_u \simeq \begin{pmatrix} u_{11}\lambda^7 & u_{12}\lambda^6 & e^{-i\delta_u} u_{13}\lambda^4 \\ u_{12}\lambda^6 & u_{22}\lambda^4 & u_{23}\lambda^4 \\ e^{i\delta_u} u_{13}\lambda^4 & u_{23}\lambda^4 & 1 \end{pmatrix} \tag{3.27}$$

$$\tilde{M}_d \simeq \begin{pmatrix} d_{11}\lambda^4 & d_{12}\lambda^3 & e^{-i\delta_d} d_{13}\lambda^4 \\ d_{12}\lambda^3 & d_{22}\lambda^2 & d_{23}\lambda^2 \\ e^{i\delta_d} d_{13}\lambda^4 & d_{23}\lambda^2 & 1 \end{pmatrix} \tag{3.28}$$

where the coefficients u_{ij} and d_{ij} are functions of the CKM parameters A and σ, quark mass ratios $\xi' s$ and λ expansion coefficients $\{\alpha_1, \beta_2, \gamma_4\}$. The elements of any general mass matrix, following the hierarchy

$$(1,1),(1,3),(3,1) \lesssim (1,2),(2,1) \lesssim (2,3),(3,2),(2,2) \lesssim (3,3)$$

can be considered to be natural mass matrix.

Chapter 4

Possibility of Unique Textures for Quark Mass Matrices

4.1 Introduction

The idea of this chapter is to explore a finite set of viable texture specific mass matrices. The general recipe to achieve this needs three essential ingredients namely, texture-zero approach, WB transformations and the condition of "naturalness". The WB transformations help in reducing the number of free parameters of the hermitian mass matrices and imposing the condition of naturalness puts a constrain on the parameter space available to these elements. We start with the most general hermitian mass matrices and explore the possibilities of viable texture- specific mass matrices using the tools, WB transformations and requirement of naturalness.

4.2 The Methodology

We start with the general hermitian mass matrices

$$M_q = \begin{pmatrix} E_q & A_q & F_q \\ A^*{}_q & D_q & B_q \\ F^*{}_q & B^*{}_q & C_q \end{pmatrix} \; (q = U, D) \tag{4.1}$$

The next step is to introduce texture zeroes in these matrices using the weak basis transformation. In principle one can always find a unitary matrix U transforming $M_U \to U^\dagger M_U U$ and $M_D \to U^\dagger M_D U$, which results in

$$M_U = \begin{pmatrix} E_U & A_U & 0 \\ A_U^* & D_U & B_U \\ 0 & B_U^* & C_U \end{pmatrix}, \; M_D = \begin{pmatrix} E_D & A_D & 0 \\ A_D^* & D_D & B_D \\ 0 & B_D^* & C_D \end{pmatrix} \; (q = U, D) \tag{4.2}$$

Here, $A_i = |A_i| \exp^{\iota\alpha_i}$ and $B_i = |B_i| \exp^{\iota\beta_i}$ with $i = U, D$. Each of the above matrix is texture 2 zero type. The condition of naturalness on these mass matrices reads

$$(1, i) < (2, j) \leq (3, 3);\ i = 1, 2, 3,\ j = 2, 3 \tag{4.3}$$

The compatibility of the obtained matrices have to be checked against the CKM matrix to ensure whether they are viable or not. This requires the know-how of constructing CKM matrix from the mass matrices M_U and M_D. The matrix M_q can be written as $M_q = Q_q^\dagger M_q^r Q_q$ where M_q^r is a symmetric matrix and Q_q is a diagonal phase matrix.

$$M_q^r = \begin{pmatrix} E_q & A_q & 0 \\ |A_q| & D_q & B_q \\ 0 & |B_q| & C_q \end{pmatrix},\ Q = \begin{pmatrix} e^{-i\alpha_q} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\beta_q} \end{pmatrix} \tag{4.4}$$

The matrix M_q^r can be diagonalized using the transformation

$$M_q^{diag} = O_q^T Q_q M_q Q_q^\dagger O_q = Diag(m_1, -m_2, m_3) \tag{4.5}$$

where 1, 2, 3 refer to u, c, t for the up sector and d, s, b for the down sector. The diagonalization of M_q^r is achieved by the following matrix [8]

$$O_q = \begin{pmatrix} \sqrt{\frac{(E_q+m_2)(m_3-E_q)(C_q-m_1)}{(C_q-E_q)(m_3-m_1)(m_2+m_1)}} & \sqrt{\frac{(m_1-E_q)(m_3-E_q)(C_q+m_2)}{(C_q-E_q)(m_3+m_2)(m_2+m_1)}} & \sqrt{\frac{(m_1-E_q)(m_2+E_q)(m_3-C_q)}{(C_q-E_q)(m_3+m_2)(m_3-m_1)}} \\ \sqrt{\frac{(C_q-m_1)(m_1-E_q)}{(m_3-m_1)(m_2+m_1)}} & -\sqrt{\frac{(C_q+m_2)(m_2+E_q)}{(m_3+m_2)(m_2+m_1)}} & \sqrt{\frac{(m_3-E_q)(m_3-C_q)}{(m_3+m_2)(m_3-m_1)}} \\ -\sqrt{\frac{(m_1-E_q)(m_3-C_q)(C_q+m_2)}{(C_q-E_q)(m_3-m_1)(m_2+m_1)}} & \sqrt{\frac{(E_q+m_2)(m_3-C_q)(C_q-m_1)}{(C_q-E_q)(m_3+m_2)(m_2+m_1)}} & \sqrt{\frac{(m_3-E_q)(C_q-m_1)(C_q+m_2)}{(C_q-E_q)(m_3+m_2)(m_3-m_1)}} \end{pmatrix} \tag{4.6}$$

The relation between the CKM Matrix and the diagonalizing matrix is the following

$$V_{CKM} = O_U^T Q_U Q_D^\dagger O_D \tag{4.7}$$

We have considered E_U, E_D, D_U, D_D as free parameters for the construction of CKM matrix.

The inputs used for the calculation were [9]

$$\begin{aligned} &m_u = 1.3^{+0.42}_{-0.41}\,\text{MeV},\ m_d = 2.82 \pm 0.48\,\text{MeV}, \\ &m_d = 57^{+18}_{-12}\,\text{MeV},\ m_c = 0.638^{+0.043}_{-0.084}\,\text{GeV}, \\ &m_b = 2.86^{0.16}_{-0.06}\,\text{GeV},\ m_t = 172.1 \pm 1.2\,\text{GeV}, \\ &\frac{m_u}{m_d} = 0.553 \pm 0.043,\ \frac{m_s}{m_d} = 18.9 \pm 0.8 \end{aligned} \tag{4.8}$$

The parameters ϕ_1 and ϕ_2 are related to the phases of mass matrices as $\phi_1 = \alpha_U - \alpha_D$ and $\phi_2 = \beta_U - \beta_D$. The parameters ϕ_1 and ϕ_2 have been given full variation from 0 to 2π. The free parameters E_U, E_D, D_U, D_D have also varied over a wide range ensuring that O_U and O_D remain real.

4.3 Results & Discussion

The resultant CKM matrix obtained is

$$V_{CKM} = \begin{pmatrix} 0.9739 - 0.9745 & 0.2246 - 0.2259 & 0.00337 - 0.00365 \\ 0.2224 - 0.2259 & 0.9730 - 0.9990 & 0.0408 - 0.0422 \\ 0.0076 - 0.0101 & 0.0408 - 0.0422 & 0.9990 - 0.9999 \end{pmatrix} \tag{4.9}$$

which is compatible with one given by the Particle Data Group (PDG) [10]. The magnitudes of the element of the mass matrices which reproduce the CKM matrix of Eqn. (4.9) are

$$M_U = \begin{pmatrix} 0 - 0.00138 & 0.006 - 0.042 & 0 \\ 0.006 - 0.042 & 26.46 - 102.68 & 62.82 - 86.10 \\ 0 & 62.82 - 86.10 & 68.78 - 145.00 \end{pmatrix} \tag{4.10}$$

$$M_D = \begin{pmatrix} 0 - 0.00127 & 0.011 - 0.019 & 0 \\ 0.011 - 0.019 & 0.36 - 1.66 & 1.03 - 1.44 \\ 0 & 1.03 - 1.44 & 1.16 - 2.44 \end{pmatrix} \tag{4.11}$$

The structure of these mass matrices reveal that their (1, 1) element is very small in comparison with the rest of non-zero elements. This indicates the redundancy of the (1, 1) element. The plots of the (1, 1) elements, which are shown below, with CKM parameters confirm their redundancy.

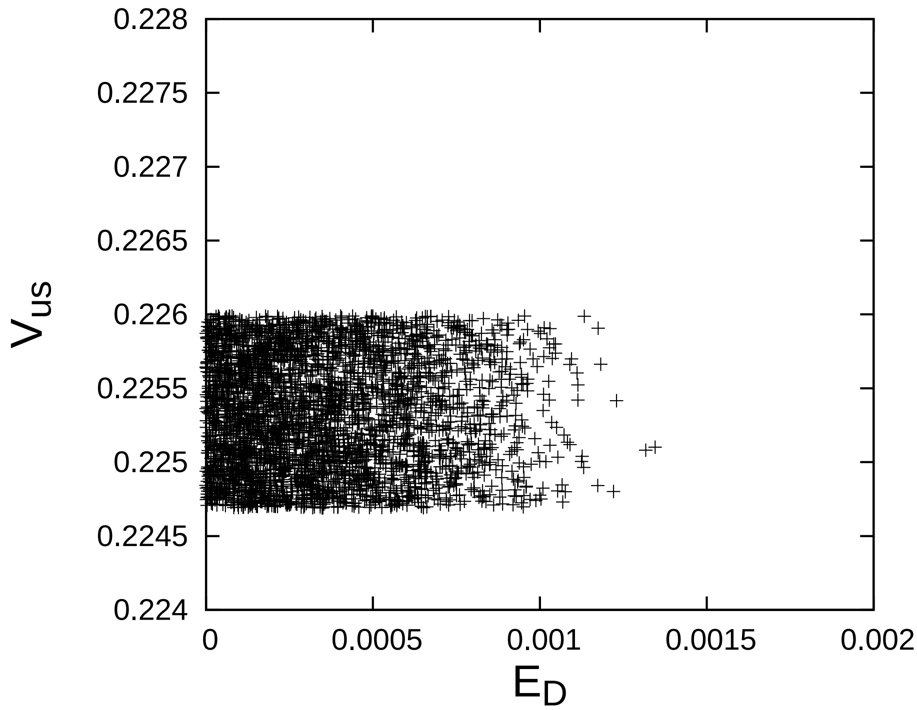

Figure 4.1: Dependence of V_{us} on E_D

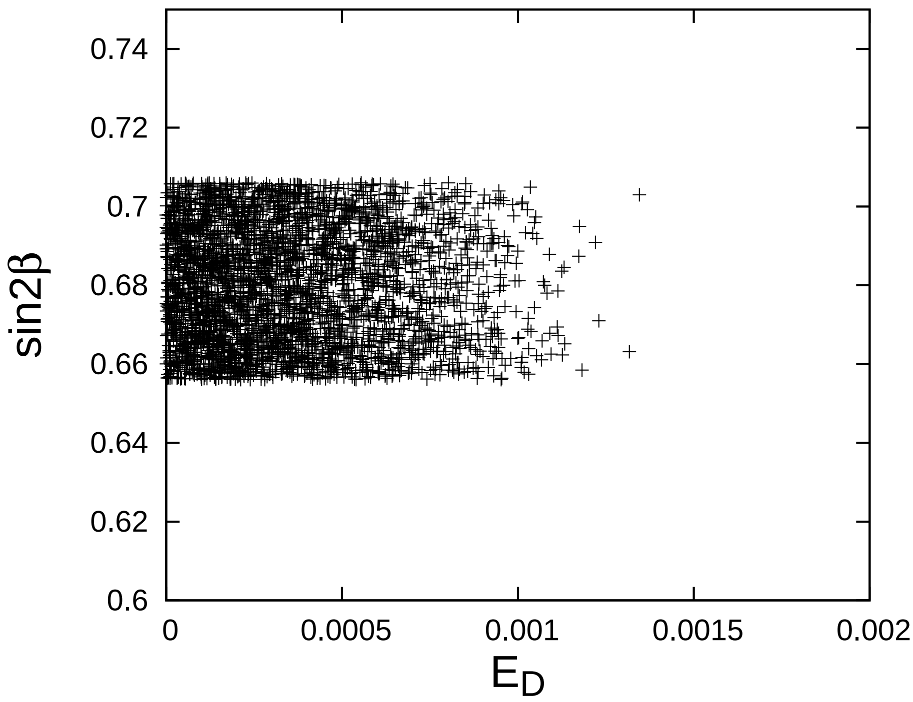

Figure 4.2: Dependence of $\sin 2\beta$ on E_D

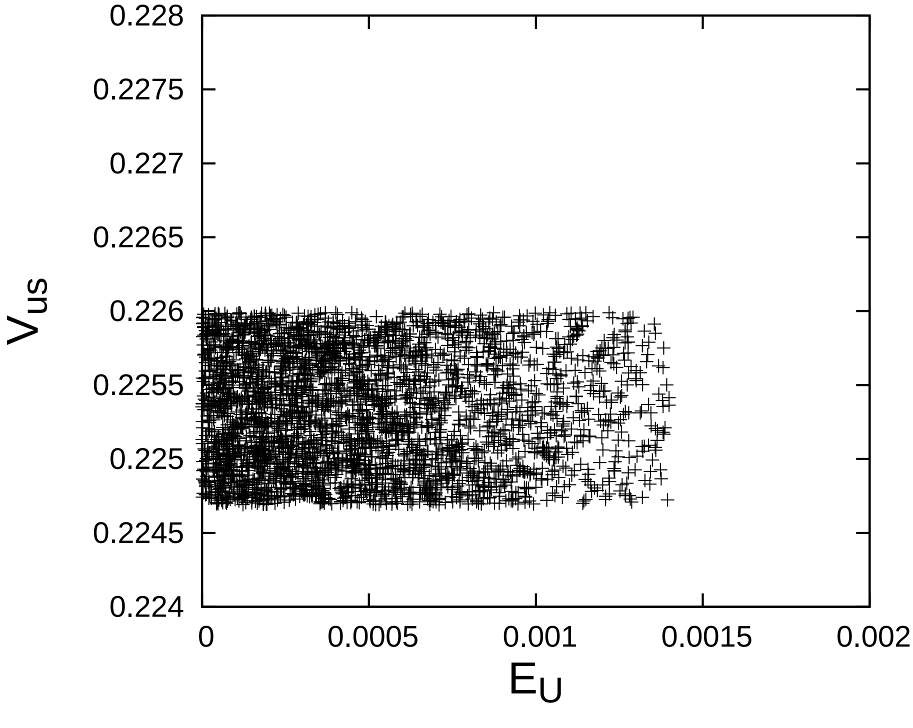

Figure 4.3: Dependence of V_{us} on E_U

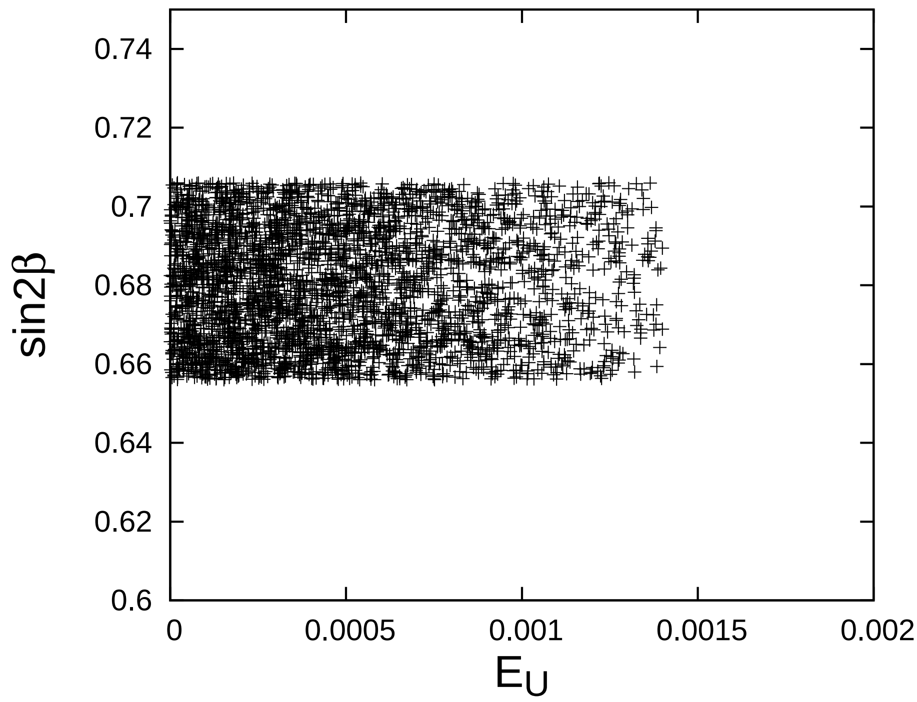

Figure 4.4: Dependence of $\sin 2\beta$ on E_U

The plots show that parameters E_U and E_D assume quite small values for producing the experimental range of CKM parameters. These parameters are essentially redundant. This indicates a transition from texture-2 zero mass matrices to texture-4 zero mass matrices. A similar analysis for these matrices results in the following CKM matrix

$$V_{CKM} = \begin{pmatrix} 0.9741-0.9744 & 0.2246-0.2259 & 0.00337-0.00365 \\ 0.2245-0.2258 & 0.9732-0.9736 & 0.0407-0.0422 \\ 0.0071-0.0100 & 0.0396-0.0417 & 0.9990-0.9992 \end{pmatrix} \tag{4.12}$$

which is in agreement with the quark mixing matrix by PDG [10]. It has been shown by Sharma et al. [3] that the following matrices

$$\begin{pmatrix} D & A & 0 \\ A^* & 0 & B \\ 0 & B^* & C \end{pmatrix}, \begin{pmatrix} 0 & A & D \\ A^* & 0 & B \\ D^* & B^* & C \end{pmatrix}, \begin{pmatrix} A & 0 & 0 \\ 0 & D & B \\ 0 & B^* & C \end{pmatrix} \tag{4.13}$$

and their permutations are not viable for the description of quark mixing data. Therefore, we are left with only the following form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & D & B \\ 0 & B^* & C \end{pmatrix} \tag{4.14}$$

and its permutations as a viable option.

The following plots show the viability of texture four zero matrices.

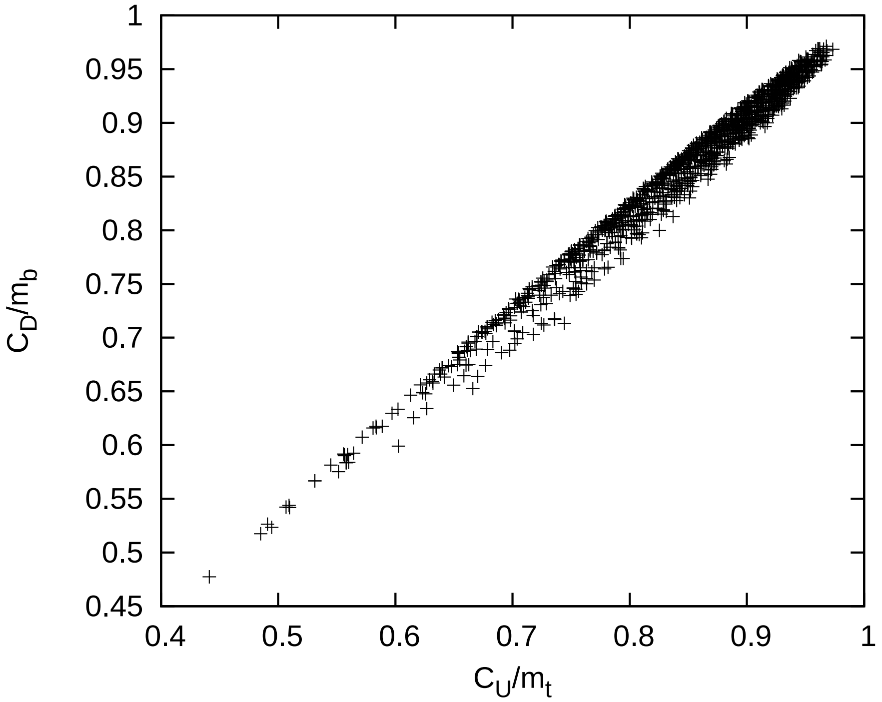

Figure 4.5: C_D/m_b versus C_U/m_t

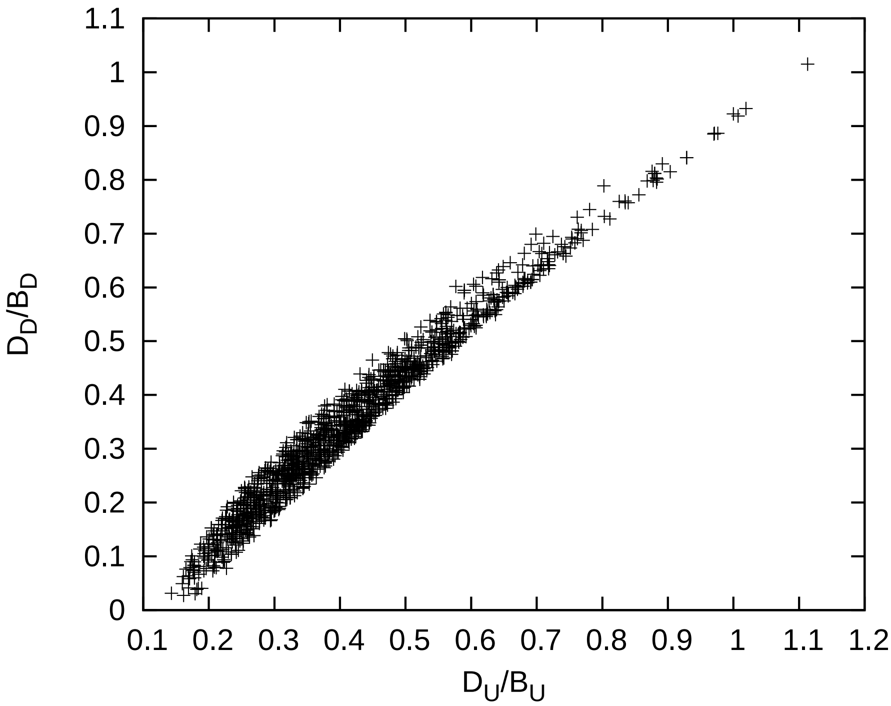

Figure 4.6: D_D/B_D versus D_U/B_U

From the above plots we found that there is a good range of values of C_U and D_U for which the data can be fitted.

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