# Prior information in inference of performance in constrained thermodynamic processes

A thesis

submitted by

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*dedicated*

*to*

*my parents*

# Declaration

The work presented in this thesis has been carried out by me under the guidance of Dr. Ramandeep S. Johal at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, diploma or a fellowship to any other University or Institute. Whenever contributions of others are involved, every effort has been made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

**Preety**

Place :

Date :

In my capacity as the supervisor of the candidate's PhD thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

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## List of publications

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2. Informative priors and analogy between quantum and classical heat engines.
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3. Prior information and inference of optimality in thermodynamic processes.
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4. Incomplete information and expected performance characteristics of heat engines.
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6. Optimal engine performance using inference for non-identical finite source and sink.
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# Contents

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# Abstract

In the present thesis, we explore the relevance of the prior information in classical thermodynamic processes with limited information to estimate their performance characteristics. We followed the Bayesian approach where all uncertainty is treated probabilistically and a probability may be assigned to an uncertain parameter taking up a possible value. The corresponding probability distribution is simply known as a prior. In the present context, we propose appropriate priors in case of limited information about the thermodynamic coordinates of the process. First we consider the process of reversible work extraction with identical thermodynamic systems in which input heat from the source is converted into work with delivery of the waste heat into sink, preserving the total entropy of the composite system. The work extracted and efficiency of the engine is estimated. The estimates show good agreement with the optimal work extracted and the corresponding efficiency especially near equilibrium. The inference approach also extended to non-identical systems reproduces the optimal behavior to a good extent. Next, we consider the well-known process of pure thermal interaction between the two systems with fixed total energy. The main quantity of interest is the estimated net entropy production which matches with the corresponding optimal value upto third order. An intuitive interpretation for the prior is also proposed.

# Chapter 1

# Introduction

Inference may be defined as a mode of reasoning which seeks to arrive at logical conclusions from the premises known or assumed to be true. We perform inference in many every day activities, for example, we infer that it will rain when we see the sky is covered with clouds. Inference is based upon one's prior state of knowledge about the nature of the system and/or information from some measurement. Inferences can be obtained with deductive as well as inductive reasoning.

## Deductive inference

Deductive inference specifies assertions or premises in terms of sentences which take the truth values and thus provides a procedure which leads to conclusions that are certain, based on the given assertions. Here, we can conclude that some event or hypothesis is either true or false.

## Inductive inference

This kind of inference is traditionally associated with the word "probably". It specifies assertions or premises in terms of sentences which takes the possible (plausible) values and thus provides an inference procedure which leads to the highly plausible conclusions based on the given assertions. Here, we infer that some event or hypothesis may be true only with a certain probability.

## 1.1 Statistical inference

Statistical inference [1, 2, 3] is based on a probabilistic modelling of the observed phenomenon. So, statistical inference is basically a process of deriving conclusions regarding the phenomenon about which we have partial knowledge. It plays an essential role in understanding the world around us and thus helps us to make decisions. However, the field of statistical inference has remained a subject of debate since there exist many competing approaches. There are two major schools of thought in statistical inference: frequentism and Bayesianism.

### 1.1.1 Nature of probability and modes of inference

#### What is probability?

Probability measures the degree of uncertainty that whether an event will occur or not. There are two major competing interpretations of probability, which we discuss below:

- **Frequentist probability**: This interpretation of probability is based on *objective belief* of the observer. Frequentist probability interprets probability as the limit of relative frequency of occurence of a certain event in a large number of trials. The school of thought in statistical inference associated with frequentist interpretation of probability is termed as **frequentist inference**.

- **Bayesian probability**: This interpretation of probability is based on *subjective belief* or rational degree of belief of the observer in the occurence or non-occurence of the events. However, to identify or to measure the observer's belief in numerical terms, an operational approach is mandatory.

Thus, Bayesian probability includes one's rational belief as well as sample data to form probability judgements. The school of thought in statistical inference associated with subjective interpretation of probability is termed as **Bayesian inference**.

### 1.1.2 Limitation of frequentist inference over Bayesian inference

In frequentist approach to inference, almost all prior knowledge i.e. the knowledge already existing before any experimental data is collected, is ignored as we are only concerned with the relative frequency of the outcomes. The frequentist approach can be used only in cases where trials are repeatable. Further, the definition of frequentist probability is true for an infinite sequence of repeatable trials while in real situations, we are constrained to deal with finite number of trials. But, there are certain events which are not repeatable and so it becomes difficult to apply the frequentist interpretation of probability. Consider the question: What is the probability that it will rain today? Now, the occurence of rain which although is a repeatable event, does not occur under identical conditions as a large number of 'todays' is not feasible for the application of frequentism. Similarly, some other examples of such events are: Will Sachin score a century in the next match against Australia? Is there any life on Mars? etc. Such kind of events can be dealt probabilistically only within Bayesian approach where probability is defined as the observer's degree of rational belief in the event, based on the knowledge she is having. If the observer finds conditions appropriate for rain such as clouds, thunderstorm etc., she will be more sure that it will rain. Thus, for a frequentist, the locus of uncertainty described by the probability lies in the events but for a Bayesian, the locus of uncertainty is in the agent/observer. Actually, more precisely, we can argue that Bayesians consider each event as a unique event and what is sought is how plausible a certain inference can be with respect to that event.

## 1.2 Bayesian inference

Bayesian inference is based on the subjective interpretation of probability as discussed in the preceding Section. In recent years, Bayesian inference, also known as "*Science of Prior information*" [4, 5], has gained immense popularity over the established 'Frequentist approach' to inference. Bayesian methods have found applications in many different areas of research, such as physics [6, 7, 8], economics [9], machine learning [10], human cognition [11], quantum probabilities [12, 13] and so on. Even physical theories such as classical and quantum mechanics can be regarded as a manifestation of Bayesian inference [14, 15]. It has also become a major part of statistics. Let us now discuss an overview of the Bayesian approach to inference [16]:

### 1.2.1 Bayes methodology

We begin the discussion by presenting the famous theorem known as *Bayes' Theorem* which underpins the entire Bayesian approach.

#### A brief history and philosophy

Bayesian approach was introduced by Thomas Bayes in a well-known paper published posthumously [17]. Richard Price discovered the notes of Bayes and sent them to the Royal Society under the title "Note on the Solution of a Problem in a Doctrine about an event". This work was reprinted later in 1958 [17]. Later in 1959, R. A. Fisher gave a detailed account of Bayes' work in [18]. The philosophy of Bayes' approach is based on the connection between inductive and deductive reasoning (or inference). Jeffreys considered deductive reasoning as a special case of inductive reasoning [19]. He gave the argument that knowledge gained from deductive logic contains the information of past observations which are going to be used for the prediction of future events and this act of generalisation of past experiences and prediction of future is actually *Inductive inference.*

**Bayes' theorem**

Bayes' theorem gives a general description for inversion of probabilities and thus introduced the concept of Bayesian inference. The theorem is the foundation for Bayesian inference and can be illustrated with the following statements:

- **Prior information** is a piece of knowledge about the system before any measurement or an experiment is performed to collect data. It happens rarely that we do not know anything about the system or its control parameters which define the conditions of the experiment or the measurement. We assign a probability distribution for the system's parameter on the basis of certain constraints governing the system, such as the parameter may be positive valued or restricted to a finite range.

- The probabilities proposed for a parameter on the basis of prior information in the problem at hand are called *prior probabilities* or simply *prior.* The choice of a prior in Bayesian analysis is a crucial step as it incorporates the available partial information as well as the uncertainty underlying the problem to be studied. Prior information plays a significant role in the derivation of a prior.

- The modified or updated probabilities for the parameter obtained after unifying the prior with the observed data by Bayes theorem are called *posterior probabilities.*

This can be illustrated with the application of *Bayes' rule* as follows:

$$p(\theta|x) = \frac{f(x|\theta)p(\theta)}{\int f(x|\theta)p(\theta)\, d\theta}, \tag{1.1}$$

$p(\theta)$ is continuous probability density function for $\theta$ to quantify the uncertainty in the parameter $\theta$, and thus is called *prior*. The function $f(x|\theta)$ is called likelihood function which is basically a formalization for all observed data $x$ given $\theta$, and finally, $p(\theta|x)$, which is an updated or modified version of $p(\theta)$ in the light of new data $x$ is called *posterior distribution*.

### 1.2.2 Types of priors

The choice of an appropriate prior [19, 20, 21] has been a crucial issue in the Bayesian epistemology, which has hampered its development and general acceptance for a long time [22]. So its appropriate determination is the most important step in the whole inference procedure. Even if the uniqueness of a prior corresponding to a given problem is hard to establish, however, an appropriate prior can still be proposed based on certain principles of coherence or from symmetry contained within the problem. Priors can be broadly divided into two categories :

- **Informative priors**

  When the priors for the uncertain parameters are assigned by making use of the prior information, then such priors are termed as informative priors. Depending upon the available prior information, priors can be categorised as:

  (1) **MaxEnt priors**

  MaxEnt prior was proposed by Jaynes [20]. This kind of priors are derived from the Maximum Entropy Principle (MEP) [23, 24]. The basic idea of MEP is to assign a prior distribution which maximizes the Shannon-information entropy [25], subject to the given information:

$$S = -\sum_{i=1}^{n} P(x_i) \ln P(x_i), \tag{1.2}$$

where the quantity $x$ can take on discrete values $\{x_1 ... x_n\}$. The information about the quantity $x$ places a number of constraints on the probability distribution $P(x_i)$. MaxEnt priors are generally derived by the constraints on the moments of the distribution ($M_R$), where $M_R = \sum_{i=1}^{n} x_i^R P(x_i)$ is $R^{\text{th}}$ moment of the probability distribution. A few examples of distributions are discussed below:

– Assume the zeroth moment, $M_0 = 1$, which is in fact the normalization condition for the distribution. This constraint will lead to a uniform distribution over the interval as:

$$P(x_i|M_0) = \frac{1}{(b-a)}. \tag{1.3}$$

– In addition to $M_0 = 1$, suppose we have been given another constraint on the first moment of the distribution, $M_1$. Maximizing Shannon entropy with these constraints will lead to an exponential distribution as:

$$P(x_i|M_0, M_1) = \frac{\exp(-\beta x_i)}{\sum_{i=1}^{n} \exp(-\beta x_i)}, \tag{1.4}$$

where Lagrange multiplier ($\beta$) is evaluated from the constraint on the first moment of the distribution.

(2) **Conjugate priors**

Conjugate priors were introduced by Howard Raiffa and Robert Schlaifer [26]. When the posterior probability distribution $p(\theta|x)$ happens to belong to the same family as the prior distribution $p(\theta)$, we say that prior is a "conjugate prior" or prior and posterior are conjugate distributions. For example, the normal distribution is a conjugate prior since it leads to a normal posterior distribution for a given likelihood function. This can be understood from the equation (1.1), in which, we can see that posterior is the output of the product of prior and the likelihood function. If we represent our prior knowledge about $\lambda$ by a gamma distribution [4]

$$P(\lambda|\alpha,\beta) = \frac{\beta^{\alpha}\lambda^{\alpha-1}}{\Gamma(\alpha)}e^{-\beta\lambda}, \tag{1.5}$$

where $\alpha > 0$, $\beta > 0$ are the specified parameters of the distribution, then the posterior distribution $P(\lambda|x_1, x_2...x_n)$ is also a gamma distribution with parameters $\alpha + n\overline{x}$ and $\beta + n$; and $\overline{x} = (1/n)\sum_{i=1}^{n} x_i$ is the sample mean. The sample data, $X^n = \{x_1, x_2, ...x_n\}$, has been chosen from the Poisson distribution, with $\lambda$ as the mean of the distribution.

- **Non-informative priors**[27]

When no prior information is available to us, then prior distributions must be derived from the sample distribution and hence, the priors derived are called non-informative priors. Non-informative priors can be divided into:

(1) **Laplace prior**

Laplace priors are based on the "Principle of Insufficient Reason" [28]. This principle says that if there is no reason to believe that out of a set of possible, mutually exclusive events, no event is preferable over any other, then one should assign equal probabilities to all the allowed events. For example, consider that you are throwing a die [29]. Since we do not believe that one side is more likely to occur than any other, we regard all probabilities as equal. And indeed, a large number of times throwing shows that this is correct. Laplace priors were addressed with a criticism of the problem of *non-invariance under bijective reparametrization* [30], since a one-to-one transformation from one parameter to another does not lead to a uniform prior in new parameter. To illustrate, consider a one-to-one transformation $g$ where we switch from $\theta$ to $\phi$ as $\phi = g(\theta)$. If prior for $\theta$ is $\pi(\theta) = 1$, then corresponding prior for $\phi$ is given by the Jacobian formula as:

$$\pi^*(\phi) = \left| \frac{d}{d\phi} g^{-1}(\phi) \right|, \quad (1.6)$$

But, $\pi^*(\phi)$ is usually not a uniform distribution and thus Laplace prior does not satisfy reparametrization invariance.

(2) **Jeffreys prior**

Harold Jeffreys proposed a class of non-informative priors [31] in order to meet the demand of invariance [30]. Jeffreys prior is constructed with the *Fisher information* contained in the model. The Fisher information ($I(\theta)$), is a measure of the amount of information, contained in the probability distribution ($f(X|\theta)$) of random variable $X$, given the parameter $\theta$:

$$I(\theta) = -\int \left[ \frac{\partial^2 \ln f(X|\theta)}{\partial \theta^2} \right] f(X|\theta)\, dX, \quad (1.7)$$

in the one-dimensional case. The Jeffreys prior is given as

$$\pi(\theta) \propto I^{1/2}(\theta), \quad (1.8)$$

defined up to normalization constant. Thus Jeffreys prior is defined as the square root of the Fisher information. The main motivation behind the choice of Jeffreys prior is that it satisfies the *invariant reparametrization* requirement. Jeffreys prior can be generalized to the case, when $\theta$ is a multidimensional parameter. In such a case, prior is proportional to the square root of the determinant of Fisher information matrix as shown:

$$\pi(\theta) \propto [\det(I(\theta))]^{1/2}, \quad (1.9)$$

where $I(\theta)$ has the following elements,

$$I_{ij}(\theta) = -\int \left[ \frac{\partial^2}{\partial\theta_i \partial\theta_j} \ln f(X|\theta) \right] f(X|\theta)\, dX, \tag{1.10}$$

for $i, j = 1, 2...k$. However, in the multidimensional case, Jeffreys approach leads to some problems and incoherences [32]. Thus Jeffreys approach works well only in case of single parameter models.

(3) **Reference prior**

Reference priors were introduced by Bernardo [33]. It was basically a modification of the Jeffreys approach, This method leads to Jeffreys' prior in the one-dimensional case, but it works well even for multidimensional parameters as well. These priors are estimated by maximizing the intrinsic discrepancy between the posterior and prior probabilities. For this, Bernardo defines a reference prior as $\Pi$ maximizing

$$K^*(\Pi) = \langle K(\Pi) \rangle, \tag{1.11}$$

where $K(\Pi)$ is the Kullback-Leibler divergence between the prior and the corresponding posterior as :

$$K(\Pi) = \int \Pi(\theta|x) \ln \left( \frac{\Pi(\theta|x)}{\Pi(\theta)} \right) d\theta, \tag{1.12}$$

and the expectation is taken over the marginal distribution of $x$. In Eq. (1.12), $\Pi(\theta)$ is a prior distribution proposed for $\theta$ and the $\Pi(\theta|x)$ is an updated or posterior probability distribution for $\theta$.

### 1.2.3 Information processing and Bayes theorem

The information processing approach in statistical inference involves the formulation of some criterion functions. The latter, when optimized, leads to certain optimal information processing rules (IPRs) and one of which is Bayes' Theorem

[34]. These optimal IPRs are 100% efficient in the sense that output information is exactly equal to the input information. The following criterion function was employed:

$$\Delta[g(\theta|y)] = \text{Output Information - Input Information}, \tag{1.13}$$

where $g(\theta|y)$ is a proper density function for $\theta$, $y$ being the observed data. After optimizing the functional (1.13), we obtain the Bayes' theorem as optimal IPR:

$$g^*(\theta|y) = \frac{\pi(\theta|I)l(y|\theta)}{h(y)}, \tag{1.14}$$

where $\pi(\theta|I)$ is the prior density and $I$ is the prior information, $l(y|\theta)$ is the likelihood function and $h(y)$ is the marginal density of the observations.

### Quality adjusted priors and likelihood function

Zellner [35] introduced the terms, *quality adjusted priors and likelihood function*, namely, $q_1(\pi)$ and $q_2(l)$ and these two terms are introduced in (1.13). Minimizing this equation with respect to the choice of $g$, subject to the constraint of its being proper, we obtain the result as :

$$g_a \propto q_1(\pi)q_2(l). \tag{1.15}$$

In the above case, Zellner has assumed that $q_1(\pi) \propto \pi^a$ and $q_2(l) \propto l^b$ such that $a$ and $b$ take values in the interval [0, 1]. Now, if $a = 0$ then $g_a \propto l^b$ and it indicates that prior information is of very low quality and we consider information in the sample data or likelihood function only. On the other hand, consider the case when sample information is of poor quality i.e. $b = 0$, in that case $g_a \propto \pi^a$. This shows that the information is in the prior density only. Thus one can choose the values of $a$ and $b$ accordingly.

## 1.3 Information in thermodynamics

There are two fundamentally different concepts of information : *Symbolic information* and *Physical information.* Symbolic information is based on the symbols created to incorporate knowledge, facts, data etc. While on the other hand, physicists mostly consider the term information to indicate conditions which do not involve any symbols and it is Physical information. It refers to the concept of physical differences. These physical differences can be any type of non-uniformity in the physical parameters, say, for example, this difference may include the distinction between two objects, difference between energy states etc. There exists a relationship between *Physical information* and *Thermodynamic entropy*, however, it is still controversial. This link specifically appears suggestive because of a significant likeness between *Shannon entropy* (Eq. (1.2)) and *Boltzmann's thermodynamic entropy*, although the origins and motivations behind these two kinds of entropy are different. Thus, the definition of entropy based on thermodynamics can be considered as a subset of physical information. In this Section, we discuss the role of information in thermodynamics.

### 1.3.1 Maxwell's demon and its history

Maxwell's demon was first introduced by *James Clerk Maxwell* in 1871 in his book *"Theory of Heat"* [36]. It played a key role in establishing a connection between information and thermodynamics. It was introduced as a contradiction to the second law of thermodynamics or to illustrate the statistical nature of second law which is based on the Clausius's premise that no spontaneous process can lead to the transfer of heat from a body at lower temperature to a body at higher temperature without the help of any external agency. But Maxwell thought an experiment in which he showed a violation of the second law by introducing a hypothetical creature which became popular as *Maxwell's Demon.*

**What is Maxwell's demon?**

Maxwell's Demon [37] is a hypothetical intelligent being of molecular size who can observe the individual molecules without expenditure of any work. To describe this intelligent being, Maxwell thought of an experiment which consists of a vessel, containing air molecules at uniform temperature, divided into two portions $A$ and $B$ (shown in the Fig. 1.1). There is a friction-free trap door in the division from where molecules from one half can pass to the other half. Now, Maxwell assumed a demon sitting near the trap door and trying to control the movement of molecules. The demon is intelligent in the sense that it observes the molecules so sharply so as to allow the faster molecules from $B$ to $A$, and only the slower molecules to go from $A$ to $B$. Thus a tiny intelligent being would create a temperature difference by allowing only the swifter molecules to enter one half and only slower molecules to leave it. Hence, in this way, the second law of thermodynamics can be violated. However, according to Maxwell, the second law has only a statistical nature according to which a mechanical interpretation based on dynamical laws would not be feasible.

### 1.3.2 Landauer's principle: A saviour for the Second law

The puzzle of Maxwell's demon has been approached from the viewpoint of information theory. Initial arguments were given by Szilard [38] and Brillioun [39, 40] which gained wide acceptance: In order to operate, the demon must accumulate information and process of information acquisition is sufficiently dissipative to save the second law. Later, Bennett [41] proposed the demon requires a memory to observe the molecules and must clear the memory periodically. Thus the second law is saved by the argument that erasing and resetting the memory of the demon is accompanied by an entropy increase in the environment. He argued that the information acquisition need not to be dissipative, but information erasure is a dissipative operation. This idea follows from *Landauer's Erasure Principle*

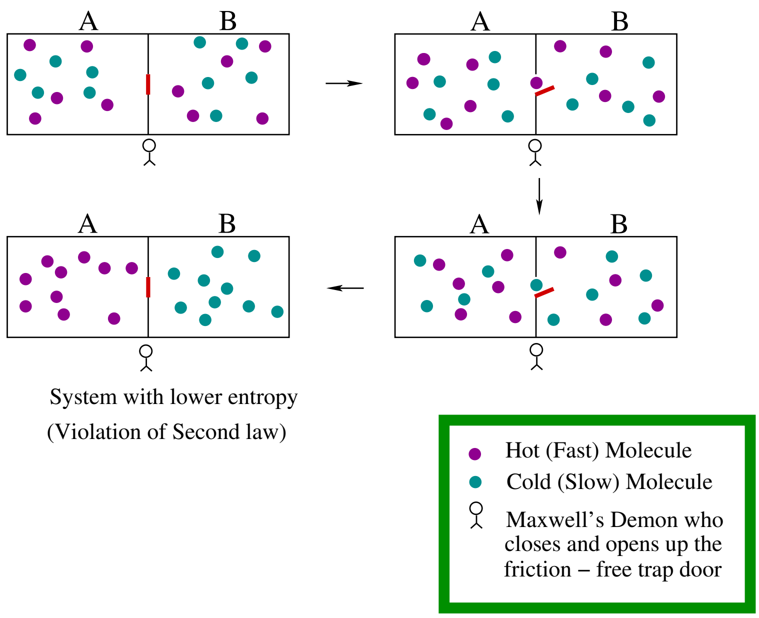


Figure 1.1: *Maxwell's Demon violating the second law by creating a temperature gradient from which work can be extracted*

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[42], according to which there is a minimal energy required to erase one bit of information, which in turn, results in heat generation in the environment, given by $kT \ln 2$, where $k$ is Boltzmann's constant and $T$ is the temperature of the environment. There is a net increase in entropy of the system plus environment, after the completion of each thermodynamic cycle without violating the second law. Hence, Landauer's principle restores the validity of second law and results in exorcism of Maxwell's demon [41, 43]. With this advent, a specific relationship between information theory and physics (especially thermodynamics) has been explored.

### 1.3.3 Information and stochastic thermodynamics

In recent years, the role of information in thermodynamics has become interesting specially in the realm of *Stochastic Thermodynamics* [44, 45]. The idea of extracting maximum work within a finite time by employing the information gained from measurements has been explored in [46, 47, 48, 49, 50, 51, 52]. The idea is based on the profound connection between statistical mechanics and information theory [23, 53, 54, 55]. In paper [56], the proposition was revisited within the framework of stochastic thermodynamics. The model considered was a Brownian particle in a time-dependent harmonic trap. The notion that work can be extracted from a single bath by exploiting the information available through measurements was investigated further in [57, 58, 59, 60]. Thus, it is possible to extract more work than the corresponding free-energy difference in a thermodynamic process with feedback. Mathematically,

$$W^{ext} \leq -\Delta F + kTI, \tag{1.16}$$

where $W^{ext}$ is the work extracted from the system in contact with a heat bath at temperature $T$. $\Delta F$ is the free-energy difference between initial and final equilibrium states. The relation (1.16) is known as *Generalized Second Law inequality* which involves mean information ($I$) acquired through measurements. The equality in Eq.(1.16) is achieved for a feedback-reversible process.

## 1.4 Finite-time thermodynamics

Sadi Carnot, a French engineer, in 1826, found that the maximum efficiency of a heat engine operating between a hot and a cold reservoir is attained only for a reversible process. The power of such a reversible heat engine is zero as it takes infinite time to complete the cycle. In finite-time models, heat engines are optimized with respect to maximum power output although the Carnot efficiency is compromised to get work done in finite time.

The breakthrough in Classical thermodynamics came with a new universal efficiency called *Curzon-Ahlborn* efficiency [61] in 1975. This work was done by

F.L. Curzon and B. Ahlborn in which they made an attempt to optimise the real heat engines. They considered a model of heat engine which they called as '*endoreversible heat engine*'. The term 'endoreversible' originates from the assumption that engine is subjected to only external irreversibilities and does not allow for any internal irreversibility. The finite-rate of heat transfer during isothermal expansion/compression branches of Carnot cycle are the causes for the external irreversibilities. This results in efficiency of engine which is less than the Carnot efficiency. The heat engine is optimized with respect to maximum power and the efficiency obtained at maximum power output is called *Curzon-Ahlborn* (CA) efficiency given by the expression $\eta_{CA} = 1 - \sqrt{T_c/T_h}$, where $T_h$ and $T_c$ are the temperatures of hot and cold baths respectively. Near-equilibrium ($T_h \approx T_c$), CA efficiency behaves as:

$$\eta_{CA} = 1 - \sqrt{1 - \eta_c} \approx \frac{\eta_c}{2} + \frac{\eta_c^2}{8} + O[\eta_c]^3, \tag{1.17}$$

where $\eta_c = 1 - T_c/T_h$ is the Carnot bound. After the work of Curzon and Ahlborn, many people studied more realistic models of heat engines by taking into account the internal friction, finite reservoirs, finite-rate and thus a new branch called "Finite-time Thermodynamics" (FTT) was introduced in 1975.

The validity and generality of CA efficiency were the questions of long-standing debate since the result obtained was model specific using on endoreversible approximation. Recently, CA efficiency was shown to be a universal efficiency at maximum power output, without any approximation, from the theory of linear irreversible thermodynamics [62, 63]. In the linear response regime, it was proven that efficiency at maximum power is bounded from above by $\eta_{CA}$ as $\eta \leq \eta_{CA} \approx \eta_c/2$. The upper limit is reached for a specific class of models which are perfectly coupled i.e. heat flux is directly proportional to the work-generating flux [64, 65, 66]. Thus, for strongly coupled systems, universality of coefficient 1/2 in the near-equilibrium expansion of efficiency can be recovered.

The efficiency at maximum power has also been studied in the context of stochastic heat engines [67], Feynman's ratchet as a heat engine [68], nanosystems with perfectly coupled fluxes such as quantum dot [69], and so on. The striking feature of these studies is the emergence of universal character of efficiency up to second order (Eq. (1.17)) when expanded close to equilibrium. In Ref. [70], it was investigated that universal feature of the efficiency at maximum power in the second order of $\eta_c$ can be attributed to a kind of left-right symmetry in the strong-coupling models. When there exists a certain kind of symmetry in the model, only then the term $\eta_c^2/8$ is recovered. For example, depending upon the value of Einstein coefficients, the efficiency of a maser [71] at maximum power may be different from CA efficiency but, if the Einstein coefficients are equal, then efficiency shows universality up to quadratic order. This universality has also been observed in low-dissipation engines performing finite-time Carnot cycles [72] between hot and cold reservoirs. The efficiency at maximum power of such heat engines in near-equilibrium regime is:

$$\eta_o = \frac{\eta_c}{2} + \frac{\eta_c^2}{4 + 4\sqrt{\frac{\sigma_c}{\sigma_h}}} + O[\eta_c]^3, \tag{1.18}$$

where $\sigma_c$ ($\sigma_h$) is the entropy production in the cold (hot) reservoir, when the engine is in contact with the reservoir(s) for a finite-time. When there is a symmetric dissipation in hot and cold reservoirs i.e. $\sigma_c = \sigma_h$, we recover the coefficient 1/8 in the quadratic term of efficiency (Eq. (1.18)). Hence, symmetric dissipation in time-dependent cycles of heat engines is analogous to the left-right symmetry on the fluxes.

## 1.5 Thesis overview

The motivation to study the role of prior information in heat engines comes from the paper [73], in which a two-level quantum system [74] is studied from the Bayesian approach and the interesting result obtained is that the inferred efficiency at maximum expected work turns out to be CA efficiency [61]. CA efficiency has been observed as a universal efficiency as discussed in Section 1.4. But in Bayesian approach, optimization has not been done explicitly, only the ignorance about the parameters of the engine is quantified in terms of probability distribution which is further used to estimate the thermodynamic quantities like work and efficiency. Such type of behavior has also been observed in some other models of quantum heat engine within Bayesian framework [75]. Motivated by this work, we implemented this approach to infer optimality in constrained thermodynamic processes [76, 77, 78, 79]. In particular, we have studied *entropy-conserving* and *energy-conserving* processes. Following the inference approach, we propose appropriate priors based on limited information about the thermodynamic coordinates of the process under consideration. We estimate the thermodynamic quantities like work, efficiency and entropy production depending upon the process under consideration. An analogy between the quantum thermodynamic machines and their corresponding classical models has also been studied by using this approach [80], where quantum machines display classical thermodynamic behavior with suitable choice of prior for given information. This Section comprises the layout of present thesis work.

In Chapter 2, we discuss the inference procedure for constrained thermodynamic process of maximum work extraction for a pair of identical finite reservoirs. The total entropy of the whole system remains conserved. The ignorance is assumed about the intermediate temperatures, $T_1$ and $T_2$, of the finite reservoirs. The constraint of entropy conservation gives a one-to-one relation between $T_1$ and $T_2$ as $T_1 = F(T_2)$, which serves as a prior information. Using this constraint, we assign a prior probability distribution to the unknown temperature(s). Thus, we may imagine two observers, one of whom chooses $T_1$ as an uncertain variable while the other chooses $T_2$. But, since the two reservoirs are identical, estimates made by both the observers are equally preferable. We estimate the work and efficiency of the heat engine. The inferred quantities are compared with their corresponding optimal values and the results shown with derived prior are in good agreement with their optimal values. This was really a striking feature in the estimation procedure as we have also shown the estimated results with uniform prior and the deviations observed for the results are quite significantly far from equilibrium. However, near-equilibrium, both priors are equally good. The universal feature of efficiency beyond the linear term, $\eta \approx \eta_c/2 + {\eta_c}^2/8$, is also inferred within this approach, where $\eta_c$ is Carnot efficiency.

In Chapter 3, we apply the inference approach to another constrained thermodynamic process in which the two finite reservoirs are in thermal contact with each other such that the total energy of the whole system remains conserved. The prior for the unknown temperatures is derived by making use of the constraint of energy conservation. The thermodynamic quantity to be estimated is the net entropy production in the two reservoirs. For this process also, we are able to reproduce the optimal characteristics within this inference approach.

In Chapter 4, we reconsider the entropy conserving process with maximum work extracted from the finite source/sink set-up obeying the relation of the form $S = \kappa U^{\omega_1}$, where $\kappa$ is a proportionality constant. In this case, source and sink are non-identical [79] unlike in Chapter 2. Thus, the problem of inference is not symmetric with respect to $T_1$ and $T_2$. In Chapter 2, systems were identical and hence $\gamma = 1$, where $\gamma = \kappa_2/\kappa_1$, $\kappa_1$ and $\kappa_2$ are the two proportionality constants for finite source and sink respectively. We perform the inference for different values of $\gamma$. It has been observed that estimates of efficiency by the two observers are not symmetric about efficiency at optimal work, instead one observer gives better estimate as compared to the other depending upon the value of $\gamma$. In the limiting cases when one reservoir becomes very large as compared to the other, exact optimal behavior of the system is recovered.

In Chapter 5, the summary of present work and the conclusions have been discussed with remarks on the future perspectives of present work.

# Chapter 2

# Inference in entropy-conserving process

## 2.1 Introduction

We propose inference procedure for entropy-conserving process in which a system is taken from a specified initial configuration to a final configuration while preserving entropy of the whole system. The system consists of two finite reservoirs as subsystems which interact via a reversible work source to deliver work from this set-up. Thus whole set-up works like a heat engine in which one reservoir acts as a source while the other acts as a sink. Heat is extracted quasi-statically from the the source, converted into useful work and the rest is ejected into the sink. According to **Maximum work theorem** [81], *of all the thermodynamic processes, the work delivered is maximum and the delivery of rejected heat is minimum for a reversible process.* The process which delivers the maximum work correspondingly rejects the minimum amount of heat to the sink and this results in the least possible increase in the entropy of the sink or we can say of the entire system. The least possible change in entropy of total system implies $\Delta S_{tot} = 0$. Thus, an entropy preserving process yields the upper bound for the work. Maximum work extraction problem from a pair of finite source/sink has been investigated in the literature before in papers [81, 82, 83, 84, 85, 86, 87]. In this chapter, we will investigate the maximum work extraction process from a probabilistic point of view motivated by Bayesian reasoning (Section 1.2) which serves as a powerful tool to treat the situations with incomplete information. Bayesian inference methods seek to quantify uncertainty due to incomplete prior knowledge about the system [19, 30]. The uncertainty may be in regard to specific values taken by the system parameters. The incomplete information is quantified as a prior probability distribution, or simply known as a prior and interpreted in the sense of degree of belief about the likely values of the uncertain parameter. Thus, to formulate the problem in the present context, we assume the ignorance of certain thermodynamic coordinates of the process due to limited information about them. Then we develop procedures to assign priors to quantify the uncertainty in the likely values of these thermodynamic coordinates. Once priors have been assigned, we draw inferences for the thermodynamic quantities like extracted work, efficiency and so on.

To perform inference, consider a pair of thermodynamic systems acting as finite reservoirs as mentioned above. We consider a situation in which we are ignorant about the final state of the two reservoirs. This ignorance corresponds to the ignorance of the exact value of any parameter(s) of the system or in the process governing the interaction between the reservoirs. In our case, it is the temperatures of the reservoirs which are the uncertain parameters since final temperatures of the reservoirs are assumed to be not known to us. So treating this situation from Bayesian point of view, we treat an unknown parameter as a random variable. A prior probability distribution is assigned for the unknown temperatures by making use of the prior knowledge about the constraints of the process. We define the estimates of temperatures as the average values with respect to proposed prior. These expected values of the final temperatures are used to estimate work and efficiency. Our main result of this study is that the estimates for these quantities show remarkable agreement with their corresponding optimal values which are obtained by extremum principles. Particularly, near equilibrium, interesting feature of this study is the emergence of agreement with efficiency at optimal work beyond linear response, when the estimated efficiency is expanded near equilibrium.

To draw a comparison, we followed the same procedure with uniform prior for the unknown temperature. This prior is a *minimally informative prior* as it does not incorporate any other information except for the parameter's range. The significance of taking into account the prior information can be seen from the results with uniform prior in far from equilibrium regime, where the proposed prior gives better agreement with optimal results than uniform prior which involves minimal information.

This Chapter is organized as follows. In Section 2.2, we discuss the outline of the inference procedure applied to entropy-conserving process with finite reservoirs. We list the various assumptions and the prior information we possess to derive the prior for the uncertain parameter(s). The prior is derived in general form under conservation of total entropy of the reservoirs. In Section 2.3, we present a model for the finite thermodynamic reservoirs obeying the fundamental relation $S \propto U^{\omega_1}$. In Section 2.4, the prior is assigned with this model and then we discuss the method and results of estimation of thermodynamic quantities like work, efficiency with their comparative plots in succeeding subsections. Section 2.5 is devoted to the discussion of another model of reservoirs as $N$ spin-1/2 systems. In the succeeding subsections, we discuss the numerical as well as analytical results in high-temperature limit for the estimation of work and efficiency. Further in Section 2.6, we consider classical Otto cycle as our model and perform the inference to obtain certain well-known thermal efficiencies. Finally, in Section 2.7, we make some concluding remarks regarding our inference approach in thermodynamic processes with uncertainty in the control parameters of the process.

## 2.2 Outline

### 2.2.1 Entropy-conserving process

Consider a pair of identical finite thermodynamic systems, each in its own equilibrium state, acting as source and sink. The systems are identical in all aspects, except that their initial temperatures are $T_+$ and $T_-$, respectively. Assume that $T_+ > T_-$. The pictorial representation for the entropy-conserving process is shown in Fig. 2.1.

### 2.2.2 Assignment of prior

Let us assume the situation in which we are uncertain about the extent to which the work extraction process has proceeded so that we are ignorant about the final state of the composite system. Here, this ignorance implies that we are lacking information about the final temperatures of the reservoirs, say $T_1$ and $T_2$ respectively. In view of incomplete information about the temperatures, we first propose prior probability distribution for these temperatures.

To identify the prior information, consider a system which depends on two parameters $T_1$ and $T_2$. Assume that the constraint in the problem leads to a one-to-one relation between the values of $T_1$ and $T_2$ given by :

$$T_1 = F(T_2). \tag{2.1}$$

Given a value of one parameter, it implies a specific value of the other due to the relation (2.1). However, if we assume the values to be uncertain, due to the relation $F(\cdot)$, essentially there is only one uncertain parameter in the problem. Our approach is to assign prior probabilities for the likely values of $T_1$ or $T_2$ since there is only one uncertain parameter due to (2.1). This information is essentially the prior information about the problem i.e. information already exists in the

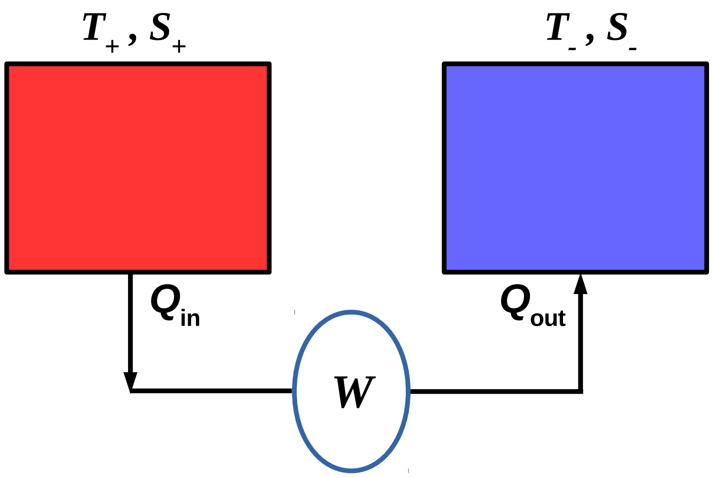



Figure 2.1: *Set-up works like a heat engine, where one reservoir at temperature $T_+$ acts as source from which a small amount of heat ($Q_{\rm in}$) is ejected quasi-statically, converted into work, $W = Q_{\rm in} - Q_{\rm out}$, and rest waste amount of heat ($Q_{\rm out}$) is discarded into the other reservoir at temperature $T_-$ acting as a sink.*

considered problem before any measurement is performed over the system.

**To proceed, it is convenient to consider two observers $A$ and $B$ for $T_1$ and $T_2$ respectively and make the following assumptions:**

(1) Two observers are in the same state of knowledge since each parameter has been specified within the same interval and also the nature of $T_1$ and $T_2$ is assumed to be identical. This is because both $T_1$ and $T_2$ represent the same physical quantity, but for different subsystems. Thus the problem is symmetric with respect to $T_1$ and $T_2$. So, we assume the same form of prior distribution, $P(T_1)$ and $P(T_2)$, for each parameter.

(2) For a certain pair of values, related by $T_1 = F(T_2)$, one may assign the same probabilities corresponding to these values of $T_1$ and $T_2$. This principle implies the following:

$$P(T_1)dT_1 = P(T_2)dT_2. \tag{2.2}$$

The above relation assigns the equal probabilities for $T_1$ and $T_2$ to be in the range $[T_1, T_1 + dT_1]$ and $[T_2, T_2 + dT_2]$ respectively. The task ahead of us is to solve for the function, $P$, by making use of the prior information using the relation $F(\cdot)$.

(3) The range of possible values for uncertain temperature is decided by invoking the information that the set up works like an engine and work must be extracted, $W = -\Delta U \geq 0$. From this constraint, we obtain the range for allowed values of $T_i$, say $[T_{i,min}, T_{i,max}]$, where $i = 1, 2$.

It is apparent from Eq. (2.2), that a relation between $T_1$ and $T_2$, should determine the form of prior. In particular, we should know the rate of change of $T_2$ with respect to $T_1$ as:

$$P(T_2) = P(T_1) \left| \frac{dT_1}{dT_2} \right|, \tag{2.3}$$

when we assume $P(T)$ to be monotonic function of $T$. Let us illustrate by assuming a particular constraint on the process.

An entropy-conserving process requires $dS = 0$, where $S$ is the total entropy of the reservoirs. Due to the additive property of entropy, we can write

$$dS_1 + dS_2 = 0, \tag{2.4}$$

and further as:

$$\left(\frac{\partial S_1}{\partial U_1}\right)_{V_1} \left(\frac{\partial U_1}{\partial T_1}\right)_{V_1} dT_1 + \left(\frac{\partial S_2}{\partial U_2}\right)_{V_2} \left(\frac{\partial U_2}{\partial T_2}\right)_{V_2} dT_2 \quad = \quad 0. \tag{2.5}$$

We assume that no work is performed on or by the heat reservoirs. Using the definition of temperature, $(\partial S/\partial U)_V = 1/T$ and heat capacity at constant volume, $(\partial U/\partial T)_V = C(T)$ in the above equation, we get:

$$\frac{dT_1}{dT_2} \quad = \quad -\frac{C_2(T_2)}{T_2} \left(\frac{C_1(T_1)}{T_1}\right)^{-1}. \tag{2.6}$$

The above equation relates an infinitesimal change in one of the temperatures to a corresponding change in the other. The negative sign indicates the fact that if one temperature decreases the other must increase. Clearly, the above ratio is suggested by the constraint on the physical process, and forms part of the prior information. The next step in the assignment of a prior, is to identify this ratio with the rate of change as Eq. (2.3). Thus a plausible prior, consistent with the rate of change of Eq. (2.6), may be given by $P(T_i) \propto C_i(T_i)/T_i$, where $i = 1, 2$. To satisfy normalisation, we must have

$$P(T_i) = N^{-1}\frac{C_i(T_i)}{T_i}, \tag{2.7}$$

where $N = \int_{T_{i,min}}^{T_{i,max}} C_i(T_i)/T_i \, dT_i$. With this prior, we make estimates for temperatures and other thermodynamic quantities like work and efficiency.

## 2.3 Model

Let the fundamental thermodynamic relation of each reservoir be given by $S \propto U^{\omega_1}$, where the constant of proportionality may depend on some universal constants and/or volume, particle number of the system and so on. Using $(\partial S/\partial U)_V = 1/T$, we get: $U \propto T^{1/(1-\omega_1)}$ and $C(T) \propto T^{\omega}$, where $\omega = \omega_1/(1-\omega_1)$. Alternately, we can write $S \propto T^{\omega}$. We restrict to the case $0 < \omega_1 < 1$, which implies that the systems have a positive heat capacity and this is consistent with the third law of thermodynamics. Some well-known physical examples in this framework are the ideal Fermi gas ($\omega_1 = 1/2$), the degenerate Bose gas ($\omega_1 = 3/5$) and the black body radiation ($\omega_1 = 3/4$)[88]. Classical ideal gas can also be treated as the limit, $\omega_1 \to 0$.

Consider the process of maximum work extraction in which two subsystems interact reversibly so as to conserve the total entropy of the composite system [81]. This implies $\Delta S = \Delta S_1 + \Delta S_2 = 0$, where $\Delta S \equiv S_{\text{final}} - S_{\text{initial}}$. While interacting, the temperatures of the two systems take on the values $T_1^{'}$ and $T_2^{'}$

respectively.

One can extract work ($W'$) in this process, which is given as $W' = Q_{\text{in}} - Q_{\text{out}}$. The expression for work (up to a constant of proportionality) is:

$$W' = (T_+^{\frac{1}{1-\omega_1}} - T_1'^{\frac{1}{1-\omega_1}}) - (T_2'^{\frac{1}{1-\omega_1}} - T_-^{\frac{1}{1-\omega_1}}), \tag{2.8}$$

where $Q_{\text{in}} = T_+^{\frac{1}{1-\omega_1}} - T_1'^{\frac{1}{1-\omega_1}}$, and $Q_{\text{out}} = T_2'^{\frac{1}{1-\omega_1}} - T_-^{\frac{1}{1-\omega_1}}$. Defining $\theta = T_-/T_+$, $T_1 = T_1'/T_+$, $T_2 = T_2'/T_+$ and $W = W'/T_+^{\frac{1}{1-\omega_1}}$, Eq. (2.8) can be written as:

$$W = \left(1 + \theta^{\frac{1}{1-\omega_1}} - T_1^{\frac{1}{1-\omega_1}} - T_2^{\frac{1}{1-\omega_1}}\right). \tag{2.9}$$

The constraint of entropy conservation leads to a relation of the form (Eq. (2.1)):

$$T_1 = (1 + \theta^{\omega} - T_2{}^{\omega})^{\frac{1}{\omega}}. \tag{2.10}$$

For this class of systems, there exists a closed form solution for the constraint equation. We derived a general prior (Eq. (2.7)), which is useful, in particular, when there does not exist explicit functional relation between $T_1$ and $T_2$ as we discuss in Section 2.5.

Substituting the value of $T_1$ from Eq. (2.10) into Eq. (2.9), we may regard $W$ as a function of $T_2$ only:

$$W(T_2) = \left(1 + \theta^{\frac{1}{1-\omega_1}} - (1 + \theta^{\omega} - T_2{}^{\omega})^{\frac{1}{\omega_1}} - T_2^{\frac{1}{1-\omega_1}}\right), \tag{2.11}$$

for fixed $\theta$. Same expression can be written in terms of $T_1$ also and this does not reveal to which reservoir $T_1$ or $T_2$ belong.
The efficiency of the engine is given as $\eta = W/Q_{\text{in}}$ and can be written as:

$$\eta = 1 - \frac{(T_2^{\frac{1}{1-\omega_1}} - \theta^{\frac{1}{1-\omega_1}})}{(1 - T_1^{\frac{1}{1-\omega_1}})}. \tag{2.12}$$

One may continue to extract more work till the two reservoirs achieve a common temperature $T_c$ i.e. $T_1 = T_2 = T_c$. We call this as the optimal work extractable from the initial set-up, where the final temperature of the reservoirs are given by:

$$T_c = \left(\frac{1+\theta^\omega}{2}\right)^{\frac{1}{\omega}}, \tag{2.13}$$

and the optimal extracted work can be written by substituting Eq. (2.13) in Eq. (2.9) to obtain:

$$W_o = 1 + \theta^{\frac{1}{1-\omega_1}} - 2\left(\frac{1+\theta^\omega}{2}\right)^{\frac{1}{\omega_1}}. \tag{2.14}$$

Similarly, efficiency at optimal work ($\eta_o$) is given by substituting Eq. (2.13) in Eq. (2.12) as:

$$\eta_o = 1 - \frac{\left((\frac{1+\theta^\omega}{2})^{\frac{1}{\omega_1}} - \theta^{\frac{1}{1-\omega_1}}\right)}{\left(1 - (\frac{1+\theta^\omega}{2})^{\frac{1}{\omega_1}}\right)}. \tag{2.15}$$

## 2.4 Inference

### 2.4.1 Prior

As discussed in Section 2.2.2, we are ignorant about the final temperatures $T_1$ and $T_2$ and hence make a rational guess about their likely values by assigning a prior. The range for $T_i$ ($i = 1, 2$) is decided by the constraint $W(T_i) \geq 0$ by using Eqs. (2.9) and (2.10). Then we find that an uncertain temperature ($T_i$) is allowed to take values in the interval $[\theta, 1]$. This can be shown graphically as well in the Fig. 2.2. The heat capacity, $C(T) = (\partial U/\partial T)_V$, for our model ($U \propto T^{1/(1-\omega_1)}$) is given as:

$$C_i(T_i) \propto T_i^\omega. \tag{2.16}$$

Using Eq. (2.7) and (2.16), we can write the normalized prior as:

$$P(T_i) = \frac{\omega T_i^{\omega-1}}{(1-\theta^\omega)}. \tag{2.17}$$

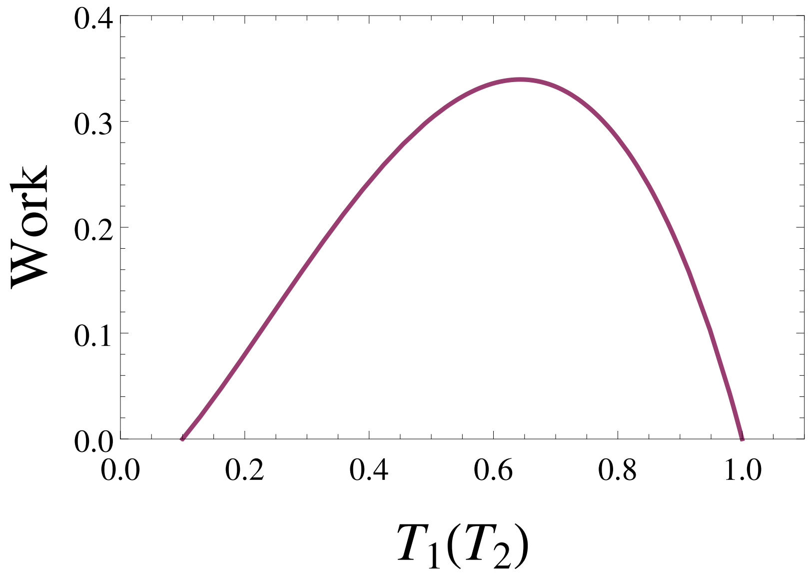



Figure 2.2: *Work as a function of* $T_1$ *and* $T_2$ *for* $\omega_1 = 3/5$, $\theta = 0.1$. *The two curves merge together.* $W(T_i) \geq 0$ *in the interval [*$\theta$*,1].*

The prior obtained is of power-law form.

The prior will be directly used to estimate temperatures of the subsystems. The estimate for $T_i$ is defined as its average value:

$$\overline{T}_i = \int_{\theta}^{1} T_i P(T_i)\, dT_i. \tag{2.18}$$

After solving the above integral, we obtain

$$\overline{T}_i = \omega_1 \frac{(1 - \theta^{\frac{1}{1-\omega_1}})}{(1 - \theta^{\frac{\omega_1}{1-\omega_1}})}. \tag{2.19}$$

### Minimally informative prior

It is expected that estimated behavior depends on the prior used. For comparison, we consider uniform prior as an alternative. The power-law prior is derived by making use of Eq. (2.10) in Eq. (2.2). However, if we ignore this assumption and consider only assumption 1, then we may use uniform prior which is a prior with minimal information. This prior might appear as a natural choice in case the only information about the uncertain parameter is its range, $[\theta, 1]$. Thus, uniform prior is written as:

$$P(T_i)dT_i = \frac{dT_i}{(1-\theta)}. \tag{2.20}$$

The expected value of $T_i$ with uniform prior is given as:

$$\overline{T}_i = \frac{1+\theta}{2}. \tag{2.21}$$

Once the prior for the unknown temperature(s) is assigned, the next step is to make the estimation for work and efficiency.

### 2.4.2 Estimation of work

With *incomplete information*, estimation of work is done by two methods as discussed below:

#### First method

The usual choice for the estimates of work is the average value in the standard way as, $\overline{W(T_2)} = \int_\theta^1 W(T_2)P(T_2)dT_2$, where $W(T_2)$ is given by Eq. (2.11). For power-law prior,

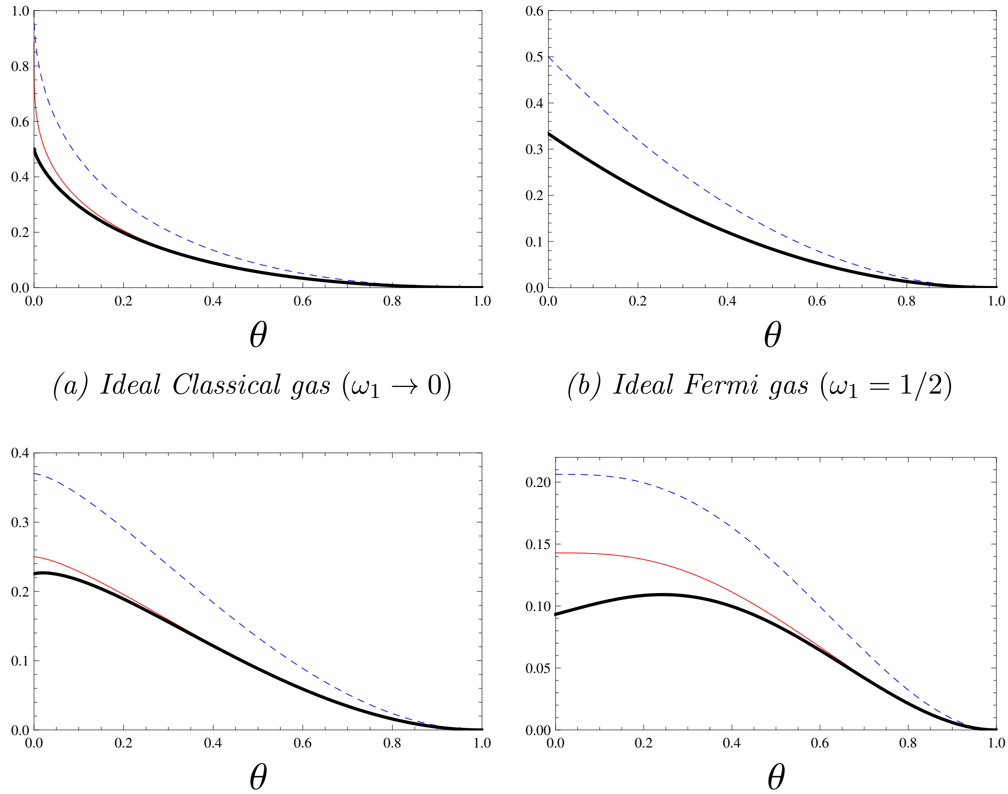
$$\overline{W}_p = 1 + \theta^{\frac{1}{1-\omega_1}} - 2\left(\frac{\omega_1}{1+\omega_1}\right)\left(\frac{1-\theta^{\frac{1+\omega_1}{1-\omega_1}}}{1-\theta^{\frac{\omega_1}{1-\omega_1}}}\right), \tag{2.22}$$

and for uniform prior,

$$\begin{aligned}\overline{W}_u \;=\; & 1 + \theta^{\frac{1}{1-\omega_1}} - \left(\frac{1-\omega_1}{2-\omega_1}\right)\left(\frac{1-\theta^{\frac{2-\omega_1}{1-\omega_1}}}{1-\theta}\right) - \frac{(1+\theta^{\omega})^{\frac{1}{\omega_1}}}{(1-\theta)} \\ & \left[{}_2F_1\left(-1+\frac{1}{\omega_1}, -\frac{1}{\omega_1}; \frac{1}{\omega_1}; \frac{1}{1+\theta^{\omega}}\right) - \theta\ {}_2F_1\left(-1+\frac{1}{\omega_1}, -\frac{1}{\omega_1}; \frac{1}{\omega_1}; \frac{\theta^{\omega}}{1+\theta^{\omega}}\right)\right],\end{aligned} \tag{2.23}$$

where ${}_2F_1(a, b; c; z)$ is the ordinary hypergeometric function [89]. Here subscript $p$ refers to the power-law prior while $u$ refers to uniform prior.

The next step is to compare these estimates of work with the optimal extracted. In case of *complete information*, the maximum work extracted from the set-up is the optimal work ($W_o$) given by Eq. (2.14). Figure 2.3 shows the results for $W_o$, $\overline{W}_p$, and $\overline{W}_u$ for different values of $\omega_1$. However, near-equilibrium ($\theta$




*(a) Ideal Classical gas* ($\omega_1 \to 0$)

*(b) Ideal Fermi gas* ($\omega_1 = 1/2$)

*(c) Degenerate Bose Gas* ($\omega_1 = 3/5$)

*(d) Black Body radiation* ($\omega_1 = 3/4$)

Figure 2.3: *Work as a function of $\theta$. The dashed curve is for $W_o$, thin curve is for $\overline{W}_p$, and thick curve is for $\overline{W}_u$.*

close to unity), estimation with this method show identical behavior for both the priors as shown:

$$\overline{W}_p \approx \overline{W}_u \approx \frac{1}{(1-\omega_1)} \frac{(1-\theta)^2}{6} + \frac{(1-2\omega_1)}{(1-\omega_1)^2} \frac{(1-\theta)^3}{12} + O[1-\theta]^4. \tag{2.24}$$

Similary, expanding $W_o$ about $\theta$ close to 1, we get:

$$W_o \approx \frac{1}{(1-\omega_1)}\frac{(1-\theta)^2}{4} + \frac{(1-2\omega_1)}{(1-\omega_1)^2}\frac{(1-\theta)^3}{8} + O[1-\theta]^4. \qquad (2.25)$$

Thus, with this method of estimation, we observe that power-law estimates are slightly better than estimates with uniform prior. However, near-equilibrium (Eq. (2.24)), both the priors are equally good. Still, these estimates do not estimate the optimal behavior as clear from Eq. (2.25). We will now discuss another method of estimation which estimates the optimal behavior of the process to a remarkable extent.

**Second method**

As shown in Fig. 2.2, the work expression is a concave function in the range [$\theta$, 1], so that we can apply Jensen's inequality (See Appendix A), $\overline{W(T)} \leq W(\overline{T})$. This gives an upper bound to the usual estimate for work. In this sense, we estimate maximum work as the maximal value of the average work. Let us now discuss the estimation procedure for work with the other method in which the estimation is done by replacing the value of a variable with its average value in the work expression.

Note that when an observer estimates a temperature (say of reservoir 2) by the above averaged value with any of the prior distributions, then his/her estimate for the temperature of the other reservoir (labelled 1) will be given as $\tilde{T}_1 = F(\overline{T}_2)$, see Fig. 2.4. This pair of estimated values of temperatures $(\tilde{T}_1, \overline{T}_2)$ automatically satisfy the entropy conserving condition. Then using these estimates for the final temperatures, each observer can estimate other thermal quantities, like the work extracted and efficiency of the process. We briefly summarise the main steps in the estimation procedure.

1) Assign a prior for the uncertain temperatures and calculate the expected value of temperature from Eq. (2.18).

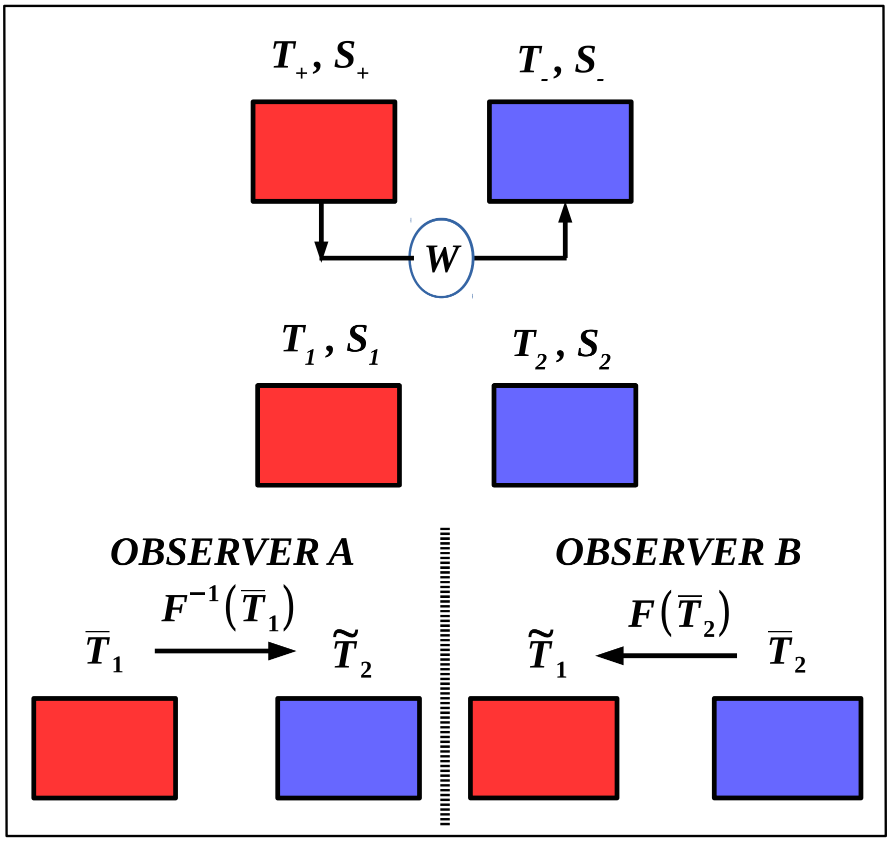



Figure 2.4: *Schematic of the work extraction process, in which given one final temperature, say $T_2$, the other temperature is determined as $T_1 = F(T_2)$, where the map $F$ is invertible. The same map is used to make estimate by an observer if the estimate of one temperature is taken as the average value of temperature over the prior, $\overline{T}_i$. This leads to identical estimates of work by each observer, but their estimates for heat exchanged by the reservoirs are in general different.*

2) Using the constraint of entropy conservation, various quantities such as extracted work, are written as function of one of the temperatures, say $T_i$.

3) The estimate of a physical quantity such as work, $\tilde{W}$, is obtained by replacing the uncertain temperature with its expected value as $\tilde{W} = W(\overline{T}_i)$.

4) Once work is estimated, efficiency ($\eta = W/Q_{\text{in}}$) can be estimated.

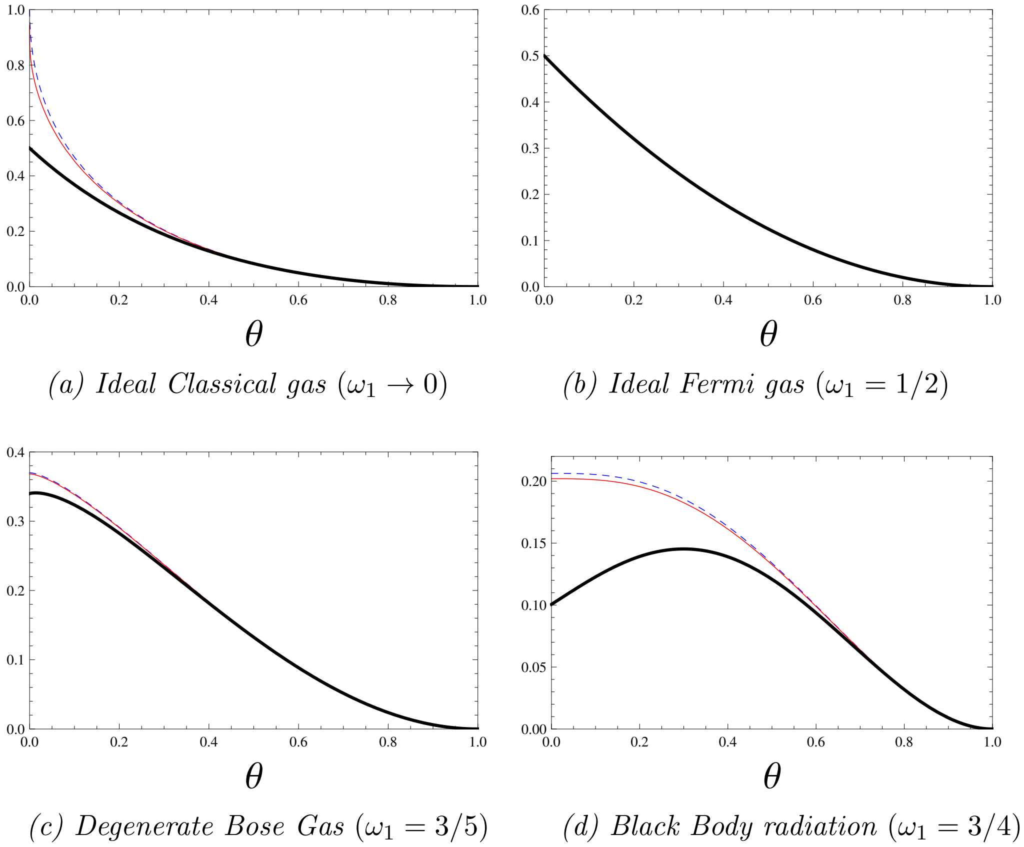
The estimates of work with power-law prior as well as with uniform prior are written as:

$$\tilde{W}_p = 1+\theta^{\frac{1}{1-\omega_1}} - \left[1+\theta^{\omega} - \left(\omega_1 \frac{1-\theta^{\frac{1}{1-\omega_1}}}{1-\theta^{\frac{\omega_1}{1-\omega_1}}}\right)^{\omega}\right]^{\frac{1}{\omega_1}} - \left(\omega_1 \frac{1-\theta^{\frac{1}{1-\omega_1}}}{1-\theta^{\frac{\omega_1}{1-\omega_1}}}\right)^{\frac{1}{1-\omega_1}}, \tag{2.26}$$

and

$$\tilde{W}_u = 1+\theta^{\frac{1}{1-\omega_1}} - \left[1+\theta^{\omega} - \left(\frac{1+\theta}{2}\right)^{\omega}\right]^{\frac{1}{\omega_1}} - \left(\frac{1+\theta}{2}\right)^{\frac{1}{1-\omega_1}}. \tag{2.27}$$

Figure 2.5 shows comparative plots for $W_o$, $\tilde{W}_p$, and $\tilde{W}_u$ for different values of $\omega_1$.




*(a) Ideal Classical gas* ($\omega_1 \to 0$)

*(b) Ideal Fermi gas* ($\omega_1 = 1/2$)

*(c) Degenerate Bose Gas* ($\omega_1 = 3/5$)

*(d) Black Body radiation* ($\omega_1 = 3/4$)

Figure 2.5: *Work as a function of $\theta$. The dashed curve is for $W_o$, thin curve is for $\tilde{W}_p$, and thick curve is for $\tilde{W}_u$.*

From the plots, it is clear that estimates with power-law prior show remarkable agreement with the optimal behavior as compared to the uniform prior. However, near-equilibrium, deviations between the results inferred with both the priors are quite insignificant. In fact, expanding $\tilde{W}_p$, $\tilde{W}_u$ and $W_o$ about $\theta = 1$ up to third order, we obtain, in *each* case

$$\tilde{W}_p \approx \tilde{W}_u \approx \frac{1}{(1-\omega_1)} \frac{(1-\theta)^2}{4} + \frac{(1-2\omega_1)}{(1-\omega_1)^2} \frac{(1-\theta)^3}{8} + O[1-\theta]^4. \tag{2.28}$$

We have observed that in the near-equilibrium regime, estimates of work with both priors approach maximum or optimal value.

Thus, apart from considering the inference with different priors, we have also seen the deviations observed in the results obtained with the two methods of estimation. The ratio of the estimates of work up to third order (Eq. (2.24) and Eq. (2.28)) comes out to be 2/3. The either estimate of work, $\tilde{W}$ or $\overline{W}$, is less than or equal to the optimal work which seems reasonable as the uncertainties in the temperatures will yield less work than the maximum work extracted in the presence of full available information ($T_c$). Even so, irrespective of the method of estimation, the estimation with power-law prior is better than a uniform prior. However, for the Fermi gas, derived power-law prior is equivalent to uniform prior in the range $[\theta, 1]$ and hence the estimates coincide in Figures 2.3 and 2.5.

### 2.4.3 Estimation of efficiency

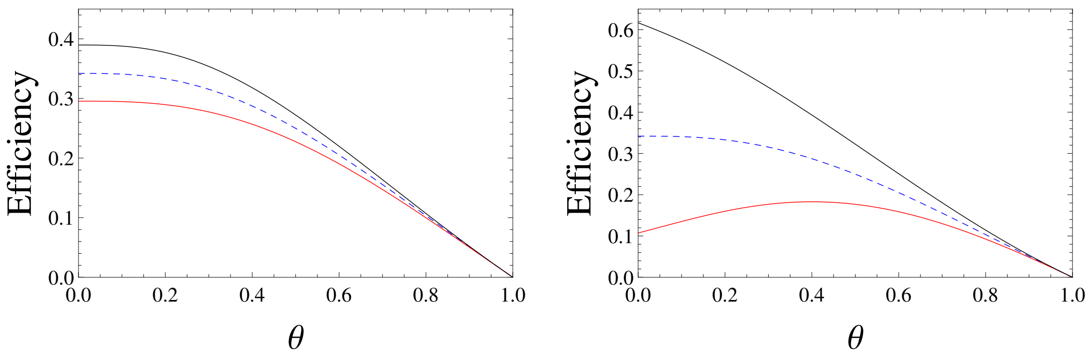
To estimate efficiency, we rewrite the input heat ($Q_{\text{in}}$) which is given by the difference of the initial and the final energies of the initially hotter system. It can be rewritten (dimensionless form) in terms of scaled temperatures, $T_1$ and $T_2$, in two equivalent ways as:

$$Q_{\text{in}}(T_1) = 1 - T_1^{\frac{1}{1-\omega_1}}, \tag{2.29}$$

or in terms of $T_2$ from Eq. (2.10):

$$Q_{\text{in}}(T_2) = 1 - (1 + \theta^{\omega} - {T_2}^{\omega})^{\frac{1}{\omega_1}}. \tag{2.30}$$

Unlike the expression for work, which is symmetric in $T_1$ and $T_2$, $Q_{\text{in}}$ is asymmetric with respect to $T_1$ and $T_2$. In contrast to the work expression, the expressions for heat exchanged require a specific knowledge about the labels on the energies/temperatures. However, this information is not available here. So there is no unique way to assign $Q_{\text{in}}$ or $Q_{\text{out}}$, from the expression for work [90]. Thus efficiency, will depend on the choice of uncertain parameter ($T_1$ or $T_2$). Now, from the assumption 1 in Section 2.2.2 which states that $T_1$ and $T_2$ represent same physical quantities, the form of prior for $T_1$ or $T_2$ is same. The expectation value for each uncertain parameter is same (2.19). If we do the analysis either with $T_1$ or $T_2$, we obtain $Q_{\text{in}}(\overline{T}_1)$ and $Q_{\text{in}}(\overline{T}_2)$ as two different estimates for input heat, respectively. These two estimates will lead to different estimates for efficiency unlike in case of work, where estimates of work are not affected by the choice of uncertain parameter. It follows that the efficiency can be estimated in two ways: $\tilde{\eta}_1 = \tilde{W}/Q_{\text{in}}(\overline{T}_1)$ or $\tilde{\eta}_2 = \tilde{W}/Q_{\text{in}}(\overline{T}_2)$. The complete behavior of efficiency estimates for different systems is shown in Fig. 2.6. For efficiency, let us focus on




*With Power-law prior* *With Uniform prior*

Figure 2.6: *Blue dashed curve is for efficiency at optimal work* ($\eta_o$), *red curve is for* $\tilde{\eta}_{p,2}(\tilde{\eta}_{u,2})$ *and black curve is for* $\tilde{\eta}_{p,1}(\tilde{\eta}_{u,1})$ *for* $\omega_1$=*3/4.*

the behavior near-equilibrium. Treating $T_1$ as an uncertain parameter and then expanding $\tilde{\eta}_{p,1}$ and $\tilde{\eta}_{u,1}$ about $\theta = 1$ up to 2nd order, we obtain

$$\tilde{\eta}_{p,1} = \frac{\eta_c}{2} + \frac{(4-5\omega_1)}{(1-\omega_1)}\frac{\eta_c^2}{24} + \frac{(8-19\omega_1+9{\omega_1}^2)}{(1-\omega_1)^2}\frac{\eta_c^3}{96} + O[\eta_c]^4, \tag{2.31}$$

$$\tilde{\eta}_{u,1} = \frac{\eta_c}{2} + \frac{(2-3\omega_1)}{(1-\omega_1)}\frac{\eta_c^2}{8} + \frac{(12-31\omega_1+17{\omega_1}^2)}{(1-\omega_1)^2}\frac{\eta_c^3}{96} + O[\eta_c]^4. \tag{2.32}$$

Treating $T_2$ as an uncertain parameter and then expanding $\tilde{\eta}_{p,2}$ and $\tilde{\eta}_{u,2}$ about $\theta = 1$ up to 2nd order, we get

$$\tilde{\eta}_{p,2} = \frac{\eta_c}{2} + \frac{(2-\omega_1)}{(1-\omega_1)}\frac{\eta_c^2}{24} + \frac{(4-7\omega_1+{\omega_1}^2)}{(1-\omega_1)^2}\frac{\eta_c^3}{96} + O[\eta_c]^4, \tag{2.33}$$

$$\tilde{\eta}_{u,2} = \frac{\eta_c}{2} + \frac{\omega_1}{(1-\omega_1)}\frac{\eta_c^2}{8} + \frac{(5-7{\omega_1}^2)}{(1-\omega_1)^2}\frac{\eta_c^3}{96} + O[\eta_c]^4. \tag{2.34}$$

If we compare the above estimates of efficiency with the efficiency at optimal work ($\eta_o$) in the near equilibrium regime, then equations match only up to first order in $\eta_c$ as can be seen from the expression of $\eta_o$ (Eq. (2.12)) :

$$\eta_o = \frac{\eta_c}{2} + \frac{\eta_c^2}{8} + \frac{(6-13\omega_1+5{\omega_1}^2)}{(1-\omega_1)^2}\frac{\eta_c^3}{96} + O[\eta_c]^4. \tag{2.35}$$

Thus, we have seen that the optimal work can be estimated up to third order (Eq. (2.28)) while in case of efficiency, estimates match only up to linear term. This discrepancy is observed due to the asymmetric nature of the input heat $Q_{\rm in}$ in $T_1$ and $T_2$. On the other hand, if we define a mean estimate for efficiency as $\tilde{\eta}_m = (\tilde{\eta}_1 + \tilde{\eta}_2)/2$, then the agreement of this mean with the optimal behavior is up to third order. The use of a mean estimate can be justified as follows. We have two hypotheses, whether the heat extracted is given by Eq. (2.29) or (2.30). According to Laplace's principle of insufficient reason [28], when we do not have a specific reason to prefer one hypothesis over another, then we should assign equal weights to each inference following from these hypotheses. In our case, we have assumed a complete ignorance about the labels attached with the final temperatures and so each expression for $Q_{\rm in}$ above is equally valid. In this sense, it is reasonable that the most unbiased estimate be an equally-weighted mean of the different estimates. The near-equilibrium behavior of mean estimate of efficiency is:

$$\tilde{\eta}_p = \frac{\eta_c}{2} + \frac{\eta_c^2}{8} + \frac{(17 - 35\omega_1 + 11\omega_1{}^2)}{(1-\omega_1)^2}\frac{\eta_c^3}{288} + O[\eta_c]^4, \tag{2.36}$$

$$\tilde{\eta}_u = \frac{\eta_c}{2} + \frac{\eta_c^2}{8} + \frac{(3 - \omega_1 - 7\omega_1{}^2)}{(1-\omega_1)^2}\frac{\eta_c^3}{96} + O[\eta_c]^4. \tag{2.37}$$

In this fashion, near-equilibrium behavior of efficiency at optimal work can be reproduced beyond linear term in $\eta_c$ in the inference approach. Apart from these quasi-static processes, such a universal behavior has been observed before in different finite-time models of heat engines [61, 67, 68, 69, 70, 72], where this response occurs when efficiency is optimized at maximum power output as discussed in Section 1.4.

On the other hand, if the estimates of work and input heat are calculated as $\overline{W}$ and $\overline{Q}_{\rm in}$ respectively, then the efficiency may also be estimated as the ratio, $\eta^{av} = \overline{W}/\overline{Q}_{\rm in}$. However, this definition for the estimation of efficiency does not reproduce the efficiency at optimal work even in first order. This can be seen clearly from the near-equilibrium expansions of $\eta_p^{av}$ and $\eta_u^{av}$ as below:

$$\eta_p^{av} = \frac{\eta_c}{3} + \frac{\eta_c^2}{9} + O[\eta_c]^3, \tag{2.38}$$

$$\eta_u^{av} = \frac{\eta_c}{3} + \frac{\eta_c^2}{9} + O[\eta_c]^3, \tag{2.39}$$

where $\eta^{av}$ is the mean estimate of efficiency defined as $(\eta_1^{av} + \eta_2^{av})/2$ and $\eta_1^{av}$ $(\eta_2^{av})$ is the estimated value of efficiency by the observer $A$ $(B)$. The interesting thing which is to be noted here is that the power-law prior ensures that $\overline{Q}_{\rm in}$ has the same value irrespective of whether $T_1$ is the variable of integration or $T_2$. Thus the information regarding the labels of the temperatures, which can be distinguished in the function $Q_{\text{in}}$ (Eqs. (2.29) and (2.30)), is lost with the use of power-law prior [90].

## 2.5 Results for spin-reservoirs

Let us now focus our study on the inference procedure with spin-reservoirs as now we do not have closed form relation between $T_1$ and $T_2$, but which could in principle, be determined numerically. However, apart from numerical calculations, we also perform the analytical calculations in high-temperature limit ($a \ll T$) i.e. when parameter $a$ is quite small compared to the reservoir temperatures. Let us first illustrate the model below.

We consider two *finite* heat reservoirs at temperatures $T_+$ and $T_-$, each consisting of $N$ non-interacting localized spin-1/2 particles. A spin-1/2 particle can be regarded as a two-level system with energy levels $(0, a)$. The mean energy for such reservoir at temperature $T$, is given by [91]:

$$U = \frac{Nae^{-a/kT}}{1 + e^{-a/kT}}, \tag{2.40}$$

where $k$ is Boltzmann's constant.
The heat capacity at constant volume, is given by:

$$C = Nk \left(\frac{a}{kT}\right)^2 \frac{e^{-a/kT}}{(1 + e^{-a/kT})^2}. \tag{2.41}$$

The entropy of a reservoir can be written as:

$$S = Nk \left[\ln (1 + e^{-a/kT}) + \frac{a}{kT} \frac{e^{-a/kT}}{1 + e^{-a/kT}}\right]. \tag{2.42}$$

In the following, we set $k = 1$. Using such systems as the heat source and the sink at hot ($T_+$) and cold ($T_-$) temperatures respectively, we reconsider the process of maximum work extraction by coupling them to reversible work source.

The entropy conservation condition, $\Delta S_{tot} = 0$, for spin-reservoirs reads as:

$$f_1 f_2 \ln\left(\frac{f_1 f_2}{(f_1-1)(f_2-1)}\right) + f_1 \ln(f_2 - 1) + f_2 \ln(f_1 - 1) + c f_1 f_2 = 0, \tag{2.43}$$

where

$$f_1 = 1 + e^{-a/T_1}, \tag{2.44}$$

$$f_2 = 1 + e^{-a/T_2}, \tag{2.45}$$

$$c = -\left[\ln\left[(1+e^{-a/T_+})(1+e^{-a/T_-})\right] + \frac{a}{T_+}\left(\frac{e^{-a/T_+}}{1+e^{-a/T_+}}\right) + \frac{a}{T_-}\left(\frac{e^{-a/T_-}}{1+e^{-a/T_-}}\right)\right]. \tag{2.46}$$

Eq. (2.43) cannot be solved in a closed form for $T_1$ in terms of $T_2$ for arbitrary value of $a$. However, in the limit of high temperatures or $a$ to be small compared with the energy scales set by the reservoir temperatures i.e. $a \ll T$, we have analytic approximations.

## High-temperature limit

In this limit, the constraint equation can be solved in a closed form for $T_1$, in terms of $T_2$. Keeping terms only up to $(a/T)^2$, we get simplified forms for various quantities as follows:

$$U \approx N\left[\frac{a}{2} - \frac{a^2}{4T}\right], \tag{2.47}$$

$$S \approx N\left[\ln 2 - \frac{a^2}{8T^2}\right], \tag{2.48}$$

$$C \approx N\left[\frac{a^2}{4T^2}\right]. \tag{2.49}$$

The explicit form of relation $T_1 = F(T_2)$ can be found in the high temperatures limit, by solving $S_1 + S_2 = S_+ + S_-$, to obtain:

$$T_1 = \frac{1}{\sqrt{1 + \frac{1}{\theta^2} - \frac{1}{T_2{}^2}}}. \tag{2.50}$$

Here, for brevity we use scaled temperatures: $T_+ = 1$ and $T_- \equiv \theta$, such that $0 \leq \theta \leq 1$.

Now, by using Eq. (2.47), we can write the work extracted ($W$) as:,

$$W = \frac{Na^2}{4}\left(\frac{1}{T_1} + \frac{1}{T_2} - \frac{(1+\theta)}{\theta}\right). \tag{2.51}$$

In terms of a single variable, using (2.50) we have

$$W(T_2) = \frac{Na^2}{4}\left(\sqrt{1 + \frac{1}{\theta^2} - \frac{1}{T_2{}^2}} + \frac{1}{T_2} - \frac{(1+\theta)}{\theta}\right). \tag{2.52}$$

At the optimality condition, $T_1 = T_2 = T_c$, and $T_c = \theta\sqrt{2/(1+\theta^2)}$. So the optimal value of work, $W_o$, in the high temperature limit, is given by:

$$W_o = \frac{Na^2}{4\theta}\left[\sqrt{2(1+\theta^2)} - (1+\theta)\right]. \tag{2.53}$$

The efficiency at optimal work, $\eta_o$, in this limit is

$$\eta_o = \left(\frac{\sqrt{2(1+\theta^2)} - (1+\theta)}{\sqrt{\frac{1+\theta^2}{2}} - \theta}\right). \tag{2.54}$$

### 2.5.1 Prior

We write the normalized prior for $T_i$ ($i = 1, 2$) by using Eqs. (2.7) and (2.49) as

$$P(T_i) = \frac{2\theta^2}{(1-\theta^2)}\frac{1}{T_i{}^3}, \tag{2.55}$$

which can further be used in Eq. (2.18) to obtain the estimate for one of the temperatures ($T_2$) as:

$$\overline{T}_2 = \frac{2\theta}{1+\theta}. \tag{2.56}$$

The range for $T_i$'s is $[\theta, 1]$ as determined from the condition $W \geq 0$. Then the other temperature $T_1$ is estimated from (2.50), by substituting $T_2 = \overline{T}_2$, yielding $\tilde{T}_1 = 2\theta/(\sqrt{3\theta^2 - 2\theta + 3})$.

### 2.5.2 Estimation of work

The estimate for work, $\tilde{W}$, is obtained by substituting the expected value of $T_2$ in Eq. (2.52). Thus using (2.56), we obtain:

$$\tilde{W}_p = \frac{Na^2}{8\theta}\left[\sqrt{3\theta^2 - 2\theta + 3} - (1+\theta)\right]. \tag{2.57}$$

For a comparative study, we also consider the uniform prior over the range $[\theta, 1]$ as $P(T_2)dT_2 = dT_2/(1-\theta)$, which gives $\overline{T}_2 = (1+\theta)/2$. For the choice of a uniform prior, Eq. (2.52) yields:

$$\tilde{W}_u = \frac{Na^2}{4\theta(1+\theta)}\left[\sqrt{(1+\theta^2)(1+\theta)^2 - 4\theta^2} - (1+\theta^2)\right]. \tag{2.58}$$

Note that the above estimates for work are the same for both the observers, again due to symmetry in the work expression (2.51), with respect to. $T_1$ and $T_2$. Fig. 2.7 illustrates the comparison of the ratio of estimated to optimal work, using different priors. Further, the agreement between different estimates of work in the near-equilibrium regime, can be studied by expanding the work estimates about $\theta = 1$:

$$\tilde{W}_p \approx \tilde{W}_u = \frac{Na^2}{16}(1-\theta)^2 + \frac{3Na^2}{32}(1-\theta)^3 + O[1-\theta]^4. \tag{2.59}$$

These estimates of work also agree with the optimal work, Eq. (2.53), up to third

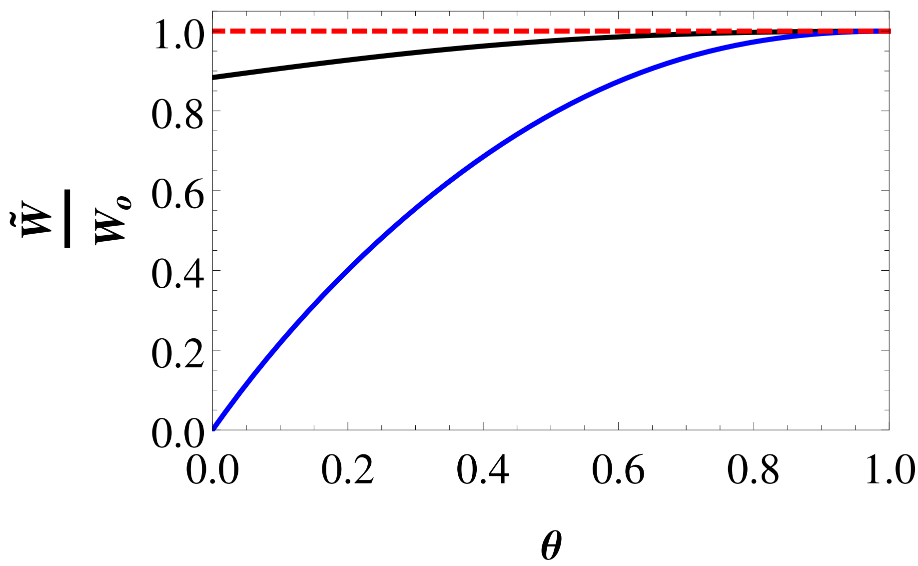



Figure 2.7: *Ratio of estimated work ($\tilde{W}$) to the optimal work $W_o$ (Eq. (2.53)), as function of $\theta$, in the high temperature limit. The lower curve is with estimate $\tilde{W}_u$ using Eq. (2.58), while the upper curve is by using the estimate $\tilde{W}_p$, Eq. (2.57). The estimates with the derived power-law prior are closer to the optimal work. The curves match in the near equilibrium regime ($\theta \approx 1$), where the estimates agree with optimal work, as shown by Eq. (2.59).*

order in $(1-\theta)$. Thus, we see that in the near equilibrium regime for small $a$ values, the uniform prior as well as the non-uniform derived power-law prior both replicate the optimal properties of the work to terms beyond linear response.

As discussed in Section 2.4.2, $W$ given by Eq. (2.51) has a unique maximum and thus a concave function in the interval $[\theta, 1]$. Hence, Jensen inequality implies $\tilde{W} \geq \overline{W}$. We have discussed the results for the work estimates as $\tilde{W}$ since it gives better approximation to optimal values than the usual definition of estimate as $\overline{W} = \int_\theta^1 W(T_i)P(T_i)dT_i$. The explicit expressions for the estimated work defined as $\overline{W}$ are:

$$\overline{W}_p = \frac{(1-\theta)^2}{3\theta(1+\theta)}, \tag{2.60}$$

and

$$\overline{W}_u = \frac{1}{1-\theta}\left[\ln\left(\frac{1}{\theta}\right) + \tan^{-1}(\theta) - \tan^{-1}\left(\frac{1}{\theta}\right)\right]. \tag{2.61}$$

Near-equilibrium, above estimates of work can be expanded as:

$$\overline{W}_p \approx \overline{W}_u \approx \frac{Na^2}{24}(1-\theta)^2 + \frac{Na^2}{16}(1-\theta)^3 + \ O[1-\theta]^4. \tag{2.62}$$

On comparing Eq. (2.62) with Eq. (2.59), which reproduces optimal behavior up to third order, we observe that estimates defined by standard averaging procedure do not show better agreement with optimal features.

### 2.5.3 Estimation of efficiency

In order to estimate efficiency, we have to first evaluate the amount of heat exchanged with the hot reservoir, $Q_{\rm in}$. Following Section 2.4.3, $Q_{\rm in}$ can be written in two alternate ways as

$$Q_{\rm in}(T_1) = \frac{Na^2}{4}\left[\frac{1}{T_1} - 1\right], \tag{2.63}$$

and, using Eq. (2.50)

$$Q_{\rm in}(T_2) = \frac{Na^2}{4}\left[\sqrt{1+\frac{1}{\theta^2} - \frac{1}{{T_2}^2}} - 1\right]. \tag{2.64}$$

It follows that the efficiency can be estimated in two ways: $\tilde{\eta}_1 = \tilde{W}/Q_{\rm in}(\overline{T}_1)$ or $\tilde{\eta}_2 = \tilde{W}/Q_{\rm in}(\overline{T}_2)$, where $\tilde{W}$ is given by Eq. (2.57). Explicitly, we obtain

$$\tilde{\eta}_1 = \frac{\sqrt{3\theta^2 - 2\theta + 3} - (1+\theta)}{1-\theta}, \tag{2.65}$$

$$\tilde{\eta}_2 = \frac{\sqrt{3\theta^2 - 2\theta + 3} - (1+\theta)}{\sqrt{3\theta^2 - 2\theta + 3} - 2\theta}. \tag{2.66}$$

We now compare the above estimates with the efficiency at optimal work given by Eq. (2.54), for which we can write near-equilibrium expansion as:

$$\eta_o \ \approx \ \frac{\eta_c}{2} + \frac{{\eta_c}^2}{8} + O[{\eta_c}^4]. \tag{2.67}$$

The estimates expanded near equilibrium are as follows:

$$\tilde{\eta}_1 \approx \frac{\eta_c}{2} + \frac{{\eta_c}^2}{4} + \frac{{\eta_c}^3}{16} + O[{\eta_c}^4], \tag{2.68}$$

$$\tilde{\eta}_2 \approx \frac{\eta_c}{2} - \frac{{\eta_c}^3}{16} + O[{\eta_c}^4], \tag{2.69}$$

In this situation, Eqs. (2.68) and (2.69) agree with the optimal behavior only up to first order. If we define a mean estimate for efficiency as $\tilde{\eta}_m = (\tilde{\eta}_1 + \tilde{\eta}_2)/2$, then the agreement of this mean with the optimal behavior is up to third order in $\eta_c$.

It is to be noted that this property also emerges with the use of a uniform prior. Thus we can see analytically that in the near equilibrium case and for small $a$ values, the uniform prior as well as the non-uniform prior both replicate the optimal properties of the work as well as efficiency to terms beyond linear response.

Let us now define the estimate of efficiency as $\eta^{av} = \overline{W}/\overline{Q}_{\rm in}$, where $\overline{W}$ is given by Eqs. (2.60) and 2.61 for derived power-law prior and uniform prior respectively. However, efficiency estimated in this way does not reproduce $\eta_o$ even in first order and close to equilibrium, $\eta^{av}$ behaves as:

$$\eta^{av} = \frac{\eta_c}{3} + \frac{\eta_c^2}{9} + O[\eta_c]^3. \tag{2.70}$$

where $\eta^{av}$ is the mean estimate of efficiency as discussed in Section 2.4.3.

### 2.5.4 General case: Numerical results

Now we turn to the general solution of the estimation problem, for arbitrary values of $a$ and $\theta$. In general, the relation between $T_1$ and $T_2$ is transcendental following from the entropy conservation condition. Thus even if the average temperature, Eq. (2.18), can be calculated analytically, the value of the other has to be evaluated numerically (Appendix B). The optimal temperature is also evaluated numerically. Fig. 2.8 compares different expressions for work, such as optimal value $W_o$, and the estimates $\tilde{W}_p$ and $\tilde{W}_u$, for general values of parameter $a$. As is expected, the estimates are close to the optimal work in the near-

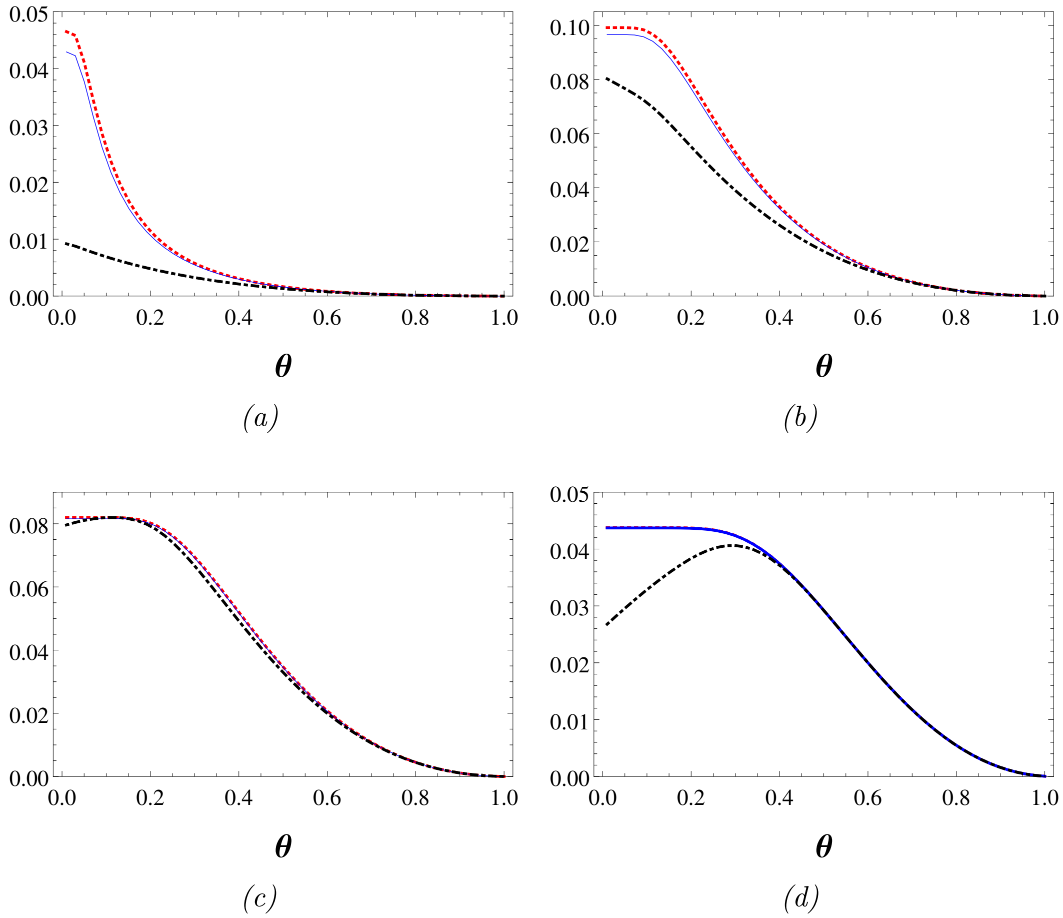



Figure 2.8: *Work, scaled by $N$, as a function of $\theta$ for different values of $a$; (a) a = 0.2, (b) a = 0.8, (c) a = 1.5, (d) a = 2.4. The dotted curve is for $W_o$, solid curve is for $\tilde{W}_p$, and dotdashed curve is for $\tilde{W}_u$.*

equilibrium regime. However, far from equilibrium, only the estimates from the derived power-law prior provide, in general, a better estimation of the optimal work than those obtained from a uniform prior, thus signifying the use of prior information in the assignment of the prior.

Similarly, we can extend the calculation to the estimation of efficiency. We evaluate the estimates for each observer and also the mean estimate by using equal weights, similar to the procedure in the high temperature limit. Fig. 2.9 compares the estimates using the derived power-law prior with the efficiency at optimal work. The numerical calculations for the general case show that the

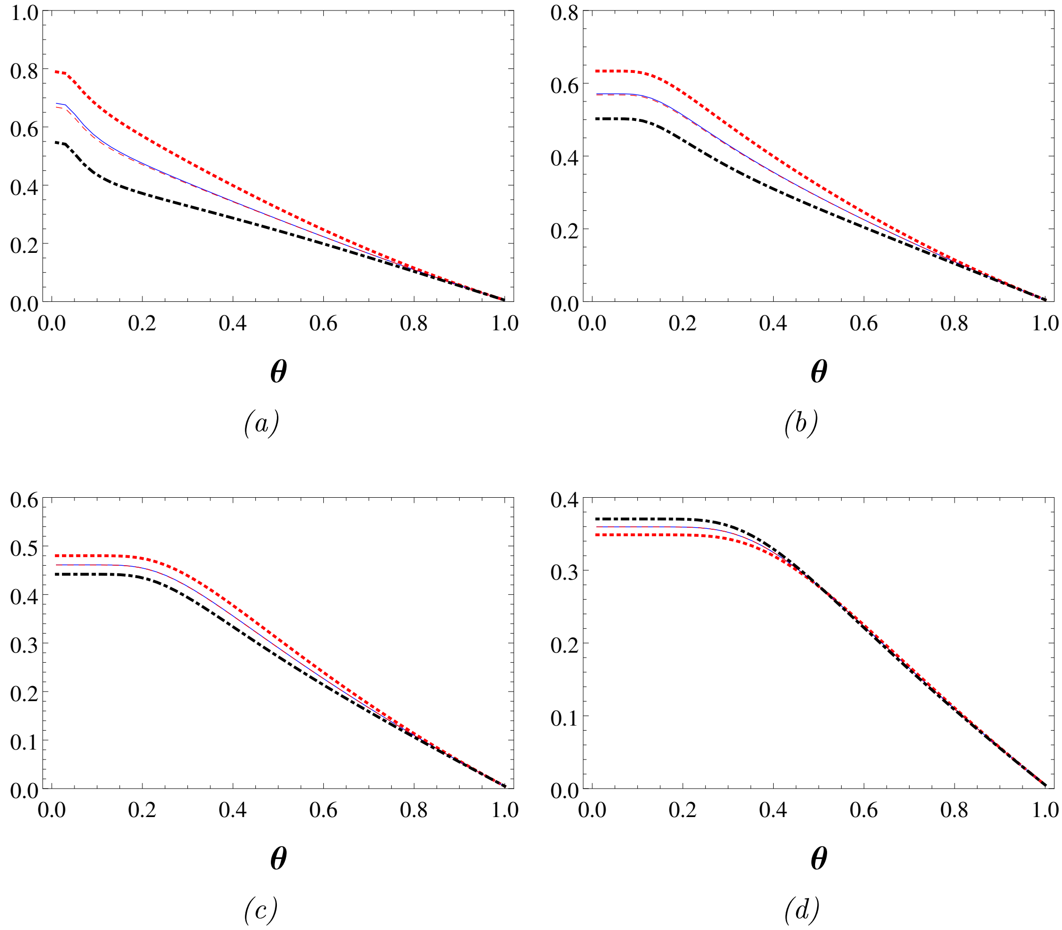



Figure 2.9: *Efficiency as a function of $\theta$ for different values of a; (a) $a = 0.2$, (b) $a = 0.8$, (c) $a = 1.5$, (d) $a = 2.4$. The dotted curve is for $\tilde{\eta}_1$, and dotdashed curve is for $\tilde{\eta}_2$. The middle, solid curve is for $\eta_o$ and the thin, dashed curve closely following it is the mean estimate of efficiency.*

estimates with the non-uniform prior provide visibly better agreement with the behavior of efficiency at optimal work, than the uniform prior.

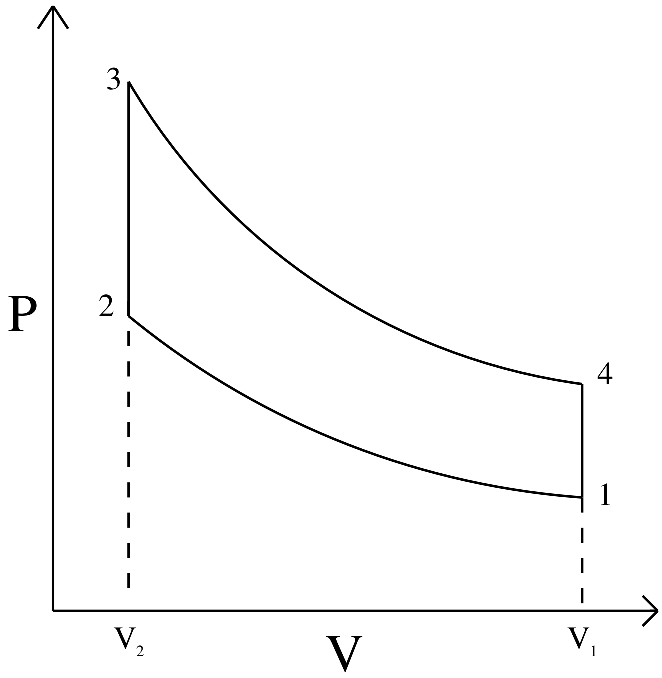



Figure 2.10: *Pressure-Volume diagram of a reversible Classical Otto cycle*

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## 2.6 Inference in classical Otto cycle

### 2.6.1 Introduction

The Otto cycle was built by a German engineer, Nikolaus Otto in 1876. It is a four-stroke internal combustion engine. Classical Otto cycle is a reversible model of heat engine which operates at maximum work output per cycle. The Otto cycle consists of four branches, two of which are adiabatic while two others are isochoric (constant volume). The working fluid is an ideal gas with constant heat capacity ($C_v$). The cycle is as shown in Fig. 2.10 [92]:

The idealized cycle consists of two reversible adiabatic segments $1 \rightarrow 2$ and $3 \rightarrow 4$ and two reversible constant volume segments $2 \rightarrow 3$ and $4 \rightarrow 1$. The cycle does not have isothermal segments and the reservoir temperatures $T_3$ and $T_1$ correspond to maximum and minimum temperatures, respectively, along each cycle. The temperature varies from $T_2$ to $T_3$ and from $T_4$ to $T_1$ along the heating and cooling segments respectively. The heat transfers to the fluid along the paths $2 \rightarrow 3$ and $4 \rightarrow 1$ are:

$$\begin{aligned} Q_{\text{in}} &= C_v(T_3 - T_2), \\ Q_{\text{out}} &= C_v(T_4 - T_1). \end{aligned}$$

Because the cycle is reversible, the fluid's entropy change per cycle is zero:

$$\Delta S = C_v \ln \frac{T_3}{T_2} + C_v \ln \frac{T_1}{T_4} = 0. \tag{2.71}$$

This gives

$$T_4 = \frac{T_1 T_3}{T_2}. \tag{2.72}$$

Since $T_1$ and $T_3$ are the fixed temperatures, thus only variables are $T_2$ and $T_4$. But due to the relation (2.72), there is only one independent parameter. The work done per cycle, $W = Q_{\text{in}} - Q_{\text{out}}$, is given by:

$$W = C_v(T_3 + T_1 - T_4 - T_2). \tag{2.73}$$

Similarly efficiency, $\eta = W/Q_{\text{in}}$, using (2.72) can be written as:

$$\eta = 1 - \frac{T_1}{T_2}. \tag{2.74}$$

For fixed values of $T_1$ and $T_3$, we will see for what values of $T_2$ and $T_4$, $W$ is maximized. Using Eq. (2.72) in (2.73) and setting $\partial W/\partial T_2 = 0$ to give $T_2^*$ as:

$$T_2^* = (T_1 T_3)^{\frac{1}{2}}, \tag{2.75}$$

which can further be used in Eq. (2.72) to obtain $T_4^*$ as [81]:

$$T_4^* = (T_1 T_3)^{\frac{1}{2}}. \tag{2.76}$$

Using the above values of $T_2^*$ and $T_4^*$ in (2.73), we obtain the expression for maximum work:

$$W_{\max} = C_v T_3 [1 - \sqrt{\theta}]^2, \tag{2.77}$$

where $\theta = T_1/T_3$.

Efficiency at maximum work is given by $\eta^* = 1 - \sqrt{\theta}$. This is CA-efficiency.

In general, we can write

$$\frac{W}{T_3 C_v} = \frac{\eta(\eta_c - \eta)}{1 - \eta}. \tag{2.78}$$

where $\eta_c = 1 - T_1/T_3$ is the Carnot efficiency. This expresses the relation between W and $\eta$. For a given $\eta_c$,

$$\lim_{\eta \to 0} [W] = \lim_{\eta \to \eta_c} [W] = 0. \tag{2.79}$$

Thus the Otto cycle gives zero work output at both its maximum and minimum efficiencies.

### 2.6.2 Inference

To consider a situation where we have partial or incomplete information about the system, we have to identify the thermodynamic control parameters of the problem. The expression for work (2.78) is only a function of efficiency $\eta$, if we assume that the reservoir temperatures $T_1$ and $T_3$ (or $\eta_c$) are held fixed. Thus if efficiency is also specified then there is no uncertain parameter in the problem. However, there are intermediate temperatures ($T_2$ and $T_4$) of the working fluid which vary during the heat cycle. Due to the cyclic process, there is only one independent temperature, which we take as $T_2$. Now in the case of complete knowledge of the value of $T_2$, we can calculate all thermal quantities of interest for the cycle. Here we want to consider the other extreme situation where we do not specifically know $T_2$, except for the fact that it lies in the range $[T_1, T_3]$. We treat this as a problem in statistical inference and follow the Bayesian approach. At this point, there is no general rule available to assign the prior. It may seem reasonable to assume a uniform distribution for $T_2$ in the range $[T_1, T_3]$, in the absence of any other information or one may adopt *Jeffreys' prior* (Section 1.2.2) for $T_2$.

Now, we argue for assigning Jeffreys' prior for $T_2$ or $T_4$. Again, consider two observers $A$ and $B$ who assign priors for $T_2$ and $T_4$ respectively. Each parameter lies in the range $[T_1, T_3]$. As the state of knowledge of the two observers is same, they can assign same functional form for their priors. Further due to Eq. (2.72), we have a one-to-one relation between $T_2$ and $T_4$, hence Eq. (2.2) can be rewritten as:

$$P(T_2)dT_2 = P(T_4)dT_4. \tag{2.80}$$

By using Eq. (2.72) in above expression, we obtain the prior as $P(T_i)dT_i \propto dT_i/T_i$ $(i = 2, 4)$, which is Jeffreys' prior. This is the prior for classical ideal gas with the constraint of entropy conservation.

Of interest here is the expected value of efficiency which is defined as $\langle\eta\rangle = \int_{T_1}^{T_3} \eta P(T_2)dT_2$. Then for Jeffreys' choice, we have

$$\langle\eta\rangle = 1 + \frac{(1-\theta)}{\ln\theta}, \tag{2.81}$$

whereas with the uniform prior, it is given by

$$\langle\eta\rangle = 1 + \frac{\theta\ln\theta}{(1-\theta)}. \tag{2.82}$$

Finally, it is interesting to observe the effect of using a generalised power-law prior for classical case as for quantum case [73], this prior serves to incorporate the uniform prior and the Jeffreys' prior in a unified way. Thus assigning a power-law probability distribution for the unknown temperature $T_2$:

$$\begin{aligned} P(T_2) &= \frac{1/T_2^b}{\int_{T_1}^{T_3} 1/T_2^b \, dT_2} \\ P(T_2) &= \left(\frac{1-b}{1-\theta^{1-b}}\right)\frac{1}{T_2^b}. \end{aligned} \tag{2.83}$$

where $b = 0$ (uniform) and $b = 1$ (Jeffreys) represent special cases.

The expected efficiency corresponding to the power-law prior is given by:

$$\langle\eta\rangle = 1 + \frac{(1-b)}{b}\frac{(\theta - \theta^{1-b})}{(1-\theta^{1-b})}. \tag{2.84}$$

Table I shows the expressions for $\langle\eta\rangle$ for different values of $b$.

**Table I**

| $b$ | $\langle\eta\rangle$ | Comments |
|---|---|---|
| -∞ | $1-\theta$ | Carnot efficiency. |
| -1 | $\frac{1-\theta}{1+\theta}$ | $\eta_c/(2-\eta_c)$ [72]. |
| 0 | $1+\frac{\theta\ln\theta}{1-\theta}$ | Finite source and infinite sink[83, 84]. |
| $\frac{1}{2}$ | $1-\sqrt{\theta}$ | CA efficiency [61]. |
| 1 | $1+\frac{1-\theta}{\ln\theta}$ | Finite heat sink and infinite source [84]. |
| 2 | $\frac{1-\theta}{2}$ | $\eta_c/2$. |
| ∞ | 0 | Zero efficiency. |

It has been seen that $\langle\eta\rangle$ is a monotonic function of $b$, and interpolates between Carnot efficiency and zero value as $b$ ranges from $-\infty$ to $\infty$. Figure 2.11 shows

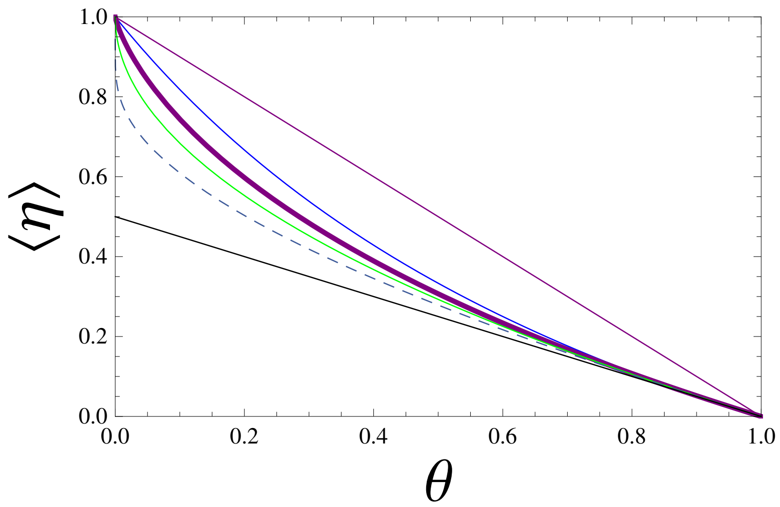



Figure 2.11: $\langle\eta\rangle$ *vs.* $\theta$ *for different b's values.* $\langle\eta\rangle$ *is a monotonic decreasing function of b as shown in the table. Here, the uppermost curve corresponds to* $\eta_c$ *and then plotted for b = -1, 0, 0.5, 1, 2. The dashed line shows the efficiency with Jeffreys' prior (b = 1) and solid curve is for uniform prior (b = 0).*

the behavior of expected efficiency of Otto cycle for different values of $b$. For $\theta$ close to equilibrium, $\langle\eta\rangle$ can be expanded as:

$$\langle\eta\rangle = \frac{\eta_c}{2} + \frac{1}{12}(2-b)\eta_c^2 + O[\eta_c]^3. \tag{2.85}$$

Hence, near-equilibrium ($\theta \approx 1$), $\langle\eta\rangle$ exhibits a universal form independent of $b$ in first order. Thus, it is interesting to note that the efficiency for the Otto cycle is estimated by the standard definition of averaging since this method yields some of the well-known thermal efficiencies of heat engines obtained in different contexts.

## 2.7 Conclusion

Thus, we have discussed the relevance of prior information in estimating the optimal performance characteristics of heat engines. The prior information is incorporated in terms of prior probabilities which reflect the degree of belief of an observer. In the present context, the uncertainty is introduced through lack of knowledge about the exact values of the thermodynamic control parameters of the process. It is really prominent to observe that the estimated behavior obtained by quantifying ignorance of the control parameters in a process are actually very close to the optimal behavior seen in the case of complete information.

In our study, we followed the inference procedure to estimate the optimality in classical thermodynamic process of entropy conservation. We consider a completely reversible model of heat engine with identical finite reservoirs acting as heat source and sink. Then we assume the ignorance of the final temperatures, $T_1$ and $T_2$, of the reservoirs. Uncertainty in the likely values of $T_1$ and $T_2$ is treated from probabilistic point of view by assigning prior probabilities. The prior probability is assigned by taking into account the prior information about the functional relation between $T_1$ and $T_2$. This yields an explicit formula for the prior. The estimated values of the thermal quantities like work, efficiency with the use of derived power-law prior show remarkable agreement with their optimal behavior.

In our analysis, we have discussed the two methods of estimating the thermal quantities. The standard method defines the estimates as the average value of the quantity over the prior $P(T_i)$. The other method, we explored, is to estimate the quantities by replacing the uncertain parameter $T_i$ with its average value $\overline{T}_i$. It is observed that latter method of estimation yields estimates of the quantities which are much closer to their optimal values than the usual method of averaging.

To show the relevance of prior information, we derive estimates with uniform prior as well and it has been noted that the uniform prior estimates are always lower than the optimal values. This is consistent with the reasoning of incorporating the prior information so that utilizing more information makes us to expect higher work output, in contrast to uniform prior which involves minimal information. However, near equilibrium, both the priors show similar behavior up to third order.

In near-equilibrium regime, universal behavior in efficiency as $\eta \approx \eta_c/2+\eta_c^2/8$ is reproduced within inference approach. Such type of behavior has been observed in many different models of heat engines where this response occurs for efficiency at maximum power output. The coefficient 1/2 is observed in case of perfectly coupled systems [64, 65, 66]. The coefficient 1/8 is indeed universal for strong coupling models that possess a left-right symmetry on the fluxes [70] and some other finite-time models where there is a symmetric dissipation with respect to hot and cold reservoirs [72]. However, our process is quasi-static and has no analogy to finite-time models but it is interesting to observe the optimal features of efficiency. Our analysis shows that this universality can be anticipated from inference approach applied to reversible thermodynamic model but with incomplete information of the thermodynamic coordinates in the concerned physical process. The interesting thing is that this universal behavior in efficiency in the present context is also attributed to certain symmetry, which is to assign equal weightage to estimated input heat. Without this symmetry, the universality holds up to linear response only.

We also implemented the inference procedure to Classical Otto cycle in which classical ideal gas is a working medium. The intermediate temperatures of the ideal gas during the cycle are treated probabilistically and thus priors were assigned for their likely values. For the classical Otto cycle, the power-law type of priors upon averaging suggest many well known expressions for efficiency which have been previously observed from very different approaches such as finite-time thermodynamics, finite heat source/sink set-up for engines and so on. Thus we have discussed an intriguing connection between prior probabilities and thermal efficiencies of heat engines in the context of Classical Otto cycle.

While concluding this Chapter, we can emphasise on the point that inference approach yield good results for estimates in constrained thermodynamic processes. The whole analysis can also be looked at in terms of macrostates. Consider the basic question in equilibrium thermodynamics [81]. Given that entropy is conserved for a bipartite system whose total energy is allowed to vary, what is the most likely state of the system? The answer is given as the equilibrium state which has minimum total energy for the given value of the total entropy. In terms of work, it translates into an extraction of maximum work. The agreement of our estimates with the optimal work and the corresponding efficiency, shows that we are able to estimate the equilibrium state consistent with the constraints, without explicitly doing an optimization. Optimization techniques formulate the problems in a mathematical model where the variables to be optimized are the control parameters of the problem which requires concrete realisation for all control parameters. As we know that lack of full information is a greatest hindrance in the optimization, so to deal with the situations with incomplete information, inference techniques can be implemented for optimization. Thus inference based approach seems to be applicable to more general situations with analogous form of constraints. It may be useful in estimating the optimal performance characteristics in a more efficient way, where the optimal solution cannot be determined in a closed form and one usually has to resort to numerical optimization.

# Chapter 3

# Inference in energy-conserving process

## 3.1 Introduction

In this chapter, we propose inference procedure for another well-known thermodynamic process with a constraint on total energy conservation. In this process, two systems interact directly in such a way so as to preserve the total energy of the composite system. This corresponds to an increase in the total entropy of the whole system. This process involves the spontaneous transfer of heat from hot reservoir to cold reservoir. The rate at which heat flows is sufficiently slow (quasi-static) such that the temperature remains spatially homogeneous within each reservoir. While interacting thermally, the entropy of initially hotter system decreases and that of the colder system increases. The main physical effect of thermal contact between the reservoirs is the entropy production in the whole isolated system. Thus, the point which is to be noted is that, although the interaction progresses quasi-statically still it is an irreversible interaction.

Inference is performed for energy-conserving process in an analogous manner for entropy-conserving process in Chapter 2. A prior is derived for the thermodynamic coordinates of the process about which we do not have full information. We consider that the final state of the two reservoirs, in the considered process, is not specified and the task at hand is to derive the prior probabilities for the final temperature. The prior for the uncertain temperature(s) is assigned by incorporating the prior information in the energy conservation constraint. The estimate of the temperature(s) is defined as the average value over the proposed prior. For this process, the thermodynamic quantity to be inferred is the total entropy production in the two systems by using the estimated values of the unknown temperatures. The estimation is performed with the derived prior and with the uniform prior which only incorporates the information about the range of the parameter. Near-equilibrium, both types of priors replicate the optimal behavior up to third order in $(1-\theta)$. However, far from equilibrium, estimated behavior with the derived prior matches closely with the optimal behavior, thus signifying the use of prior by incorporating the prior information about the process. We also discuss the estimation of entropy production by the standard averaging method like in Chapter 2.

This Chapter is organized as follows. Section 3.2 outlines the framework for inference procedure applied to energy-conserving process by taking into consideration the various assumptions and constraints. The prior is proposed for the unknown final temperature(s) of the reservoir. Thus, the uncertainty in the thermodynamic coordinates of the process can be formulated in terms of the proposed prior and this is helpful in estimating the optimal features of the process. In Section 3.3, we present the model for the reservoirs obeying the fundamental relation $S \propto T^{\omega}$. In Section 3.4, we assign the prior for this model and discuss the results obtained for the estimated entropy production in succeeding subsections. In Section 3.5, another model of $N$ spin-1/2 systems is studied within this prior-based framework and then we discuss analytical as well as numerical results for estimated entropy production. Finally, Section 3.6 is devoted to discussion and summary of the inference procedure applied to the energy-conserving process.

## 3.2 Outline

### 3.2.1 Energy-conserving process

Consider a pair of identical finite thermodynamic systems at initial temperatures $T_+$ and $T_-$ ($< T_+$), respectively. A small amount of heat ($Q$) is quasi-statically removed from the reservoir at $T_+$ and deposited in the same manner with other reservoir at $T_-$. As is known, there is a net entropy production in the reservoirs, which is also the main quantity of interest here. The pictorial representation for the energy-conserving process is shown in Figure 3.1.

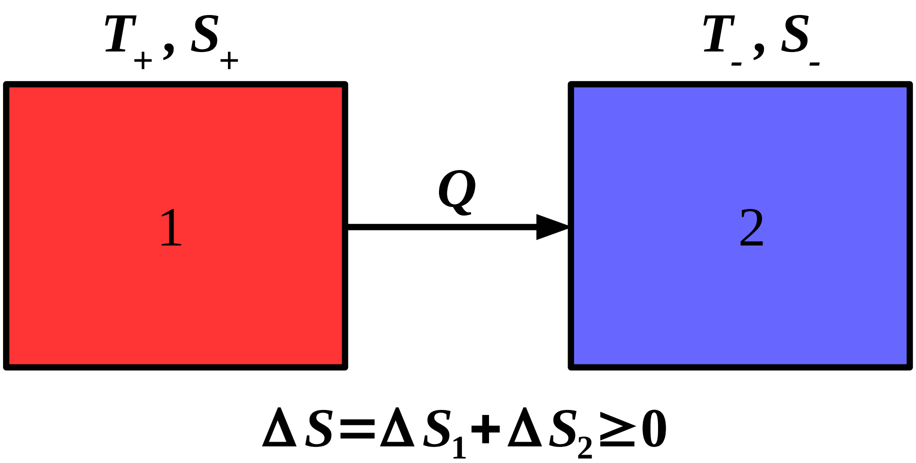



Figure 3.1: *Set-up illustrating the transfer of heat from one reservoir to another at different initial temperatures but with fixed total energy of the combined system.*

### 3.2.2 Prior

We assume ignorance of final temperatures, $T_1$ and $T_2$, of the reservoirs and propose prior probabilities for their likely values. The constraint of energy conservation in the problem leads to a one-to-one relation, $F(\cdot)$, between the values of

$T_1$ and $T_2$ analogous to Eq. (2.1). As discussed in Section 2.2.2, imagine two observers $A$ and $B$ interpreting uncertainty in terms of $T_1$ and $T_2$ respectively. The first two assumptions involved in the derivation of prior in Section 2.2.2 remain the same while the third assumption to determine the range for $T_i$'s ($i = 1, 2$) is different. The range for $T_i$ is determined from the condition $\Delta S \geq 0$, which specifies the process and yields a permissible range for $T_i$, say $[T_{i,min}, T_{i,max}]$.

The prior $\pi(T)$ is derived from the prior information that the transfer of quasi-static transfer of heat energy from one reservoir to another, does not change the total energy i.e. $dU = 0$, which further can be written as

$$dU_1 + dU_2 = 0, \tag{3.1}$$

which can be written as:

$$\left(\frac{\partial U_1}{\partial T_1}\right)_{V_1} dT_1 + \left(\frac{\partial U_2}{\partial T_2}\right)_{V_2} dT_2 = 0. \tag{3.2}$$

This yields $|dT_1/dT_2| = C_2(T_2)/C_1(T_1)$, using the definition of heat capacity at constant volume, $C(T) = (\partial U/\partial T)_V$. Thus, constraint on energy conservation yields a ratio for the infinitesimal changes in the two temperatures and thus forms a part of the prior information for this process. Again analogous to Eq. (2.3), we have

$$\pi(T_2) = \pi(T_1)\left|\frac{dT_1}{dT_2}\right|. \tag{3.3}$$

Identifying the two rates of change and using the separation of variables, we obtain the prior for each temperature as

$$\pi(T_i) = \frac{C_i(T_i)}{\int C_i(T_i) dT_i}, \tag{3.4}$$

where $i = 1, 2$ and $N = \int C_i(T_i)\, dT_i$ is the normalisation constant.

## 3.3 Model

Let us first discuss the model where entropy of each reservoir satisfies the relation of the form $S \propto T^{\omega}$ as considered in Chapter 2, where $\omega = \omega_1/(1-\omega_1)$ and $0 < \omega_1 < 1$, and $T$ is the temperature of the reservoir.

Now, in the considered process, from the initial temperatures $T_+$ and $T_-$, assume that the temperatures of the two reservoirs take on values $T_1^{'}$ and $T_2^{'}$ respectively. The net entropy produced, $\Delta S^{'} = S_1 + S_2 - S_+ - S_-$, (up to constant of proportionality) can be written as:

$$\Delta S^{'} = (T_1^{'\omega} + T_2^{'\omega}) - (T_+^{\omega} + T_-^{\omega}), \tag{3.5}$$

or

$$\Delta S = (T_1^{\omega} + T_2^{\omega}) - (1 + \theta^{\omega}), \tag{3.6}$$

where we define $\theta = T_-/T_+$, $T_1 = T_1^{'}/T_+$, $T_2 = T_2^{'}/T_+$ and $\Delta S = \Delta S^{'}/T_+{}^{\omega}$. The constraint of energy conservation, $\Delta U = 0$, gives a relation $F(\cdot)$ of the form:

$$T_1 = (1 + \theta^{1+\omega} - T_2^{1+\omega})^{\frac{1}{1+\omega}}. \tag{3.7}$$

For this process also, we have an explicit relation between $T_1$ and $T_2$ similar to the case of entropy-conserving process in Chapter 2 (Section 2.3). Substituting the value of $T_1$ in Eq. (3.6), we can write:

$$\Delta S(T_2) = (1 + \theta^{1+\omega} - T_2^{1+\omega})^{\frac{\omega}{1+\omega}} + T_2{}^{\omega} - (1 + \theta^{\omega}). \tag{3.8}$$

Similarly, $\Delta S$ can be expressed as a function of $T_1$ only. Note that the expression for entropy change (Eq. (3.6)) is symmetric with respect to $T_1$ and $T_2$. The heat will continue to flow from hot system to cold system till the two systems reach a common final temperature, $T_1 = T_2 = T_c$, which is given as:

$$T_c = \left(\frac{1+\theta^{1+\omega}}{2}\right)^{\frac{1}{1+\omega}}, \tag{3.9}$$

which is the *optimal* process. The maximal or optimal entropy production ($\Delta S_o$) is given by substituting Eq. (3.9) in Eq. (3.6):

$$\Delta S_o = 2\left(\frac{1+\theta^{1+\omega}}{2}\right)^{\frac{\omega}{1+\omega}} - (1+\theta^{\omega}). \tag{3.10}$$

## 3.4 Inference

### 3.4.1 Prior

The heat capacity of the reservoir is of the form $C(T) \propto T^{\omega}$. Hence, using Eq. (3.4), we can write the normalized prior as:

$$\pi(T_i) = \frac{(1+\omega)}{(1-\theta^{\omega+1})} T_i^{\omega}, \tag{3.11}$$

where the range of scaled temperature $T_i$, is $[\theta, 1]$ as determined from the constraint $\Delta S \geq 0$ (Eq. (3.8)). The expected value of $T_i$ with the above prior is given by:

$$\overline{T}_i = \left(\frac{\omega+1}{\omega+2}\right)\left(\frac{1-\theta^{\omega+2}}{1-\theta^{\omega+1}}\right), \tag{3.12}$$

whereas for a uniform prior, the expected value is simply

$$\overline{T}_i = \frac{(1+\theta)}{2}. \tag{3.13}$$

### 3.4.2 Estimation of entropy production

Similar to Section 2.4.2, we perform the estimation of entropy production by two methods as follows:

**Standard averaging**

Let us first define the estimates of entropy production by their average values as $\overline{\Delta S} = \int_\theta^1 \Delta S \pi(T_i) dT_i$. The corresponding expression for the estimates with power-law prior is

$$\overline{\Delta S}_p = 2\left(\frac{\omega+1}{2\omega+1}\right)\left(\frac{1-\theta^{2\omega+1}}{1-\theta^{\omega+1}}\right) - (1+\theta^\omega), \tag{3.14}$$

and with uniform prior,

$$\begin{aligned}\overline{\Delta S}_u \;=\; & \left[{}_2F_1\left(\frac{1}{\omega+1}, -\frac{\omega}{\omega+1}; \frac{\omega+2}{\omega+1}; \frac{1}{1+\theta^{\omega+1}}\right)\right. \\ & \left. -\theta\; {}_2F_1\left(\frac{1}{\omega+1}, -\frac{\omega}{\omega+1}; \frac{\omega+2}{\omega+1}; \frac{\theta^{\omega+1}}{1+\theta^{\omega+1}}\right)\right] \\ & \left(\frac{1}{\omega+1}\right)\left(\frac{1-\theta^{\omega+1}}{1-\theta}\right) - (1+\theta^\omega). \end{aligned} \tag{3.15}$$

Figure 3.2 illustrates the results obtained with the two prios their comparison with the optimal entropy production (Eq. (3.10)).

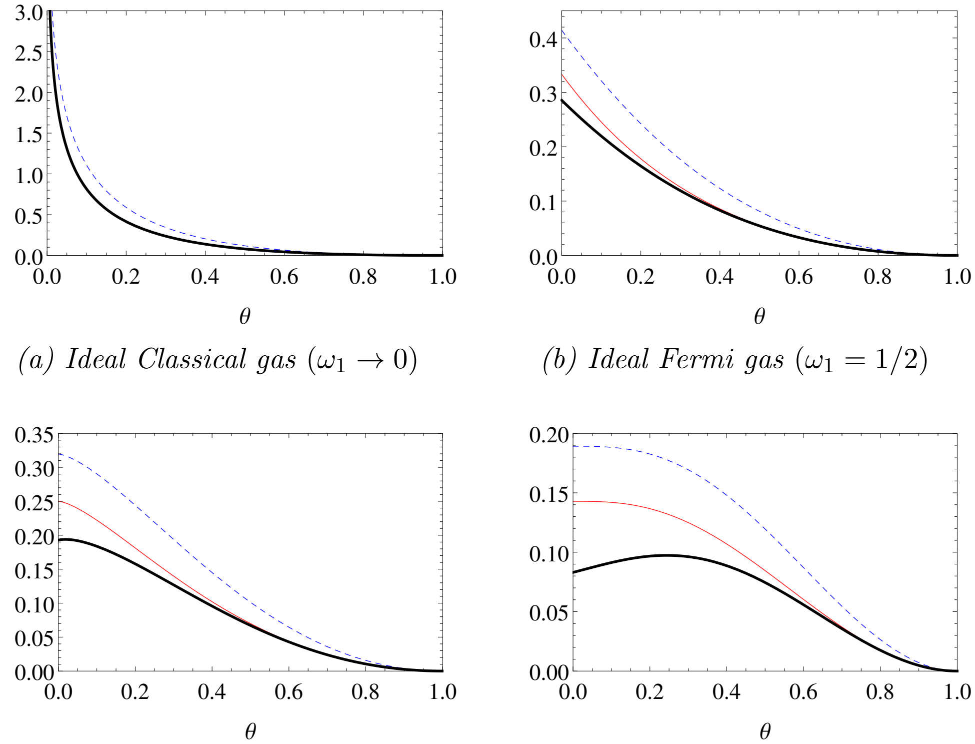
From the plots, it becomes clear that power-law prior yields better results as compared to uniform prior, however, when $\theta$ is close to 1 (near-equilibrium), estimation with both the priors agree up to third order as shown:

$$\overline{\Delta S}_p \approx \overline{\Delta S}_u \approx \frac{\omega}{6}(1-\theta)^2 + \frac{\omega(2-\omega)}{12}(1-\theta)^3 + O[1-\theta]^4. \tag{3.16}$$

Near-equilibrium expansion of $\Delta S_o$ is given as:

$$\Delta S_o \approx \frac{\omega}{4}(1-\theta)^2 + \frac{\omega(2-\omega)}{8}(1-\theta)^3 + O[1-\theta]^4. \tag{3.17}$$

On comparing Eqs. (3.16) and (3.17), we observe a scale factor of 2/3 similar to the work estimation in Section 2.4.2. Thus, with this method, optimal behavior of the process is not reproduced to a good extent although non-uniform prior




*(a) Ideal Classical gas* ($\omega_1 \to 0$) *(b) Ideal Fermi gas* ($\omega_1 = 1/2$)

*(c) Degenerate Bose Gas* ($\omega_1 = 3/5$) *(d) Black Body radiation* ($\omega_1 = 3/4$)

Figure 3.2: *Entropy production as a function of* $\theta$*. The dashed curve is for* $\Delta S_o$*, thin curve is for* $\overline{\Delta S}_p$*, and thick curve is for* $\overline{\Delta S}_u$*.*

gives slightly better approximation to the optimal values than uniform prior.

### Second method

The other method to estimate the entropy production is given by replacing $T_i$ with $\overline{T_i}$ (Eq. (3.12)) in the expression 3.8. The process of estimation is implemented as follows.

1) Calculate the expected value of one of the temperatures, (say $\overline{T}_2$).

2) From constraint of energy conservation, the estimate of $T_1$ (denoted as $\tilde{T}_1$) corresponding to estimate of $T_2$ equal to $\overline{T}_2$, is made.

3) From the knowledge of $\tilde{T}_1$ and $\overline{T}_2$, entropy production is estimated as $\widetilde{\Delta S} = \Delta S(\overline{T}_2)$.

The estimated value of entropy production with both the priors are given as follows:

$$\widetilde{\Delta S}_p = \left[1+\theta^{1+\omega} - \left(\frac{\omega+1}{\omega+2}\right)^{1+\omega}\left(\frac{1-\theta^{\omega+2}}{1-\theta^{\omega+1}}\right)^{1+\omega}\right]^{\frac{\omega}{1+\omega}} + \left(\frac{\omega+1}{\omega+2}\right)^{\omega}\left(\frac{1-\theta^{\omega+2}}{1-\theta^{\omega+1}}\right)^{\omega} - (1+\theta^{\omega}), \tag{3.18}$$

and

$$\widetilde{\Delta S}_u = \left[1+\theta^{1+\omega} - \left(\frac{1+\theta}{2}\right)^{1+\omega}\right]^{\frac{\omega}{1+\omega}} + \left(\frac{1+\theta}{2}\right)^{\omega} - (1+\theta^{\omega}). \tag{3.19}$$

Figure 3.3 shows the results for estimated entropy production for different values of $\omega_1$. The results show that estimation with the use of power-law prior closely

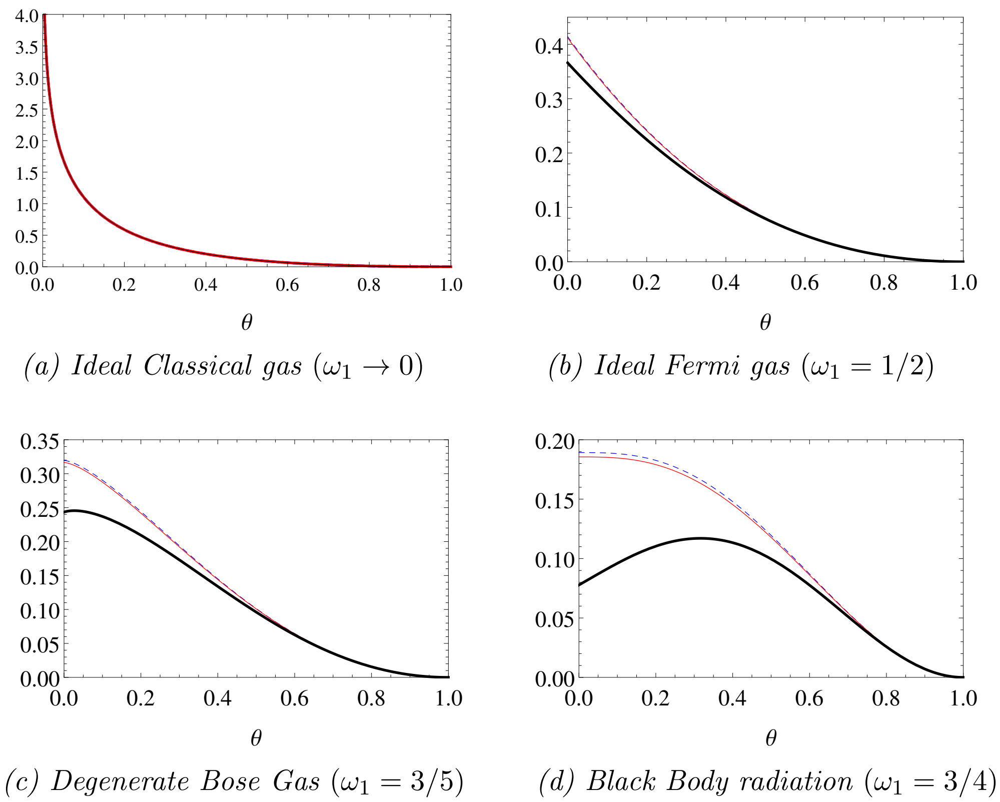



Figure 3.3: *Entropy production as a function of $\theta$. The dashed curve is for $\Delta S_o$, thin curve is for $\widetilde{\Delta S}_p$, and thick curve is for $\widetilde{\Delta S}_u$.*

matches with the optimal behavior in comparison with the uniform prior. But, near-equilibrium, estimates with both the priors agree up to third order with the optimal entropy production (Eq. (3.17)) as shown:

$$\widetilde{\Delta S}_p \approx \widetilde{\Delta S}_u \approx \frac{\omega}{4}(1-\theta)^2 + \frac{\omega(2-\omega)}{8}(1-\theta)^3 + O[1-\theta]^4. \tag{3.20}$$

With both of the methods of estimation, it has become clear that estimates obtained with the second method show remarkable agreement with the optimal features than the estimates defined as the expected value of the entropy production. This can be attributed to the fact that $\Delta S$ is a concave function in the interval $[\theta, 1]$ and thus from Jensen inequality, $\widetilde{\Delta S} \geq \overline{\Delta S}$. Thus, similar to work expression in Chapter 2, this inequality gives upper bound to the usual estimate for entropy production.

## 3.5 Entropy production with spin-reservoirs

We study another model for the reservoirs for which no explicit relation exists between $T_1$ and $T_2$. So, we have to carry out numerical calculations to solve constraint equation. We consider the reservoirs as $N$ non-interacting, localized spin-1/2 particles at temperatures $T_+$ and $T_-$. A spin-1/2 particle can be regarded as a two-level system, with energy levels $(0, a)$. The expressions for mean internal energy, heat capacity and entropy of the spin reservoirs have been given earlier in Eqs. (2.40), (2.41) and (2.42). We reconsider the energy-conserving process with these reservoirs.

Using Eq. (2.40) in the constraint $dU = 0$ for energy conservation in pure thermal interaction, we obtain:

$$2f_1f_2 - (f_1 + f_2) + c'f_1f_2 = 0, \tag{3.21}$$

where $f_1$, $f_2$ are given as

$$f_1 = 1 + e^{-a/T_1}, \tag{3.22}$$

$$f_2 = 1 + e^{-a/T_2}, \tag{3.23}$$

and

$$c' = -\left(\frac{e^{-a}}{1+e^{-a}} + \frac{e^{-a/\theta}}{1+e^{-a/\theta}}\right). \tag{3.24}$$

Here also, we use scaled temperatures as $T_+ = 1$ and $T_- \equiv \theta$. Eq. (3.21) is also an implicit equation and cannot be solved for $T_1$ in terms of $T_2$. For general $a$'s values, only numerical solution can be found. In this case also, we perform the estimation in the high-temperature limit to illustrate the utility of incorporating the prior information. The numerical results for the estimated entropy production also show good agreement with the optimal entropy production as we discuss in Section 3.5.3.

**High-temperature limit**

In the case of high temperatures, i.e. $a << T_i$, we can perform analytic calculations. The expressions for mean internal energy, heat capacity and entropy of the spin reservoirs in this limit have been given earlier in Eqs. (2.47), (2.48) and (2.49). The explicit relation between $T_1$ and $T_2$ is obtained by using Eq. (2.47) and energy conservation condition on the reservoirs. This yields

$$T_1 = \left(1 + \frac{1}{\theta} - \frac{1}{T_2}\right)^{-1}. \tag{3.25}$$

The entropy produced can be written using Eq. (2.48) as:

$$\Delta S = \frac{Na^2}{8}\left(1 + \frac{1}{\theta^2} - \frac{1}{{T_1}^2} - \frac{1}{{T_2}^2}\right). \tag{3.26}$$

Alternately, for high temperature case, we can express Eq. (3.26) as function of one variable, using Eq. (3.25). For the optimal process, $T_1 = T_2 = T_c$, where $T_c = 2\theta/(1+\theta)$. In this case, we have

$$\Delta S_o = \frac{Na^2}{16\theta^2}(1-\theta)^2. \tag{3.27}$$

### 3.5.1 Prior

The normalized form of prior can be written using Eqs. (3.4) and (2.49) as:

$$P(T_i) = \frac{\theta}{(1-\theta)}\frac{1}{T_i^2}. \tag{3.28}$$

The average value, $\overline{T_i}$, is calculated as:

$$\overline{T}_i = \frac{\theta}{(1-\theta)} \ln\left(\frac{1}{\theta}\right), \tag{3.29}$$

and with uniform prior, the expected value is simply $(1+\theta)/2$. The condition $\Delta S \geq 0$ determines the range of $T_i$ as $[\theta, 1]$.

### 3.5.2 Estimation of entropy production: High-temperature limit

The estimate for entropy production, $\widetilde{\Delta S}$, is given by replacing, say, $T_2$ with $\overline{T}_2$ in Eqs. (3.25) and (3.26), which gives:

$$\widetilde{\Delta S}_p = 2\left[\frac{1-\theta^2}{\theta^2 \ln\left(\frac{1}{\theta}\right)} - \frac{(1-\theta)^2}{\theta^2 \ln\left(\frac{1}{\theta}\right)^2} - \frac{1}{\theta}\right], \tag{3.30}$$

$$\widetilde{\Delta S}_u = \frac{2(1-\theta)^2}{\theta(1+\theta)^2}. \tag{3.31}$$

Fig. 3.4 illustrates the comparison between the estimated and optimal entropy production, with the use of different priors. The derived prior estimates the

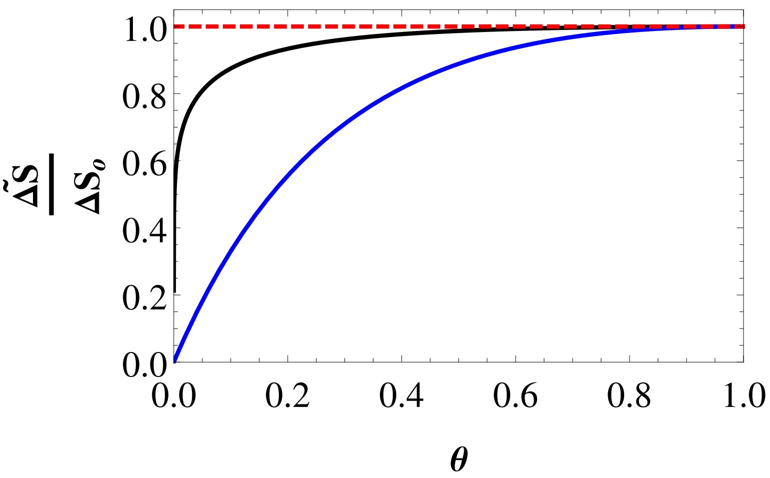



Figure 3.4: *Ratio of estimated entropy production* ($\widetilde{\Delta S}$) *to the optimal value* $\Delta S_o$ *(Eq. (3.27)), as a function of* $\theta$*, in the high temperature limit. The lower curve is for estimate* $\widetilde{\Delta S}_u$ *using a uniform prior, while the upper curve uses the estimate* $\widetilde{\Delta S}_p$ *due to the derived prior. The latter estimates are closer to the maximal entropy production. The curves agree in the near equilibrium regime, where the lower order terms match with optimal values, as given by Eq. (3.32).*

maximal entropy production much more closely than the non-informative uniform prior. Further, when we expand the estimates for entropy production in near-equilibrium regime, we get:

$$\widetilde{\Delta S}_p \approx \widetilde{\Delta S}_u = \frac{Na^2}{16}(1-\theta)^2 + \frac{Na^2}{8}(1-\theta)^3 + O[1-\theta]^4. \tag{3.32}$$

The above estimates match with the maximal entropy production, up to third order.

Following Section 3.4.2, here also, we discuss the estimates defined as the corresponding expected values with the use of non-uniform as well as uniform prior as follows:

$$\overline{\Delta S}_p = \frac{(1-\theta)^2}{3\theta^2}, \tag{3.33}$$

$$\overline{\Delta S}_u = \frac{2\left[(1+\theta)\ln(\frac{1}{\theta}) - 2(1-\theta)\right]}{\theta(1-\theta)}. \tag{3.34}$$

Near-equilibrium behavior of the estimates as

$$\overline{\Delta S}_p \approx \overline{\Delta S}_u \approx \frac{Na^2}{24}(1-\theta)^2 + \frac{Na^2}{12}(1-\theta)^3 + O[1-\theta]^4, \tag{3.35}$$

clearly shows that these estimates do not match with the maximal entropy production (Eq. (3.32)) due to a scale factor of 2/3 between these two expansions.

### 3.5.3 General case: Numerical estimation

Now we turn to the general solution of the constraint equation, for arbitrary values of $a$ and $\theta$. In general, there is no explicit relation between $T_1$ and $T_2$ following from the constraint of energy conservation condition. Thus even if we calculate the average temperature, $\overline{T}_i$, analytically, the value of the other has to be evaluated numerically. The optimal temperature has to be evaluated numerically. The net entropy production, in general, can be written as:

$$\begin{aligned} \Delta S &= N\left[\ln\left(\frac{(1+e^{-a/T_1})(1+e^{-a/T_2})}{(1+e^{-a})(1+e^{-a/\theta})}\right) + \frac{a}{T_1}\frac{e^{-a/T_1}}{(1+e^{-a/T_1})}\right. \\ &\left. + \frac{a}{T_2}\frac{e^{-a/T_2}}{(1+e^{-a/T_2})} - a\frac{e^{-a}}{(1+e^{-a})} - \frac{a}{\theta}\frac{e^{-a/\theta}}{(1+e^{-a/\theta})}\right]. \end{aligned} \tag{3.36}$$

Fig. 3.5 shows the comparative numerical plots for entropy production. Thus for general values of $a$, the numerical calculations using the derived prior show that the estimated entropy production is in good agreement with the optimal entropy production, in a manner similar to the findings of the previous section.

## 3.6 Discussion and summary

In this Chapter, we have discussed the inference based approach to estimate the performance of a thermodynamic process in which the two identical finite systems at different initial temperatures are in thermal contact with each other

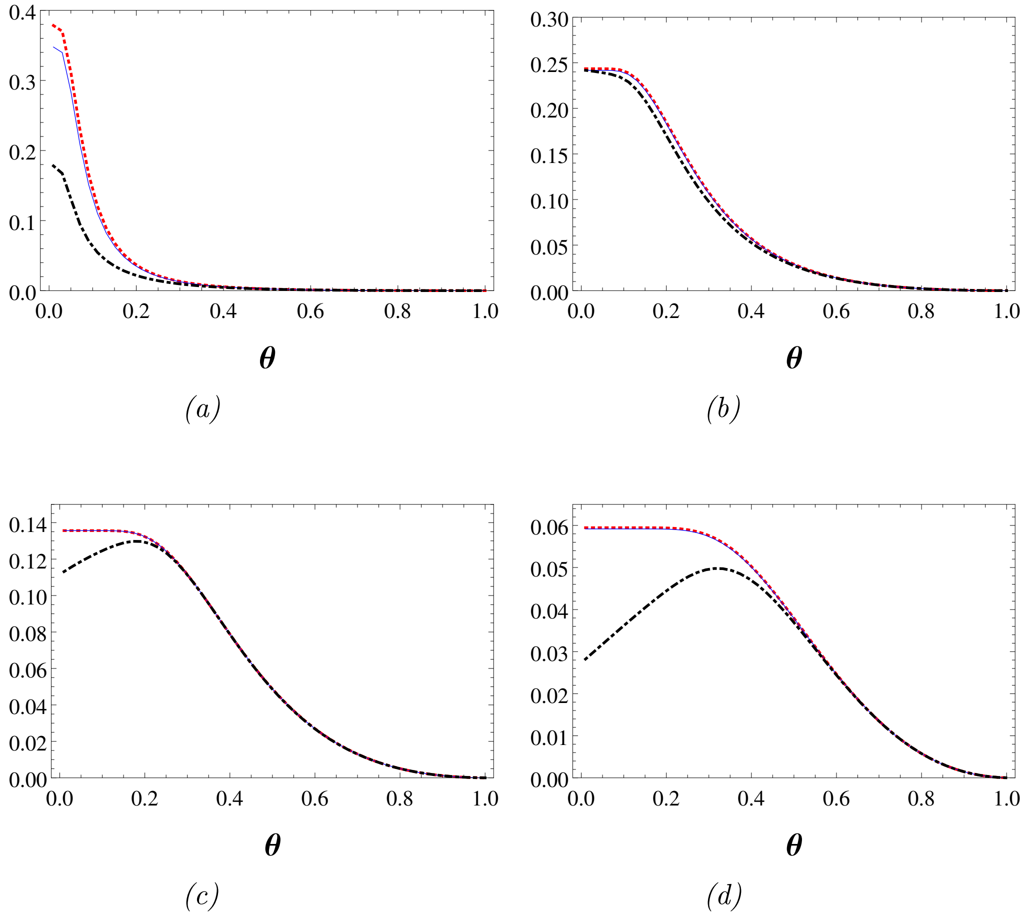



Figure 3.5: *Entropy production as a function of $\theta$ for different values of a; (a) a = 0.2, (b) a = 0.8, (c) a = 1.5, (d) a = 2.4. The top, dotted curve is the optimal entropy production, the middle, solid curve is the estimate $\widetilde{\Delta S}_p$, and lower, dotdashed curve is for the estimate $\widetilde{\Delta S}_u$.*

with no change in the total energy of the composite system. We assign priors for the unknown final temperatures, $T_1$ and $T_2$, of the reservoirs. We argued for and proposed a general prior while incorporating the prior information about the process which is the constraint of energy conservation. We proposed a procedure to assign prior while incorporating two essential elements of the prior information: (i) inference of the rate of change $dT_1/dT_2$ from the constraint; (ii)the range of allowed values for the uncertain temperature. For this process, the range for allowed values of $T_1$ and $T_2$ is determined from the physical condition of $\Delta S \geq 0$

i.e. there will be a net entropy production in the system. It is $[\theta,\ 1]$ for both $T_1$ and $T_2$.

Once the priors are assigned, then we estimated the entropy produced in the whole system with the derived prior as well as with uniform prior. Similar to entropy-conserving process in Chapter 2, we followed the two methods of estimation. And due to concave nature of the function $\Delta S$ in the interval $[\theta, 1]$, entropy production estimated as $\Delta S(\overline{T}_i)$ is always higher and hence, closely matches with maximal entropy production as compared to the estimates defined as expected values, $\langle \Delta S(T_i) \rangle$, of the entropy production. We consider two models for the finite reservoirs. For spin-reservoirs, analytical solution of the constraint equation does not exist and thus we perform numerical calculations for general values of $a$ (energy spacing) as well as analytical calculations in the high temperatures limit. The agreement of estimated values with the optimal values are shown in high temperatures limit, as well as by numeric calculations.

It is well-known that the optimal process considered can be regarded in terms of respective equilibrium state of the bipartite system subject to the given constraint of energy conservation. Thus under conservation of total energy, the equilibrium state is the one with the maximum total entropy. The agreement of our estimates with the optimal values shows that we are able to estimate the equilibrium state without following an optimization procedure. Concluding, we can say that inference approach works well to estimate the optimal behavior in energy-conserving process also.

# Chapter 4

# Inference of engine performance with non-identical finite source and sink

## 4.1 Introduction

In Chapter 2, we addressed the problem of maximum work extraction [81, 82, 83, 84, 85, 86, 87] with finite source/sink within inference approach motivated by Bayesian reasoning. A pair of identical finite systems were considered for the purpose of a source and a sink with different initial energies and hence temperatures. The fundamental thermodynamic relation obeyed by the systems is taken to be $S = \kappa U^{\omega_1}$, where $\kappa$ may depend on some universal constants and/or volume, particle number of the system. The range $0 < \omega_1 < 1$, implied systems with a positive heat capacity. The optimal or maximum work extracted from this set-up by coupling it to some work source was estimated. The efficiency at optimal work was also inferred up to second order as $\eta \approx \eta_c/2 + {\eta_c}^2/8$ [61, 67, 68, 69, 70, 72], in near-equilibrium regime. A generalisation of this approach can be thought of by considering the non-identical systems as reservoirs. In Ref. [93], finite reservoirs are modelled by perfect gas systems with different constant heat capacities. Thus, the new information about relative sizes of source and sink can be utilized in the assignment of prior for the uncertain temperatures. The ranges of allowed values for $T_1$ and $T_2$ are different in this case. In this Chapter, we consider two dissimilar systems obeying a thermodynamic relation of the form $S_i = \kappa_i U_i^{\omega_1}$ $(i = 1, 2)$ and reconsider the maximum work extraction process within inference approach. We derive the temperature and efficiency estimates for this model which show remarkable agreement with their optimal values.

This Chapter is organized as follows. In Section 4.2, we discuss the model for finite reservoirs. Section 4.3 outlines the discussion on the permissible range and thus the form of prior for $T_1$ and $T_2$. In Section 4.4, we estimate the temperature of the source and the sink. It also comprises of the discussion on inference of special cases when one system becomes very large in comparison to the other. Section 4.5 describes the near-equilibrium analytical estimation for the arbitrary values of $\gamma$ (see the line below Eq. (4.1)). Section 4.6 discusses the numerical results for the estimates of efficiency for arbitrary sizes of reservoirs. Finally in Section 4.7, we make some concluding remarks on our extended inference approach in the case of non-identical systems.

## 4.2 Model

To model finite reservoirs, consider a pair of thermodynamic systems obeying a relation of the form $S_i = \kappa_i U_i^{\omega_1}$, where $U$ is the internal energy of the system and $\omega_1$ is some known constant. Using the basic definition : $(\partial S/\partial U)_V = 1/T$, we get: $U = (\omega_1 \kappa T)^{1+\omega}$. Alternately, we can write : $S = \kappa^{1+\omega}(\omega_1 T)^{\omega}$, where $\omega = \omega_1/(1-\omega_1)$.

Since the thermodynamic relation obeyed by the two systems remains the same, the two may be non-identical only if they differ in their volumes, number/nature of particles etc. Thus, it is the constant of proportionality, $\kappa$, which is different for the two systems. Let $T_+$ and $T_-$ $(< T_+)$ be the initial temperatures of, say, source and sink with $\kappa_1$ and $\kappa_2$ as the proportionality constants respectively.

To perform inference, examine an arbitrary intermediate stage of the process when the temperatures of the two are $T_1'$ and $T_2'$ respectively. The work extracted from the engine is $W = Q_{\text{in}} - Q_{\text{out}}$, which can be written as:

$$W = (\kappa_2\omega_1)^{1+\omega}\left[\gamma^{-(1+\omega)}({T_+}^{1+\omega} - {T_1'}^{1+\omega}) - ({T_2'}^{1+\omega} - {T_-}^{1+\omega})\right], \tag{4.1}$$

where $\gamma = \kappa_2/\kappa_1$, and
$Q_{\text{in}} = (\kappa_1\omega_1)^{1+\omega}({T_+}^{1+\omega} - {T_1'}^{1+\omega})$; $Q_{\text{out}} = (\kappa_2\omega_1)^{1+\omega}({T_2'}^{1+\omega} - {T_-}^{1+\omega})$.
For convenience, we define $\gamma^{1+\omega} = \sigma$, $\theta = T_-/T_+$, $T_1 = T_1'/T_+$ and $T_2 = T_2'/T_-$. Thus, work can be rewritten as:

$$W = (\kappa_2\omega_1 T_+)^{1+\omega}\Big[\sigma^{-1}(1 - {T_1}^{1+\omega}) + (\theta^{1+\omega} - {T_2}^{1+\omega})\Big]. \tag{4.2}$$

The constraint of entropy conservation, $S_1 + S_2 = S_+ + S_-$, gives

$$T_1 = \Big[1 + \sigma(\theta^\omega - {T_2}^\omega)\Big]^{\frac{1}{\omega}}, \tag{4.3}$$

or equivalently,

$$T_2 = \Big[\theta^\omega + \sigma^{-1}(1 - {T_1}^\omega)\Big]^{\frac{1}{\omega}}. \tag{4.4}$$

By making use of above equations, work can be written as a function of one variable (say $T_2$) only:

$$W(T_2) = (\kappa_2\omega_1 T_+)^{1+\omega}\left[\sigma^{-1}\left(1 - \left(1 + \sigma(\theta^\omega - {T_2}^\omega)\right)^{\frac{1+\omega}{\omega}}\right) + \left(\theta^{1+\omega} - {T_2}^{1+\omega}\right)\right]. \tag{4.5}$$

A similar expression of work can be written as a function of $T_1$ also.

The optimal work can be extracted from the engine when the two systems reach a common temperature. The common temperature, $T_c$ is given from Eq. (4.3) with $T_1 = T_2 = T_c$ as:

$$T_c = \left( \frac{1 + \sigma\theta^{\omega}}{1 + \sigma} \right)^{\frac{1}{\omega}}. \tag{4.6}$$

The efficiency of the engine is given as $\eta = 1 - Q_{\text{out}}/Q_{\text{in}}$. Thus, for any arbitrary value of $\gamma$, efficiency at any intermediate stage of the process can be given as:

$$\eta_{\gamma} = 1 - \sigma \frac{({T_2}^{1+\omega} - \theta^{1+\omega})}{(1 - {T_1}^{1+\omega})}. \tag{4.7}$$

For efficiency at optimal work ($\eta^*_{\gamma}$), we substitute $T_1 = T_2 = T_c$ in above equation to obtain:

$$\eta^*_{\gamma} = 1 - \sigma \frac{({T_c}^{1+\omega} - \theta^{1+\omega})}{(1 - {T_c}^{1+\omega})}. \tag{4.8}$$

Let us discuss the efficiency in the *limiting* cases:

(a) In the limit $\gamma \to 0$ i.e. when heat source is very large as compared to heat sink, temperature of source remains constant at $T_1^{'} = T_+$ or $T_1 = 1$ while the temperature of sink approaches this value at optimal work extraction. We write efficiency as a function of $T_2$ as

$$\eta_0 = 1 - \left( \frac{\omega}{1 + \omega} \right) \left( \frac{{T_2}^{1+\omega} - \theta^{1+\omega}}{{T_2}^{\omega} - \theta^{\omega}} \right). \tag{4.9}$$

Efficiency at optimal work in this limit is given by substituting $T_2 = 1$ in above equation as:

$$\eta^*_0 = 1 - \left( \frac{\omega}{1 + \omega} \right) \left( \frac{1 - \theta^{1+\omega}}{1 - \theta^{\omega}} \right). \tag{4.10}$$

(b) In the limit $\gamma \to \infty$ i.e. when heat sink is very large in comparison to heat source, the sink stays at temperature ($T_- \equiv \theta$) and source approaches this value at optimal work extraction. The efficiency can be expressed in terms of $T_1$ as:

$$\eta_\infty = 1 - \theta\left(\frac{1+\omega}{\omega}\right)\left(\frac{1 - {T_1}^{\omega}}{1 - {T_1}^{1+\omega}}\right). \tag{4.11}$$

For the optimal process, substitute $T_1 = \theta$ in above expression to obtain:

$$\eta^*_\infty = 1 - \theta\left(\frac{1+\omega}{\omega}\right)\left(\frac{1 - \theta^{\omega}}{1 - \theta^{1+\omega}}\right). \tag{4.12}$$

## 4.3 Assignment of prior

The inference is performed by assigning the prior probability distributions for the uncertain parameters $T_1$ and $T_2$, since we assume ignorance of the actual values of $T_1$ and $T_2$ or the extent to which the process has proceeded. As before, we assume two observers, $A$ and $B$, for $T_1$ and $T_2$ respectively. Let us summarise the prior information we possess before making the inference:
(i) There exists a one-to-one relation between $T_1$ and $T_2$ given by Eq. (4.3) or (4.4) which suggests that probability of $T_1$ to lie in small range $[T_1, T_1 + dT_1]$ is same as the probability of $T_2$ to lie in $[T_2, T_2 + dT_2]$, so we can write :

$$P_1(T_1)dT_1 = P_2(T_2)dT_2, \tag{4.13}$$

or we can identify the ratio of rate of change of temperatures as $P_2(T_2) = P_1(T_1)\,|dT_1/dT_2|$, where $P_1$ and $P_2$ are the normalised prior distribution functions for $T_1$ and $T_2$.
(ii) The set-up works like a heat engine and thus $W \geq 0$.

With identical systems, we had an additional assumption on the identical form of prior distribution, $P$, for $T_1$ and $T_2$. However, since now the two systems are not identical, this information has to be incorporated while assigning the prior. This information can be incorporated in the allowed values of the range of $T_1$ and $T_2$. Now, the allowed range for $T_1$ and $T_2$ is not $[\theta, 1]$. It will be different for both the parameters for different values of $\gamma$ ($\neq 1$). Thus, say, $T_1$ ranges in

$[T_m, 1]$ and $T_2$ ranges in $[\theta, T_M]$ respectively satisfying the constraint $W \geq 0$ [93]. This is shown graphically in Figure 4.1 for the two cases with $\gamma < 1$ and $\gamma > 1$.

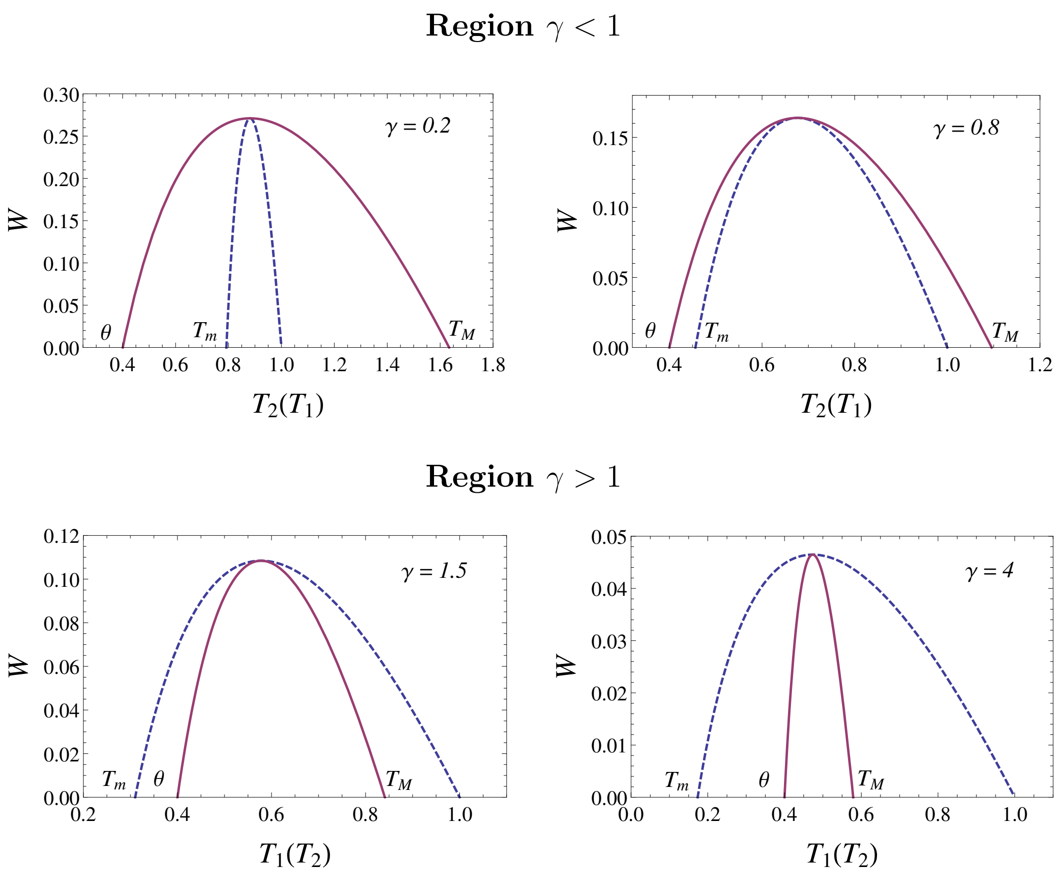



Figure 4.1: *Work is plotted as a function of* $T_1$ *(Dashed Curve) and* $T_2$ *(Solid Curve) for* $\omega_1 = 0.1$, $\theta = 0.4$. *Region* $\gamma < 1$ *corresponds to the case of larger source as compared to sink while the region* $\gamma > 1$ *corresponds to the case of larger sink as compared to source.*

For $\gamma < 1$ (larger source in comparison to sink), the range of allowed values of $T_1$ is narrower than the range for $T_2$. In the limiting case of $\gamma \to 0$ (infinite source and finite sink), $[T_m, 1]$ shrinks to a point $T_1 = 1$ which is expected for an infinite source as now the temperature of the source stays at $T_+ = 1$. Similarly, for $\gamma > 1$ (larger sink in comparison to source), the range for $T_2$ shrinks in comparison to the range for $T_1$, and $[\theta, T_M]$ shrinks to a point $T_2 = \theta$ for $\gamma \to \infty$ (infinite sink and finite source). This information on the range of uncertain parameters will be incorporated to determine the normalisation constants for prior distributions and thus we can write $P_1$ and $P_2$ as [93]:

$$P_1(T_1) = \frac{f(T_1)}{\int_{T_m}^{1} f(T_1)dT_1}, \tag{4.14}$$

$$P_2(T_2) = \frac{f(T_2)}{\int_{\theta}^{T_M} f(T_2)dT_2}, \tag{4.15}$$

where the form of $f$ is common to both the priors and the required prior is determined as discussed in Chapter 2 as

$$P_i(T_i) = \frac{C_i(T_i)/T_i}{N_i}, \tag{4.16}$$

for $i = 1, 2$ and $N_i = \int C_i(T_i)/T_i \, dT_i$ is the normalisation constant.
With our model ($C_i \propto {T_i}^{\omega}$), the functional form of the prior distribution can be written as

$$P_i(T_i) \propto T_i^{\omega-1}. \tag{4.17}$$

## 4.4 Estimation of temperature

The expected value of a temperature is:

$$\overline{T_i} = \int_{T_{i,min}}^{T_{i,max}} T_i P(T_i) dT_i, \tag{4.18}$$

where $i = 1, 2$. Taking into account the respective ranges of allowed values of $T_1$ and $T_2$, identified above, we obtain:

$$\overline{T}_1 = \left(\frac{\omega}{1+\omega}\right)\left(\frac{1-{T_m}^{1+\omega}}{1-{T_m}^{\omega}}\right), \tag{4.19}$$

$$\overline{T}_2 = \left(\frac{\omega}{1+\omega}\right)\left(\frac{{T_M}^{1+\omega}-\theta^{1+\omega}}{{T_M}^{\omega}-\theta^{\omega}}\right). \tag{4.20}$$

To determine $T_m$ or $T_M$, we solve Eq. (4.2) by setting $W(T_1) = 0$ or $W(T_2) = 0$ respectively.

In general, Eq. (4.2) has to be solved numerically for arbitrary values of $\omega_1$. For ideal Fermi gas ($\omega_1 = 1/2$), it can be solved analytically [94] and thus we obtain:

$$T_1 \in \left[\frac{1-\gamma^2+2\gamma^2\theta}{1+\gamma^2}, 1\right], \tag{4.21}$$

$$T_2 \in \left[\theta, \frac{2-\theta+\gamma^2\theta}{1+\gamma^2}\right]. \tag{4.22}$$

Due to Eq. (4.3), we can write one-to-one relation between $T_m$ and $T_M$ as:

$$1 - {T_m}^{\omega} = \sigma({T_M}^{\omega} - \theta^{\omega}). \tag{4.23}$$

Using above equation in $W(T_M) = 0$, we obtain:

$$1 - {T_m}^{1+\omega} = \sigma({T_M}^{1+\omega} - \theta^{1+\omega}). \tag{4.24}$$

From Eqs. (4.19), (4.20), (4.23) and (4.24), we can write:

$$\overline{T}_1 = \overline{T}_2. \tag{4.25}$$

However, firstly we will solve Eq. (4.2) for the limiting cases when one of the systems become very large in comparison to the other system.

### 4.4.1 Infinite source and finite sink

This case corresponds to the limit $\gamma \to 0$. Here the only uncertain parameter is $T_2$ as temperature of source stays at $T_+ = 1$ while the temperature of sink approaches $T_+$ at optimal work extraction. To discuss this limit, we set Eq. (4.5) as $W(T_2) = 0$ and obtain:

$$T_2{}^{1+\omega} - \theta^{1+\omega} = \frac{1}{\sigma}\left[1 - \Big(1 + \sigma(\theta^{\omega} - T_2{}^{\omega})\Big)^{\frac{1+\omega}{\omega}}\right]. \tag{4.26}$$

Taking the limit $\gamma \to 0$, the above equation gets simplified to:

$$\omega\left(T_2{}^{1+\omega} - \theta^{1+\omega}\right) = (1+\omega)\left(T_2{}^{\omega} - \theta^{\omega}\right), \tag{4.27}$$

whose trivial solution is $T_2 = \theta$. The other solution is $T_M$ so we write:

$$\omega\left(T_M{}^{1+\omega} - \theta^{1+\omega}\right) = (1+\omega)\left(T_M{}^{\omega} - \theta^{\omega}\right). \tag{4.28}$$

Consistency between Eqs. (4.20) and (4.28) demands that we must have:

$$\overline{T}_2 = 1. \tag{4.29}$$

Thus expected sink temperature exactly matches with temperature of heat source for optimal process. The efficiency is estimated by replacing $T_2$ in Eq. (4.9) by Eq. (4.29) and estimate for efficiency is same as Eq. (4.10). Hence, inference approach reproduces the optimal behaviour exactly in the limit $\gamma \to 0$.

### 4.4.2 Finite source and infinite sink

Consider the case of infinite sink in comparison to source ($\gamma \to \infty$). Here, the sink stays at temperature $T_-(\equiv \theta)$ and the temperature of source approaches $T_-$ for optimal work extraction. Hence $T_1$ is the only uncertain parameter in this limiting case. The range for $T_1$ is determined by using Eq. (4.4) in Eq. (4.2) and then setting $W(T_1) = 0$, we get:

$$1 - T_1{}^{1+\omega} = \sigma\left[\Big(\theta^{\omega} + \sigma^{-1}(1 - T_1{}^{\omega})\Big)^{\frac{1+\omega}{\omega}} - \theta^{1+\omega}\right]. \tag{4.30}$$

In the limit $\gamma \to \infty$, the above equation gets simplifies to:

$$\omega\left(1 - {T_1}^{1+\omega}\right) = \theta(1+\omega)\left(1 - {T_1}^{\omega}\right). \tag{4.31}$$

The trivial root of above equation is $T_1 = 1$ and other root ($T_m$) satisfies:

$$\omega\left(1 - {T_m}^{1+\omega}\right) = \theta(1+\omega)\left(1 - {T_m}^{\omega}\right). \tag{4.32}$$

From Eqs. (4.19) and (4.32), we obtain:

$$\overline{T}_1 = \theta. \tag{4.33}$$

It is clear from the Eq. (4.33) that the average temperature of the source exactly matches with the temperature of the infinite sink which happens in case of maximum work extraction. Further, efficiency at optimal work ($\eta^*_\infty$) is also inferred exactly due to Eqs. (4.11) and (4.33). Thus, we are able to infer exactly the optimal behaviour of the system when sink is infinitely large in comparison with source.

## 4.5 Near-equilibrium estimation

In this Section, we approximate the values of $T_m$ and $T_M$ when $\theta$ is close to unity. For this, consider the case $0 < \gamma < 1$. Let us examine the observer $B$. Since close to equilibrium, $T_M$ is also close to unity so we can introduce a small parameter $\epsilon > 0$ such that

$$T_M = \theta\ (1+\epsilon). \tag{4.34}$$

Rewriting Eq. (4.5) as $W(T_M) = 0$:

$$1 + \sigma\theta^{1+\omega}[1 - (1+\epsilon)^{1+\omega}] = \Big(1 + \sigma\theta^{\omega}[1 - (1+\epsilon)^{\omega}]\Big)^{\frac{1+\omega}{\omega}}. \tag{4.35}$$

Making series expansion in $\epsilon$ and keeping terms only up to second order, we get a quadratic equation in $\epsilon$ (See Appendix C):

$$(1-\omega)\left[\sigma^2\theta^{2\omega}+3\sigma\theta^{\omega}-\omega(1-\theta)+2\right]\epsilon^2-3\left[\sigma-\omega(1-\theta)+1\right]\epsilon+6(1-\theta)=0. \tag{4.36}$$

For instance, if we take limit $\omega \to 0$ in above equation, we reproduce the case for perfect gas as:

$$(\gamma+1)(\gamma+2)\epsilon^2-3(\gamma+1)\epsilon+6(1-\theta)=0, \tag{4.37}$$

whose acceptable solution [93] is approximated up to second order in $\eta_c$ as:

$$\epsilon=\frac{2}{1+\gamma}\eta_c+\frac{4(2+\gamma)}{3(1+\gamma)^2}\eta_c^2. \tag{4.38}$$

Similarly, Eq. (4.36) can be solved for $\epsilon$ and the solution can be approximated as:

$$\epsilon=\frac{2}{1+\sigma}\eta_c+\frac{2[4-\omega+\sigma(\omega+2)]}{3(1+\sigma)^2}\eta_c^2. \tag{4.39}$$

From the value of $\epsilon$, we can determine $T_M$, which in turn determines $T_m$ due to Eq. (4.3). Let us now discuss the near-equilibrium estimation of the estimated temperatures of the reservoirs.

Suppose $\tilde{T}_1$ ($\tilde{T}_2$) are the estimates for $T_1$ ($T_2$) by the observer B (A) by making use of Eqs. (4.3) and (4.4). Close to equilibrium, estimated temperatures of reservoirs behave as

$$\overline{T}_2 = 1-\frac{\sigma}{1+\sigma}\eta_c-\frac{\sigma(1-\omega)}{3(1+\sigma)^2}\eta_c^2+O[\eta_c^3], \tag{4.40}$$

$$\tilde{T}_2 = 1-\frac{\sigma}{1+\sigma}\eta_c-\frac{(1+3\sigma)(1-\omega)}{6(1+\sigma)^2}\eta_c^2+O[\eta_c^3], \tag{4.41}$$

$$\tilde{T}_1 = 1 - \frac{\sigma}{1+\sigma}\eta_c - \frac{\sigma(3+\sigma)(1-\omega)}{6(1+\sigma)^2}\eta_c^2 + O[\eta_c^3], \quad (4.42)$$

while the optimal temperature, $T_c$ behaves as

$$T_c = 1 - \frac{\sigma}{1+\sigma}\eta_c - \frac{\sigma(1-\omega)}{2(1+\sigma)^2}\eta_c^2 + O[\eta_c^3]. \quad (4.43)$$

In the above case, we have seen that both estimates, $\overline{T}_2$ (by observer $B$) and $\tilde{T}_2$ (by observer $A$), match with $T_c$ only up to first order. Let us define the estimated value of the temperature of one reservoir (say sink) as the weighted mean of the estimates by two observers. It will be given as

$$T_{2,m} = w_1\tilde{T}_2 + w_2\overline{T}_2, \quad (4.44)$$

where $w_1$, $w_2$ are the weights satisfying the condition $w_1 + w_2 = 1$. If we choose weights $w_1$ and $w_2$ so as to obtain agreement up to second order, then the weights to be assigned are:

$$w_1 = \frac{\sigma}{1+\sigma}, \quad (4.45)$$

$$w_2 = \frac{1}{1+\sigma}. \quad (4.46)$$

Interestingly, this weighted mean ($T_{2,m}$) for the the sink shows remarkable agreement with $T_c$ up to third order in $\eta_c$ close to equilibrium. Further, for $\gamma \to 0$, $T_{2,m}$ becomes exactly equal to $\overline{T}_2$ showing that estimation is done only by observer $B$ as the other reservoir corresponding to observer $A$ (source) becomes infinite in comparison and hence, its temperature stays constant at $T_+$ and no uncertainty exists in its value. Similarly in the limit of $\gamma \to \infty$, mean estimate of temperature becomes exactly equal to the expected temperature of the source.

### 4.5.1 Estimation of efficiency

Efficiency is estimated by replacing $\overline{T}_2$ in Eqs. (4.3) and (4.5) to obtain efficiency estimate by observer $B$ ($\tilde{\eta}_2$). Similarly, denote the efficiency estimate by observer $A$ as ($\tilde{\eta}_1$). Expanding the estimates of efficiency close to equilibrium as

$$\tilde{\eta}_1 = \frac{\eta_c}{2} + \frac{2(1+\sigma)+\omega(\sigma-2)}{12(1+\sigma)}\eta_c^2 + O[\eta_c^3], \tag{4.47}$$

$$\tilde{\eta}_2 = \frac{\eta_c}{2} + \frac{(1+\sigma)+\omega(2\sigma-1)}{12(1+\sigma)}\eta_c^2 + O[\eta_c^3], \tag{4.48}$$

while the efficiency at optimal work behaves as

$$\eta_\gamma^* = \frac{\eta_c}{2} + \frac{(1+2\sigma)+\omega(\sigma-1)}{12(1+\sigma)}\eta_c^2 + O[\eta_c^3]. \tag{4.49}$$

where $\eta_c = 1-\theta$ is Carnot efficiency. It is clear from the above expressions that estimates of efficiency either by observer $A$ or by observer $B$ matches with efficiency at optimal work ($\eta_\gamma^*$) only up to first order in $\eta_c$. With non-identical systems, quadratic term ($\eta_c^2/8$) in the efficiency at optimal work does not appear and becomes dependent on value of $\gamma$. However, mean efficiency ($\tilde{\eta}_m$) defined as

$$\tilde{\eta}_m = w_1\tilde{\eta}_1 + w_2\tilde{\eta}_2, \tag{4.50}$$

matches with efficiency at optimal work up to second order, where $w_1$ and $w_2$ are the weights assumed for the estimation done by observer $A$ and observer $B$ respectively, similar to the case of temperature estimation. For the special case of identical systems ($\gamma = 1$), the assigned weights are equal to 1/2 as in Chapter 2. In this case, quadratic term is also recovered. The non-identical weights chosen above are also consistent in the extreme cases, when one system becomes very large as compared to the other, say, for example, $\gamma \to 0$ makes $w_1 = 0$ and thus estimation is performed over the temperature of sink only, as the temperature of source stays at its initial value. Similarly, in the limit $\gamma \to \infty$, estimation is performed over the temperature of source since $w_2 = 0$. For these limiting cases, we obtain the *exact* estimates for efficiency at optimal work respectively.

## 4.6 Numerical results for arbitrary $\gamma$

We have observed that optimal behavior of the process can be estimated exactly for the case when one system is very large in comparison to other. It is then of interest to compare the estimated values with the optimal values for arbitrary values of $\gamma$. For this, Eq. (4.5) has to be solved to numerically. The trivial root is $\theta$ while the other root, $T_M$, can be determined numerically for given values of $\omega_1$, $\gamma$, and $\theta$. Then, we obtain numerical estimates of efficiency by observer $A$ and observer $B$ for arbitrary values of $\gamma$. Figure 4.2 shows the comparative plots of efficiency for different values of $\gamma$.

From the numerical plots, it becomes clear that in the $0 < \gamma < 1$ regime, estimates made by observer $B$ give better estimation to optimal values as compared to the observer $A$, the agreement is exact in the limit $\gamma \to 0$. Similarly, in the region $\gamma > 1$, observer $A$ reproduces the optimal behavior to a remarkable extent than observer $B$. In the limit $\gamma \to \infty$, the exact optimal behavior is estimated by observer $A$. However, close to equilibrium, estimates by both the observers (Eqs. (4.47) and (4.48)) agree up to first order with optimal behavior (Eq. (4.49)). As discussed in Section 4.5.1, mean efficiency matches with efficiency at optimal work up to second order in $\eta_c$. It can be easily seen from the plots also, where the mean efficiency closely matches with optimal value in both the regions.

## 4.7 Conclusion

Earlier in Chapter 2, we observed that prior probabilities play an important role in the estimation of optimal performance characteristics of quasi-static thermodynamic process with constraints where two identical finite systems acting as

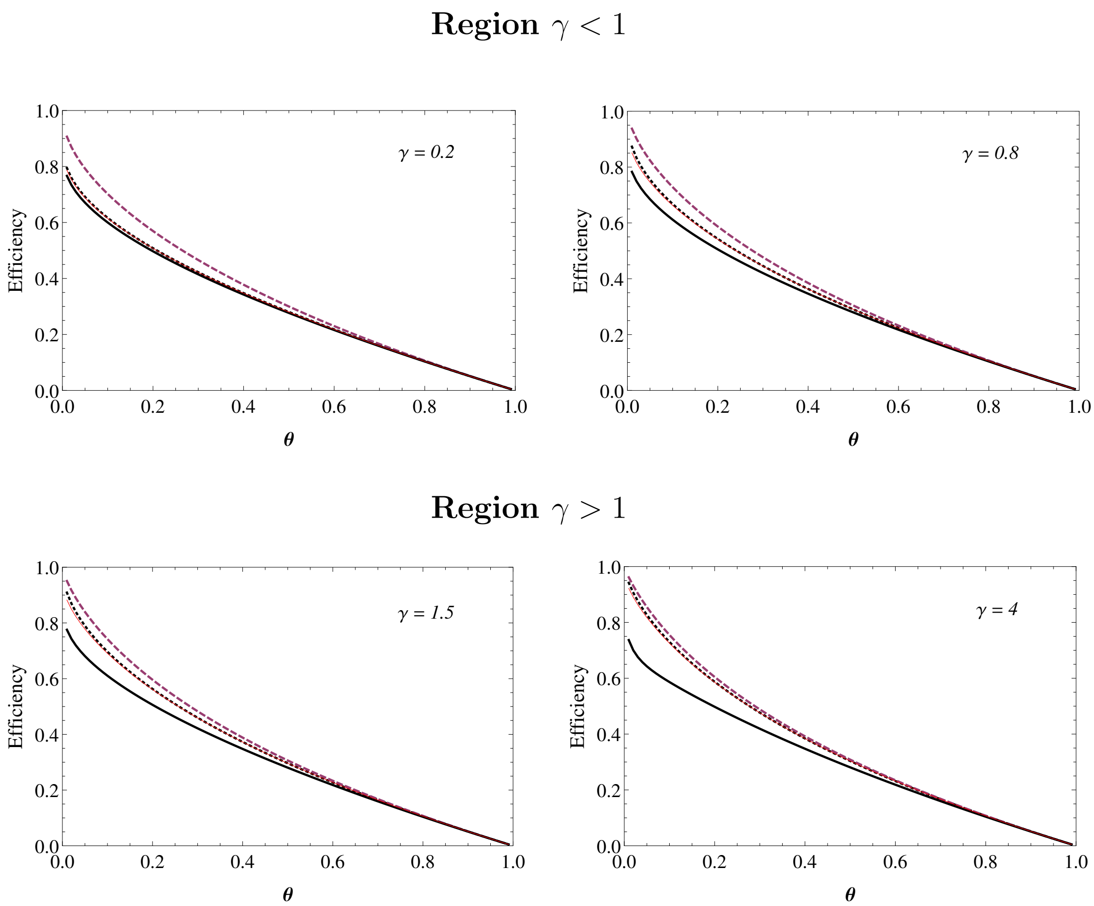



Figure 4.2: *Efficiency as a function of $\theta$ for $\omega_1 = 0.1$. The uppermost dashed curve is for $\tilde{\eta}_1$, lowermost curve is for $\tilde{\eta}_2$, dotted curve is for efficiency at optimal work while the middle solid curve is for mean efficiency ($\tilde{\eta}_m$) which closely follows $\eta_o$.*

source and sink undergo a reversible process of maximum work extraction. In this Chapter, we have extended our inference-based approach to the process of reversible work extraction from a set-up of non-identical finite heat source/sink. The entropy of each reservoir satisfies the relation $S \propto T^{\omega}$. Then ignorance is assumed about the intermediate temperatures $T_1$ and $T_2$ of source and sink respectively and uncertainty in the values of $T_1$ and $T_2$ is quantified in terms of prior probabilities. With non-identical sizes of systems, we can distinguish and label the two systems acting as source and sink. Note that this information was missing in earlier studies with similar reservoirs. This also results in the ranges of allowed values of $T_1$ and $T_2$ with the constraint $W \geq 0$ being different unlike with identical sizes of the systems, where the ranges for $T_1$ and $T_2$ are same and equal to $[\theta, 1]$.

We observed that estimates match exactly with their optimal values when one of the reservoir becomes very large as compared to the other. For arbitrary values of $\gamma$, we performed the numerical calculations. These calculations show that information incorporated in the prior distributions reproduce the optimal behaviour of the system, however, now the efficiency estimates made by two observers are not symmetrically distributed about the efficiency at optimal work unlike in the case with similar reservoirs ($\gamma = 1$). Instead, estimates made by one observer lie closer to the optimal value as compared to the other depending upon the value of $\gamma$. While generalising this approach, we focused mainly on the estimation of efficiency at optimal work. Recently, the near-equilibrium universality of efficiency at maximum power in finite-time models of heat engines has attracted a lot of attention as mentioned in Section 1.4. With non-identical reservoirs, it has been observed that universality with term $\eta_c^2/8$ in efficiency, close to equilibrium, does not hold and it becomes system dependent. It can be attributed to the fact that now the two reservoirs are not identical and hence the symmetry in the problem of inference has lost since equal weights (1/2) for the mean efficiency were assigned due to identical nature of the reservoirs. Further, the efficiency at optimal work can be reproduced up to second order by defining mean efficiency with non-identical weights for the efficiency estimates by the two observers. Thus, with non-identical systems also, we have been able to quantify the prior information and use it to estimate the optimal performance in thermodynamic process of entropy conservation.

# Chapter 5

# Discussion and future directions

The main motivation of the present thesis is to quantify the prior information in the estimation of the optimal state in a bipartite system undergoing a constrained thermodynamic process. In particular, we have considered entropy-conserving and energy-conserving processes. Our approach is motivated by inductive reasoning. In this kind of reasoning, as the premises provide only a partial information, we cannot make conclusions that are certain but only the ones that seem highly plausible. Bayesian inference, an extension of this approach, regards the probability of occurrence/truth of an event or a hypothesis, as a measure of our degree of rational belief in the truth of that hypothesis [95]. The prior distribution is sought as a measure of the *a priori* state of knowledge of the observer. In my work, the limited information on thermodynamic coordinates of the process is quantified as an uncertainty in the problem and thus appropriate priors have been proposed for these coordinates.

In Chapters 2 and 3, entropy-conserving and energy-conserving processes were discussed respectively. The reservoirs acting as source and sink were modelled by a pair of thermodynamic finite identical systems but at different initial temperatures. The priors were proposed for the final unknown temperatures of the two systems which were related by a one-to-one relation due to the constraint in the problem. We estimated the optimal work as well as efficiency at optimal work within the inference procedure quite accurately for entropy conserving process while for energy conserving process, the main quality to be inferred was the net entropy production.

In Chapter 4, we re-examined the entropy-conserving process to generalise inference approach to finite non-identical thermodynamic systems. Interesting results were obtained when one of the system becomes very large in comparison to the other and we reproduced the exact optimal behavior of the process. For arbitrary sizes of the reservoirs also, we were able to estimate the optimal features.

We have also tried to gain an insight into the form of prior proposed for the constrained thermodynamic processes. In the following, we discuss the form of prior and meaning of the temperature estimates. First we note that the prior for final temperature in the entropy-conserving process is

$$P(T)dT = N^{-1}\frac{C(T)dT}{T}, \tag{5.1}$$

where $N = \int_{T_-}^{T_+} C(T)/T \, dT$. Choosing entropy as an uncertain parameter, above prior can be re-expressed as:

$$p(S)dS = \frac{dS}{(S_+ - S_-)}. \tag{5.2}$$

Thus the derived prior is equivalent to a uniform prior in terms of the entropy, defined over the interval $[S_-, S_+]$. Similarly, the prior for the energy-conserving process,

$$\pi(T)dT = \frac{C(T)dT}{\int_{T_-}^{T_+} C(T)dT}, \tag{5.3}$$

implies a uniform prior over the energy of a reservoir:

$$p(U)dU = \frac{dU}{(U_+ - U_-)}. \tag{5.4}$$

Thus our particular choice of prior for temperature, implies a uniform prior density for the quantity being conserved in the process.

Secondly, the proposed prior lends a specific meaning to the final common temperature ($T_c$). For the optimal entropy-conserving process, the change in entropy of a reservoir, $\Delta S_1 = -\Delta S_2$, is given by: $S_+ - S_c = S_c - S_-$. This can be written in integral form as:

$$\int_{S_c}^{S_+} dS = \int_{S_-}^{S_c} dS. \tag{5.5}$$

As the prior density is uniform in terms of entropy, so we can write

$$\int_{S_c}^{S_+} p(S)dS = \int_{S_-}^{S_c} p(S)dS, \tag{5.6}$$

where $p(S) = 1/(S_+ - S_-)$. Thus our choice of prior implies that we are assigning *equal* probability (one-half each) that entropy $S$ of a reservoir may lie in the interval $[S_-, S_c]$, or in the interval $[S_c, S_+]$. A similar statement can be made in terms of $T_c$. Thus $T_c$ is the median of prior $P(T)$, on either side of which we expect equal chances that the final temperature may lie.

Finally, let us analyse the expected value of temperature as defined by $\overline{T} = \int_{T_-}^{T_+} TP(T)\, dT$. For the entropy conserving process, by using Eq. (5.1), the estimate for temperature has the general form:

$$\begin{aligned} \overline{T} &= \frac{1}{N}\int_{T_-}^{T_+} C(T)\, dT \\ &= \frac{1}{N}\int_{U_-}^{U_+} dU \\ &= \frac{(U_+ - U_-)}{(S_+ - S_-)}, \end{aligned} \tag{5.7}$$

where $N = \int_{T_-}^{T_+} C/T\, dT$. This suggests that $\overline{T}$ is the estimate for the derivative of the function $U(S)$ whose values at two points, $U_+(S_+)$ and $U_-(S_-)$, have been given. We note that the above intuitive meaning arises naturally within the energy representation [81]. Similarly, if we consider pure thermal interaction, while there is no simple interpretation for the expected value of $T$ in this case, however the expected value of the inverse temperature $\beta = 1/T$, is given simply as

$$\overline{\beta} = \frac{(S_+ - S_-)}{(U_+ - U_-)}. \tag{5.8}$$

So here, $\overline{\beta}$ can be regarded as an estimate for the derivative of the function $S(U)$, when its values at two points, $S_+(U_+)$ and $S_-(U_-)$, have been given. It is interesting to note that whereas $T$ serves as the fundamental variable in the energy representation, it is the inverse temperature $\beta$, which may be regarded as more fundamental in the entropy representation defined by the fundamental relation $S(U)$ [81, 96].

An integral part of Bayesian analysis is updating using Bayes theorem, in which the prior probabilities may be updated to posterior probabilities by incorporating fresh information from experiment or data. Our point of view is Bayesian insofar as we seek to assign an appropriate prior for our incomplete knowledge about the system. This is where we make use of the prior information about the given process. But our approach differs from the usual Bayesian inference where the priors can be updated to posterior probabilities. For instance, within the Bayesian approach for estimating the temperature of a system in contact with a heat reservoir [96, 97], it is appreciated that temperature itself is not an observable. The inference about temperature may be made indirectly by measuring the energy of the system. Then the quantity of interest is the posterior probability $p(T|E)dT$, for the temperature given that the measured energy of the system is $E$. This quantity is obtained from Bayes' theorem as

$$p(T|E)dT = \frac{p(E|T)\pi(T)dT}{p(E)}, \tag{5.9}$$

where $p(E) = \int p(E|T)\pi(T)dT$, and $p(E|T)$ is the probability (likelihood function) for the energy of the system to be $E$, if the temperature of the reservoir is $T$. Thus to do the analysis, one has to assign a prior $\pi(T)$ for temperature, that reflects our initial state of knowledge, for example, use of uniform prior [96] and $1/T$ prior [97] have been argued with ideal gas systems.

However, our purpose in this thesis is to highlight the estimates obtained only from the priors. Inference based on priors may be regarded as a particular case of Bayesian inference theorem using quality-adjusted priors and likelihood functions, as discussed in Section 1.2.3. The latter procedure may be justified when the authenticity or "quality" of the information from the prior or the likelihood function is not ensured. Then the posterior probabilities in a generalized form may be given by $p(T|E) \propto [p(E|T)]^b[\pi(T)]^a$, where $0 \leq a, b \leq 1$. The standard Bayes' theorem corresponds to the choice $a = b = 1$. If the prior information is of very low quality, then one sets $a \to 0$, and so any prior essentially turns into a uniform prior. In the present paper, we discount any information from the likelihood function ($b = 0$). As observed in the applications [73, 76, 77, 78, 79], this approach leads to estimates which are comparable to the optimal features of the concerned process. We also observed that the estimate of the temperature, say, in an entropy-conserving process, takes on a natural meaning of an estimate for the derivative, $(\partial U/\partial S)_V$. It turns out that the assigned prior is equivalent to a uniform prior over the quantity held constant in the constraint on the process. Moreover, the prior is such that the optimal temperature $T_c$ emerges as the median of the prior distribution.

It will be interesting to extend this approach to general scenarios to further investigate the relevance of prior information for the optimal characteristics of constrained systems. We have given concrete proposals for two extremal processes: entropy-conserving and energy-conserving. For more general processes which lie between these two limits, it would be interesting to formulate priors which interpolate between these two extremes, and thus to estimate the amount of dissipation. This may have implications for situations where due to non-equilibrium environment, the temperature is not well defined [98, 99]. So a prior for temperature can be motivated from the nature of energy exchange between system and environment as well as taking into account the possible range of temperature values. Other possible lines of investigation can be to perform inference using the posterior probabilities that incorporate the information from some measurement, to generalize the priors to multipartite systems or the treatment of non-identical reservoirs which may obey different fundamental relations.

# Appendix A

# Jensen's inequality

Jensen's inequality was proved by Johan Jensen, a Danish mathematician in 1906. This inequality relates the value of a convex/concave function of an expectation value to the expectation value of the convex/concave function. In other words, for a convex function $f$, we can write:

$$f(\langle x\rangle) \leq \langle f(x)\rangle, \tag{A.1}$$

where $x$ is a random variable and $\langle ...\rangle$ denotes the average or expectation value. The above inequality gets reversed for a concave function.

A function is said to be convex on some interval, say, $I$ if the secant line of that function lies on or above the graph of the function over $[x_1, x_2]$, where $x_1$, $x_2 \in I$ as shown in Fig. A.1. Thus, we write:

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2). \tag{A.2}$$

The above inequality can also be regarded as Jensen's inequality for two points. This can be generalized to $n$ points as follows:

$$f(\lambda_1 x_1 + \lambda_2 x_2 + ....\lambda_n x_n) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + ....\lambda_n f(x_n), \tag{A.3}$$

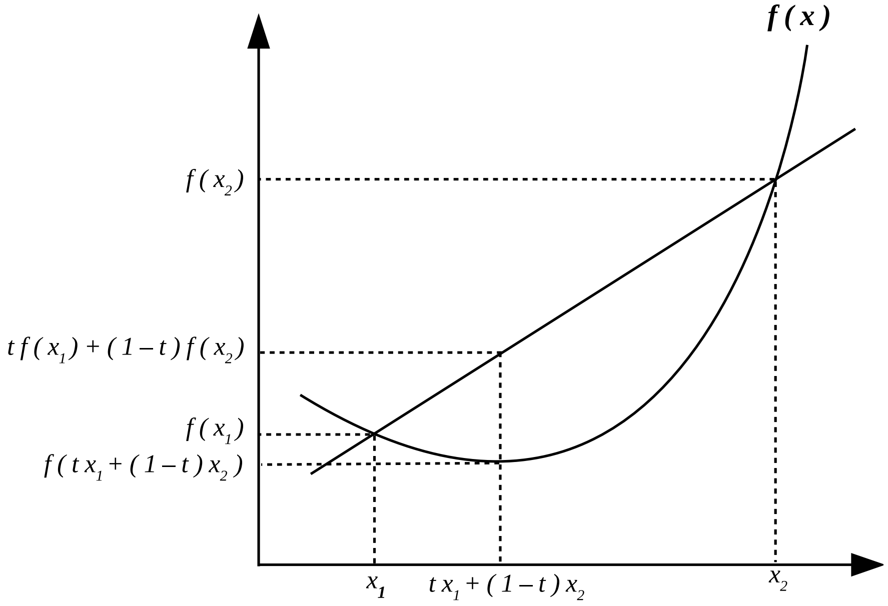



Figure A.1: *Jensen's inequality for two points*

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where $f$ is some convex function on some interval $I$, $x_i \in I$, $\lambda_i \geq 0$, $i = 0, 1.....n$ and $\sum_i \lambda_i = 1$. In context of probability theory, Jensen's inequality is generally stated in the following form: If $x$ is some random variable and $f$ is some convex function, then

$$f(\mathbb{E}[x]) \leq \mathbb{E}[f(x)], \tag{A.4}$$

where $\mathbb{E}$ denotes the expectation value with respect to probability distribution of $x$.

# Appendix B

# Numerical estimation

As discussed in Section 2.5, the entropy-conservation condition for spin-reservoirs cannot be solved analytically. Rewriting the equation as:

$$f_1 f_2 \ln\left(\frac{f_1 f_2}{(f_1-1)(f_2-1)}\right) + f_1 \ln(f_2-1) + f_2 \ln(f_1-1) + c f_1 f_2 = 0, \tag{B.1}$$

where

$$f_1 = 1 + e^{-a/T_1}, \tag{B.2}$$

$$f_2 = 1 + e^{-a/T_2}, \tag{B.3}$$

$$\begin{aligned} c = &-\Bigg[\ln\left[(1+e^{-a/T_+})(1+e^{-a/T_-})\right] \\ &+ \frac{a}{T_+}\left(\frac{e^{-a/T_+}}{1+e^{-a/T_+}}\right) + \frac{a}{T_-}\left(\frac{e^{-a/T_-}}{1+e^{-a/T_-}}\right)\Bigg], \end{aligned} \tag{B.4}$$

where $T_+$ ($T_-$) is the temperature of the hot (cold) reservoir and $a$ is the energy-level spacing. We find the numerical solution of constraint equation for $T_1$ when $T_2$ is substituted by its average value ($\overline{T}_2$) and then we get the corresponding estimate of $T_1$ as $\tilde{T}_1$.

# Mathematica code

We use scaled temperatures: $T_+ = 1$ and thus $T_- \equiv \theta$, such that $0 \leq \theta \leq 1$. For some arbitrary of $a$, calculate $\overline{T}_2$ which is given as

$$\overline{T}_2 = \frac{\frac{1}{1+e^a} - \frac{1}{1+e^{a/\theta}}}{\left(\frac{1}{\theta} - 1\right) + \left(\frac{1}{1+e^a} - \frac{1}{\theta(1+e^{a/\theta})}\right) + \frac{1}{a}\ln\left(\frac{1+e^a}{1+e^{a/\theta}}\right)}, \tag{B.5}$$

and set $T_2 = \overline{T}_2$; $T_1 = \tilde{T}_1$ in Eq. (B.1). To determine $\tilde{T}_1$ :

$$\begin{aligned}\text{For}\Big[\theta = 0.01, \theta < 1, \theta = \theta + 0.01, \text{FindRoot}\Big[f_1 f_2 \ln\left(\frac{f_1 f_2}{(f_1-1)(f_2-1)}\right) \\ + f_1 \ln(f_2 - 1) + f_2 \ln(f_1 - 1) + c f_1 f_2 == 0, \\ \left\{\tilde{T}_1, \theta + 0.01, \theta, 1\right\}\Big]\Big].\end{aligned} \tag{B.6}$$

Thus $\tilde{T}_1$ is obtained numerically by running iterations over small intervals of $\theta$ for different values of $a$. The work can be estimated as $\tilde{W}$ by replacing ($T_1$, $T_2$) with ($\tilde{T}_1$, $\overline{T}_2$) in the expression of work given below:

$$W = Na\left[\left(\frac{e^{-a}}{1+e^{-a}} + \frac{e^{-a/\theta}}{1+e^{-a/\theta}}\right) - \left(\frac{e^{-a/T_1}}{1+e^{-a/T_1}} + \frac{e^{-a/T_2}}{1+e^{-a/T_2}}\right)\right]. \tag{B.7}$$

Similary, with uniform prior, one may use $\overline{T}_2 = (1+\theta)/2$ to estimate work.

For optimal process, $T_1 = T_2 = T_c$ and thus Eq. (B.1) gets simplified as:

$$2f\ln\left(\frac{f}{f-1}\right) + 2\ln(f-1) + cf = 0, \tag{B.8}$$

where $f = 1+e^{-a/T_c}$. A program similar to B.6 can be developed for the numerical solution of Eq. (B.8) to obtain $T_c$ and hence we can calculate optimal work, $W_o$. Similarly, efficiency at optimal work can be estimated numerically.

# Appendix C

# Near-equilibrium estimation

Following the inference with non-identical finite source and sink as discussed in Section 4.5, we estimate the near-equilibrium optimal behavior of the engine analytically. Consider the case $0 < \gamma < 1$. Close to equilibrium, we can write $T_M = \theta\ (1+\epsilon)$, where $\epsilon > 0$ is a small parameter. To calculate $T_M$ analytically, we determine $\epsilon$ in the near-equilibrium regime. For this, rewrite $W(T_M) = 0$ and substitute the value of $T_M$ in terms of $\epsilon$

$$\begin{aligned}
1+\sigma[\theta^{1+\omega} - T_M^{1+\omega}] &= \Big(1+\sigma[\theta^{\omega} - T_M^{\omega}]\Big)^{\frac{1+\omega}{\omega}}, \\
1+\sigma[\theta^{1+\omega} - \theta^{1+\omega}(1+\epsilon)^{1+\omega}] &= \Big(1+\sigma[\theta^{\omega} - \theta^{\omega}(1+\epsilon)^{\omega}]\Big)^{\frac{1+\omega}{\omega}}, \\
1+\sigma\theta^{1+\omega}[1-(1+\epsilon)^{1+\omega}] &= \Big(1+\sigma\theta^{\omega}[1-(1+\epsilon)^{\omega}]\Big)^{\frac{1+\omega}{\omega}}. \qquad \text{(C.1)}
\end{aligned}$$

Expanding both sides as a series in $\epsilon$, we get :

$$\begin{aligned}
&1-\epsilon(1+\omega)\sigma\theta^{1+\omega}\Big[1+\frac{\epsilon}{2}\omega+\frac{\epsilon^2}{6}\omega\,(\omega-1)\Big] \\
&= \Big(1-\epsilon\omega\sigma\theta^{\omega}\Big[1+\frac{\epsilon}{2}\,(\omega-1)+\frac{\epsilon^2}{6}\,(\omega-1)\,(\omega-2)\Big]\Big)^{\frac{1+\omega}{\omega}}. \qquad \text{(C.2)}
\end{aligned}$$

Expanding the right hand side of above equation and keeping terms only up to second order in $\epsilon$, we get :

$$\begin{aligned} 1 &- \epsilon(1+\omega)\sigma\theta^{1+\omega}\left[1+\frac{\epsilon}{2}\omega+\frac{\epsilon^2}{6}\omega\,(\omega-1)\right] \\ =1 &- \epsilon(1+\omega)\sigma\theta^{1+\omega}\left[\frac{1}{\theta}\left(1+\frac{\epsilon}{2}\,(\omega-1)+\frac{\epsilon^2}{6}\,(\omega-1)\,(\omega-2)\right)\right. \\ &- \left.\frac{\epsilon\sigma\theta^{\omega-1}}{2}\Big(1+\epsilon\,(\omega-1)\Big)-\frac{\epsilon^2\sigma^2\theta^{2\omega-1}}{6}\,(\omega-1)\right]. \end{aligned} \tag{C.3}$$

After simplifying the above equation, we get quadratic equation in $\epsilon$ as :

$$(1-\omega)\left[\sigma^2\theta^{2\omega}+3\sigma\theta^{\omega}-\omega(1-\theta)+2\right]\epsilon^2-3\left[\sigma-\omega(1-\theta)+1\right]\epsilon+6(1-\theta)=0. \tag{C.4}$$

Above equation is quadratic of the form $a\epsilon^2+b\epsilon+c=0$, where

$$a = (1-\omega)\left[\sigma^2\theta^{2\omega}+3\sigma\theta^{\omega}-\omega(1-\theta)+2\right], \tag{C.5}$$

$$b = -3\left[\sigma-\omega(1-\theta)+1\right], \tag{C.6}$$

$$c = 6(1-\theta). \tag{C.7}$$

Eq. (C.4) can be solved for $\epsilon$ and whose solution is given as:

$$\epsilon_{\pm}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}. \tag{C.8}$$

Near-equilibrium, $\eta_c = 1-\theta$ is a small parameter, so the solution $\epsilon$ can be expanded as series in the powers of $\eta_c$. Substitute the values of $a$, $b$ and $c$ in above solution and expanding $\epsilon_-$ up to second order in $\eta_c$, we get :

$$\epsilon_- = \frac{2}{1+\sigma}\eta_c+\frac{2[4-\omega+\sigma(\omega+2)]}{3(1+\sigma)^2}{\eta_c}^2. \tag{C.9}$$

The other solution, $\epsilon_+$ is not acceptable since it does not give $\epsilon$ as a small parameter as it has a constant term in the expansion about $\eta_c$ close to zero as shown below:

$$\epsilon_+ = \frac{6(1+\sigma)}{2(1-\omega)(2+3\sigma+\sigma^2)} - \left(\frac{2}{1+\sigma} + \frac{3\omega(1-\sigma)}{(1-\omega)(2+\sigma)^2}\right)\eta_c. \tag{C.10}$$

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