Quark Mass Matrices & Weyls Inequality

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Certificate of Examination

This is to certify that the dissertation titled " Quark Mass Matrices And Weyls Inequality submitted by **Mr. Shivam Umarvaishya** (Registration Number: **MS10044**) for the partial fulfillment of **BS-MS dual degree program** of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. Manmohan Gupta at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Mohali, April 22, 2016

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Mohali, April 22, 2016

Prof. Manmohan Gupta

Abstract

The issue of texture specific Quark mass matrices has been presented here by incorporating Weak Basis Transformation. We have also thought to look the same problem from another angle by using Poincare Theorem and Weyls Inequality.

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1

Introduction

The understanding of Fermion masses is one of the important problem of flavor Physics. The biggest challenge in this theory is understanding of fermion masses, spanning many orders of magnitudes in a unified framework. The theoretical understanding of fermion masses goes along two approaches.These are ’top-down’ and ’bottom-up’ approaches. There have been made large number of attempts from top-down approach like Grand Unification,Super symmetry,and super strings etc. But almost non of them provide compelling concept for flavor dynamics. Here i have adopted The ’ bottom-up’ approach to understand this problem. In bottom-up approach, we find phenomenological fermion mass matrices which are compatible with low energy data i.e. compatible with physical observable like Quark and Lepton masses. Texture specific mass matrices provide a good example bottom-up approach.Many attempts have been made to find fermion mass matrices which are compatible with low energy data and to integrate textures within grand unified theory. However the findings of fermion mass matrices which are compatible with low energy data and with texture framework has not been found yet.

I have made an attempt to understand the fermion mass matrices which are compatible with low energy data and with texture framework in agreement with weak basis transformation.

2 Weak Basis Transformation

2.1 Fermion Mass Matrices

In the standard model of strong, weak and electromagnetic interactions the elementary particles are quarks and leptons and these are categorized in three generations.

$$\text{Quarks:} \begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix},$$

$$\text{Leptons:} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}$$

The spontaneous symmetry breaking of $SU(2) \times U(1)$ gauge group to $U(1)$, through the Yukawa couplings and the vacuum expectation value of the neutral Higgs field provides masses to fermions. The Lagrangian of the Yukawa sector of the standard model is [**P1**]:

$$\mathcal{L} = Y_d^{ij} \bar{Q}_L^i \phi D_R^j + Y_u^{ij} \bar{Q}_L^i \tilde{\phi} U_R^j + Y_e^{ij} \bar{L}_L^i \phi E_R^j + h.c. \tag{2.1}$$

where ϕ is the Higgs doublet under $SU(2)$ and $\tilde{\phi} = \iota \tau_2 \phi^\dagger$

Here, Y_u, Y_d, Y_e are 3×3 matrices with 18 real parameters each. After the spontaneous symmetry breaking, the Higgs acquire a vacuum expectation value (VEV) v

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix}, \tilde{\phi} = \frac{1}{\sqrt{2}} \begin{pmatrix} v+h \\ 0 \end{pmatrix} \tag{2.2}$$

which leads to the introduction of sensationalized 3×3 quark mass matrices (at present ignore the lepton part)

$$M_u^{ij} = \frac{v}{\sqrt{2}} Y_u^{ij} \tag{2.3}$$

$$M_d^{ij} = \frac{v}{\sqrt{2}} Y_d^{ij} \tag{2.4}$$

Each mass matrix contains 18 parameters in general but we know that a complex matrix can be decomposed in to products of a hermitian and unitary matrix. so we can write our mass matrices in to products of a hermitian and unitary matrix. we can absorb the unitary part of it into right handed Quark fields. Now our Each mass matrices are just a hermitian matrix with total 9 independent parameters.

2.2 The Mass Matrices of Quark

The matrices M_U and M_D are for the up and down sector of quarks.These mass matirces total of 18 independent parameters , larger in number compared to only 10 physical observables.These ten observables are six quark masses, three mixing angles and CP violating phase. In the general case mass terms are quadratic in terms of fermion fields. The quark mass terms, below the electroweak symmetry breaking, is

$$\bar{Q}_{U_L} M_U Q_{U_R} + \bar{Q}_{D_L} M_D Q_{D_R} \tag{2.5}$$

where $Q_{U_{L(R)}}$ and $Q_{D_{L(R)}}$ are left handed (right handed) quark fields for up sector (u, c, t) and down sector (d, s, b) respectively. The matrices M_U and M_D are for the up and down sector quarks respectively. The above equation has to be re-expressed in terms of physical quark fields to make any sense. This is achieved by diagonalizing the mass matrices via bi-unitary transformations.

$$V_{U_L}^{\dagger} M_U V_{U_R} = M_U{}^{diag} \equiv \text{diag}\,(m_u, m_c, m_t) \tag{2.6}$$

$$V_{D_L}^{\dagger} M_D V_{D_R} = M_D{}^{diag} \equiv \text{diag}\,(m_d, m_s, m_b) \tag{2.7}$$

where m_u, m_d, etc. are eigenvalues of the quark mass matrices which correspond to physical quark masses. The equation (1.5) can be re-written using Eqs. (1.6) and (1.7) as

$$\bar{Q}_{U_L} V_{U_L} M_U^{diag} V_{U_R}^{\dagger} Q_{U_R} + \bar{Q}_{D_L} V_{D_L} M_D^{diag} V_{U_D}^{\dagger} Q_{U_D} \tag{2.8}$$

which in terms of physical fields are

$$\bar{Q}_{U_L}^{phys} M_U^{diag} Q_{U_R}^{phys} + \bar{Q}_{D_L}^{phys} M_D^{diag} Q_{D_R}^{phys} \tag{2.9}$$

where $Q_{U_L}^{phys} = V_{U_L}^{\dagger} Q_{U_L}$ and $Q_{D_L}^{phys} = V_{D_L}^{\dagger} Q_{D_L}$ and so on. The mismatch in the diagonalization of up and down matrices leads to the definition of quark mixing matrix, known as the Cabibbo-Kobayashi-Maskawa (CKM) matrix, given by

$$V_{CKM} = V_{U_L}^{\dagger} V_{D_L} \tag{2.10}$$

The CKM matrix describes the weak interaction eigenstates (d', s', b') of the quarks in terms of their flavour eigenstates (d, s, b), e.g.,

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{2.11}$$

The CKM matrix is a unitary matrix which describes the transition of one quark into another. A general $n \times n$ unitary matrix has n^2 parameters, $\frac{n(n-1)}{2}$

of these are the Eulers angles and remaining $\frac{n(n+1)}{2}$ are the phases. However, some of these phases can be rotated away. So, in a $n \times n$ we are left with only $\frac{(n-1)(n-2)}{2}$ measurable phases. Thus, in the case of three families of quarks, the mixing matrix is expressed in terms of three angles and one phase, the latter being responsible for CP violation.

The SM imposes the unitarity constraint on the quark mixing matrix. The unitarity of CKM matrix leads to nine relations, three being the normalization conditions and the rest six are non-diagonal relations which are defined in the follwing way

$$\sum_{\alpha=d,s,b} V_{i\alpha} V_{j\alpha}^* = \delta_{ij} \tag{2.12}$$

$$\sum_{i=u,c,t} V_{i\alpha} V_{i\beta}^* = \delta_{\alpha\beta} \tag{2.13}$$

where the Greek indices run over the down type quarks (d, s, b) and the Latin ones run over the up type quarks(u, c, t).

2.3 The Technology

The mass matrices in the Standard Model are completely arbitrary 3×3 complex matrices. However, they can be reduced to hermitian matrices with less no. of indpendent parameters . However, the above prescription still leaves us with 18 independent parameters which are still in excess when compared to the number of physical observables -six quark masses, three mixing angles and a CP violating phase. To account for this redundancy, we require some additional assumptions. In this context the concept of textures was introduced implicitly by Weinberg [**S2**] and explicitly by Fritzsch [**F1**], where in certain elements of the mass matrices are assumed to be highly suppressed or can be considered zero also. The zero elements of the mass matrices can be characterized as texture zeros defined in a particular manner.

A particular texture structure is said to be texture n zero, if it has n number of non-trivial zeros,for example, if the sum of the number of diagonal zeros and half the number of the symmetrically placed off diagonal zeros is n.

The Fritzsch's-like texture specific hermitian quark mass matrices have the following form.

$$M_U = \begin{pmatrix} 0 & A_U & 0 \\ A^*{}_U & D_U & B_U \\ 0 & B^*{}_U & C_U \end{pmatrix}, \; M_D = \begin{pmatrix} 0 & A_D & 0 \\ A^*{}_D & D_D & B_D \\ 0 & B^*{}_D & C_D \end{pmatrix} \tag{2.14}$$

Here, $A_i = |A_i| \exp^{\iota \alpha_i}$ and $B_i = |B_i| \exp^{\iota \beta_i}$ with $i = U, D$. Each of the above matrix is texture 2 zero type.

One particular facility available to achieve texture zeroes is of the Weak Basis Transformations. Branco et al [**B14**] initiated the idea of WB transformations to introduce the texture zeroes compatible with the SM so as to lend predictability to the general mass matrices. Initially, texture zeroes were introduced as ansatz. However, efforts have been made to deduce these from symmetry considerations as well as from general considerations. In this chapter we would attempt the introduction of textures though general considerations.

In the SM one has the freedom to make a unitary transformation W on the quark fields e.g.,

$$q_L \to U q_L, \; q_R \to U q_R, \; q'_L \to U q'_L, \; q'_R \to U q'_R \tag{2.15}$$

under which gauge currents

$$\mathcal{L}_W = \frac{g}{\sqrt{2}} \overline{(u, c, t)} \gamma^\mu \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L W_\mu + hc \tag{2.16}$$

remain real and diagonal but the mass matrices transform as

$$M_u \to U^\dagger M_u U \ , \ M_d \to U^\dagger M_d U \tag{2.17}$$

2.4 The (1,1) Weak Basis Zero

In this section we present the results of Branco et al. [**B14**]. We discuss the zeroes occurring at (1,1) position in up and down quark mass matrices. The most general transformation that leaves the mass matrices hermitian is:

$$M_u \longrightarrow M_u' = U^\dagger M_u U \tag{2.18}$$

$$M_d \longrightarrow M_d' = U^\dagger M_d U \tag{2.19}$$

where U is an arbitrary unitary matrix. In such a basis, we can always find a set of unitary matrices $\{U_u, U_d\}$ which can diagonalize the mass matrices such that

$$D_u' = U_u^\dagger M_u U_u \tag{2.20}$$

$$D_d' = U_d^\dagger M_d U_d \tag{2.21}$$

where $D_u \equiv$ diag (m_u, m_c, m_t) and $D_d \equiv$ diag (m_d, m_s, m_b). We choose to work in basis where M_u is diagonal and M_d is hermitian, i.e.

$$M_u = D_u \tag{2.22}$$

$$M_d = V D_d V^\dagger \tag{2.23}$$

The matrix V is an arbitrary unitary matrix. Effecting a WB transformation with U, under which M_u and M_d transform as:

$$M_u \longrightarrow M_u' = U^\dagger D_u U, \tag{2.24}$$

$$M_d \longrightarrow M_d' = U^\dagger V D_d V^\dagger U \tag{2.25}$$

that $(M'_u)_{11} = (M'_d)_{11} = 0$. This requires the solution of the following system of equations.

$$m_u \mid U_{11} \mid^2 + m_c \mid U_{12} \mid^2 + m_t \mid U_{31} \mid^2 = 0 \tag{2.26}$$

$$m_d \mid X_{11} \mid^2 + m_s \mid X_{12} \mid^2 + m_b \mid X_{31} \mid^2 = 0 \tag{2.27}$$

$$\mid U_{11} \mid^2 + \mid U_{12} \mid^2 + \mid U_{13} \mid^2 = 1 \tag{2.28}$$

where $X = V^\dagger U$ and thus:

$$|X_{i1}|^2 = |V_{1i}|^2|U_{11}|^2 + |V_{2i}|^2|U_{21}|^2 + |V_{3i}|^2|U_{31}|^2 + 2Re(V_{1i}^* U_{11} V_{2i} U_{21}^*) + 2Re(V_{1i}^* U_{11} V_{3i} U_{31}^*) + 2Re(V_{2i}^* U_{21} V_{3i} U_{31}^*), \quad \text{(i = 1, 2, 3)} \tag{2.29}$$

The system of Eqs. (2.9) has a real solution only if, at least one of the mass parameters m_u, m_c, m_t and one of the parameters m_d, m_s, m_b is negative. For the arbitrary mass matrices M_u and M_d, one has to find a unique U satisfying (2.9). It is not always possible to find analytic solutions for (Eqn 2.9). For the simple case, when $V = \mathbf{1}$, $X = U$ and we obtain the following solutions:

$$\mid U_{11} \mid^2 = \frac{m_c m_b - m_s m_t}{\Delta} \tag{2.30a}$$

$$\mid U_{21} \mid^2 = \frac{m_d m_t - m_u m_b}{\Delta} \tag{2.30b}$$

$$\mid U_{31} \mid^2 = \frac{m_u m_s - m_d m_c}{\Delta} \tag{2.30c}$$

where

$$\Delta = (m_t - m_u)(m_b - m_s) - (m_t - m_c)(m_b - m_d) \tag{2.31}$$

Next, if we choose V to be a realistic CKM matrix

$$V = \begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.32}$$

In this case, Eqs.(2.9) become

$$\begin{aligned} \mid X_{11} \mid^2 &= cos^2\theta \mid U_{11} \mid^2 + sin^2\theta \mid U_{21} \mid^2 - sin2\theta\, U_{11}U_{21} && (2.33a) \\ \mid X_{21} \mid^2 &= sin^2\theta \mid U_{11} \mid^2 + cos^2\theta \mid U_{21} \mid^2 + sin2\theta\, U_{11}U_{21} && (2.33b) \\ \mid X_{31} \mid^2 &= \mid U_{31} \mid^2 && (2.33c) \end{aligned}$$

Using unitarity, we can write

$$(m_u - m_t) \mid U_{11} \mid^2 + (m_c - m_t) \mid U_{21} \mid^2 + m_t = 0 \quad (2.34a)$$

$$\begin{aligned} (m_d cos^2\theta + m_s sin^2\theta - m_b) \mid U_{11} \mid^2 + m_d sin^2\theta + m_s cos^2\theta - m_b) \mid U_{21} \mid^2 \\ + (m_s - m_d)\, sin2\theta\, U_{11}U_{21} + m_b = 0 \end{aligned} \quad (2.34b)$$

Parametrizing the solutions as:

$$\sqrt{m_t - m_u}\, U_{11} = \sqrt{m_t}\, cos\phi \quad (2.35a)$$

$$\sqrt{m_t - m_u}\, U_{21} = \sqrt{m_t}\, sin\phi \quad (2.35b)$$

Denoting

$$\begin{aligned} a &= m_b - (m_b - m_d sin^2\theta - m_s cos^2\theta)\frac{m_t}{m_t - m_c} && (2.36a) \\ b &= (m_s - m_d)\frac{m_t sin2\theta}{\sqrt{(m_t - m_u)(m_t - m_c)}}, && (2.36b) \\ c &= m_b - (m_b - m_d cos^2\theta - m_s sin^2\theta)\frac{m_t}{m_t - m_u} && (2.36c) \end{aligned}$$

introducing $z \equiv \tan\phi$, the solution is given by the quadratic equation

$$az^2 + bz + c = 0 \quad (2.37)$$

If $\theta = 0$ and $V = \mathbf{1}$, we recover the results of Eqs. (2.11).

2.5 The (One Three, Three One) Problem

In this section we present our attempts and partial results to obtain texture two zero matrices from the most general 3×3 unitary matrix using the recipe of weak basis transformations. Fritzch in his paper [**F2**] had discussed the possibility of achieving the texture two form given below,

$$M_U = \begin{pmatrix} E_U & A_U & 0 \\ A_U^* & D_U & B_U \\ 0 & B_U^* & C_U \end{pmatrix}, M_D = \begin{pmatrix} E_D & A_D & 0 \\ A_D^* & D_D & B_D \\ 0 & B_D^* & C_D \end{pmatrix} \quad (2.38)$$

starting from the hermitian mass matrices,

$$M_q = \begin{pmatrix} E_q & A_q & F_q \\ A_q^* & D_q & B_q \\ F_q^* & B_q^* & C_q \end{pmatrix}, (q = U, D) \quad (2.39)$$

through a common unitary transformation. We tried to find out the exact form of the unitary matrix which accomplishes this task. We start by choosing a basis in which M_U is diagonal and M_D hermitian.

$$M_U = \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{pmatrix}, M_D = \begin{pmatrix} \mu_{11} & \mu_{12}e^{i\eta_{12}} & \mu_{13}e^{i\eta_{13}} \\ \mu_{12}e^{-i\eta_{12}} & \mu_{22} & \mu_{23}e^{i\eta_{23}} \\ \mu_{13}e^{-i\eta_{13}} & \mu_{23}e^{-i\eta_{23}} & \mu_{33} \end{pmatrix} \quad (2.40)$$

The unitary matrix for effecting the weak basis transformation is the following :

$$U = U_1 \begin{pmatrix} \cos\alpha\cos\gamma & \sin\alpha\cos\gamma & \sin\gamma e^{i(\alpha_3-\delta)} \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta\sin\gamma e^{i\delta} & \cos\alpha\cos\beta - \sin\alpha\sin\beta\sin\gamma e^{i\delta} & \sin\beta\cos\gamma \\ \sin\alpha\sin\beta - \cos\alpha\cos\beta\sin\gamma e^{i\delta} & -\cos\alpha\sin\beta - \sin\alpha\cos\beta\sin\gamma e^{i\delta} & \cos\beta\cos\gamma \end{pmatrix} \quad (2.41)$$

where U_1 and U_2 are given by

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i(\alpha_4-\alpha_3)} & 0 \\ 0 & 0 & e^{i(\alpha_5-\alpha_3)} \end{pmatrix}, \; U_2 = \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & e^{i\alpha_3} \end{pmatrix} \tag{2.42}$$

The result of the weak basis transformation on the matrices is the following.

$$M'_U = U^\dagger M_U U \tag{2.43a}$$

$$M'_D = U^\dagger M_D U \tag{2.43b}$$

Since, we are interested in only $(M'_U)_{13}$ and $(M'_D)_{13}$, we study the transformation of only those elements.

$$(M'_D)_{13} = {U^\dagger}_{1i}(M_D)_{ij}U_{j3} \tag{2.44}$$

, where i, j = 1, 2, 3 or

$$\begin{aligned}(M'_D)_{13} = {U^\dagger}_{11}\{M_{11}U_{13} + M_{12}U_{23} + M_{13}U_{33}\}+ \\ {U^\dagger}_{12}\{M_{21}U_{13} + M_{22}U_{23} + M_{23}U_{33}\}+ \\ {U^\dagger}_{13}\{M_{31}U_{13} + M_{32}U_{23} + M_{33}U_{33}\}\end{aligned} \tag{2.45}$$

which translates into

$$\begin{aligned}(M'_D)_{13} = 0 = \mu_{11}\cos\alpha\cos\gamma\sin\gamma e^{i(\alpha_3-\alpha_1-\delta)} + \\ \mu_{22}\sin\beta\cos\gamma(\sin\alpha\cos\beta - \cos\alpha\sin\beta\sin\gamma e^{-i\delta})e^{i(\alpha_3-\alpha_1)} \\ +\mu_{33}\cos\beta\cos\gamma(\sin\alpha\sin\beta - \cos\alpha\cos\beta\sin\gamma e^{-i\delta})e^{i(\alpha_3-\alpha_1)} \\ +\mu_{12}[\cos\alpha\cos^2\gamma\sin\beta e^{i(\alpha_4-\alpha_1+\eta_{12})} \\ +sin\gamma(\sin\alpha\cos\beta - \cos\alpha\sin\gamma e^{-i\delta})e^{i(2\alpha_3-\alpha_1-\alpha_4-\eta_{12}-\delta)}] \\ +\mu_{13}[\cos\alpha\cos\beta\cos^2\gamma e^{i(\alpha_4-\alpha_1+\eta_{13})} \\ +\sin\gamma(\sin\alpha\sin\beta - cos\alpha\sin\beta\sin\gamma e^{-i\delta})e^{i(2\alpha_3-\alpha_5-\alpha_1-\eta_13-\delta)}] \\ +\mu_{23}[\cos\beta\cos\gamma(\sin\alpha\cos\beta - \cos\alpha\sin\beta\sin\gamma e^{-i\delta})e^{i(\alpha_5+\alpha_3-\alpha_4-\alpha_1+\eta_{23})} \\ +\sin\beta\cos\gamma(\sin\alpha\sin\beta - \cos\alpha\cos\beta\sin\gamma e^{-i\delta})e^{i(\alpha_4+\alpha_3-\alpha_5-\alpha_1-\eta_{23})}]\end{aligned} \tag{2.46}$$

Similarly, the other equation is:

$$\begin{aligned}(M'_U)_{13} = 0 = m_{11}\cos\alpha\cos\gamma\sin\gamma e^{i(\alpha_3-\alpha_1-\delta)} \\ +m_{22}\sin\beta\cos\gamma(\sin\alpha\cos\beta - \cos\alpha\sin\beta\sin\gamma e^{-i\delta})e^{i(\alpha_3-\alpha_1)} \\ +m_{33}\cos\beta\cos\gamma(\sin\alpha\sin\beta - \cos\alpha\cos\beta\sin\gamma e^{-i\delta})e^{i(\alpha_3-\alpha_1)}\end{aligned} \tag{2.47}$$

Now, we have to simultaneously solve Eqs. (2.27 & 2.28). We make the following assumptions to simplify the above equations.

$$\begin{aligned}\alpha_3 &= \alpha_1 \\ \delta &= 0 \\ \alpha_4 - \alpha_3 + \eta_{12} &= 0 \\ \alpha_5 - \alpha_3 + \eta_{13} &= 0 \\ \alpha_5 - \alpha_4 + \eta_{23} &= 0\end{aligned} \tag{2.48}$$

The assumptions of Eqn. (2.29), along with $\gamma = 0$ reduces Eqn. (2.28) to

$$m_{22}\sin\alpha\sin 2\beta + m_{33}\sin\alpha\sin 2\beta = 0 \tag{2.49}$$

$\implies$ either $\sin\alpha = 0$ or $\sin 2\beta(m_{22} + m_{33}) = 0$. If $\sin\alpha \neq 0$,then

$$\sin 2\beta(m_{22} + m_{33}) = 0 \tag{2.50}$$

which gives $\beta = 0, \frac{\pi}{2}$. $\gamma = 0$ and $\beta = 0$, reduces Eqn. (2.27) to

$$\mu_{13}\cos\alpha + \mu_{23}\sin\alpha = 0 \tag{2.51a}$$

$$\tan\alpha = \frac{-\mu_{13}}{\mu_{23}} \tag{2.51b}$$

whereas $\gamma = 0$ and $\beta = \frac{\pi}{2}$, reduces Eqn. (2.27) to

$$\tan\alpha = \frac{-\mu_{12}}{\mu_{23}} \tag{2.52}$$

On the other hand, if $\sin\alpha = 0 \implies \alpha = 0$

We obtain yet another solution with $\alpha = 0$ and $\gamma = 0$ which is

$$\tan\beta = \frac{-\mu_{12}}{\mu_{13}} \tag{2.53}$$

With $\gamma = 0$ $\beta = \frac{\pi}{2}$ and $\tan\alpha = \frac{-\mu_{12}}{\mu_{23}}$, the matrix U becomes

$$U = \begin{pmatrix} \frac{\mu_{23}}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & -\frac{\mu_{12}}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & 0 \\ 0 & 0 & 1 \\ -\frac{\mu_{12}}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & -\frac{\mu_{23}}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & 0 \end{pmatrix} \tag{2.54}$$

By virtue of Eqn. (2.24a), M_U' becomes

$$M_U' = \begin{pmatrix} \frac{m_{33}\mu_{12}^2}{\mu_{12}^2+\mu_{23}^2} + \frac{m_{11}\mu_{23}^2}{\mu_{12}^2+\mu_{23}^2} & \frac{m_{33}\mu_{12}\mu_{23}}{\mu_{12}^2+\mu_{23}^2} - \frac{m_{11}\mu_{12}\mu_{23}}{\mu_{12}^2+\mu_{23}^2} & 0 \\ \frac{m_{33}\mu_{12}\mu_{23}}{\mu_{12}^2+\mu_{23}^2} - \frac{m_{11}\mu_{12}\mu_{23}}{\mu_{12}^2+\mu_{23}^2} & \frac{m_{11}\mu_{12}^2}{\mu_{12}^2+\mu_{23}^2} + \frac{m_{33}\mu_{23}^2}{\mu_{12}^2+\mu_{23}^2} & 0 \\ 0 & 0 & m_{22} \end{pmatrix} \tag{2.55}$$

Similary, Eqn. (2.24b) leads to

$$M_D' = \begin{pmatrix} \frac{\mu_{23}\left(\frac{\mu_{11}\mu_{23}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}-\frac{\mu_{12}\mu_{13}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}\right)}{\sqrt{\mu_{12}^2+\mu_{23}^2}} - \frac{\mu_{12}\left(\frac{\mu_{13}\mu_{23}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}-\frac{\mu_{12}\mu_{33}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}\right)}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & -\frac{\mu_{12}\left(\frac{\mu_{11}\mu_{23}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}-\frac{\mu_{12}\mu_{13}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}\right)}{\sqrt{\mu_{12}^2+\mu_{23}^2}} - \frac{\mu_{23}\left(\frac{\mu_{13}\mu_{23}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}-\frac{\mu_{12}\mu_{33}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}\right)}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & 0 \\ \frac{\mu_{23}\left(-\frac{\mu_{11}\mu_{12}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}-\frac{\mu_{13}\mu_{23}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}\right)}{\sqrt{\mu_{12}^2+\mu_{23}^2}} - \frac{\mu_{12}\left(-\frac{\mu_{12}\mu_{13}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}-\frac{\mu_{23}\mu_{33}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}\right)}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & -\frac{\mu_{12}\left(-\frac{\mu_{11}\mu_{12}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}-\frac{\mu_{13}\mu_{23}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}\right)}{\sqrt{\mu_{12}^2+\mu_{23}^2}} - \frac{\mu_{23}\left(-\frac{\mu_{12}\mu_{13}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}-\frac{\mu_{23}\mu_{33}}{\sqrt{\mu_{12}^2+\mu_{23}^2}}\right)}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & -\frac{\mu_{12}^2}{\sqrt{\mu_{12}^2+\mu_{23}^2}} - \frac{\mu_{23}^2}{\sqrt{\mu_{12}^2+\mu_{23}^2}} \\ 0 & -\frac{\mu_{12}^2}{\sqrt{\mu_{12}^2+\mu_{23}^2}} - \frac{\mu_{23}^2}{\sqrt{\mu_{12}^2+\mu_{23}^2}} & \mu_{22} \end{pmatrix} \tag{2.56}$$

We notice that M_d has been put in the texture two zero form (Eqn. 2.19) though the weak basis transformation but the same form couldn't be achieved for M_u. We have additional zeroes on symmetrical positions (2,3) & (3,2). Efforts were made to get rid of these zeroes using another weak basis transformation but that couldn't be achieved without destroying zeroes at (1,3) & (3,1) position.

3

Weyls Ineqaulity

Within the framework of Grandunified theories We can Express The mass matrices M_e and M_d as

$$M_d = H + F \tag{3.1}$$

$$M_e = H - 3F \tag{3.2}$$

Under the condition when CP Vialation is ignored

H and F are symmertic coupling matrices with real entries.We know M_e and M_d both are hermitian Matrices. To understand the same problem with which we have been struggling in above chapter, here we may look it from a different angle. To investigate the problem from this end we need to understand some theorem of linear algebra. The theorems are about, given two hermitian operators(Say A & B) with their know values of Eigen values.

What will be the relationship between the eigenvaules of Operator which is linear combination of those two operators (Say a A + b B) and eigenvalues of A & B.

3.1 The Poincare Theorem

Consider a Hermitian operator A acting on an 'n' Dimensional Vector Space H. Let the Eigenvectors of Hermitian Operator $A|a_i>s$ are complete basis set of space H with Eigenvalues $\alpha_1 > \alpha_2 > .... > \alpha_n$.

3.1.1 The Theorem States

A. You can not Choose a k Dimensional Subspace M in H such that all its Unit vector has

$$< X|A|X >> \alpha_k \tag{3.3}$$

Proof : The proof is simple. souppse we span the subspace M by Egenvectors of A then We need ot choose k vector out f n $|a_1>$, $|a_2>$, ...$|a_n>$.

Even if we choose first k eigen vectors, It will include $|a_k>$ hence $<a_k|A|a_k>$ = a_k. In all other cases you have to choose atleast one $\lfloor a_{i.}>$such that i > k

then in all those cases it Fallows

$$< X|A|X >\leq \alpha_k \tag{3.4}$$

.

B. You cannot choose a k Dimentional subspace in H such that all its unit vectors has < Y |A|Y > < α_{n-k+1}.

Proof : By using the same proof as above, but now from opposite end of $|a_1>, |a_2>, ...|a_n>$.

3.2 The Weyls Ineqality

Consider three Hermitian operatrs A, B, C actingn on 'n' Dimentional Vector space where C = A + B.

Here we are intrested in understanding the realationship between Eigenvalue of Operator C with eigenvalues of A & B .

Let $|a_1>, |a_2>, ...|a_n>, and |b_1>, |b_2>, ...|b_n>$and $|c_1>, |c_2>, ...|c_n>$ be the Eigen vectors and $a_1, a_2, ...a_n$, and $b_1, b_2, ...b_n$and $c_1, c_2, ...c_n$be the Eigenvalues of A , B , C reespectively . Then The Weyls ineqality is

$$c_{i+j-1} = a_i + b_j \tag{3.5}$$

where

$$i + j - 1 < n \tag{3.6}$$

Proof : To Understand this inequality consider 3 subspaces L, M, P spaned by $\{ |a_i>, ...|a_n>\}, \{|b_j>, ...|b_n>\}$and $\{|c_1>, |c_2>, ...|c_k>\}$.

The dimention of each subspace is n-i+1 , n-j+1 and k. If k = i+j-1 then L Intrscsn Mintrscsn P $\ncong \phi$ As sum of Dimentions is 2n+1 thus there are vectors in L M P say X Such that X belnong to L M P.

Then Poincare Theorem Says

$$c_{i+j-1} \leq < X|C|X > = < X|A+B|X > = < X|A|X > + < X|B|X > \quad (3.7)$$

But we know

$$< X \mid A \mid X > \epsilon[a_n, a_i] \quad (3.8)$$

,

$$< X|B|X > \epsilon[b_n, b_j] \quad (3.9)$$

and

$$< X|C|X > \epsilon[c_{i+j-1}, c_1] \quad (3.10)$$

thus

$$c_{i+j-1} \leq < X|C|X > = < X|A+B|X > = < X|A|X > + < X|B|X > \leq a_i + b_j \quad (3.11)$$

$$c_{i+j-1} \leq a_i + b_j \quad (3.12)$$

ie

$$c_1 \leq a_1 + b_1 \quad (3.13)$$

.

3.3 The More Genaralise Statement of Weyls Ineqality

Cnsidering the same situation as above.Now we will study the more genaralised form of Weyls Ineqaulity.

$$a_i + b_n \leq c_i \leq a_i + b_1 \tag{3.14}$$

Proof : The right part of this Ineqaulity is trival and can be derived from Weyls Ineqality by just putting j = 1 .

Now lets understand the left part

$$a_i + b_n \leq c_i \tag{3.15}$$

Again consider 3 subspaces P Q R spaned by { $|c_i >$, ...$|c_n>$} , {$|a_1 >$, ...$|a_i >$} and {$|bi_j >$, ...$|b_n >$}.The dimention of each subspace is n-i+1 , i , n-i+1.

The dimension of these subspaces are

$$n - i + 1 + i + n - i + 1 = 2n + 2 - i > n \tag{3.16}$$

So Intersection of these 3 subspaces is not empty. Let say X belong to all these spaces, Then

$$c_i \geq < X|C|X > = < X|A+B|X > = < X|A|X > + < X|B|X > \tag{3.17}$$

But we know

$$< X|A|X > \epsilon [a_i, a_1] \tag{3.18}$$

,

$$< X|B|X > \epsilon [b_n, b_i] \tag{3.19}$$

and

$$< X|C|X > \epsilon [c_n, c_i] \tag{3.20}$$

thus

$$c_i \geq < X|A|X > + < X|B|X > \geq a_i + b_n. \tag{3.21}$$

$$c_i \geq a_i + b_n \tag{3.22}$$

.

Thus The form more genaralised form of Weyls Ineqaulity.

$$a_i + b_n \leq c_i \leq a_i + b_1 \tag{3.23}$$

Eg:

$$c_1 \leq a_1 + b_1 \tag{3.24}$$

.

$$c_n \geq a_n + b_n \tag{3.25}$$

.

3.4 The Genaral Result

Consider Three Hermitian operatrs A, B and C actingn n dimentional linear vectr space Where C = A + B.Here we are intrested in understanding the realationship between Eigenvalue of Operator C with eigenvalues of A & B

.

Let $|a_1>$, $|a_2>$, ...$|a_n>, and |b_1>$, $|b_2>$, ...$|b_n>$and $|c_1>$, $|c_2>$, ...$|c_n>$ be the Eigen vectors and a_1,a_2, ...a_n, and b_1,b_2, ...$b_n$and c_1,c_2,......$c_n$be the Eigenvalues of A , B , C reespectively .

$$\hat{C}|c_k >= (A + B)|c_k > \tag{3.26}$$

$$< c_j|\hat{C}|c_k >=< c_j||A + B|c_k > \tag{3.27}$$

We can write

$$|c_k >= \sum_i < a_i|c_k > |a_i > And < c_j| = \sum_i < c_j|a_i >< a_i| \tag{3.28}$$

$$< c_j|\hat{C}|c_k >= \sum_i < c_j|a_i >< a_i|(A+B) \sum_l < a_l|c_k > |a_l > \tag{3.29}$$

$$= \sum_i \sum_l < c_j|a_i >< a_l|c_k >< a_i|(A+B)|a_l > \tag{3.30}$$

$$= \sum_i \sum_l < c_j|a_i >< a_l|c_k >< a_i|A|a_l > + \sum_i \sum_l < c_j|a_i >< a_l|c_k >< a_i|B|a_l > \tag{3.31}$$

$$= \sum_i \sum_l < c_j|a_i >< a_l|c_k > a_l \delta_l^i + + \sum_i \sum_l < c_j|a_i >< a_l|c_k >< a_i|B|a_l > \tag{3.32}$$

For k = j

$$c_k = \sum_l \| < c_j|a_l > \|^2 a_l + \sum_i \sum_l < c_k|a_i >< a_l|c_k >< a_i|B|a_l > \tag{3.33}$$

But note

$$< a_i|A, B|a_l >=< a_i|(AB+BA)|a_l > \tag{3.34}$$

$$=< a_i\lfloor(AB)\lfloor a_l > + < a_i\lfloor(BA)\lfloor a_l > \tag{3.35}$$

$$= a_i < a_i\lfloor B\lfloor a_l > +a_l < a_i\lfloor B\lfloor a_l > \tag{3.36}$$

Putting this result in above Eqation

$$c_k = \sum_l \| < c_k\lfloor a_l > \|^2 a_l + \sum_i \sum_l < c_k\lfloor a_i >< a_l\lfloor c_k >< a_i\lfloor A, B\lfloor a_l > \frac{1}{a_i + a_l}. \tag{3.37}$$

If we knw the anticmmutator f A & B say t then

$$c_k = \sum_l \| < c_k\lfloor a_l > \|^2 a_l + \sum_i \sum_l < c_k\lfloor a_i >< a_l\lfloor c_k >< a_i\lfloor t\lfloor a_l > \frac{1}{a_i + a_l} \tag{3.38}$$

.

$$c_k = \sum_l \| < c_k\lfloor a_l > \|^2 a_l + \sum_i \sum_l < c_k\lfloor a_i >< a_l\lfloor c_k >< a_i\lfloor a_l > \frac{t}{a_i + a_l} \tag{3.39}$$

$$.c_k = \sum_l \| < c_k\lfloor a_l > \|^2 a_l + \sum_l < c_k\lfloor a_l >< a_l\lfloor c_k > \frac{t}{a_i + a_l}\delta_l^i \tag{3.40}$$

.

$$c_k = \sum_l \| < c_k\lfloor a_l > \|^2 a_l + \sum_l \frac{t}{2a_l} \| < c_k\lfloor a_l > \|^2 \tag{3.41}$$

$$c_k = \sum_l \| < c_k \lfloor a_l > \|^2 (a_l + A, B\frac{1}{2a_l}) \qquad (3.42)$$