Spectral Sequences

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Certificate of Examination

This is to certify that the dissertation titled **"Spectral Sequences"** submitted by **Arul Ganesh S S** (Reg. No. MS10096) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: November 15, 2015

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Yashonidhi Pandey at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

Arul Ganesh S S
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Dated: November 15, 2015

In my capacity as the supervisor of the candidates' project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Yashonidhi Pandey
(Supervisor)

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Finally, I would like to acknowledge that the material presented in this thesis is based on other people's work. If I have made then it is the selection, presentation and some basic calculations of the materials from different sources which are listed in the bibliography.

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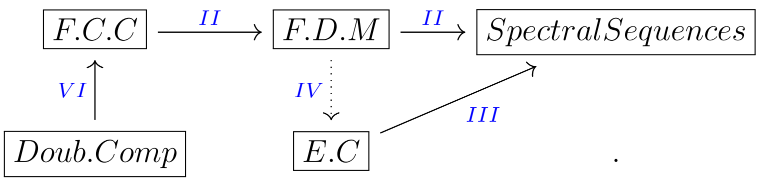
Contents

Introduction

Spectral sequences are a very powerful computational tool in Homological Algebra and Algebraic Topology. They package information about relations between homology groups. The aim of this exposition is to understand their construction and applications in certain contexts. In Chapter II we shall explicitly construct the Spectral Sequence associated with a filtered differential module. This discussion is based on Chapter XV of [2]. It has the advantage of being elementary and thus helping a novice get started. The Exact Couples of Massey, originally introduced in topology, form a broader source of Spectral Sequences. We discuss them in Chapter III, following Chapter VIII of [1]. Chapter IV explains how some of the familiar situations (Filtered differential module, Filtered Chain complex etc.) give rise to exact couples and thereby Spectral Sequences. We shall briefly discuss the question of convergence of spectral sequence in Chapter V. But an explicit discussion will be limited to spectral sequences associated with filtered chain complexes. Chapter VI discusses how double complexes give rise to two different spectral sequences. Then we discuss some applications of spectral sequences to give conceptual proofs of results proved by diagram-chasing in Homological Algebra. The sixth Chapter introduces the Grothendieck spectral sequence. The following is a schematic representation of, how the major topics of this exposition is organized between chapters [1]



One may read Chapter II independent of the rest.

We believe that from a practical point of view Exact Couples is the most efficient set-up for theoretical constructions of spectral sequences. So in various expositions, we have preferred to use them and total-degree in the construction instead of Filtered differential modules and complementary degree. We have used complementary degree when it shows up naturally such as in the Total complex of a bi-complex.

In the last few chapters, we give external applications of spectral sequences to Topology. We construct spectral sequences arising in non-abelian categories like that of groups, simplicial sets and topological spaces. To this end, in Chapter VIII, we give the most essential introduction to simplicial sets.

[1] F.C.C - Filtered Chain Complex; F.D.M - Filtered Differential Module;
E.C - Exact Couple; Doub. Comp - Double Complex.

The focus audience of this thesis is beginners in Homological Algebra. Those who are familiar with the subject may find this exposition rather lengthy. We request them to read diagonally.

Chapter I

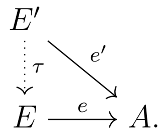
Some category theory

Here we state some definitions and results which will be used in this exposition. Nevertheless we shall assume familiarity with the definition of category and functor. For further clarifications one may refer to [1] and [2].
Let $\mathfrak{C}$ be a category. Let A, B in $\mathfrak{C}$ be any two objects. We shall denote the set of morphisms from A to B by $\mathfrak{C}(A, B)$.

Definition 1 (Equaliser). Let ϕ_1 and $\phi_2 : A \to B$ be two morphisms in $\mathfrak{C}$. An equaliser of ϕ_1 and ϕ_2 is a pair (e, E) of an object E and a morphism $e : E \to A$ such that:

(i) $\phi_1 e = \phi_2 e$,

(ii) e is universal with respect to the above property. *ie* If (e', E') is another pair of object and morphism satisfying (i). Then there exists a unique morphism $\tau : E' \to E$ such that the following diagram is commutative

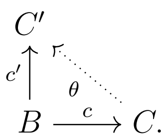


Similarly we define the dual notion, a *co-equaliser* of morphisms ϕ_1 and $\phi_2 : A \to B$.

Definition 2 (Co-equaliser). Let ϕ_1 and $\phi_2 : A \to B$ be two morphisms in $\mathfrak{C}$. Then the co-equaliser of ϕ_1 and ϕ_2 is a pair (c, C) of an object C and a morphism $c : B \to C$ such that:

(i) $c\phi_1 = c\phi_2$,

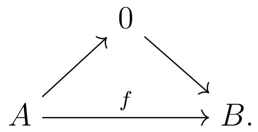
(ii) c is universal with respect to (i). In long hand this means, if (c', C') is another pair of object and morphism satisfying (i). Then there exists a unique morphism $\theta : C \to C'$ such that the following diagram is commutative



Definition 3 (Zero object)**.** Let A be an object in $\mathfrak{C}$. We say that A is a zero object if the sets $\mathfrak{C}(A, B)$ and $\mathfrak{C}(B, A)$ are singletons for any B in $\mathfrak{C}$.

Zero object, if exist will be unique.

Definition 4 (Zero morphism)**.** Let $f : A \to B$ be a morphism in $\mathfrak{C}$. We say f is a zero morphism if the following diagram is commutative



We will denote zero morphism by 0.

Suppose $\mathfrak{C}$ has zero object. Then we can define *kernel* and *co-kernel* of morphisms.

Definition 5 (Kernel)**.** Let $\phi : A \to B$ be a morphism. Let (i, K) be a pair of an object K and a morphism $i : K \to A$. We say (i, K) is the kernel of ϕ if, (i, K) is the equaliser of ϕ and 0.

Similarly *co-kernel*, (p, C) is defined as the co-equaliser of ϕ and 0.

Definition 6 (Monomorphism and Epimorphism)**.** We say a morphism $\mu : M \to A$ is a monomorphism if for every morphism $\alpha, \beta : B \to A$;

$$\mu\alpha = \mu\beta \Rightarrow \alpha = \beta.$$

Similarly $\epsilon : A \to C$ is an epimorphism if for all morphisms $\alpha, \beta : A \to B$;

$$\alpha\epsilon = \beta\epsilon \Rightarrow \alpha = \beta.$$

Definition 7 (Product)**.** Let X_i, $i \in I$, be a family of objects in the category $\mathfrak{C}$. Here I is some indexing set. Then a product $(X; p_i)$ of the objects X_i is an object X, together with:

(i) Morphisms $p_i : X \to X_i$, called projections,

(ii) Universal property: given any object Y and morphisms $f_i : Y \to X_i$, there exists a unique morphism $f = \{f_i\} : Y \to X$ with $p_i f = f_i$.

Definition 8 (Additive Category)**.** A category $\mathfrak{A}$ is said to be additive if the following are satisfied;

(i) It contains zero object,

(ii) Product exists for any finite collection of objects,

(iii) For any pair of objects $\mathfrak{A}(-,-)$ is an abelian group. Moreover given any three objects A, B, C, the composition

$$\mathfrak{A}(A,B) \times \mathfrak{A}(B,C) \to \mathfrak{A}(A,C)$$

is bilinear. That is if $f, g \in \mathfrak{A}(A,B)$ and $h, j \in \mathfrak{A}(B,C)$ then

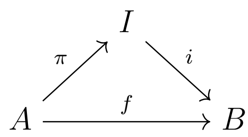
$$(f+g) \circ (h+j) = f \circ h + f \circ j + g \circ h + g \circ j$$

belongs to $\mathfrak{A}(A,C)$.

Definition 9 (Abelian Category)**.** An additive category is *abelian* if:

1. Every morphism has a kernel and a co-kernel.
2. Let $\mu : M \to A$ be any monomorphism. Let (c, C) be it's cokernel. Then the kernel of $c : A \to C$ is (μ, M). Similarly suppose $\epsilon : A \to C$ is an epimorphism such that (K, i) is it's kernel. Then the cokernel of $i : K \to A$ is (ϵ, C).

3. Any morphism $f : A \to B$ may be factored into an epimorphism $\pi : A \to I$ followed by a monomorphism $i : I \to B$.Thus we have the following commutative diagram in $\mathfrak{A}$



   where i is a monomorphism and π is an epimorphism.

Condition two is usually expressed as 'every monomorphism is the kernel of it's cokernel; every epimorphism is the cokernel of it's kernel'. Similarly condition three is often expressed as 'every morphism is expressible as the composition of an epimorphism followed by a monomorphism'.

Example 0.1. *Let R be a commutative ring with identity. Then the category of modules over R is an abelian category. We will use $R - \mathfrak{Mod}$ to denote category of R-modules. When R is $\mathbb{Z}$, the ring of integers, we have category of abelian groups. We shall denote it by $\mathfrak{Ab}$. The category of R-complexes denoted, $R - \mathfrak{Comp}$ is also an abelian category.*

Example 0.2. *The category of groups is not an abelian category.*

Example 0.3. *The category of Topological spaces denoted by $\mathfrak{Top}$ is not even additive. Similarly the category of pointed topological spaces ($\mathfrak{Top}_*$) is also not additive.*

We require the following proposition to define the notion of additive functor. For proof one may look at [1, II.9.5].

Proposition 1. Let $F : \mathfrak{A} \to \mathfrak{B}$ be a functor from an additive category $\mathfrak{A}$ to an additive category $\mathfrak{B}$. Then the following conditions are equivalent:

1. F preserves product (of two objects),
2. for each A, A' in $\mathfrak{A}$, $F : \mathfrak{A}(A, A') \to \mathfrak{B}(FA, FA')$ is a homomorphism.

Definition 10 (Additive Functor)**.** A functor which satisfies any of the conditions given by Proposition 1 is said to be an additive functor.

Example 0.4. *Let I be a fixed R-module and M an arbitrary R-module. Let $Hom(I, M) : \mathfrak{R} - \mathfrak{mod} \to \mathfrak{Ab}$ be a functor which associates M to the group of homomorphism from I to M. This is an additive functor.*

Example 0.5. *Let $\mathfrak{S}$ be the category of sets. Forgetful functor $F : \mathfrak{R} - \mathfrak{mod} \to \mathfrak{S}$ which associates every module M to it's underlying set is non-additive.*

Example 0.6. *The fundamental group functor $\pi_1 : \mathfrak{Top}_* \to \mathfrak{G}$ is also not additive.*

Proposition 2. Let $\mathfrak{A}$, $\mathfrak{B}$ be abelian categories. Let $F : \mathfrak{A} \to \mathfrak{B}$ be an additive functor. Let

$$0 \to K \xrightarrow{k} A \xrightarrow{p} B \to 0$$

be a split exact sequence in $\mathfrak{A}$. Then

$$0 \to FK \xrightarrow{F(k)} FA \xrightarrow{F(p)} FB \to 0$$

is also a split exact sequence.

Proof. By definition we have maps $\tau : A \to K$ and $\theta : B \to A$ such that

$$\tau \circ k = Id_K$$

and

$$p \circ \theta = Id_B.$$

Moreover $A \cong K \oplus B$. Let us apply F to the split exact sequence. Then we have the following

$$0 \to FK \underset{\mathrm{F}(\tau)}{\overset{\mathrm{F(k)}}{\rightleftarrows}} FA \underset{\mathrm{F}(\theta)}{\overset{\mathrm{F(p)}}{\rightleftarrows}} FB \to 0$$

in $\mathfrak{B}$. Since additive functor preserves finite sum

$$FA \cong FK \oplus FB.$$

Moreover

$$F(\tau) \circ F(k) = F(\tau \circ k) = F(Id_K) = Id_{FK}.$$

Similarly

$$F(p) \circ F(\theta) = Id_{FB}.$$

Thus the split exactness is preserved. □

Definition 11 (Differential Object)**.** Let $\mathfrak{A}$ be an abelian category. A *differential object* in $\mathfrak{A}$ is pair (A, d) consisting of an object A and an endomorphism $d : A \to A$ such that $d^2 = 0$.

We may construct a category of differential object in $\mathfrak{A}$ denoted $(\mathfrak{A}, d)$ as follows. The objects in $(\mathfrak{A}, d)$ are precisely the differential objects (A, d) in $\mathfrak{A}$. Let (A, d) and (A', d') be two differential objects in $\mathfrak{A}$. Then a morphism $f : (A, d) \to (A', d')$ is a morphism in $\mathfrak{A}(A, A')$ such that the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d'} \\ A & \xrightarrow{f} & A'. \end{array}$$

It can be shown that $(\mathfrak{A}, d)$ is an abelian category. Moreover to every object (A, d) we may associate the homology object, namely $H(A, d) = ker(d)/im(d)$. One may show that homology is an additive functor but we do not intend to elaborate on this or provide justifications here.

Definition 12 (Covariant δ Functor)**.** Let $\mathfrak{A}$ be an abelian Category and $\mathfrak{C}$ be an additive Category. Let a and b be two integers (which can be equal to $\pm\infty$) such that $a + 1 < b$. A covariant δ-functor from $\mathfrak{A}$ to $\mathfrak{C}$ in degrees $a < i < b$, is a system $T = (T^i)$ of additive covariant functors from $\mathfrak{A}$ to $\mathfrak{C}$, $(a < i < b)$, such that the following properties hold:

1. For any i such that $a < i < b - 1$ and for any exact sequence $0 \to A' \to A \to A'' \to 0$, there exist a morphism

$$\delta : T^i(A'') \to T^{i+1}(A').$$

   We call this morphism the "boundary" or "connecting" homomorphism.

2. If we have a second exact sequence $0 \to B' \to B \to B'' \to 0$ and a morphism from the first exact sequence to the second, then the corresponding diagram

$$\begin{array}{ccc} T^i(A'') & \xrightarrow{\delta} & T^{i+1}(A') \\ \downarrow & & \downarrow \\ T^i(B'') & \xrightarrow{\delta} & T^{i+1}(B') \end{array}$$

   commutes.

3. For any short exact sequence $0 \to A' \to A \to A'' \to 0$, the associated sequence of morphisms

$$\cdots \to T^i(A') \to T^i(A) \to T^i(A'') \to T^{i+1}(A') \to \cdots$$

   forms a complex. That is the composition of two consecutive morphism is zero.

Similarly one define a covariant δ^*-functor from $\mathfrak{A}$ to $\mathfrak{C}$. The only difference is that δ^*-functor decreases the degree by one. Homology functors $\{H_n\}$ from the Category of chain-complexes over a commutative ring R is clearly covariant δ^*-functor. Similarly Homology functors $\{H^n\}$ from the Category of co-chain-complexes over a commutative ring R is clearly covariant δ^*-functor

Definition 13 (Adjoint of a functor)**.** Let $\mathfrak{C}$ and $\mathfrak{D}$ be two categories. Let

$$F : \mathfrak{D} \to \mathfrak{C}$$

be a functor. Let $A \in \mathfrak{C}$ and $B \in \mathfrak{D}$ be two arbitrary objects. We say a functor

$$G : \mathfrak{C} \to \mathfrak{D}$$

is right adjoint to F if

$$\mathfrak{C}(F(A), B) \simeq \mathfrak{D}(A, G(B)),$$

as sets. In this context we may also say that F is left adjoint to G. We denote "G is right adjoint to F" by

$$F \dashv G.$$

Lemma 0.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be additive categories. Consider the functor*

$$F : \mathfrak{C} \to \mathfrak{D}.$$

Let

$$G : \mathfrak{D} \to \mathfrak{C}$$

be it's right adjoint. If F preserves monomorphisms then G preserves injectives.

Proof. Let I be an injective object in $\mathfrak{D}$. Consider the following diagram in $\mathfrak{C}$

$$\begin{array}{ccc} GI & & \\ {\scriptstyle\phi}\uparrow & & \\ A & \xrightarrow{\mu} & B \end{array}$$

where μ is a monomorphism. Now applying F to the above diagram we shall obtain

$$\begin{array}{ccc} I & & \\ {\scriptstyle F\phi}\uparrow & & \\ FA & \xrightarrow{F\mu} & FB \end{array}$$

in $\mathfrak{D}$. By our hypothesis $F\mu$ is a monomorphism. Since I is injective in $\mathfrak{C}$, so there exists

$$\psi : FB \to I$$

such that $\psi(F\mu) = F\phi$. Now since G adjoint to F this means that there exist

$$\psi^* : B \to FI$$

such that following diagram is commutative

$$\begin{array}{ccc} GI & & \\ {\scriptstyle\phi}\uparrow & \nwarrow {\scriptstyle\psi^*} & \\ A & \xrightarrow{\mu} & B. \end{array}$$

$\square$

Example 0.7. *Forgetful functor from category of groups to category of sets,*

$$F : R : \mathfrak{G} \to \mathfrak{S}$$

forgets the group structure on a group and just remembers the underlying set. It admits a left adjoint, namely the functor

$$L : \mathfrak{S} \to \mathfrak{G}$$

which sends each set to the free group over it.

A *constant functor* $D(d) : \mathfrak{C} \to \mathfrak{D}$ is a functor that maps each object of the category $\mathfrak{C}$ to a fixed object $d \in \mathfrak{D}$ and each morphism of $\mathfrak{C}$ to the identity morphism of that fixed object.

Let $\mathfrak{I}$ fixed small category and $\mathfrak{C}$ be any category. Consider the functor category $\mathfrak{C}^{\mathfrak{I}}$, whose objects are co-variant functors from $\mathfrak{I}$ to $\mathfrak{C}$ and morphisms are natural transformations between the functors.

The diagonal functor $D\colon \mathfrak{C} \to \mathfrak{C}^{\mathfrak{I}}$ is that functor which sends each object $c \in \mathfrak{C}$ to the constant functor $D(c)$, and each morphism $f\colon c \to c'$ of $\mathfrak{C}$ to the natural transformation $Df\colon Dc{\to}Dc'$.

Definition 14 (Limit and Co-limit)**.** Let $\mathfrak{I}$ fixed small category and $\mathfrak{C}$ be any category. Consider the functor category $\mathfrak{C}^{\mathfrak{I}}$. Let D be the diagonal functor from $\mathfrak{C} \to \mathfrak{C}^{\mathfrak{I}}$,

$$D : \mathfrak{C} \to \mathfrak{C}^{\mathfrak{I}}.$$

We have $(DA)_i = A$ for all i.

We define the co-limit in $\mathfrak{C}$ to be the left adjoint to D. We shall denote this functor by *colim*. Similarly we define the limit in the category $\mathfrak{C}$ to be the right adjoint to D. We shall denote it by *lim*.

Pullback in a category is a special case of limit. Similarly *Pushout* is a special case of co-limit.

Chapter II

Filtered Differential Modules

In this chapter we explicitly construct (and define) the spectral sequence associated with a Filtered Differential Module. We shall then treat the case of Filtered Chain Complexes as Filtered Differential Modules together with the additional detail of *complementary degree* q. The proofs are straightforward generalizations. Indeed, we have tried to highlight this claim by reproducing the arguments highlighting the complementary degree $\mathbf{q}$.

One may construct Spectral Sequences with increasing or decreasing filtrations. We choose to work with increasing filtrations. Similarly the case of filtered co-chain complex will only be briefly discussed. Our treatment here does not assume a background in Homological Algebra.

1 Preliminaries

Let R denote a commutative ring with identity. Let C_* be a *chain-complex* of R-modules. Recall from homological algebra the functor

$$H_n : \mathfrak{R} - \mathfrak{comp} \to \mathfrak{R} - \mathfrak{mod}$$

which associate to every complex C_* the n^{th} homology module. Recall that if

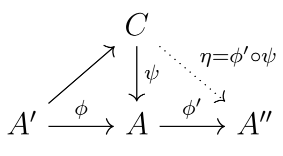
$$0 \to C_* \to D_* \to E_* \to 0$$

is a short exact sequence of R-complexes, then we have the following long exact sequence of R-Modules

$$\to H_n(C_*) \to H_n(D_*) \to H_n(E_*) \to H_{n-1}(C_*) \to .$$

The following lemma shall play a crucial role in our discussion

Lemma 1.1. *Let A, A', A'' be R-modules. Suppose they fit as in the following commutative diagram*



such that the bottom row is exact. Then the canonical morphism

$$\frac{Im(\psi)}{Im(\phi)} \to Im(\eta)$$

is an isomorphism.

Proof. The proof follows from the following observation. Let A,B,C be R-modules and $f : A \to B$, $g : B \to C$ be R-linear maps such that $ker(g) \subset Im(f)$. We have a natural morphism from $Im(f)$ to $im(g \circ f)$ which factors through $\frac{Im(f)}{Ker(g)}$. Then by the so-called first isomorphism theorem in group theory, we have $Im(g \circ f) \cong \frac{Im(f)}{Ker(g)}$. □

2 Filtration and Associated Graded

In this section we define the notion of filtration of a module and it's associated graded. We will also see when and how in the case of differential modules a filtration at the level of module induces a filtration at the level of homology.

Definition 2.1. *By a filtration F of a module M we mean a family of sub-modules $\{F_p\}_{p\in\mathbb{Z}}$ of M with the following properties:*

(i) $...F_{p-1} \subset F_p...,$

(ii) $\bigcup F_p = M.$

We call the above filtration the *increasing* filtration of M. If $F^p \subset F^{p-1}$, we call the filtration a *decreasing* filtration.

Definition 2.2. *An* ***associated graded module*** *of a Module M with respect to a filtration F is $\bigoplus_p F_p/F_{p-1}$. We shall denote this associated graded module by*

$$E^0_p(M). \tag{2.1}$$

Definition 2.3. *A R-linear map d from a R-module M to itself is called a* ***differentiation*** *if $d \circ d = 0$. We call a module equipped with a differentiation as a differential module.*

Definition 2.4. *Let M be a differential module and F be a filtration of M. We say F is* ***compatible*** *with respect to d if*

$$d(F_p) \subset F_p; \qquad \forall p.$$

We shall call such objects Filtered Differential modules.

We now make some simple but important constructions of complexes associated with any Filtered Differential Module (M, d, F):

1. Given a differential module (M, d) we may form a complex

$$\cdots \xrightarrow{d} M \xrightarrow{d} M \xrightarrow{d} M \xrightarrow{d} \cdots \ .$$

   We shall denote it by $(M, d)_*$.

2. Notice that restriction of d on F_p is a differentiation for each p. By abuse of notation we shall denote $d|_{F_p}$ also by d. Thus for a fixed p

$$\cdots \xrightarrow{d} F_p \xrightarrow{d} F_p \xrightarrow{d} F_p \xrightarrow{d} \cdots$$

   forms a complex. We will denote it by $(F_p, d)_*$.

3. Let F_p/F_{p-r}, where $r \geq 1$ be an arbitrary quotient. Let $\bar{d}$ denote the differentiation induced by d. Then,

$$\cdots \xrightarrow{\bar{d}} F_p/F_{p-r} \xrightarrow{\bar{d}} F_p/F_{p-r} \xrightarrow{\bar{d}} \cdots$$

   also forms a R-complex. We shall denote it by $(F_p/F_{p-r}, \bar{d})_*$.

Thus when we talk of homology of M, F_p and F_p/F_{p-r}, we mean the homology of any term of the corresponding complex.

2.1 Filtration at the level of Homology

The inclusion of F_p into M induces a map at the level of homology. We shall denote

$$Im(H(F_p) \to H(M))$$

by $F_p(H)$. Notice that $F_p(H)$ is an increasing filtration of $H(M)$. Therefore we may view $H(M)$ as a filtered module.

Proposition 2.1. *The sequence of modules $\{F_p(H)\}_{p\in\mathbb{Z}}$ is a filtration of $H(M)$.*

We will denote this filtration by $F(H)$.

Proof. Clearly $\bigcup F_p(H) \subset H(M)$. Let us now show the reverse inclusion, $\bigcup F_p(H) \supset H(M)$. Let $[z]$ be an element of $H(M)$. Then we have $z \in M$ such that
(i) $d(z) = 0$ and
(ii) $[z]$ is the homology class of z.
Plainly $z \in F_p$ for some p. Under inclusion $z \in F_p$ is mapped to z in M. Hence $[z]$ in $H(M)$ is the image of $[z]$ in $H(F_p)$ under the induced map. So $H(M) \subset \bigcup F_p(H)$.

Now to check that the sequence of modules is increasing, let us consider the following commutative diagram of modules

$$\begin{array}{ccc} & F_{p-r} & \\ \swarrow & & \searrow \\ F_p & \longrightarrow & M. \end{array}$$

Now we shall erect the respective complexes over each module to obtain the following commutative diagram of complexes

$$\begin{array}{ccc} & (F_{p-r}, d)_* & \\ \swarrow & & \searrow \\ (F_p, d)_* & \longrightarrow & (M, d)_*. \end{array}$$

Apply homology functor to the above diagram. Now we have the following commutative diagram at the level of homology

$$\begin{array}{ccc} & H(F_{p-r}) & \\ \swarrow & & \searrow \\ H(F_p) & \longrightarrow & H(M). \end{array}$$

From the diagram it follows that $F_{p-r}(H) \subset F_p(H), \forall r \geqslant 1$. $\square$

Let us denote the p^{th} piece $F_p(H)/F_{p-1}(H)$ by $E_p(H)$. Then the associated graded module of the homology module equipped with filtration is given by

$$\bigoplus_p E_p(H).$$

We shall refer to this as *the Graded of the Homology.*

3 Spectral Sequence associated with a Filtered Differential Module

In this section we construct the Spectral Sequence associated with a Filtered Differential Module. In the process we will see that the graded of homology namely

$$\bigoplus_p E_p(H) = \bigoplus_p F_p(H)/F_{p-1}(H)$$

can be obtained as a sub-quotient of homology of graded,

$$\oplus_p H(F_p/F_{p-1}).$$

Spectral sequences provides us with this passage from homology of graded to graded of homology. We begin by giving two explicit descriptions of $E_p(H)$ which is the p^{th} piece of graded of homology.

First description of $E_p(H)$: Consider the following diagram

$$\begin{array}{ccccc} & & H(F_p) & & \\ & \nearrow & \downarrow & \searrow\cdots & \\ H(F_{p-1}) & \longrightarrow & H(M) & \longrightarrow & H(M/F_{p-1}). \end{array} \tag{3.1}$$

Observe that the diagram is commutative and the row is exact. Now by definition

$$\begin{aligned} E_p(H) &= F_p(H)/F_{p-1}(H) \\ &= Im(H(F_p) \to H(M))/Im(H(F_{p-1}) \to H(M)). \end{aligned}$$

Applying Lemma 1.1 to the diagram (3.1) we see that

$$E_p(H) \cong Im(H(F_p) \to H(M/F_{p-1})). \tag{3.2}$$

An alternate description of $H(F_p) \to H(M/F_{p-1})$. Let us apply homology functor to the following commutative diagram of complexes with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (F_{p-1}, d)_* & \longrightarrow & (F_p, d)_* & \longrightarrow & (F_p/F_{p-1}, \bar{d})_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (F_{p-1}, d)_* & \longrightarrow & (M, d)_* & \longrightarrow & (M/F_{p-1}, \bar{d})_* & \longrightarrow & 0. \end{array}$$

Now we obtain the following commutative diagram

$$\begin{array}{ccc} H(F_p) & \longrightarrow & H(F_p/F_{p-1}) \\ \downarrow & \searrow\cdots & \downarrow \\ H(M) & \longrightarrow & H(M/F_{p-1}). \end{array} \tag{3.3}$$

Notice that the map $H(F_p) \to H(M/F_{p-1})$ is given by the composition of the left vertical arrow followed by the horizontal. Here below we make use of the up horizontal followed by the vertical to obtain the second description.

Second description of $E_p(H)$: From here on we shall adopt the following convention: the various complexes $(C, d)_*$ will be abbreviated to just C. Consider the following commutative diagram of complexes with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_p & \longrightarrow & M & \longrightarrow & M/F_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & F_p/F_{p-1} & \longrightarrow & M/F_{p-1} & \longrightarrow & M/F_p & \longrightarrow & 0. \end{array} \tag{3.4}$$

From (3.4) we may obtain the following commutative diagram

$$\begin{array}{ccccc} H(M/F_p) & \xrightarrow{\delta} & H(F_p) & & \\ \downarrow{\scriptstyle id} & & \downarrow & & \\ H(M/F_p) & \xrightarrow{\delta} & H(F_p/F_{p-1}) & \longrightarrow & H(M/F_{p-1}). \end{array} \tag{3.5}$$

Thus we have the following commutative diagram with exact row

$$\begin{array}{ccccc} & \overset{\delta\circ id^{-1}}{\nearrow} & H(F_p) & \cdots\searrow & \\ & & \downarrow{\scriptstyle \delta} & & \\ H(M/F_p) & \longrightarrow & H(F_p/F_{p-1}) & \longrightarrow & H(M/F_{p-1}). \end{array}$$

Here the dotted arrow is simply given by the composition of solid vertical followed by solid horizontal. Applying Lemma 1.1 to above diagram we obtain,

$$Im(H(F_p) \to H(M/F_{p-1})) \cong \frac{Im(H(F_p) \to H(F_p/F_{p-1}))}{Im(H(M/F_p) \to H(F_p/F_{p-1}))}. \tag{3.6}$$

To summarize, by definition

$$E_p(H) = Im(H(F_p) \to H(M))/Im(H(F_{p-1}) \to H(M)).$$

By (3.2) it is isomorphic to

$$Im(H(F_p) \to H(M/F_{p-1})).$$

On the other hand by (3.3) and (3.6) it isomorphic also to

$$\frac{Im(H(F_p) \to H(F_p/F_{p-1}))}{Im(H(M/F_p) \to H(F_p/F_{p-1}))}.$$

With it's new description $E_p(H)$ is clearly a sub-quotient of $H(F_p/F_{p-1})$, that is quotient of sub-objects of $H(F_p/F_{p-1})$. If you recall, $H(F_p/F_{p-1})$ is homology of the (p^{th} piece of) associated graded. Thus we have obtained ***the graded of homology as a sub-quotient of homology of graded!***.

Spectral sequences is a machine grinding these ideas to construct $Gr \circ H(M)$ from $H \circ Gr(M)$. To carry out this grinding let us give a name to key objects that have appeared in the two constructions,

$$Z_p^\infty(M) := Im(H(F_p) \to H(F_p/F_{p-1})), \tag{3.7}$$
$$B_p^\infty(M) := Im(H(M/F_p) \to H(F_p/F_{p-1})), \tag{3.8}$$
$$E_p^\infty(M) := Z_p^\infty/B_p^\infty. \tag{3.9}$$

Notice

$$E_p(H) \cong E_p^\infty(M).$$

The Construction: Consider the following short exact sequence of complexes

$$0 \to F_{p-1}/F_{p-r} \to F_p/F_{p-r} \to F_p/F_{p-1} \to 0. \tag{3.10}$$

Now generalizing Z_p^∞, we define

$$Z_p^r(M) := Im(H(F_p/F_{p-r}) \to H(F_p/F_{p-1})); \qquad r \geq 1. \tag{3.11}$$

Consider the following short exact sequence of complexes

$$0 \to F_p/F_{p-1} \to F_{p+r-1}/F_{p-1} \to F_{p+r-1}/F_p \to 0. \tag{3.12}$$

Then at the level of homology we have a connecting homomorphism

$$H(F_{p+r-1}/F_p) \xrightarrow{\delta} H(F_p/F_{p-1}). \tag{3.13}$$

Now generalizing B_p^∞ we define

$$B_p^r(M) := Im(H(F_{p+r-1}/F_p) \xrightarrow{\delta} H(F_p/F_{p-1})); \qquad r \geq 1. \tag{3.14}$$

Now we make a few simple observations. Setting $r = \infty$, the equations (3.11) and (3.14) reduce to (3.7) and (3.8) are respectively. Setting $r = 1$, (3.11) becomes the identity morphism on $H(F_p/F_{p-1})$ and (3.14) is zero. Therefore $Z_p^1(M)/B_p^1(M)$ is isomorphic to $H(F_p/F_{p-1})$.

Proposition 3.1. *Let the objects be defined as above, then we have the following increasing sequence of objects*

$$...B_p^r \subset B_p^{r+1} \subset ... \subset B_p^\infty \subset Z_p^\infty \subset ... \subset Z_p^{r+1} \subset Z_p^r \subset ....$$

Proof. Proof shall be divided into five steps:
i)$B_p^r \subset B_p^{r+1}$, ii)$B_p^r \subset B_p^\infty$, iii)$B_p^\infty \subset Z_p^\infty$, iv)$Z_p^\infty \subset Z_p^r$, v)$Z_p^{r+1} \subset Z_p^r$. In view of discussion thus far, we need not prove (iii). We will prove (i) now. Consider the following commutative diagram of complexes with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (F_p/F_{p-1}) & \longrightarrow & (F_{p+r-1}/F_{p-1}) & \longrightarrow & (F_{p+r-1}/F_p) & \longrightarrow & 0 \\
 & & \downarrow id & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (F_p/F_{p-1}) & \longrightarrow & (F_{p+r}/F_{p-1}) & \longrightarrow & (F_{p+r}/F_p) & \longrightarrow & 0.
\end{array}$$

We may apply homology functor to the above and obtain the following commutative diagram

$$\begin{array}{ccc} H(F_{p+r-1}/F_p) & \longrightarrow & H(F_p/F_{p-1}) \\ \downarrow & & \downarrow{\scriptstyle id} \\ H(F_{p+r}/F_p) & \longrightarrow & H(F_p/F_{p-1}). \end{array}$$

Notice, that the horizontal arrows are given by connecting homomorphisms. Given the diagram above we clearly have $B_r^p \subset B_{r+1}^p$. Similar arguments can be constructed to prove other inclusions. □

In view of Proposition 3.1 we define

$$E_p^r = Z_p^r/B_p^r. \tag{3.15}$$

Notice that E_p^r is defined in such a way that $E_p^1 = H(F_p/F_{p-1})$, the homology of graded.

We now let r run over values between 1 and ∞ in the equations (3.10) and (3.12). Spectral sequences is an organizational principle that records the various relations that emerge between E_p^r. Now we make these remarks precise. Consider the commutative diagram of complexes with exact rows given below,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{p-r}/F_{p-r-1} & \longrightarrow & F_p/F_{p-r-1} & \longrightarrow & F_p/F_{p-r} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow & & \\ 0 & \longrightarrow & F_{p-1}/F_{p-r-1} & \longrightarrow & F_p/F_{p-r-1} & \longrightarrow & F_p/F_{p-1} & \longrightarrow & 0. \end{array} \tag{3.16}$$

Taking homology, we get

$$\begin{array}{ccccc} H(F_p/F_{p-r-1}) & \longrightarrow & H(F_p/F_{p-r}) & \xrightarrow{\delta} & H(F_{p-r}/F_{p-r-1}) \\ \downarrow{\scriptstyle Id} & \nearrow & \downarrow & & \downarrow \\ H(F_p/F_{p-r-1}) & \longrightarrow & H(F_p/F_{p-1}) & \xrightarrow{\delta} & H(F_{p-1}/F_{p-r-1}) \end{array} \tag{3.17}$$

where the dotted arrow is obtained by taking inverse of the identity morphism. Solidifying the dotted morphism, we get the following diagram which satisfies the conditions required by Lemma 1.1 where the dotted arrow is taken to be the composition of vertical followed by the horizontal and the bottom exact row comes from the bottom exact row of diagram (3.17)

$$\begin{array}{ccccc} & & H(F_p/F_{p-r}) & & \\ & \nearrow & \downarrow & \searrow & \\ H(F_p/F_{p-r-1}) & \longrightarrow & H(F_p/F_{p-1}) & \longrightarrow & H(F_{p-1}/F_{p-r-1}). \end{array} \tag{3.18}$$

Now consider the following commutative diagram of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_{p-r}/F_{p-r-1} & \longrightarrow & F_{p-1}/F_{p-r-1} & \longrightarrow & F_{p-1}/F_{p-r} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle id} & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & F_{p-r}/F_{p-r-1} & \longrightarrow & F_{p}/F_{p-r-1} & \longrightarrow & F_{p}/F_{p-r} & \longrightarrow & 0.
\end{array}$$

Taking homology we get

$$\begin{array}{ccccc}
H(F_{p-1}/F_{p-r}) & \xrightarrow{\delta} & H(F_{p-r}/F_{p-r-1}) & \longrightarrow & H(F_{p-1}/F_{p-r-1}) \\
\Big\downarrow & {\scriptstyle Id^{-1}\circ\delta} \nearrow & \Big\downarrow{\scriptstyle id} & & \Big\downarrow \\
H(F_p/F_{p-r}) & \xrightarrow{\delta} & H(F_{p-r}/F_{p-r-1}) & \longrightarrow & H(F_p/F_{p-r-1}).
\end{array} \tag{3.19}$$

Solidifying the dotted arrow, we get the diagram

$$\begin{array}{ccccc}
H(F_{p-1}/F_{p-r}) & \xrightarrow{\delta} & H(F_{p-r}/F_{p-r-1}) & \longrightarrow & H(F_{p-1}/F_{p-r-1}) \\
\Big\downarrow & \nearrow & & \nearrow & \\
H(F_p/F_{p-r}) & & & &
\end{array} \tag{3.20}$$

where the dotted arrow is given simply by the composing the North-East morphism followed by horizontal. Further, the top exact row comes from the top exact row of diagram (3.19). We are in a position to apply Lemma 1.1 to (3.20). Following proposition plays a crucial role in the construction of spectral sequences.

Proposition 3.2. *Both the quotients* Z_p^r/Z_p^{r+1} *and* B_{p-r}^{r+1}/B_{p-r}^r *are canonically isomorphic to*

$$Im(H(F_p/F_{p-r}) \to H(F_{p-1}/F_{p-r-1})).$$

Hence we have $Z_p^r/Z_p^{r+1} \cong B_{p-r}^{r+1}/B_{p-r}^r$.

Proof. Recall by definition,

$$Z_p^r = Im(H(F_p/F_{p-r}) \to H(F_p/F_{p-1})).$$

So

$$Z_p^r/Z_p^{r+1} = \frac{Im(H(F_p/F_{p-r}) \to H(F_p/F_{p-1}))}{Im(H(F_p/F_{p-r-1}) \to H(F_p/F_{p-1}))}.$$

Now in view of (3.18) and Lemma 1.1 we have

$$Z_p^r/Z_p^{r+1} \cong Im(H(F_p/F_{p-r}) \to H(F_{p-1}/F_{p-r-1})).$$

Similarly by definition

$$B_p^r = Im(H(F_{p+r-1}/F_p) \to H(F_p/F_{p-1}))$$

so,

$$B^{r+1}_{p-r}/B^r_{p-r} = \frac{Im(H(F_p/F_{p-r}) \to H(F_{p-r}/F_{p-r-1}))}{Im(H(F_{p-1}/F_{p-r} \to H(F_{p-r}/F_{p-r-1}))}.$$

Now by (3.20) and Lemma 1.1 we have

$$B^{r+1}_{p-r}/B^r_{p-r} \cong Im(H(F_p/F_{p-r}) \to H(F_{p-1}/F_{p-r-1})).$$

It remains to check that the two morphisms

$$Im(H(F_p/F_{p-r}) \to H(F_{p-1}/F_{p-r-1}))$$

corresponding to diagrams (3.18) and (3.20) are the same. From the diagram (3.17), it follows that the following diagram commutes

$$\begin{array}{ccc} H(F_p/F_{p-r}) & \xrightarrow{\delta} & H(F_{p-r}/F_{p-r-1}) \\ \downarrow & \searrow & \downarrow \\ H(F_p/F_{p-1}) & \xrightarrow{\delta} & H(F_{p-1}/F_{p-r-1}). \end{array} \tag{3.21}$$

We see that the dotted arrows of both diagrams (3.18) and (3.20) are equal because they are equal to the dotted arrow of the diagram above.

□

Proposition 3.3. *Let $\epsilon : Z^r_p/B^r_p \to Z^r_p/Z^{r+1}_p$ denote canonical epimorphism. Let $\mu : B^{r+1}_{p-r}/B^r_{p-r} \to Z^r_{p-r}/B^r_{p-r}$ denote the canonical monomorphism. Define $d^r_p \colon E^r_p \to E^r_{p-r}$ as the composition of morphisms*

$$E^r_p = Z^r_p/B^r_p \xrightarrow{\epsilon} Z^r_p/Z^{r+1}_p \cong B^{r+1}_{p-r}/B^r_{p-r} \xrightarrow{\mu} Z^r_{p-r}/B^r_{p-r} = E^r_{p-r}.$$

Let E^r denote $\oplus_p E^r_p$. Now $d^r = \{d^r_p\}_p$ is a differentiation of degree $-r$ on E^r.

Proof. It is enough to show that $Im(d^r_p) \subset ker(d^r_{p-r})$. We have

$$Im(d^r_p) = B^{r+1}_{p-r}/B^r_{p-r}$$

and

$$ker(d^r_{p-r}) = Z^{r+1}_{p-r}/B^r_{p-r}.$$

Clearly $Im(d^r_p)$ is a subset of $ker(d^r_{p-r})$. Since the choice of r, so this holds for all r. □

Remark 1. *Notice that the information about degree of d^r is actually captured in the isomorphism given by Proposition 3.2.*

Now we compute the homology of E^r with respect to the differentiation d^r.

$$H(E^r_p) = ker(d^r_p)/im(d^r_{p+r}) = (Z^{r+1}_p/B^r_p)/(B^{r+1}_p/B^r_p) = Z^{r+1}_p/B^{r+1}_p = E^{r+1}_p.$$

Our discussion so far shall be summarized in the following theorem.

Theorem 3.4. *For each $r \geq 1$, d^r is a differentiation of degree $-r$ on E^r. Moreover E^{r+1} is isomorphic to the homology of E^r with respect to d^r.*

Now we are in a position to define spectral sequence associated with a filtered(increasing) differential module.

Definition 3.1. *By a spectral sequence of a differential module corresponding to a compatible filtration, we mean the sequence of graded modules*

$$E^2, E^3, E^4, ...$$

and differentiations $d^2, d^3, d^4...$ satisfying the relation $H(E^r, d^r) = E^{r+1}$.

3.1 Decreasing Filtration

In Definition 2.1 set $p = -p$. Then

$$M \supset ... \supset F^p \supset F^{p+1} \supset ...$$

is a decreasing filtration of M. To indicate the difference we opt for a slightly different notation. We shall raise the index p and lower the index r. Now if we carry out similar construction as in Section 3, we would obtain:

$$Z_r^p := Im(H(F^p/F^{p+r}) \to H(F^p/F^{p+1})), \tag{3.22}$$
$$B_r^p := Im(H(F^{p-r+1}/F^p) \to H(F^p/F^{p+1})), \tag{3.23}$$
$$E_r^p := Z_r^p/B_r^p; \qquad r \geq 1. \tag{3.24}$$

Notice that Z_∞^p and B_∞^p shall remain the same. That is

$$Z_\infty^p = Z_p^\infty \quad \text{and} \quad B_\infty^p = Z_p^\infty.$$

Proposition 3.3 can be now rewritten as follows.

Proposition 3.5. *Let E_r be $\oplus_p E_r^p$. Define $d_r^p \colon E_r^p \to E_r^{p+r}$ as composition of the maps in the following diagram*

$$E_r^p = Z_r^p/B_r^p \xrightarrow{\epsilon} Z_r^p/Z_{r+1}^p \cong B_{r+1}^{p+r}/B_r^{p+r} \xrightarrow{\mu} Z_r^{p+r}/B_r^{p+r} = E_r^{p+r}.$$

Now $d_r = \{d_r^p\}_p$ is a differentiation of degree r on E_r.

Similarly Theorem 3.4 in the context of decreasing filtration will read as:

Theorem 3.6. *For each $r \geq 1$, d_r is a differentiation of degree r on E_r. Moreover E_{r+1} is isomorphic to the homology of E_r with respect to d_r.*

3.2 Cartan-Eilenberg system

The reader may check that the following axiomatization known by the name of *Cartan-Eilenberg system* is a straightforward abstraction of our discussion thus far. A Cartan-Eilenberg system consists of a module $H(p,q)$ for each pair of integers, $-\infty \leq p \leq q \leq \infty$ along with

1. a homomorphisms $\eta : H(p',p) \to H(p,q)$ whenever $p \leq p'$ and $q \leq q'$;

2. for $-\infty \leq p \leq q \leq r \leq \infty$, we have a connecting homomorphism $\delta : H(p,q) \to H(q,r)$;

3. the morphism $H(p,q) \to H(p,q)$ is identity;

4. if $p \leq p' \leq p''$ and $q \leq q' \leq q''$, then the following diagram commutes:

$$\begin{array}{ccc} H(p'',q'') & \longrightarrow & H(p,q) \\ & \searrow \quad \nearrow & \\ & H(p',q') & \end{array}$$

5. if $p \leq p'$, $q \leq q'$ and $r \leq r'$, then the following diagram commutes:

$$\begin{array}{ccc} H(p',q') & \longrightarrow & H(q',r') \\ \downarrow & & \downarrow \\ H(p,q) & \longrightarrow & H(q,r) \end{array}$$

6. for $-\infty \leq p \leq q \leq r \leq \infty$, the following sequence is exact:

$$\cdots \to H(q,r) \to H(p,r) \to H(p,q) \xrightarrow{\delta} H(q,r) \to \cdots$$

7. $H(-\infty,q)$ is the direct limit of the system

$$H(q,q) \to H(q-1,q) \to H(q-2,q) \to \cdots ...$$

With this definition we get a spectral sequence by letting

$$\begin{aligned} Z_r^p = & \quad im(H(p,p+r) \to H(p,p+1)) \\ B_r^p = & \quad im(H(p-r+1,p) \to H(p,p+1)) \\ E_r^p = & \quad Z_r^p / B_r^p. \end{aligned} \tag{3.25}$$

We see that a Filtered Differential Module (M,d,f) gives rise to a Cartan-Eilenberg system by setting

$$H(p,q) = H(F^p/F^q).$$

4 The Graded Case

Let $M = (\cdots \to M_p \to M_{p-1} \to \cdots)$ be a chain complex. In other words, M is a graded module equipped with a differentiation of degree -1. Let M_p denote the p^{th} graded piece of M. As before, we shall work with an increasing filtration

$$F_{p-1} \subset F_p \subset ... \subset M$$

of M. We shall suppose that the filtration on M and the chain-complex structure are compatible in the following sense: each module F_p in the filtration is the direct sum of modules $M_{p+q} \bigcap F_p$ i.e

$$F_p = \oplus_q M_{p+q} \bigcap F_p.$$

This compatibility condition has the advantage that the sub-module F_p of M with the induced differentiation becomes a chain-complex in its own right! Further, F_p/F_{p-r} will also become a chain-complex.

Here we call p the degree of filtration, q the complementary degree and $p+q$ the total degree. We will use bold face for the newly introduced index q. *Henceforth, in this section, we will work with the complementary degree.* The motivation for doing so comes from the case of double complexes where one wants to visualize $E^r_{*,*}$ as the r-th page or sheet over the double complex with $E^r_{p,q}$ lying over $B_{p,q}$.

The module $E^0_p(M)$ may be identified as the direct sum $\oplus_q E^0_{p,\mathbf{q}}(M)$. We introduce the following notations: the piece with total degree $p+q$ of the chain-complex F_p will be denoted as $F_{p,\mathbf{q}}$. It is given by the formula

$$F_{p,\mathbf{q}} := M_{p+\mathbf{q}} \cap F_p. \tag{4.1}$$

Thus the piece of total degree $p+q$ of F_{p-1} is given by $F_{p-1,\mathbf{q+1}} = M_{p+\mathbf{q}} \cap F_{p-1}$. The p-th piece of the graded object of M is by definition the chain-complex F_p/F_{p-1}. Its piece of total degree $p+q$ is given by

$$E^0_{p,\mathbf{q}}(M) := F_{p,\mathbf{q}}/F_{p-1,\mathbf{q+1}}. \tag{4.2}$$

In the graded case, the homologies of M also acquire a degree. We define the p-th filtered object of the homology $H_{p+q}(M)$ which has total degree $p+q$ as

$$F_p H_{p+\mathbf{q}} = Im(H_{p+\mathbf{q}}(F_p) \to H_{p+\mathbf{q}}(M)). \tag{4.3}$$

Let $E^0_{p,\mathbf{q}}(H)$ denote the p-th piece of the graded of homology $H_{p+q}(M)$. It is given by the formula

$$E^0_{p,\mathbf{q}}(H) = F_p H_{p+q}/F_{p-1}H_{p+\mathbf{q}}.$$

Now the module $E_p(H)$ is graded compatibly with the grading of $E_{p,\mathbf{q}}$. Indeed, summing the morphism $H_{p+\mathbf{q}}(F_p) \to H_{p+\mathbf{q}}(M)$ of (4.3) over q, we see that

$$F_p H = F_p(\oplus_q H_{p+\mathbf{q}}) = Im(H(F_p) = \oplus_q H_{p+\mathbf{q}}(F_p) \to \oplus_q H_{p+\mathbf{q}}(M) = H(M)).$$

As in Section 3, and working as always with complementary degree, for $1 \le r \le \infty$ we define:

$$Z^r_{p,\mathbf{q}} := Im(H_{p+q}(F_p/F_{p-r}) \to H_{p+q}(F_p/F_{p-1})), \tag{4.4}$$
$$B^r_{p,\mathbf{q}} := Im(H_{p+q+1}(F_{p+r-1}/F_p) \to H_{p+q}(F_p/F_{p-1})), \tag{4.5}$$
$$E^r_{p,\mathbf{q}} := Z^r_{p,\mathbf{q}}/B^r_{p,\mathbf{q}}. \tag{4.6}$$

Recall that the definition of B^r_p involved a connecting morphism. This justifies the change by -1 of total degree in the definition of $B^r_{p,\mathbf{q}}$. Now the module Z^r_p is given by the direct sum of modules $Z^r_{p,\mathbf{q}}$ over $\mathbf{q}$. Similarly the module B^r_p is given by the sum of modules $B^r_{p,\mathbf{q}}$ over $\mathbf{q}$. Hence we shall identify each of the module E^p_r with the direct sum $\oplus_q E^r_{p,\mathbf{q}}$. Setting $r = \infty$, we may verify that the isomorphism $E^\infty_{p,\mathbf{q}} \cong E_{p,\mathbf{q}}(H)$ still holds.

Let us recall that in Proposition 3.3, we had defined the differentiation

$$d^p_r \colon E^r_p \to E^r_{p-r}$$

as the composition of the morphisms in the following diagram

$$E^r_p = Z^r_p/B^r_p \xrightarrow{\epsilon} Z^r_p/Z^{r+1}_p \cong B^{r+1}_{p-r}/B^r_{p-r} \xrightarrow{\mu} Z^r_{p-r}/B^r_{p-r} = E^r_{p-r}.$$

The degree of d^r is dictated by the isomorphism $Z^r_p/Z^{r+1}_p \cong B^{r+1}_{p-r}/B^r_{p-r}$. Similarly in the graded case also we would like to determine the degree of d^r. Here we know that Z^r_p and B^r_p split as direct sums of Z and B with two indices. Looking closely at the proofs of Section 3, for each q we would like to determine (if possible) a q' such that

$$Z^r_{p,\mathbf{q}}/Z^{r+1}_{p,\mathbf{q}} \simeq B^{r+1}_{p-r,\mathbf{q}'}/B^r_{p-r,\mathbf{q}'}$$

canonically. So let us revisit the constructions we carried out in the case of Filtered Differential Modules book-keeping, this time, the degree of the homology.

Consider the commutative diagram of complexes with exact rows given below,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{p-r}/F_{p-r-1} & \longrightarrow & F_p/F_{p-r-1} & \longrightarrow & F_p/F_{p-r} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow & & \\ 0 & \longrightarrow & F_{p-1}/F_{p-r-1} & \longrightarrow & F_p/F_{p-r-1} & \longrightarrow & F_p/F_{p-1} & \longrightarrow & 0. \end{array} \tag{4.7}$$

Taking homology we get

$$\begin{array}{ccccc} H_{p+q}(F_p/F_{p-r-1}) & \longrightarrow & H_{p+q}(F_p/F_{p-r}) & \xrightarrow{\delta} & H_{p+q-1}(F_{p-r}/F_{p-r-1}) \\ \downarrow{\scriptstyle Id} & \nearrow & \downarrow & & \downarrow \\ H_{p+q}(F_p/F_{p-r-1}) & \longrightarrow & H_{p+q}(F_p/F_{p-1}) & \xrightarrow{\delta} & H_{p+q-1}(F_{p-1}/F_{p-r-1}). \end{array} \tag{4.8}$$

Solidifying the dotted arrow, we may construct the following diagram

$$
\begin{array}{ccccc}
 & \nearrow & H_{p+q}(F_p/F_{p-r}) & \overset{\cdots}{\searrow} & \\
 & & \downarrow & & \\
H_{p+q}(F_p/F_{p-r-1}) & \longrightarrow & H_{p+q}(F_p/F_{p-1}) & \longrightarrow & H_{p+q-1}(F_{p-1}/F_{p-r-1})
\end{array} \tag{4.9}
$$

which satisfies the conditions required by Lemma 1.1. By definition we have,

$$
\begin{aligned}
Z^r_{p,\mathbf{q}} &= Im(H_{p+q}(F_p/F_{p-r}) \to H_{p+q}(F_p/F_{p-1})),\\
Z^{r+1}_{p,\mathbf{q}} &= Im(H_{p+q}(F_p/F_{p-r-1}) \to H_{p+q}(F_p/F_{p-1})).
\end{aligned}
$$

Thus by (4.9) and Lemma 1.1 we have

$$
Z^r_{p,\mathbf{q}}/Z^{r+1}_{p,\mathbf{q}} \cong Im(H_{p+q}(F_p/F_{p-r}) \to H_{p+q-1}(F_{p-1}/F_{p-r-1})). \tag{4.10}
$$

Now consider the following commutative diagram of complexes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F_{p-r}/F_{p-r-1} & \longrightarrow & F_{p-1}/F_{p-r-1} & \longrightarrow & F_{p-1}/F_{p-r} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{p-r}/F_{p-r-1} & \longrightarrow & F_p/F_{p-r-1} & \longrightarrow & F_p/F_{p-r} & \longrightarrow & 0.
\end{array}
$$

Taking homology we get,

$$
\begin{array}{ccccc}
H_{p-r+q'+1}(F_{p-1}/F_{p-r}) & \xrightarrow{\delta} & H_{p-r+q'}(F_{p-r}/F_{p-r-1}) & \longrightarrow & H_{p-r+q'}(F_{p-1}/F_{p-r-1})\\
\downarrow & \overset{Id^{-1}\circ\delta}{\nearrow} & \downarrow{\scriptstyle id} & & \downarrow\\
H_{p-r+q'+1}(F_p/F_{p-r}) & \xrightarrow{\delta} & H_{p-r+q'}(F_{p-r}/F_{p-r-1}) & \longrightarrow & H_{p-r+q'}(F_p/F_{p-r-1}).
\end{array} \tag{4.11}
$$

Solidifying the dotted arrow, we can apply Lemma 1.1 to the following diagram

$$
\begin{array}{ccccc}
H_{p-r+q'+1}(F_{p-1}/F_{p-r}) & \xrightarrow{\delta} & H_{p-r+q'}(F_{p-r}/F_{p-r-1}) & \longrightarrow & H_{p-r+q'}(F_{p-1}/F_{p-r-1})\\
\downarrow & \nearrow & & \overset{\cdots}{\nearrow} & \\
H_{p-r+q'+1}(F_p/F_{p-r}). & & & &
\end{array} \tag{4.12}
$$

By definition we have that

$$
\begin{aligned}
B^{r+1}_{p-r,\mathbf{q'}} &= Im(H_{p-r+q'+1}(F_p/F_{p-r}) \to H_{p-r+q'}(F_{p-r}/F_{p-r-1})), \qquad (4.13)\\
B^{r}_{p-r,\mathbf{q'}} &= Im(H_{p-r+\mathbf{q'+1}}(F_{p-1}/F_{p-r}) \to H_{p-r+q'}(F_{p-r}/F_{p-r-1})).
\end{aligned}
$$

Now in view of diagram (4.12) and Lemma 1.1

$$B^{r+1}_{p-r,\mathbf{q'}}/B^{r}_{p-r,\mathbf{q'}} \cong Im(H_{p-r+q'+1}(F_p/F_{p-r}) \to H_{p-r+q'}(F_{p-1}/F_{p-r-1})). \quad (4.14)$$

We want the RHS of (4.10) and (4.14) to be the same. Clearly this is possible if and only if $p - r + q' = p + q - 1$, i.e

$$q' = q + r - 1.$$

Thus the bidegree of $d^r : E^r \to E^r$ is $(-r, r-1)$.

So in the case of filtered chain-complex the spectral sequence is a sequence of bi-graded differential objects

$$\cdots (E^r, d^r) \cdots$$

with *complementary* bi-degree of d^r given by $(-r, r-1)$.

Definition 4.1 (Homologically graded spectral sequence)**.** *A homologically graded spectral sequence is a family of doubly graded differential modules* $\{E^r, d^r\}$ *with*

(i) bi-degree $(d^r) = (-r, r-1)$,

(ii) $H(E^r, d^r) = E^{r+1}$.

4.1 Filtered Co-chain Complex

Having constructed spectral sequences associated with filtered chain complexes, we would like to understand it in the context of filtered co-chain complexes. The immediate question is should one work with a decreasing or an increasing filtration. We address this question first.

Choice of Filtration. Let

$$\cdots \longrightarrow C^{p-1} \longrightarrow C^p \longrightarrow C^{p+1} \longrightarrow \cdots$$

be a co-chain complex. Let us denote it by M. Clearly M is a graded module of degree of 1. A natural way to filter complexes (both chain and co-chain) is to truncate. The following is M truncated from left, notice that the objects to the left of p are zero

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow C^p \longrightarrow C^{p+1} \longrightarrow \cdots \quad .$$

Similarly if M is truncated from right we would have

$$\cdots \longrightarrow C^{p-1} \longrightarrow C^p \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \quad .$$

Let us set F^p to be the complex truncated at p. Then we have a filtration of M. As p increases the filtration is increasing if truncation is from right and deceasing if from left. Most of the filtrations we encounter involves truncation in one way or other. Clearly, for a co-chain complex, only the filtration obtained by left truncation is compatible with differentiation. Similarly one may justify the choice of increasing filtration for chain complex.

Spectral Sequence It is enough for us to specify the degree of the differentiation d_r. We already know that the degree change with respect to p in case of differential module with decreasing filtration is r. If you compare a chain complex with a co-chain complex all that has happened is a reversal in the direction of arrows. From which we may infer that the degree change with respect to q will be $1 - r$. Thus we have the following definition.

Definition 4.2. *[Co-Homologically graded spectral sequence] A co-homologically graded spectral sequence is a family of doubly graded differential modules $\{E_r, d_r\}$ with*

(i) bi-degree $(d_r) = (r, 1 - r)$,

(ii) $H(E_r, d_r) = E_{r+1}$.

Notice that here we have lowered the index r and raised the indices p, q. So each graded piece is represented as $E_r^{p,q}$. Thus for each value of p and q we have the map

$$d_r^{p,q} : E_r^{p,q} \to E^{p+r,q-r+1}.$$

Chapter III

Exact couples

Our aim in this chapter is to understand the more general setting of exact couples introduced by Massey. Exact couples are pairs of objects and morphisms in abelian category which forms an exact triangle. They are a natural source of spectral sequences arising in Topology. The material in this chapter must be accessible to anybody familiar with proofs involving diagram chases. This chapter doesn't depend on chapter II. One may very well treat this as a new beginning. Even our definition of spectral sequence in here will be independent of any prior construction.

1 Exact couples and Spectral sequence

We give a general definition of a Spectral sequence.

Definition 1.1 (Spectral Sequence)**.** *Let $\mathfrak{A}$ be an abelian category. A spectral sequence in $\mathfrak{A}$ is a sequence of differential objects of $\mathfrak{A}$*

$$..., (E_n, d_n), (E_{n+1}, d_{n+1}), ... \tag{1.1}$$

such that $E_{n+1} = H(E_n, d)$.

We shall denote the above spectral sequence by E. Let E, E' be two spectral sequences. Let $\phi_n : E_n \to E'_n$ be a morphism from E_n to E'_n as differential objects. Then $\phi = \{\phi_n\} : E \to E'$ is morphism of spectral sequences if $H(\phi_n) = \phi_{n+1}\ \forall n$. It can be easily verified that spectral sequences in $\mathfrak{A}$ forms a category. We shall represent the category of spectral sequences in $\mathfrak{A}$ by $\mathfrak{E}(\mathfrak{A})$ or in short by $\mathfrak{E}$.

To generalise the construction of spectral sequences we define the following.

Definition 1.2 (Exact Couple)**.** *Let D,E be objects in $\mathfrak{A}$. Let $\alpha : D \to D$, $\beta : D \to E$ and $\gamma : E \to D$ be morphisms. Then we say $\boldsymbol{EC} := \{D, E, \alpha, \beta, \gamma\}$ is an exact couple in $\mathfrak{A}$ if the diagram*

$$\begin{array}{ccc} D & \xrightarrow{\alpha} & D \\ & {\scriptstyle\gamma}\nwarrow & \downarrow{\scriptstyle\beta} \\ & & E \end{array} \tag{1.2}$$

is exact at each object.

A morphisms $\phi : \mathbf{EC} \to \mathbf{EC}'$ is a pair of morphisms $\kappa : D \to D'$ and $\lambda : E \to E'$ such that

$$\kappa\alpha = \alpha'\kappa; \quad \lambda\beta = \beta'\kappa; \quad \gamma'\lambda = \kappa\gamma; \tag{1.3}$$

It can be verified that exact couples in $\mathfrak{A}$ also forms a category. But we won't prove it here. Consider the map $d = \beta \circ \gamma : E \to E$. Since $\gamma \circ \beta = 0$ we have $d \circ d = (\beta \circ \gamma) \circ (\beta \circ \gamma) = 0$. Thus d is a differential on E.

1.1 Deriving exact couples

Given an exact couple $\{D, E, \alpha, \beta, \gamma\}$ we construct another exact couple called it's derived couple. Set $D_1 = im(\alpha : D \to D)$ and E_1 as the homology of E with respect to d. Now define morphisms $\alpha_1, \beta_1, \gamma_1$ as follows:

1. By definition $D_1 = im(\alpha)$ is a submodule of D. The morphism $\alpha_1 : D_1 \to D_1$ is defined as the restriction of α to D_1. In this sense α_1 is 'induced' by α.
2. Let $x_1 \in D_1$. Now there exists $x \in D$ such that $x_1 = \alpha(x)$. We define $\beta_1 : D_1 \to E_1$ by
$$\beta_1(x_1) = \beta_1(\alpha(x)) = \text{the class of}\, \beta(x) \,\text{in}\, E_1 =: [\beta(x)].$$
In this sense β_1 is 'induced' by $\beta\alpha^{-1}$. We shall later verify that this map is well defined i.e it is independent of the choice of preimage of x_1.
3. Let $[z] \in E_1$. Now $[z]$ is the homology class of some $z \in E$. The morphism $\gamma_1 : E_1 \to D_1$ is now defined by $\gamma_1([z]) = \gamma(z)$. So one may say γ_1 is 'induced' by γ.

Before proceeding one need to justify these definitions which apriori seem to depend on choices.

Clearly α_1 is a morphism from D_1 to D_1.

Well-definedness of β_1*:* Now suppose $x_1 \in D_1$. Then by definition $x_1 = \alpha(x)$ for some $x \in D$. Define

$$\beta_1(x_1) = \beta_1(\alpha(x)) = [\beta(x)]. \tag{1.4}$$

Where $[\beta(x)]$ denotes the homology class of $\beta(x)$. We claim $\beta_1(x_1) \in E_1$. It is enough to show that $\beta(x)$ is a cycle in E. Now

$$d\beta(x) = \beta\gamma\beta(x),$$

but since $\gamma\beta = 0$. Hence we have the claim. Now we must show that the definition is dependent only on $\alpha(x)$ and not on x. Equivalently we need to show that if $\alpha(x) = 0$ then $[\beta(x)]$ is a boundary. Suppose $\alpha(x) = 0$ then we have $y \in E$ such that $x = \gamma(y)$. Thus $\beta(x) = \beta(\gamma(y))$. Hence

$$\beta_1(\alpha(x)) = [\beta(\gamma(y))] = [d(y)]$$

is a boundary.

Well-definedness of γ_1: We need to show that $\gamma(z) \in D_1$ and the definition is independent of choice of representative. Since z is a cycle we have $d(z) = \beta\gamma(z) = 0$. But

$$ker\beta = \alpha D = D_1.$$

So $\gamma(z) \in D_1$. Now suppose $[z_1] = [z_2]$, then $z_2 = z_1 + \beta\gamma(y)$ for some $y \in E$. So

$$\gamma(z_2) = \gamma(z_1 + \beta\gamma(y)) = \gamma(z_1).$$

Thus the definition make sense and we have the following diagram

$$\begin{array}{ccc} D_1 & \xrightarrow{\alpha_1} & D_1 \\ & {}_{\gamma_1}\nwarrow & \downarrow{\scriptstyle \beta_1} \\ & & E_1. \end{array} \tag{1.5}$$

Theorem 1.1. *The above diagram is exact.*

Proof. Exactness at top left D_1: Let $[z] \in E_1$. Then,

$$\alpha_1\gamma_1([z]) = \alpha\gamma(z) = 0.$$

Conversely, let $x \in D_1$ be such that $\alpha_1(x) = 0$. We know that α_1 is induced by α so $\alpha(x) = 0$. Hence $x = \gamma(z)$ for some $z \in E$. Now $x \in D_1 = ker\beta$. So $d(z) = \beta\gamma(z) = \beta(x) = 0$. Thus z is a cycle in E. Hence we have $[z] \in E_1$ such that $\gamma_1([z]) = x$.

Exactness at top right D_1: Let $x \in D_1$. Then $\beta_1\alpha_1(x) = [\beta(x)]$. But $x \in D_1 = ker(\beta)$ so $[\beta(x)] = [0]$. Conversely, suppose $\beta_1(x)$ is a boundary for some $x \in D_1$, that is, for $x' \in D$ such that $\alpha(x') = x$ we have $y \in E$ such that $\beta(x') = \beta\gamma(y)$. Thus $x' = \gamma(y) + x_0$ for some $x_0 \in D_1$. Hence we have

$$x = \alpha\gamma(y) + \alpha(x_0) = \alpha(x_0).$$

Thus given $\beta_1(x)$ is boundary we have produced a $x_0 \in D_1$ such that $\alpha_1(x_0) = \alpha(x_0) = x$.

Exactness at E_1: Let $\alpha(x) \in D_1$ where $x \in D$. Then

$$\gamma_1\beta_1(\alpha(x)) = \gamma_1[\beta(x)] = \gamma(\beta(x)) = 0.$$

Conversely, let $[z] \in E_1$ be such that $\gamma_1[z] = 0$. Thus we have $\gamma(z) = 0$ so $z = \beta(x)$ for some $x \in D$. Hence we have

$$[z] = [\beta(x)] = \beta_1(\alpha(x)).$$

Thus given $\gamma_1[z] = 0$ we have produced $y = \alpha(x)$ in D_1 such that $\beta_1(y) = [z]$. □

Now if we express E_1 in terms of cycles and boundary we have,

$$E_1 = ker(\beta\gamma)/Im(\beta\gamma) = \gamma^{-1}\alpha(D)/\beta\alpha^{-1}(0). \tag{1.6}$$

By iterating the process we shall obtain a sequence of (derived)exact couples $\mathbf{EC_1}$, $\mathbf{EC_2}$, $\mathbf{EC_3}$,..., $\mathbf{EC_n}$,... where

$$\mathbf{EC_n} = \{D_n, E_n, \alpha_n, \beta_n, \gamma_n\}. \tag{1.7}$$

Further one makes the following observations:

Clearly $(E_n, d_n); n = 1, 2, 3...$ where $d_n = \beta_n\gamma_n$ is a spectral sequence.

Theorem 1.2. *$E_n = \gamma^{-1}(\alpha^n D)/\beta\alpha^{-n}(0)$ and $d_n : E_n \to E_n$ is induced by $\beta\alpha^{-n}\gamma$.*

Proof. We shall establish the claim by induction. Let us state our induction hypothesis. We claim for any $n \geq 0$ that

1. $\mathbf{EC_n} = \{D_n, E_n, \alpha_n, \beta_n, \gamma_n\}$ forms an exact couple.
2. By construction D_n is a sub-module of D. The morphism $\alpha_n : D_n \to D_n$ is the restriction of α to D_n.
3. The morphism $\beta_n : D_n \to E_n$ has the following property. Let $x_n \in D_n$. Now there exists $x \in D$ such that $x_n = \alpha^n(x)$. We have $\beta_n(x_n) = \beta_n(\alpha^n(x))$ equals the class of $\beta(x)$ in E. Actually the class of $\beta(x)$ belongs to E_n. In this sense β_1 is 'induced' by $\beta\alpha^{-n}$.
4. Let $[z] \in E_n$. We obtain E_n by taking homology of E repeatedly. So there exists a $z \in E$ such that it is a representative of the class $[z]$. The morphism $\gamma_n : E_n \to D_n$ has the property that $\gamma_n([z]) = \gamma(z)$. Actually $\gamma(z)$ lies in $D_n \subset D$. We may say thus that γ_n is induced by γ.
5. The last three properties characterize the morphisms α_n,β_n and γ_n.

From Theorem 1.1 and subsequent discussion we clearly have the theorem for $n = 0$. Assume it's true for indices upto n. So, we have derived exact couples only upto $n-1$. Therefore we have the n$^{\text{th}}$ derived couple $\mathbf{EC}_n := \{D_n, E_n, \alpha_n, \beta_n, \gamma_n\}$ of $\mathbf{EC}$. Let us derive it. So we set

$$D_{n+1} = \alpha_n(D_n)$$

and

$$E_{n+1} = H(E_n, \beta_n\gamma_n).$$

Consider the derived couple

$$\begin{array}{ccc} D_{n+1} & \xrightarrow{\alpha_{n+1}} & D_{n+1} \\ & {\scriptstyle \gamma_{n+1}} \nwarrow & \downarrow {\scriptstyle \beta_{n+1}} \\ & & E_{n+1}. \end{array} \tag{1.8}$$

Let us check the induction claim for $n+1$:

1. the couple is exact because it is obtained by deriving the exact couple $\mathbf{EC}_n$.

2. the morphism $\alpha_{n+1} : D_{n+1} \to D_{n+1}$ is induced by α_n. This just means that α_{n+1} is α_n restricted to D_{n+1} by the definition of deriving an exact couple. By induction hypothesis α_n is just α restricted to D_n. Thus afortiori α_{n+1} is also α restricted to D_{n+1}.

3. the morphism $\beta_{n+1} : D_{n+1} \to E_{n+1}$ is induced by $\beta_n\alpha_n^{-1}$ by definition of deriving an exact couple. By induction hypothesis β_n itself is $\beta \circ \alpha^{-n}$ and α_n^{-1} is just α. Thus $\beta_n\alpha_n^{-1}$ equals $\beta \circ \alpha^{-n} \circ \alpha^{-1} = \beta \circ \alpha^{-(n+1)}$,

4. the morphism $\gamma_{n+1} : E_{n+1} \to D_{n+1}$ is induced by γ_n. By definition of deriving an exact couple, this means that for any $e_{n+1} \in E_{n+1}$ we take a pre-image *class* $e_n \in E_n$ and define $\gamma_{n+1}(e_{n+1}) = \gamma_n(e_n)$. By induction hypothesis, $\gamma_n(e_n)$ equals $\gamma(e)$ where $e \in E$ is any element representing the *class* $e_n \in E_n$.

5. we have expressed α_{n+1}, β_{n+1} and γ_{n+1} in terms of these properties, above, by the definition of deriving an exact couple combined with the characterizations of α_n,β_n and γ_n themselves in terms of these properties.

Hence $E_{n+1} = H(E_n, d_n)$ equals $ker(\beta_n\gamma_n)/Im(\beta_n\gamma_n)$. Now, by exactness at D_n on the right, we have $\ker(\beta_n) = \alpha_n(D_n)$, which equals D_{n+1}. Thus $\ker(\beta_n\gamma_n) = \gamma_n^{-1}(D_{n+1})$. Recall that $\gamma_n : E_n \to D_n$ and $D_{n+1} \hookrightarrow D_n$ is a sub-module. Now clearly $D_{n+1} = \alpha^{n+1}D$. Further let us recall the description of γ_n from the induction hypothesis: let $[z] \in E_n$ be a class. Let $z \in E$ be a representative of this class. We have $\gamma_n([z])$ which belongs to D_n is given by $\gamma(z)$. Thus the inverse-image of D_n through γ_n may be computed by taking the inverse image of D_n through γ. So

$$\gamma_n^{-1} = \gamma^{-1}|_{D_n}.$$

Thus

$$\ker(\beta_n\gamma_n) = \gamma^{-1}\alpha^{n+1}D.$$

By definition $Im(\beta_n\gamma_n) = \beta_n(Im(\gamma_n)$. Since $Im(\gamma_n) = ker(\alpha_n) = \alpha_n^{-1}(0)$, so we have

$$Im(\beta_n\gamma_n) = \beta_n\alpha_n^{-1}(0).$$

Now the description of E_{n+1} follows easily.

Finally, $d_{n+1} = \beta_{n+1} \circ \gamma_{n+1}$ by definition. By induction hypothesis β_{n+1} is induced by $\beta\alpha^{-(n+1)}$ and γ_{n+1} is induced by γ. So d_{n+1} is induced by $\beta\alpha^{-(n+1)}\gamma$. □

Proposition 1.3. *The process of associating an exact couple with a spectral sequence defines a functor*

$$SS : \mathfrak{EC} \to \mathfrak{E}, \tag{1.9}$$

from category of exact couples to category of spectral sequences.

Proof. omitted □

2 A note on Limits E^{∞}

Let us define the limit E^{∞} of a spectral sequence arising from an exact couple. Consider the n^{th} term of the spectral sequence E^n. Let us denote the cycles of d^n by $E^{n,n+1}$. So $E^{n,n+1} \hookrightarrow E^n$ is a sub-module. Now

$$\sigma^{n,n+1} \colon E^{n,n+1} \to E^{n+1}$$

is a surjection. Similarly let $E^{n,n+2}$ be the sub-module of $E^{n,n+1}$ consisting of those elements whose image by $\sigma^{n,n+1}$ become cycles for d^{n+1}. Thus we have a surjection $\sigma^{n,n+2} : E^{n,n+2} \to E^{n+2}$.

By abuse of language we may say that $x \in E^{n,n+1}$ *is a cycle for* d^n, $x \in E^{n,n+2}$ *is a cycle for both* d^n *and* d^{n+1}. More generally, using this abuse of language, we define $E^{n,n+r}$ as the sub-module of E^n consisting of those elements whose images are the cycles for various d^{n+k} for $0 \leq k \leq r$. We shall denote by $\sigma^{n,n+r} : E^{n,n+r} \to E^{n+r}$ the surjection.

Let $E^{n,\infty}$ be such that it is the collection of all those x which are cycles for all d^{n+r}, $r \geq 0$.
Restrict $\sigma^{n,n+1}$ to $E^{n,\infty}$ then it's a surjection on to $E^{n+1,\infty}$. Thus we have the following system,

$$... \to E^{n,\infty} \to E^{n+1,\infty} \to E^{n+2,\infty} \to ...$$

where each arrow is an epimorphism. Define

$$E^{\infty} = \varinjlim_{n} (E^{n,\infty}, \sigma). \tag{2.1}$$

This object essentially is the collection of 0 and those x which is a cycle for every n and boundary for no n.

Chapter IV

From Filtered Differential Objects to Exact couples

This chapter shall serve as the link between Chapters II and III. We would see how the content of Chapter II fits inside the general setting developed in Chapter III. The passage from Homology of graded to graded of homology which we briefly discussed before shall be discussed in detail. One may notice a mismatch between indices here and chapter II. A small note at the end of the chapter shall clarify this discrepancy. To minimize confusion we would use $\{C_p\}_{p\in\mathbb{Z}}$ to represent filtration of a differential object. For most part of the discussion we will restrict ourselves to category of filtered differential modules. Towards the end we shall show how does spectral sequences associated with filtered chain complexes can be viewed as a special case under the filtered differential modules. We recommend familiarity with the material discussed in preceding Chapters.

1 Filtered Differential Objects and Spectral Sequences

Consider $\mathfrak{R}-\mathfrak{mod}$, the category of R-modules. Let (C, d) be a differential object in $\mathfrak{R}-\mathfrak{mod}$. Let

$$... \subset C_{p-1} \subset C_p ... \subset C \tag{1.1}$$

be a compatible filtration of (C, d). Recall that we say a filtration is compatible with differentiation if

$$dC_{(p)} \subset C_{(p)} \quad \forall p.$$

We shall denote this differential object equipped with compatible filtration by (C, d, f).

Let (C, d, f) and (C', d', f') be two different filtered differential objects. Say ϕ is a morphism from (C, d) to (C', d') as differential objects. Now ϕ is morphism from (C, d, f) to (C', d', f') if

$$\phi(C_p) \subset C'_p; \qquad \forall p.$$

Filtered differential objects in $\mathfrak{R}-\mathfrak{mod}$ with morphism as defined above forms a category. We shall denote this category by $(\mathfrak{R}-\mathfrak{mod}, d, f)$. Given a filtered differential object C, we have the following exact sequence of differential objects

$$0 \longrightarrow C_{p-1} \longrightarrow C_p \longrightarrow C_p/C_{p-1} \longrightarrow 0 \quad . \tag{1.2}$$

We know that Homology functor associates exact sequence of differential objects to exact triangles of Homology objects. Thus we have the following exact triangle,

$$\begin{array}{ccc} H(C_{(p-1)}) & \xrightarrow{\alpha_{(p)}} & H(C_{(p)}) \\ & {\nwarrow}{\scriptstyle \gamma_{(p)}} & \downarrow{\scriptstyle \beta_{(p)}} \\ & & H(C_{(p)}/C_{(p-1)}). \end{array} \tag{1.3}$$

Let us suppress the indices and denote the graded modules $\oplus_p H(C_{(p)})$ by D and $\oplus_p H(C_{(p)}/H(C_{(p-1)})$ by E. Let $\mathfrak{R} - \mathfrak{mod}^{\mathbb{Z}}$ denote the category of graded R-modules. Then we have the following exact couple in $\mathfrak{R} - \mathfrak{mod}^{\mathbb{Z}}$

$$\begin{array}{ccc} D & \xrightarrow{\alpha} & D \\ & {\nwarrow}{\scriptstyle \gamma} & \downarrow{\scriptstyle \beta} \\ & & E. \end{array} \tag{1.4}$$

Remark. This process defines a functor say

$$\bar{H} : (\mathfrak{R} - \mathfrak{mod}, d, f) \to \mathfrak{EC}(\mathfrak{R} - \mathfrak{mod}^{\mathbb{Z}}).$$

When we extract E out of the couple together with $\beta\gamma$ we get a functor

$$E : (\mathfrak{R} - \mathfrak{mod}, d, f) \to ((\mathfrak{R} - \mathfrak{mod}, d)^{\mathbb{Z}}).$$

Notice that E is spectral sequence functor.

Let $\mathfrak{B} = (\mathfrak{R} - \mathfrak{mod}, d)$ denote the category of differential objects in $\mathfrak{R} - \mathfrak{mod}$. Let us denote the category of filtered objects in $\mathfrak{B}$ by $(\mathfrak{B}, f)$. To each object C in $(\mathfrak{B}, f)$ we shall associate it's associated graded object. Plainly this defines a functor say, Gr from $(\mathfrak{B}, f)$ to $\mathfrak{B}^{\mathbb{Z}}$. We also know that the associated graded of a filtered differential module is also a differential module(graded). Hence we can apply homology functor on $Gr(C)$, where C is an object in $(\mathfrak{R} - \mathfrak{mod}, d)$. Observe that $H \circ Gr$ yields $E = E_0$, first term of a spectral sequence.

On the other hand starting with a filtered differential object C in $(\mathfrak{R} - \mathfrak{mod}, d, f)$ we may pass directly to homology. Let us recall how $H(C)$ gets filtered. The inclusion $C_p \subset C$ induces a map $H(C_p) \to H(C)$. Let $FH_{(p)}$ denote $Im(H(C_{(p)}) \to H(C))$. Thus the homology object $H(C)$ gets filtered as shown below

$$... \subseteq FH_{(p-1)} \subseteq FH_{(p)} \subseteq ... \subseteq H(C). \tag{1.5}$$

Taking the associated graded of $H(C)$ with respect to this filtration, we have a functor $Gr \circ H$ from $(\mathfrak{R} - \mathfrak{mod}, d, f)$ to $\mathfrak{R} - \mathfrak{mod}^{\mathbb{Z}}$, which to C associates the graded object $\oplus_p(FH_{(p)})$.

Let's say we can determine $H(C_{(p)})/C_{(p-1)})$ to a significant extent. Then spectral sequences shall provide us with information about the graded object associated with $H(C)$ filtered by it's sub-object $FH_{(p)}$. Then a question arises, as to how much information about $H(C)$ we can recover. We shall have an informal discussion now. For the graded object to adequately represent $H(C)$, we would want the filtration to satisfy two conditions:

$$i) \bigcup_p FH_{(p)} = H(C) \qquad ii) \bigcap_p FH_{(p)} = 0. \tag{1.6}$$

For if (i) fails then there would be non-zero elements in $H(C)$ which are not present in any FH_p. And if (ii) fails then there would be non-zero elements in $H(C)$ which are present in $FH_{(p)}$ for all p. Hence they will be lost in the graded object. If both conditions are met then $\forall\ x \in H(C)$ we have a p such that

$$x \notin FH_{(r)} \quad \text{for} \quad r < p$$

and

$$x \in FH_{(r)} \quad \text{for} \quad r \geq p.$$

Thus for every x in $H(C)$ we have an unique representative in $Gr(H(C))$. Conversely every non-zero element in $Gr(H(C))$ represents an unique element in $H(C)$. Thus all we may lose is the information about the inclsions $FH_{(p-1)} \hookrightarrow FH_p$ of modules for various p. Helas, there is no way we can determine these inclusions!

In Chapter V we shall determine conditions which will ensure finite convergence of spectral sequence associated with a filtered chain complex, such that the above conditions are met. That shall help us with a tool to determine homology of filtered chain complexes.

1.1 Spectral Sequence of a Filtered chain complex via Exact couples

Consider the category of graded modules, $\mathfrak{R} - \mathfrak{mod}^{\mathbb{Z}}$. Let $\mathfrak{C}$ denote the category of differential objects in $\mathfrak{R} - \mathfrak{mod}^{\mathbb{Z}}$. Let C be an object in $\mathfrak{C}$ which is equipped with a differentiation of degree -1. Then (1.1) is filtration of a *chain complexes*. Thus C is a filtered chain complex. The following is an illustration of (1.2) in the context of chain complex. Notice that it is a short exact sequence of chain complexes. Here p is the index of filtration and q is the index within each chain complex

$$\begin{array}{ccccccccc} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_{\mathbf{p-1},q+1} & \longrightarrow & C_{\mathbf{p},q+1} & \longrightarrow & (C_{\mathbf{p}}/C_{\mathbf{p-1}})_{q+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_{\mathbf{p-1},q} & \longrightarrow & C_{\mathbf{p},q} & \longrightarrow & (C_{\mathbf{p}}/C_{\mathbf{p-1}})_{q} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_{\mathbf{p-1},q-1} & \longrightarrow & C_{\mathbf{p},q-1} & \longrightarrow & (C_{\mathbf{p}}/C_{\mathbf{p-1}})_{q-1} & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \end{array} \tag{1.7}$$

This give rise to the following long exact sequence at the level of homology

$$\longrightarrow H_q(C_{p-1}) \xrightarrow{\alpha_{p,q}} H_q(C_p) \xrightarrow{\beta_{p,q}} H_q(C_p/C_{p-1}) \xrightarrow{\gamma_{p,q}} H_{q-1}(C_{p-1}) \longrightarrow \quad . \tag{1.8}$$

Let us set

$$\begin{aligned} D_{p,q} &:= H_q(C_p) \\ E_{p,q} &:= H_q(C_p/C_{p-1}). \end{aligned} \tag{1.9}$$

Thus we may define the following graded modules

$$\begin{aligned} E &= \bigoplus_{p,q} E_{p,q} \\ D &= \bigoplus_{p,q} D_{p,q}. \end{aligned} \tag{1.10}$$

To abbreviate, let us denote $\{\alpha_{p,q}\}$ by α, $\{\beta_{p,q}\}$ by β and $\{\gamma_{p,q}\}$ by γ. Now we have the following exact couple in category of doubly graded modules denoted $\mathfrak{R} - \mathfrak{mod}^{\mathbb{Z}\times\mathbb{Z}}$

$$\begin{array}{ccc} D & \xrightarrow{\alpha} & D \\ & {\scriptstyle\gamma}\nwarrow & \downarrow{\scriptstyle\beta} \\ & & E. \end{array} \tag{1.11}$$

Notice that the bi-degrees of α, β, γ are as follows;

$$deg\, \alpha = (1,0), deg\, \beta = (0,0), deg\, \gamma = (-1,-1). \tag{1.12}$$

Then it follows from Theorem III.1.2 that in the n[th] derived couple associated with the spectral sequence, we have

$$deg\, \alpha_n = (1,0), \tag{1.13}$$

$$deg\, \beta_n = (-n,0), \tag{1.14}$$

$$deg\, \gamma_n = (-1,-1), \tag{1.15}$$

$$deg\, d_n = deg\, \beta_n + deg\, \gamma_n = (-n-1,-1). \tag{1.16}$$

In the next chapter we shall explore the conditions under which the associated spectral sequence converges in finite number of steps.

1.2 Notational differences between Chapters II and IV

In this chapter and the previous one we have indexed spectral sequence to begin with E_0. Whereas in the first chapter we begun with E_1. Recall the statement that d_n is induced by $\beta\alpha^{-n}\gamma$. It is convenient to have index n to be power of n. This is essentially the reason to begin the indexing with 0. The difference is that E_{r+1} in chapter II has become the new E_r. One may conclude this by comparing the degrees of differentiations as well. Another difference is in q. In chapter I, q was the complementary degree. Then we also had a notion of total degree $(p+q)$. It is the total degree of Chapter I which has become q in Chapter III. Suppose we denote the q in Chapter I by q_1 and the one in Chapter III by q_3 then $q_1 = q_3 - p$. These changes in q and r change the bi-degree of differential as well owing to the formula in (1.16). In chapter I bi-degree of d^r was $(-r, r-1)$ but now it is $(-r-1, -1)$.

Chapter V

Finite convergence of spectral sequence associated with filtered chain complexes

In Section 2 of Chapter III we defined the limit E^∞ of a spectral sequence arising from an exact couple. It is defined by taking limits over a possibly infinite indexing set. So it becomes interesting to investigate conditions under which this limit may be computed in finitely many steps. When this happens, we say that the spectral sequence *converges finitely* to its limit. Our focus shall remain on finite convergence of spectral sequences associated with filtered chain complexes. Our aim is to describe sufficient conditions for finite convergence.

1 Finite convergence conditions

Consider a spectral sequence ..., $(E^r, d^r), (E^{r+1}, d^{r+1}), \ldots$. Suppose $d^r = 0$ for $r \geq n$ for some positive integer n. Now the sequence is stationary ($E^r = E^{r+1}$) for $r \geq n$. We may call E^n the limiting term of sequence. We shall adopt a similar strategy to make sure the spectral sequence converge in finitely many steps.

The problem of finite convergence of spectral sequence associated with a filtered chain complex C can be stated as follows: to search for sufficient conditions under which

(i) $Gr \circ H(C) = E^\infty$;

(ii) $\bigcup_p FH_p = H(C)$ and $\bigcap_p FH_p = 0$ are satisfied;

(iii) The spectral sequence converges finitely, that is E^∞ term is reached after finitely many steps.

Let us recall the exact couple of IV.(1.11). We have

$$D = \bigoplus_{p,q} H_q(C_p), \tag{1.1}$$

$$E = \bigoplus_{p,q} H_q(C_p/C_{p-1}) \tag{1.2}$$

such that the following is an exact couple

$$\begin{array}{ccc} D & \xrightarrow{\alpha} & D \\ & {}_{\gamma}\nwarrow & \downarrow{\scriptstyle\beta} \\ & & E. \end{array}$$

Definition 1.1. *Suppose for any given q, there exist a p_0 (possibly dependent on q) such that $\alpha_{p,q} : D_{p,q} \xrightarrow{\sim} D_{p+1,q}$ is an isomorphism for $p \geq p_0$, then we say that $\alpha : D \to D$ is positively stationary. Similarly if for any given q, there exist a p_0 (possibly dependent on q) such that $\alpha_{p,q} : D_{p,q} \xrightarrow{\sim} D_{p+1,q}$ is an isomorphism for $p \leq p_0$, then we say $\alpha : D \to D$ is negatively stationary. We say α is stationary if it is both positively and negatively stationary.*

Theorem 1.1. *If α is stationary, the spectral sequence associated with the exact couple converges finitely: that is, given p, q there exists r such that $E^r_{p,q} = E^{r+1}_{p,q} = ...E^\infty_{p,q}$.*

Proof. Consider the following exact sequence

$$\cdots \to D_{p-1,q} \xrightarrow{\alpha} D_{p,q} \xrightarrow{\beta} E_{p,q} \xrightarrow{\gamma} D_{p-1,q-1} \xrightarrow{\alpha} D_{p,q-1} \to \cdots . \tag{1.3}$$

Fix q and assume p is sufficiently large. Since α is positively stationary, $\alpha_{p,q}$ is an isomorphism. Now $ker(\beta) \cong D_{p,q}$ and $im(\gamma) = 0$. Since the sequence is exact we have

$$E_{p,q} = ker(\gamma) = im(\beta) = 0.$$

That is, for fixed q and p sufficiently large we have $E_{p,q} = 0$. Similarly since α is negatively stationary we have $E_{p,q} = 0$ for p sufficiently small as well. Now fix p, q and consider

$$\cdots \to E^r_{p+r+1,q+1} \xrightarrow{d^r} E^r_{p,q} \xrightarrow{d^r} E^r_{p-r-1,q-1} \to \cdots . \tag{1.4}$$

Now for r sufficiently large we have $E^r_{p+r+1,q+1} = 0$ and $E^r_{p-r-1,q-1} = 0$. Since $E^n_{*,*}$ is always a sub-quotient of $E^r_{*,*}$ for $n \geq r$, so we have $E^n_{p+r+1,q+1} = 0$ and $E^n_{p-r-1,q-1} = 0$ for all $n \geq r$. Thus for r sufficiently large we have, $E^{r+1}_{p,q} = H(E^r_{p,q}, d^r) = E^r_{p,q}$. Observe that the whole of $E^s_{p,q}$ is a cycle for every d^s, $s \geq r$ and only 0 is a boundary for some d^s, $s \geq r$. Hence $E^r_{p,q} = E^{r+1}_{p,q} = ...$ Further these groups are isomorphic to $E^\infty_{p,q}$ by subsection III2. □

Let

$$... \subset C_{p-1} \subset C_p ... \subset C$$

be a filtered chain complex as given in IV.(1.1). We want to ensure, for the spectral sequence associated with this filtered chain complex

(i) $E^\infty_{p,q}$ as defined in III.2 equals $im(H_q(C_p) \to H_q(C))/im(H_q(C_{p-1}) \to H_q(C))$;

(ii) α is stationary.

To this end, we begin by considering another exact couple. Consider the following exact sequence of chain complexes

$$0 \longrightarrow C_p/C_{p-1} \longrightarrow C/C_{p-1} \longrightarrow C/C_p \longrightarrow 0 \quad .$$

Setting $\bar{D}$ be the object given by

$$\bar{D} = \bigoplus_{p,q} H_q(C/C_{p-1}), \tag{1.5}$$

the above sequence gives rise to an exact couple of bigraded objects

$$\begin{array}{ccc} \bar{D} & \xrightarrow{\bar{\alpha}} & \bar{D} \\ & {\scriptstyle\bar{\gamma}}\nwarrow & \downarrow{\scriptstyle\bar{\beta}} \\ & & E. \end{array} \tag{1.6}$$

We have

$$deg\bar{\alpha} = (1,0), deg\bar{\beta} = (-1,-1), deg\bar{\gamma} = (0,0). \tag{1.7}$$

Since the constructions are very similar to the ones we have seen before, we simply refer the reader to Subsection 1.1 of Chapter IV.

We now make a definition which will be applied to D,E and $\bar{D}$.

Definition 1.2. *A bigraded object A is said to be positively graded if for any given q there exists p_0 (possibly dependent on q) such that $A_{p,q} = 0$ if $p < p_0$. Similarly we say A is negatively graded if for any q there exists p_0 (possibly dependent on q) such that $A_{p,q} = 0$ for every $p > p_0$.*

Remark 1. Observe that if D is positively graded, then α is negatively stationary. Similarly if $\bar{D}$ is negatively graded, then $\bar{\alpha}$ is positively stationary.

Theorem 1.2. *The following conditions are equivalent:*
(i) α is positively stationary;
(ii) E is negatively graded;
(iii) $\bar{\alpha}$ is positively stationary.

Proof. In the course of proving Theorem 1.1 we had established that if α is positively stationary then for a fixed q, and p sufficiently large $E_{p,q} = 0$. Thus (i) $\Rightarrow$ (ii). Conversely, consider the exact sequence

$$\cdots \to E_{p,q+1} \xrightarrow{\gamma} D_{p-1,q} \xrightarrow{\alpha} D_{p,q} \xrightarrow{\beta} E_{p,q} \to \cdots . \tag{1.8}$$

Suppose E is negatively graded. Fix q, then for p sufficiently large

$$E_{p,q} = 0 \quad \text{and} \qquad E_{p,q+1} = 0.$$

Thus α is an isomorphism for p sufficiently large. So we have (ii) $\Rightarrow$ (i).
If we replicate the arguments for the exact couple (1.6) consisting of $\bar{D} = H_q(C/C_{p-1})$ then we shall obtain the implication (ii) $\Leftrightarrow$ (iii). ☐

Definition 1.3. *Let*

$$... \subseteq C_{p-1} \subseteq C_p \subseteq ... \subseteq C, \qquad -\infty < p < \infty \tag{1.9}$$

be a filtration of a chain complex C. We say this filtration is finite, if for each q, there exists p_0, p_1 such that

$$\begin{aligned} &(i)\ C_{p,q} = 0 \qquad \text{for } p \le p_0, \\ &(ii)\ C_{p,q} = C_q \qquad \text{for } p \ge p_1. \end{aligned} \tag{1.10}$$

Definition 1.4. *We say that the filtration is homologically finite, if, for each q, there exist p_0, p_1 such that*

$$\begin{aligned} &(i)\ H_q(C_p) = 0 \qquad &\text{for } p \le p_0, \\ &(ii)\ H_q(C_p) = H_q(C) \qquad &\text{for } p \ge p_1. \end{aligned} \tag{1.11}$$

Proposition 1.3. *If the filtration of a chain complex C is finite, it is homologically finite.*

Proof. Clealy, (1.10) (i) implies (1.11) (i). Now (1.10) (ii) implies that, given q,

$$C_{q-1,p} = C_{q-1}, \quad C_{q,p} = C_q, \quad C_{q+1,p} = C_{q+1}; \qquad \text{for p large.}$$

Thus $H_q(C_p) = H_q(C)$ for p large. □

Theorem 1.4. *If the filtration of the chain complex C is homologically finite, then:*
(i) the associated spectral sequence converges finitely;
(ii) the induced filtration of $H(C)$ is finite;
(iii) $E^\infty \cong Gr \circ H(C)$. More precisely,

$$E^\infty_{p,q} \cong (Gr \circ H_q(C))_p = im(H_q(C_p) \to H_q(C))/im(H_q(C_{p-1}) \to H_q(C)).$$

Proof. We claim that $\bar{D}_{p,q} = H_q(C/C_p)$ is negatively graded. This can easily be verified in view of (ii) of equation (1.11). Consider the following short exact sequence of comlexes

$$0 \to C_p \to C \to C/C_p \to 0.$$

Let us apply homology to this short exact sequence to obtain the following long exact sequence

$$\cdots \to H_q(C_p) \to H_q(C) \to H_q(C/C_p) \to \cdots .$$

Fix q. Now for p sufficiently large we have $H_q(C_p) = H_q(C)$ and $H_{q-1}(C_p) = H_q(C)$. Hence $\bar{D}_{p,q} = H_q(C/C_p) = 0$ for large p. Thus $\bar{D}$ is negatively graded.

Let us check that the filtration is homologically finite. Fix q. Now for p sufficiently small we have $D_{p,q} = H_q(C_p) = 0$. So D is positively graded.

Now by Remark 1, $\bar{\alpha} : \bar{D} \to \bar{D}$ is positively stationary. So by Theorem 1.2 α is positively stationary. Since D is positively graded so α is also negatively stationary. Thus α is stationary. Now we apply Theorem 1.1 to obtain (i).

We now prove (ii). The filtration is homologically finite. So given any q there exist p_0 and p_1 such that

$$
\begin{aligned}
H_q(C)_p &= im(H_q(C_p) \to H_q(C)) = 0; && p \leq p_0 \\
H_q(C)_p &= im(H_q(C_p) \to H_q(C)) = H_q(C); && p \geq p_1.
\end{aligned}
$$

Thus the induced filtration is finite.

We now prove (iii). Consider the following exact sequence from the n^{th} derived couple of the exact couple given by IV.1.11,

$$\cdots \to D^n_{p+n-1,q} \xrightarrow{\alpha_n} D^n_{p+n,q} \xrightarrow{\beta_n} E^n_{p,q} \xrightarrow{\gamma_n} D^n_{p-1,q-1} \to \cdots. \tag{1.12}$$

We fix p, q. Suppose n is large so that $E^n_{p,q} = E^\infty_{p,q}$ by (i). Now

$$D^n_{p+n,q} = \alpha^n D_{p,q} = im(H_q(C_p) \to H_q(C_{p+n})).$$

Since filtration of C is homologically finite so $H_q(C_{p+n}) = H_q(C)$ for large n. Thus for large enough n we have

$$D^n_{p+n,q} = im(H_q(C_p) \to H_q(C)).$$

Let us denote $im(H_q(C_p) \to H_q(C))$ by $H_q(C)_p$. Similarly for large n we have

$$D^n_{p+n-1,q} = im(H_q(C_{p-1}) \to H_q(C)).$$

We denote $im(H_q(C_{p-1}) \to H_q(C))$ by $H_q(C)_{p-1}$. It then follows that

$$\alpha_n : D^n_{p+n-1,q} \to D^n_{p+n,q}$$

for large values of n induces the inclusion $H_q(C)_{p-1} \hookrightarrow H_q(C)_p$.

Similarly we have

$$D^n_{p-1,q-1} = \alpha^n D^0_{p-n-1,q-1} = im(H_{q-1}(C_{p-n-1}) \to H_{q-1}(C_{p-1})).$$

For n large this is zero by (1.11) (i). For any value of n, by the exact couple $(D_n, E_n, \alpha_n, \beta_n, \gamma_n)$ we have the following exact sequence

$$\cdots \to D^n_{p+n-1,q} \xrightarrow{\alpha_n} D^n_{p+n,q} \xrightarrow{\beta_n} E^n_{p,q} \xrightarrow{\gamma_n} D^n_{p-1,q-1} \to \cdots.$$

Now for large n we denoted $D^n_{p+n-1,q}$ as $H_q(C)_{p-1}$ and $D^n_{p+n,q}$ as $H_q(C)_p$ and further we proved that α_n corresponds to the inclusion $H_q(C)_{p-1} \hookrightarrow H_q(C)_p$ and $D^n_{p-1,q-1} = 0$. Thus the above exact sequence reduces to the following short exact sequence

$$0 \to H_q(C)_{p-1} \xrightarrow{\alpha_n} H_q(C)_p \xrightarrow{\beta_n} E^n_{p,q} \xrightarrow{\gamma_n} 0.$$

From which it follows that for large n,

$$E^n_{p,q} \cong im\beta_n \cong H_q(C)_p / H_q(C)_{p-1} = (Gr \circ H_q(C))_p.$$

□

2 Only two column or two row spectral sequences

To explicitly illustrate the importance of convergence we discuss two rather simple instances. These examples shows how convergence of spectral sequences helps us understand homology objects better.

We begin by treating the case of two columns.

Proposition 2.1. *Let* $\mathbf{C}$ *be a filtered chain complex. Suppose the associated spectral sequence,* $\{E^r, d^r\}_r$ *converges. Further assume* $E^r_{p,q} = 0$ *if* $p \notin \{0, 1\}$*. Then we have the following short exact sequence for each* q

$$0 \to E^1_{0,q} \to H_q(\mathbf{C}) \to E^1_{1,q} \to 0.$$

Proof. Recall by equation (IV1.16) that bidegree of $d^r = (-r-1, -1)$. For $r = 1$, we get bidegree of $d^1 = (-2, -1)$. Since by our hypothesis $E^r_{p,q} \neq 0$ if and only if $p \in \{0, 1\}$, so

$$d^r_{p,q} = 0 \qquad \forall \quad p, q \quad \text{and} \quad r \geq 1.$$

Thus

$$E^1 = E^2 = \cdots = E^\infty.$$

Since the spectral sequence converges, we have

$$F_p(H_q)/F_{p-1}(H_q) \cong E^\infty_{p,q}.$$

Let us drop $' \cong'$ and write $' ='$ instead. Thus we have

$$F_p(H_q)/F_{p-1}(H_q) = \begin{cases} E^1_{0,q}, & p = 0 \\ E^1_{1,q}, & p = 1 \\ 0, & \text{otherwise.} \end{cases}$$

From above we may infer that the induced filtration on $H_q(\mathbf{C})$ has following form

$$\cdots = F_{-2}(H_q) = F_{-1}(H_q) \subset F_0(H_q) \subset F_1(H_q) = F_2(H_q) = \cdots \subset H_q(\mathbf{C}).$$

We had assumed that the spectral sequence converges, hence we have $\cap F_p(H_q) = 0$ and $\cup F_p(H_q) = H_q(\mathbf{C})$. This implies $F_{-1}(H_q) = 0$ and $F_1(H_q) = H_q(\mathbf{C})$. Thus we have

$$\begin{aligned} F_0(H_q) &= E^1_{0,q}; \\ H_q(\mathbf{C})/F_0(H_q) &= E^1_{1,q}. \end{aligned}$$

In view of above we have the desired short exact sequence

$$0 \to E^1_{0,q} \to H_q(\mathbf{C}) \to E^1_{1,q} \to 0.$$

□

Corollary 2.2. *Suppose $E^r_{p,q} = 0$ if $p \neq 0$ then $E^1_{0,q} = H_q(\mathbf{C})$.*

This is description in terms of $E^1_{p,q}$ only. We would like to express $E^1_{p,q}$ in terms of $E^0_{p,q}$. By definition

$$E^1_{p,q} = \frac{ker(d^0_{p,q} : E^0_{p,q} \to E^0_{p-1,q-1})}{im(d^0_{p+1,q+1} : E^0_{p+1,q+1} \to E^0_{p,q})}.$$

Now $E^0_{p,q} \neq 0$ only if $p \in \{0, 1\}$. So

$$E^1_{0,q} = \frac{ker(d^0_{0,q} : E^0_{0,q} \to E^0_{-1,q-1} = 0)}{im(d^0_{1,q+1} : E^0_{1,q+1} \to E^0_{0,q})} = cokernel(d^0_{1,q+1} : E^0_{1,q+1} \to E^0_{0,q});$$

and

$$E^1_{1,q} = \frac{ker(d^0_{1,q} : E^0_{1,q} \to E^0_{0,q-1})}{im(d^0_{2,q+1} : E^0_{2,q+1} = 0 \to E^0_{1,q})} = ker(d^0_{1,q} : E^0_{1,q} \to E^0_{0,q-1}).$$

From degree considerations one may see that similar situation can arise if we set $E^r_{p,q} = 0$ if $q \neq p$ or $q \neq p + 1$. This is the case of two rows.

Proposition 2.3. *Let $\mathbf{C}$ be a filtered chain complex. Suppose that the associated spectral sequence, $\{E^r, d^r\}_r$ converges. Further assume $E^r_{p,q} = 0$ if $q - p \notin \{0, 1\}$. Then we have a long exact sequence as shown below*

$$\begin{array}{ccccccccc} \longrightarrow & E^1_{p-1,p} & \longrightarrow & H_p(\mathbf{C}) & \longrightarrow & E^1_{p,p} & \xrightarrow{d} & E^1_{p-2,p-1} & \\ & & & & & & & \downarrow & . \\ \longleftarrow & H_{p-2}(\mathbf{C}) & \longleftarrow & E^1_{p-3,p-2} & \xleftarrow[d]{} & E^1_{p-1,p-1} & \longleftarrow & H_{p-1}(\mathbf{C}) & \end{array}$$

Proof. Recall by equation (1.16) that the bi-degree of $d^r_{p,q}$ for any p and q is given by $(-1 - r, -1)$. Thus

$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r-1,q-1}.$$

Now $E^r_{p,q}$ is non-zero only when $q = p$ or $q = p+1$. Thus for $r \geq 2$ we will always have either $E^r_{p,q} = 0$ or $E^r_{p-r-1,q-1} = 0$. Thus $d^r_{p,q} = 0$ whenever $r \geq 2$. Thus $E^2_{p,q} = E^\infty_{p,q}$ for all p and q. We proceed to compute $E^2_{p,q}$. Given our hypothesis we only need to compute $E^2_{p,p}$ and $E^2_{p,p+1}$. We first compute $E^2_{p,p}$. Consider the following diagram

$$\cdots \to E^1_{p+2,p+1} \xrightarrow{d^1_{p+2,p+1}} E^1_{p,p} \xrightarrow{d^1_{p,p}} E^1_{p-2,p-1} \to \cdots .$$

By definition $E^2_{p,p} = ker(d^1_{p,p})/im(d^1_{p+2,p+1})$. But $E^1_{p+2,p+1} = 0$ because $(p + 1) - (p + 2) \notin \{0, 1\}$. Hence

$$E^2_{p,p} = ker(d^1_{p,p}). \tag{2.1}$$

To compute $E^2_{p,p+1}$ we appeal to

$$\cdots \to E^1_{p+2,p+2} \xrightarrow{d^1_{p+2,p+2}} E^1_{p,p+1} \xrightarrow{d^1_{p,p+1}} E^1_{p-2,p} \to \cdots .$$

Notice that $E^1_{p-2,p} = 0$ because $p-(p-2) \notin \{0,1\}$. So

$$E^2_{p,p+1} = E^1_{p,p+1}/im(d^1_{p+2,p+2}). \tag{2.2}$$

The spectral sequence is assumed to be convergent and since $d^3_{p,q} = 0$ for all p, q , so for each p we have,

(i) $F_{p-1}(H_p)/F_{p-2}(H_p) = E^\infty_{p-1,p}$ by definition. By (2.1) this is isomorphic to $E^2_{p-1,p} = E^1_{p-1,p}/im(d^1_{p+1,p+1})$;

(ii) $F_p(H_p)/F_{p-1}(H_p) = E^\infty_{p,p}$ by definition. By (2.2) this is isomorphic to $E^2_{p,p} = ker(d^1_{p,p})$.

Now $F_{p+r}(H_p)/F_{p+r-1}(H_p) = E^\infty_{p+r,p}$ by definition. We have shown that this is isomorphic to $E^2_{p+r,p}$. Thus by our hypothesis $E^\infty_{p+r,p} \neq 0$ only if $r \in \{-1,0\}$. Now for each value of p we have the following

$$\cdots = F_{p-2}(H_p) \subset F_{p-1}(H_p) \subset F_p(H_p) = F_{p+1}(H_q) = F_{p+2}(H_q) = \cdots \subset H_q(\mathbf{C}).$$

Our spectral sequence converges, hence we have $\cup F_p(H_q) = H_q(C)$ and $\cap F_p(H_q) = 0$. Thus it follows that

$$F_p(H_p) = H_p(\mathbf{C})$$

and $F_{p-2}(H_p) = 0$. We summarise all relations we have obtained so far:

i) $F_p(H_p) = H_p(\mathbf{C})$,

ii) $F_p(H_p)/F_{p-1}(H_p) = ker(d^1_{p,p})$,

iii) $F_{p-1}(H_p) = E^1_{p-1,p}/im(d^1_{p+1,p+1})$.

Using the obvious short exact sequences

$$0 \to F_{p-1}(H_p) \to F_p(H_p) \to F_p(H_p)/F_{p-1}(H_p) \to 0$$

and

$$0 \to ker(d^1_{p,p}) \to E^1_{p,p} \to im(d^1_{p,p}) \to 0$$

for various p, we may construct the following long exact sequence using the above relations:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & im(d^1_{p+1,p+1}) & \xrightarrow{\mu} & E^1_{p-1,p} & \longrightarrow & H_p(\mathbf{C}) & \xrightarrow{\epsilon} & ker(d^1_{p,p}) \\ & & & & & & & & \downarrow{\scriptstyle\mu} \\ & & & & \cdots & \underset{\mu}{\longleftarrow} & im(d^1_{p,p}) & \underset{\epsilon}{\longleftarrow} & E^1_{p,p} \end{array}.$$

Here μ are monomorphisms and ϵ are epimorphisms. We may suppress them to obtain the desired long exact sequence. □

Corollary 2.4. *Suppose $E^r_{p,q} = 0$ if $q \neq p$, then $E^1_{p,p} = H_p(\mathbf{C})$.*

3 Only three column or row spectral sequences

Similarly we investigate the case of three column spectral sequence.

Proposition 3.1. *Let* $\mathbf{C}$ *be a filtered chain complex. Suppose the associated spectral sequence,* $\{E^r, d^r\}_r$ *converges. Further assume* $E^r_{p,q} = 0$ *if* $p \notin \{0,1,2\}$*. Then we have the following short exact sequences for each* q

$$0 \to F_1(H_q) \to H_q(\mathbf{C}) \to E^2_{2,q} \to 0, \tag{3.1}$$

$$0 \to E^2_{0,q} \to F_1(H_q) \to E^2_{1,q} \to 0. \tag{3.2}$$

Proof. Recall by equation (IV1.16) that bidegree of $d^r = (-r-1,-1)$. For $r = 2$, we get bidegree of $d^2 = (-3,-1)$. Since by our hypothesis $E^r_{p,q} \neq 0$ if and only if $p \in \{0,1,2\}$, so

$$d^r_{p,q} = 0 \qquad \forall \quad p,q \quad \text{and} \quad r \geq 2.$$

Thus

$$E^2 = E^3 = \cdots = E^\infty.$$

Since the spectral sequence converges, we have

$$F_p(H_q)/F_{p-1}(H_q) \cong E^\infty_{p,q}.$$

Let us drop $'\cong'$ and write $'='$ instead. Thus we have

$$F_p(H_q)/F_{p-1}(H_q) = \begin{cases} E^2_{0,q}, & p = 0 \\ E^2_{1,q}, & p = 1 \\ E^2_{2,q}, & p = 2 \\ 0, & \text{otherwise.} \end{cases}$$

From above we may infer that the induced filtration on $H_q(\mathbf{C})$ has following form

$$\cdots = F_{-2}(H_q) = F_{-1}(H_q) \subset F_0(H_q) \subset F_1(H_q) \subset F_2(H_q) = F_3(H_q) = \cdots \subset H_q(\mathbf{C}).$$

The spectral sequence converges, hence we have $\cap F_p(H_q) = 0$ and $\cup F_p(H_q) = H_q(\mathbf{C})$. This implies $F_{-1}(H_q) = 0$ and $F_2(H_q) = H_q(\mathbf{C})$. Thus we have:

$$\begin{aligned} F_0(H_q) &= E^2_{0,q}, \\ F_1(H_q)/F_0(H_q) &= E^2_{1,q}, \\ H_q(\mathbf{C})/F_1(H_q) &= E^2_{2,q}. \end{aligned}$$

In view of above we have the desired short exact sequences

$$0 \to F_1(H_q) \to H_q(\mathbf{C}) \to E^2_{2,q} \to 0,$$
$$0 \to E^2_{0,q} \to F_1(H_q) \to E^2_{1,q} \to 0.$$

□

As before we try to write $E^2_{p,q}$ in terms of $E^1_{p,q}$. Now the bi-degree is $(-2,-1)$. So

$$E^2_{0,q} = cokernel(d^1_{2,q+1} : E^1_{2,q+1} \to E^1_{0,q}),$$

and

$$E^2_{1,q} = E^1_{1,q},$$

and

$$E^2_{2,q} = kernel(d^1_{2,q} : E^1_{2,q} \to E^1_{0,q-1}).$$

Suppose that we carry out calculations given in proofs of Proposition 2.1 and 3.1 for an r-column spectral sequence. Then we shall obtain $r-1$ short exact sequences.

Proposition 3.2. *Let* $\mathbf{C}$ *be a filtered chain complex. Suppose that the associated spectral sequence,* $\{E^r, d^r\}_r$ *converges. Further assume* $E^r_{p,q} = 0$ *if* $q - p \notin \{0,1,2\}$*. Then for each* p *we have the following two short exact sequences*

$$0 \longrightarrow F_{p-1}(H_p) \longrightarrow H_p(\mathbf{C}) \longrightarrow ker(d^2_{p,p}) \longrightarrow 0;$$

$$0 \longrightarrow \frac{coker(d^1_{p,p+1} : E^1_{p,p+1} \to E^1_{p-2,p})}{im(d^2_{p+1,p+1} : E^2_{p+1,p+1} \to E^2_{p-2,p})} \longrightarrow F_{p-1}(H_p) \longrightarrow E^2_{p-1,p} \longrightarrow 0.$$

Proof. Recall by equation (1.16) that the bi-degree of $d^r_{p,q}$ for any p and q is given by $(-1-r,-1)$. Thus

$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r-1,q-1}.$$

Now $E^r_{p,q}$ is non-zero only when $q-p \in \{0,1,2\}$. Thus for $r \geq 3$ we will always have either $E^r_{p,q} = 0$ or $E^r_{p-r-1,q-1} = 0$. Thus $d^r_{p,q} = 0$ whenever $r \geq 3$. Thus $E^3_{p,q} = E^\infty_{p,q}$ for all p and q. We proceed to compute $E^3_{p,q}$. Given our hypothesis we only need to compute $E^3_{p,p}$ and $E^3_{p,p+1}$ and $E^3_{p,p+2}$. We first compute $E^3_{p,p}$. Consider the following diagram

$$\cdots \to E^2_{p+3,p+1} \xrightarrow{d^2_{p+3,p+1}} E^2_{p,p} \xrightarrow{d^2_{p,p}} E^2_{p-3,p-1} \to \cdots.$$

By definition $E^3_{p,p} = ker(d^2_{p,p})/im(d^2_{p+3,p+1})$. But $E^2_{p+3,p+1} = 0$ because $(p+1)-(p+3) \notin \{0,1,2\}$. Hence

$$E^3_{p,p} = ker(d^2_{p,p}). \tag{3.3}$$

To compute $E^3_{p,p+1}$ we appeal to

$$\cdots \to E^2_{p+3,p+2} \xrightarrow{d^2_{p+3,p+2}} E^2_{p,p+1} \xrightarrow{d^2_{p,p+1}} E^2_{p-3,p} \to \cdots.$$

Notice that $E^2_{p-3,p} = 0$ because $p-(p-3) \notin \{0,1,2\}$ and $E^2_{p+3,p+2} = 0$ because $(p+2)-(p+3) \notin \{0,1,2\}$. So

$$E^3_{p,p+1} = E^2_{p,p+1}. \tag{3.4}$$

Now we compute $E^3_{p,p+2}$. Consider the following diagram

$$\cdots \to E^2_{p+3,p+3} \xrightarrow{d^2_{p+3,p+3}} E^2_{p,p+2} \xrightarrow{d^2_{p,p+2}} E^2_{p-3,p+1} \to \cdots .$$

Since $(p+1)-(p-3) \notin \{0,1,2\}$, so $E^2_{p-3,p+1}=0$. Hence we have

$$E^3_{p,p+2} = E^2_{p,p+2}/im(d^2_{p+3,p+3} : E^2_{p+3,p+3} \to E^2_{p,p+2}). \tag{3.5}$$

Since spectral sequence is assumed to be convergent and since $d^3_{p,q}=0$ for all p,q, so for each p we have

(i) $F_{p-2}(H_p)/F_{p-3}(H_p) = E^{\infty}_{p-2,p}$ by definition. By (3.3) this is isomorphic to $E^3_{p-2,p} = E^2_{p-2,p}/im(d^2_{p+1,p+1})$.

(ii) $F_{p-1}(H_p)/F_{p-2}(H_p) = E^{\infty}_{p-1,p}$ by definition. By (3.4) this is isomorphic to $E^3_{p-1,p} = E^2_{p-1,p}$.

(iii) $F_p(H_p)/F_{p-1}(H_p) = E^{\infty}_{p,p}$ by definition. By (3.5) this is isomorphic to $E^3_{p,p} = ker(d^2_{p,p})$.

Now for each value of p we have the following series of inclusions

$$0 \cdots = F_{p-3}(H_p) \subset F_{p-2}(H_p) \subset F_{p-1}(H_p) \subset F_p(H_p) = F_{p+1}(H_q) = \cdots = H_q(\mathbf{C}).$$

Our spectral sequence converges, hence we have $\cup F_p(H_q) = H_q(C)$ and $\cap F_p(H_q) = 0$. Thus it follows that

$$F_p(H_p) = H_p(\mathbf{C})$$

and $F_{p-3}(H_p) = 0$. We summarise all relations we have obtained so far:

i) $F_p(H_p) = H_p(\mathbf{C})$,

ii) $F_p(H_p)/F_{p-1}(H_p) = ker(d^2_{p,p})$,

iii) $F_{p-1}(H_p)/F_{p-2}(H_p) = E^2_{p-1,p}$,

iv) $F_{p-2}(H_p) = E^2_{p-2,p}/im(d^1_{p+1,p+1})$.

Using the obvious short exact sequences

$$0 \to F_{p-1}(H_p) \to F_p(H_p) \to F_p(H_p)/F_{p-1}(H_p) \to 0$$

and

$$0 \to F_{p-2}(H_p) \to F_{p-1}(H_p) \to F_{p-1}(H_p)/F_{p-2}(H_p) \to 0$$

for various p, we construct the following short exact sequences:

$$0 \longrightarrow F_{p-1}(H_p) \longrightarrow H_p(\mathbf{C}) \longrightarrow ker(d^2_{p,p}) \longrightarrow 0;$$

$$0 \longrightarrow \frac{E^2_{p-2,p}}{im(d^2_{p+1,p+1}:E^2_{p+1,p+1}\to E^2_{p-2,p})} \longrightarrow F_{p-1}(H_p) \longrightarrow E^2_{p-1,p} \longrightarrow 0.$$

Further we can compute $E^2_{p-2,p}$ in terms of $d^1_{p,p+1}$. By definition $E^2_{p-2,p}$ is given by the equation

$$ker(d^1_{p-2,p} : E^1_{p-2,p} \to E^1_{p-4,p-1})/im(d^1_{p,p+1} : E^1_{p,p+1} \to E^1_{p-2,p}).$$

Since $(p-1)-(p-4) \notin \{0,1,2\}$, so $E^1_{p-4,p-1} = 0$. Thus we obtain $E^2_{p-2,p}$ as

$$coker(d^1_{p,p+1} : E^1_{p,p+1} \to E^1_{p-2,p}).$$

Now the proposition follows. □

Chapter VI

Double complexes

1 Filtered Double complex and Spectral sequence

Let $\mathfrak{A}$ be an abelian category.

Definition 1.1. *Let $\{B_{x,y}\}_{(x,y)\in\mathbb{Z}\times\mathbb{Z}}$ be a collection of objects in $\mathfrak{A}$ such that they fit in the following diagram*

$$
\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \leftarrow & B_{-2,2} & \xleftarrow[d^h]{} & B_{-1,2} & \xleftarrow[d^h]{} & B_{0,2} & \xleftarrow[d^h]{} & B_{1,2} & \xleftarrow[d^h]{} & B_{2,2} & \leftarrow & \cdots \\
 & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \\
\cdots & \leftarrow & B_{-2,1} & \xleftarrow[d^h]{} & B_{-1,1} & \xleftarrow[d^h]{} & B_{0,1} & \xleftarrow[d^h]{} & B_{1,1} & \xleftarrow[d^h]{} & B_{2,1} & \leftarrow & \cdots \\
 & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \\
\cdots & \leftarrow & B_{-2,0} & \xleftarrow[d^h]{} & B_{-1,0} & \xleftarrow[d^h]{} & B_{0,0} & \xleftarrow[d^h]{} & B_{1,0} & \xleftarrow[d^h]{} & B_{2,0} & \leftarrow & \cdots \\
 & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \\
\cdots & \leftarrow & B_{-2,-1} & \xleftarrow[d^h]{} & B_{-1,-1} & \xleftarrow[d^h]{} & B_{0,-1} & \xleftarrow[d^h]{} & B_{1,-1} & \xleftarrow[d^h]{} & B_{2,-1} & \leftarrow & \cdots \\
 & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \\
\cdots & \leftarrow & B_{-2,-2} & \xleftarrow[d^h]{} & B_{-1,-2} & \xleftarrow[d^h]{} & B_{0,-2} & \xleftarrow[d^h]{} & B_{1,-2} & \xleftarrow[d^h]{} & B_{2,-2} & \leftarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & &
\end{array}
\quad . \tag{1.1}
$$

We call this diagram a double complex in $\mathfrak{A}$ if the following are satisfied:

(i) $d^h \circ d^h = 0$,

(ii) $d^v \circ d^v = 0$,

(iii) $d^v d^h + d^h d^v = 0$.

We denote the above double complex by $\mathbf{B}$. We shall call d^h the horizontal and d^v the vertical differential of $\mathbf{B}$.
Each square above is anti-commutative. So we have

$$(-1)^x d^h_{x,y-1} d^v_{x,y} = (-1)^{x-1}(d^v_{x-1,y} d^h_{x,y}).$$

Remark. One may replace anti-commutative diagrams above with commutative diagrams. This shall be achieved by setting

$$d'^h := d^h \tag{1.2}$$
$$d'^v := (-1)^x d^v \quad \text{on} \quad B_{x,y}. \tag{1.3}$$

By the following equalities, the commutativity follows

$$(d'^h)_{x,y-1}(d'^v)_{x,y} = (-1)^x d^h_{x,y-1} d^v_{x,y} = (-1)^{x-1}(d^v_{x-1,y} d^h_{x,y}) = (d'^v)_{x-1,y}(d'^h)_{x,y}.$$

Given a double complex, by the total complex construction, we may construct a chain complex. First we define graded module Tot $\mathbf{B}$ as follows:

$$(\text{Tot}\mathbf{B})_n = \bigoplus_{x+y=n} B_{x,y}.$$

Observe, for example $(\text{Tot}\mathbf{B})_0$ is given by the diagonal elements of $\mathbf{B}$. Now we shall show that there is a differentiation of degree -1 on Tot$\mathbf{B}$. We define $d : \text{Tot}\mathbf{B} \to \text{Tot}\mathbf{B}$ as,

$$d = d^h + d^v.$$

Observe that $d^h(B_{x,y}) \subseteq B_{x-1,y}$ and $d^v(B_{x,y}) \subseteq B_{x,y-1}$. So

$$d(B_{x,y}) \subset B_{x-1,y} \oplus B_{x,y-1}.$$

Hence d maps $(\text{Tot}\mathbf{B})_n$ to $(\text{Tot}\mathbf{B})_{n-1}$ indeed. Further by the following equalities

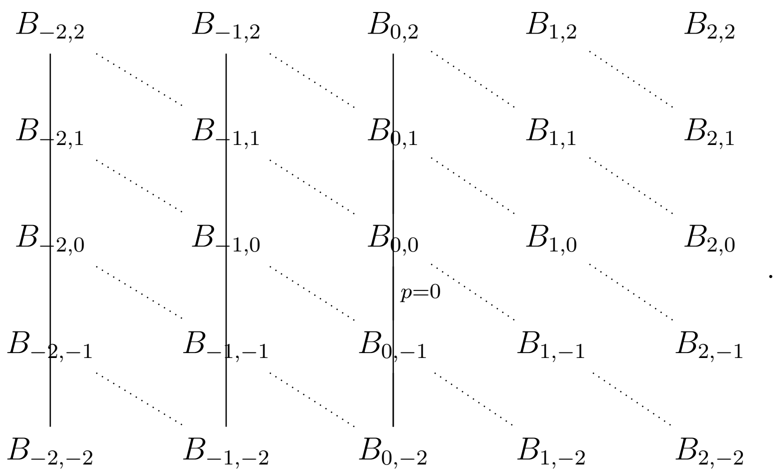
$$d \circ d = (d^h + d^v) \circ (d^h + d^v) = d^h d^h + d^v d^h + d^h d^v + d^v d^v = 0\ .$$

it follows that d is a differentiation operator of degree -1 on Tot $\mathbf{B}$. So Tot $\mathbf{B}$ is a chain complex indeed.

The complex Tot $\mathbf{B}$ may be filtered in two natural ways. The first filtration through columns is denoted ${}^C F_p(\text{Tot}\mathbf{B})$. It is given by the following,

$${}^C F_p(\text{Tot}\mathbf{B})_n = \bigoplus_{\substack{x+y=n \\ x \le p}} B_{x,y}. \tag{1.4}$$

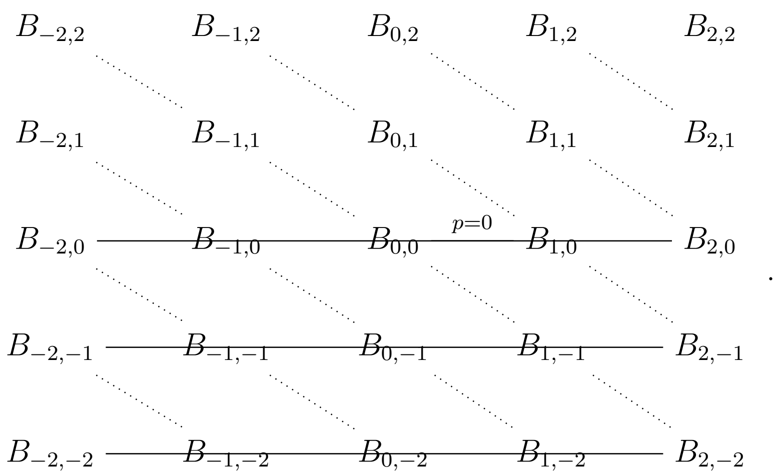
The diagram below illustrates this filtration. For each value of p only those objects which are to the left of the lines will be part of ${}^C F_p(\text{Tot}\mathbf{B})$. One may easily see that this filtration increases with p. We shall refer to this filtration as *column* filtration



Similarly we have the *row* filtration ${}^{R}F_p(\mathrm{Tot}\mathbf{B})$

$$ {}^{R}F_p(\mathrm{Tot}\mathbf{B})_n = \bigoplus_{\substack{x+y=n \\ y\leq p}} B_{x,y}. \tag{1.5}$$

Following is an illustration of (1.5)



Again, notice that the filtration increases with p.

Remark. Both, column and row filtrations of Tot$\mathbf{B}$ are increasing filtrations.

Notice that given a double complex $\mathbf{B}$ we may form partial chain complexes $(\mathbf{B}, d^h)$ and $(\mathbf{B}, d^v)$. Let $H(\mathbf{B}, d^h)$ denote the homology object of $\mathbf{B}$ with respect to d^h. Then d^v induces a differential on $H(\mathbf{B}, d^h)$. By abuse of notation let us denote this induced differentiation also by d^v. Thus we shall obtain $H(H(\mathbf{B}, d^h), d^v)$, the homology of

$H(\mathbf{B}, d^h)$ with respect to d^v. Similarly we have $H(H(\mathbf{B}, d^v), d^h)$. Proposition 1.3 below relates these homology objects to spectral sequences associated with the above filtrations. But let us prove first a technical result.

Let us abbreviate ${}^{C}F_p(\mathrm{Tot}\mathbf{B})$ as F_p. From the diagram corresponding to column filtration or otherwise one may see that the complex F_p/F_{p-1} is given by the p^{th} column of $\mathbf{B}$. That is F_p/F_{p-1} as a graded module is given by

$$F_p/F_{p-1} = \bigoplus_y B_{p,y} \tag{1.6}$$

$$(F_p/F_{p-1})_q = B_{p,q-p}. \tag{1.7}$$

The second equality follows because the total degree must be q. Further, the differential induced by $d = d^h + d^v$ on F_p/F_{p-1} is simply induced by d^v. Further, taking homology of F_p/F_{p-1}, for any y, the differentials d^h induce differentials on

$$H_{y-1}(F_{p-1}/F_{p-2}) \xleftarrow{d^h} H_y(F_p/F_{p-1}). \tag{1.8}$$

By abuse of notation, we will denote these differentials also by d^h.

Now recall the following exact sequence from Chapter IV[eq. (1.8)] which gives the differentials d on $H_q(F_p/F_{p-1})$ as $\beta \circ \gamma$:

$$H_q(F_p/F_{p-1}) \xrightarrow{\gamma} H_{q-1}(F_{p-1}) \xrightarrow{\beta} H_{q-1}(F_{p-1}/F_{p-2}) \quad . \tag{1.9}$$

Let $[b]$ be an element of $H_q(F_p/F_{p-1})$. We choose a representative b of $[b]$ in F_p/F_{p-1}. Then we have the following Lemma whose proof uses several facts specific to the case of the total complex of a double complex.

Lemma 1.1. *The homology class of $\beta\gamma([b])$ in $H_{q-1}(F_{p-1}/F_{p-2})$ is same as that of $d^h(b)$ where d^h is the differential induced by the horizontal differential on the homology as in equation (1.8).*

Proof. Consider

$$(F_{p-1}/F_{p-2})_{q-1} \xleftarrow{d^h} (F_p/F_{p-1})_q,$$

which may alternatively be written as

$$B_{p-1,q-1-(p-1)} \xleftarrow{d^h} B_{p,q-p}.$$

This map may be factorised as shown in the diagram below

$$\begin{array}{ccc} B_{p-1,q-1-(p-1)} & \xleftarrow{\beta'} & \bigoplus_{\substack{x+y=q-1 \\ x \le p-1}} B_{x,y} \\ & \nwarrow^{d^h} & \uparrow d^h \\ & & B_{p,q-p}. \end{array}$$

Here the morphism β' is the usual surjection. We may rewrite as follows

$$\begin{array}{ccc} (F_{p-1}/F_{p-2})_{q-1} & \xleftarrow[\beta']{} & (F_{p-1})_{q-1} \\ & {\scriptstyle d^h}\nwarrow & \uparrow{\scriptstyle d^h} \\ & & (F_p/F_{p-1})_q. \end{array} \tag{1.10}$$

From the digrams (IV 1.7) and (IV 1.8), setting our total complex as the complex C there, it follows that the map induced by β' at the level of homologies is precisely β. Thus taking homology of diagram (1.10), we get

$$\begin{array}{ccc} H_{q-1}(F_{p-1}/F_{p-2}) & \xleftarrow[\beta]{} & H_{q-1}(F_{p-1}) \\ & {\scriptstyle d^h}\nwarrow & \uparrow{\scriptstyle H(d^h)} \\ & & H_q(F_p/F_{p-1}). \end{array} \tag{1.11}$$

So to prove the lemma, it suffices to show that $H(d^h) = \gamma$.

To this end, let us revisit the construction of γ from (IV 1.8), where it arises as a connecting homomorphism. Consider the following commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ \leftarrow\cdots & & (F_{p-1})_{q-1} & \xleftarrow[d^h+d^v]{} & (F_{p-1})_q & & \leftarrow\cdots \\ & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & \\ \leftarrow\cdots & & (F_p)_{q-1} & \xleftarrow[d^h+d^v]{} & (F_p)_q & & \leftarrow\cdots \\ & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \\ \leftarrow\cdots & & (F_p/F_{p-1})_{q-1} & \xleftarrow[d^v]{} & (F_p/F_{p-1})_q & & \leftarrow\cdots \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \tag{1.12}$$

in which columns are exact.

Recall that $[b] \in H_q(F_p/F_{p-1}, d^v)$. We want to choose carefully a $b \in (F_p/F_{p-1})_q = B_{p,q-p}$ as its pre-image and then rather view it as an element in $(F_p)_q$. In fact, an element of $B_{p,q-p}$ may be chosen as the pre-image b of $[b]$. Now following the construction of the connecting homomorphism, in the diagram (1.12), we wish to choose a pre-image of b, viewed as an element of F_p/F_{p-1}, in $(F_p)_q$. We may in fact choose this pre-image to be b itself. Indeed, we have $b \in B_{p,q-p} \subset (F_p)_q$.

Recall that the differentiation d^v on F_p/F_{p-1} is induced by differentiation $d^h + d^v$ on F_p. Thus we have $(d^h + d^v)(b) = d^h(b)$ because $d^v(b) = 0$. Now

$$p_1 d^h(b) = d^v p_2(b) = d^v(b) = 0.$$

Hence $d^h(b)$ is in the kernel of p_1. But $ker(p_1) = im(i_1)$ because the columns are exact. So there exists a b' in $(F_{p-1})_{q-1}$ such that $i_1(b') = d^h(b)$. In fact, $b' = i_1^{-1}(d^h(b))$. By definition of the connecting homomorphism

$$\gamma([b]) = [b'] \in H((F_{p-1})_{q-1}, d^h + d^v).$$

Since i_1 is an inclusion, we may take b' as $d^h(b)$ viewed as an element of $(F_{p-1})_{q-1}$. Thus

$$[d^h(b)] = [b'] = \gamma([b]). \tag{1.13}$$

Here $b \in B_{p,q-p}$ and $B_{p-1,q-p} \xleftarrow{d^h} B_{p,q-p}$. Further $B_{p,q-p} = (F_p/F_{p-1})_q$, $B_{p-1,q-p} \hookrightarrow (F_{p-1})_{q-1}$ and we have a connecting map from $H(F_p/F_{p-1}) \to H(F_{p-1})$ because of the short exact sequence $0 \to F_{p-1} \to F_p \to F_p/F_{p-1} \to 0$ of complexes. Thus equation (1.13) means that the map

$$H_{q-1}(F_{p-1}, d^h + d^v) \xleftarrow{H(d^h)} H_q(F_p/F_{p-1}, d^v)$$

induced by d^h at the level of homologies equals γ i.e

$$H(d^h) = \gamma. \tag{1.14}$$

□

Suppose we had chosen to work with row filtrations instead. Taking homology of F_p/F_{p-1}, for any x, the differentials d^v induce differentials on

$$H_{x-1}(F_{p-1}/F_{p-2}) \xleftarrow{d^v} H_x(F_p/F_{p-1}). \tag{1.15}$$

By abuse of notation, we will denote these differentials also by d^v. Then we will see that the corresponding $\beta\gamma$ is same as the differential induced by the horizontal differential d^v. We state this as yet another Lemma. But we don't intend to give a proof.

Lemma 1.2. *The map $\beta\gamma(b)$ from $H_q({}^RF_p/{}^RF_{p-1})$ to $H_{q-1}({}^RF_{p-1}/{}^RF_{p-2})$ is same as d^v, where d^v is the differential induced by the vertical differential at the level of homology as in equation (1.15).*

Now we are in a position to state and prove the following Proposition.

Proposition 1.3. *Let B be a double complex. Let $B_{p,*}$ denote the p-th column and $B_{*,p}$ denote the p-th row. For the column spectral sequence associated with the filtration (1.4), namely*

$$^CF_p(Tot\mathbf{B})_n = \bigoplus_{\substack{x+y=n \\ x\le p}} B_{x,y},$$

we have

$$^CE_0^{p,q} = H_{q-p}(B_{p,*}, d^v), \quad {}^CE_1^{p,q} = H_p(H_{q-p}(\mathbf{B}, d^v), d^h). \tag{1.16}$$

Similarly, for the row spectral sequence associated with the filtration (1.5), namely

$$^{R}F_p(\mathit{Tot}\mathbf{B})_n = \bigoplus_{\substack{x+y=n \\ y\leq p}} B_{x,y},$$

we have

$$^{R}E_0^{p,q} = H_{q-p}(B_{*,p}, d^h), \quad ^{R}E_1^{p,q} = H_p(H_{q-p}(\mathbf{B}, d^h), d^v). \tag{1.17}$$

Proof. We prove (1.16). Let us abbreviate $^{C}F_p(\mathrm{Tot}\mathbf{B})$ as F_p. We know [See ChapterIV eq.(1.9)] that the first term of the spectral sequence associated with a filtered chain complex is given by $H_q(F_p/F_{p-1})$. We compute F_p/F_{p-1} first. The q^{th} term $(F_p/F_{p-1})_q$ of F_p/F_{p-1} is $B_{p,y}$ such that $p+y=q$. So we have

$$(F_p/F_{p-1})_q = B_{p,q-p}.$$

Thus

$$^{C}E_0^{p,q} = H_q(^{C}F_p(\mathrm{Tot}\mathbf{B})/^{C}F_{p-1}(\mathrm{Tot}\mathbf{B})) = H_{q-p}(B_{p,*}, d^v).$$

Recall that the differential at E^0 is given by $\beta\gamma$ where γ and β are as given in (1.9). Now by Lemma 1.1 the composition $\beta \circ \gamma$ agrees with the horizontal differential d^h. Thus we have

$$^{C}E_1^{p,q} = H_p(H_{q-p}(\mathbf{B}, d^v), d^h).$$

Using similar arguments and Lemma 1.2 we may prove (1.17). □

Remark. We may of course replace d^h and d^v in the above discussion with d'^h and d'^v.

Two column double complexes are the simplest examples of double complexes. To illustrate the technique of filtering, we revisit this example. The reader will notice that the approach of filtering with columns works out in a fairly elementary way. However, filtering two columns by rows tantamounts to reinventing the wheel!

Proposition 1.4. *Let* $\mathbf{B}$ *be a two column double complex. That is* $B_{p,q}$ *is non-zero only when* $p = 0$ *or* $p = 1$*. Then we have a short exact sequence:*

$$0 \longrightarrow {}^{C}E^1_{0,q} \longrightarrow H_q(\mathit{Tot}\mathbf{B}) \longrightarrow {}^{C}E^1_{1,q} \longrightarrow 0.$$

Proof. (i) By proposition 1.3 we have

$$^{C}E^0_{p,q} = H_{q-p}(B_{p,*}, d^v).$$

But $B_{p,*}$ is non-zero only when $p = 0$ or 1. Thus

$$^{C}E^0_{p,q} = \begin{cases} H_q(B_{0,*}, d^v) & \text{if} \quad p = 0 \\ H_{q-1}(B_{1,*}, d^v) & \text{if} \quad p = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote the complex $(B_{0,*}, d^v)$ by $\mathbf{X}$ and $(B_{1,*}, d^v)$ by $\mathbf{Y}$. Now

$$(\mathrm{Tot}\mathbf{B}_q) = B_{0,q} \oplus B_{1,q-1}.$$

So

$$0 \longrightarrow X_q \longrightarrow (\mathrm{Tot}\mathbf{B})_q \longrightarrow Y_{q-1} \longrightarrow 0$$

is a short exact sequence. Notice that there is a shift in the index of $\mathbf{Y}$. We shall denote the shifted complex by $\mathbf{Y}[-1]$. Notice that $Y[-1]_q = Y_{q-1}$.

Remark. The following is a short exact sequence of complexes.

$$0 \longrightarrow \mathbf{X} \xrightarrow{i} \mathrm{Tot}\mathbf{B} \xrightarrow{p} \mathbf{Y}[-1] \longrightarrow 0.$$

Whenever we have a short exact sequence of complexes we can have a long exact sequence of homology objects. The following is that long exact sequence in our context

$$\longrightarrow H_q(\mathbf{X}) \xrightarrow{i*} H_q(\mathrm{Tot}\mathbf{B}) \xrightarrow{p*} H_q(\mathbf{Y}[-1]) \xrightarrow{\delta} H_{q-1}(\mathbf{X}) \xrightarrow{i*} H_{q-1}(\mathrm{Tot}\mathbf{B}) \xrightarrow{p*} H_{q-1}(\mathbf{Y}[-1]) \longrightarrow .$$

By Lemma 1.1 the connecting homomorphism δ is same as the map induced by horizontal differential at the level quotients. Recall that $Y[-1]_q = Y_{q-1}$, hence we may rewrite the long exact sequence as follows,

$$\longrightarrow H_q(\mathbf{X}) \xrightarrow{i*} H_q(\mathrm{Tot}\mathbf{B}) \xrightarrow{p*} H_{q-1}(\mathbf{Y}) \xrightarrow{\delta} H_{q-1}(\mathbf{X}) \xrightarrow{i*} H_{q-1}(\mathrm{Tot}\mathbf{B}) \xrightarrow{p*} H_{q-2}(\mathbf{Y}) \longrightarrow .$$

Consider the following short exact sequence

$$0 \longrightarrow ker(p*) \longrightarrow H_q(\mathrm{Tot}\mathbf{B}) \longrightarrow im(p*) \longrightarrow 0 \quad .$$

The long sequence is exact. So $ker(p*) = im(i*)$. Now $im(i*) \cong H_q(\mathbf{X})/ker(i*)$. Again by exactness of long sequence $ker(i*) = im(\delta)$ and $im(p*) = ker(\delta)$. Thus we obtain a short exact sequence which is isomorphic to the one above

$$0 \longrightarrow coker(\delta) \longrightarrow H_q(\mathrm{Tot}\mathbf{B}) \longrightarrow ker(\delta) \longrightarrow 0. \tag{1.18}$$

Let us return to the spectral sequence now. The E^1 level is given by

$${}^C E^1_{p,q} = H_p(H_{q-p}(\mathbf{B}, d^v), d^h).$$

In view of discussion thus far

$$ {}^{C}E^{1}_{p,q} = \begin{cases} H_0(H_q(\mathbf{X}), d^h) & \text{if} \quad p = 0 \\ H_1(H_{q-1}(\mathbf{Y}), d^h) & \text{if} \quad p = 1 \\ 0 & \text{otherwise.} \end{cases} $$

But we know that the induced map d^h and the connecting homomorphism δ are same. Hence computing ${}^{C}E^{1}_{p,q}$ essentially amounts to computing the homology of following complex

$$ 0 \longleftarrow H_q(\mathbf{X}) \underset{\delta}{\longleftarrow} H_q(\mathbf{Y}) \longleftarrow 0. $$

Thus we have

$$ H_0(H_q(\mathbf{X}), \delta) = coker(\delta) $$

and

$$ H_1(H_q(\mathbf{Y}), \delta) = ker(\delta). $$

So the spectral sequence at E^1 level is given by

$$ {}^{C}E^{1}_{p,q} = \begin{cases} coker(\delta) & \text{if} \quad p = 0 \\ ker(\delta) & \text{if} \quad p = 1 \\ 0 & \text{otherwise} \end{cases} $$

We may now rewrite (1.18) to obtain

$$ 0 \longrightarrow {}^{C}E^{1}_{0,q} \longrightarrow H_q(\mathrm{Tot}\mathbf{B}) \longrightarrow {}^{C}E^{1}_{1,q} \longrightarrow 0. $$

□

We now come to a natural question. When does the spectral sequences associated with a double complex converge? Proposition 1.5 below gives a sufficient condition.

Definition 15. We say that the double complex $\mathbf{B}$ is *positive* if there exists n_0 such that

$$ B_{x,y} = 0 \quad \text{if} \quad x < n_0 \quad \text{or} \quad y < n_0. \tag{1.19} $$

Proposition 1.5. *Let* $\mathbf{B}$ *be positive. Then both the first and second spectral sequence converges finitely to the graded object associated with* $H_n(Tot\ \mathbf{B})$*, suitably filtered.*

Proof. In view of Theorem V.1.4 one only need to verify that the filtrations (1.4) and (1.5) are finite. Given (1.19), for a fixed n such that $n = r + s$ we have $B_{r,s} \neq 0\ iff$ $n_0 \leq r \leq n - n_0$. Thus,

$$ {}^{C}F_p(\mathrm{Tot}\mathbf{B})_n = 0 \quad \text{if} \quad p \leq n_0 - 1 $$

$$ {}^{C}F_p(\mathrm{Tot}\mathbf{B})_n = (\mathrm{Tot}\mathbf{B})_n \quad \text{if} \quad p \geq n - n_0\ ; $$

and similarly for row filtration. □

2 Some Examples

We give alternate proofs for some familiar results in homological algebra using spectral sequences. Our approach may be summarized as follows:

1. Given a diagram, one completes it to a double complex with arrows oriented as in diagram (1.1).
2. Now one considers both the filtrations - by rows and columns and deduces the two spectral sequences.
3. From E_2 of one SS, one deduces information about E_∞. From this one extracts information about the other E_2.

We recall the following Propositions and Corollaries from Chapter V. We shall use them throughout this section.

Proposition 2.1. *Let* $\mathbf{C}$ *be a filtered chain complex. Suppose the associated spectral sequence,* $\{E^r, d^r\}_r$ *converges. Further assume* $E^r_{p,q} = 0$ *if and only if* $p \notin \{0, 1\}$*. Then we have the following short exact sequence for each* q

$$0 \to E^1_{0,q} \to H_q(\mathbf{C}) \to E^1_{1,q} \to 0.$$

Corollary 2.2. *Suppose* $E^r_{p,q} = 0$ *if when* $p \neq 0$ *then* $E^1_{0,q} = H_q(\mathbf{C})$*.*

Proposition 2.3. *Let* $\mathbf{C}$ *be a filtered chain complex. Suppose that the associated spectral sequence,* $\{E^r, d^r\}_r$ *converges. Further assume* $E^r_{p,q} \neq 0$ *if and only if* $q - p \in \{0, 1\}$*. Then we have a long exact sequence as shown below*

$$\begin{array}{ccccccccc}
\longrightarrow & E^1_{p-1,p} & \longrightarrow & H_p(\mathbf{C}) & \longrightarrow & E^1_{p,p} & \xrightarrow{d} & E^1_{p-2,p-1} & \\
 & & & & & & & \downarrow & \\
\longleftarrow & H_{p-2}(\mathbf{C}) & \longleftarrow & E^1_{p-3,p-2} & \xleftarrow[d]{} & E^1_{p-1,p-1} & \longleftarrow & H_{p-1}(\mathbf{C}). &
\end{array}$$

Corollary 2.4. *Suppose* $E^r_{p,q} = 0$ *if* $q \neq p$ *then* $E^1_{p,p} = H_p(\mathbf{C})$*.*

Example 2.1 (Five Lemma)**.** *Consider the following commutative diagram*

$$\begin{array}{ccccccccc}
A & \xrightarrow{f} & B & \longrightarrow & C & \longrightarrow & D & \xrightarrow{g} & E \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\
A' & \xrightarrow{f'} & B' & \longrightarrow & C' & \longrightarrow & D & \xrightarrow{g'} & E
\end{array} \tag{2.1}$$

with exact rows. Five lemma says:

(i) If β *,* δ *are monomorphisms and* α *is an epimorphism, then* γ *is a monomorphism;*

(ii) If β , δ are epimorphisms and ϵ is a monomorphism, then γ is an epimorphism.

Proof. We prove (ii) using spectral sequences. Let us rewrite (2.1) to obtain a positive double complex;

$$\begin{array}{ccccccccccccccc}
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \longleftarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
coker(g) & \longleftarrow & E & \underset{g}{\longleftarrow} & D & \longleftarrow & C & \longleftarrow & B & \underset{f}{\longleftarrow} & A & \longleftarrow & ker(f) & \longleftarrow & 0 \longleftarrow \\
\downarrow & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow \\
coker(g') & \longleftarrow & E' & \underset{g'}{\longleftarrow} & D' & \longleftarrow & C' & \longleftarrow & B' & \underset{f'}{\longleftarrow} & A' & \longleftarrow & ker f' & \longleftarrow & 0 \longleftarrow .
\end{array}$$

Clearly the above diagram is a double complex. Let us denote it by $\mathbf{B}$. Thus by Proposition 1.5 both the spectral sequences associated to Tot$\mathbf{B}$ converges. Let us first compute the spectral sequence associated with row filtration. The E^0 level is given by

$${}^{R}E^0_{p,q} = H_{q-p}(B_{*,p}, d_h).$$

Since the rows here are exact, so ${}^{R}E^0_{p,q} = 0$ for all p, q. Now we shall apply Corollary 2.2 to obtain $H_q(\text{Tot}\mathbf{B}) = 0$ for all q.

We want to show that $coker(\gamma) = 0$. Observe that

$$coker(\gamma) = H_0(B_{3,*}, d^v).$$

Similarly

$$\begin{aligned}
coker(\beta) &= H_0(B_{4,*}, d^v);\\
coker(\delta) &= H_0(B_{2,*}, d^v);\\
ker(\epsilon) &= H_1(B_{1,*}, d^v).
\end{aligned}$$

We are given that each of these objects is zero. Now consider the spectral sequence associated with column filtration of Tot$\mathbf{B}$. We know from Proposition 1.3 that ${}^{C}E^0_{p,q} = H_{q-p}(B_{p,*}, d^v)$. Hence we see that

$$\begin{aligned}
{}^{C}E^0_{3,3} &= H_0(B_{3,*}, d^v) = coker(\gamma));\\
{}^{C}E^0_{4,4} &= H_0(B_{4,*}, d^v) = 0;\\
{}^{C}E^0_{2,2} &= H_0(B_{2,*}, d^v) = 0;\\
{}^{C}E^0_{1,2} &= H_1(B_{1,*}, d^v) = 0.
\end{aligned} \tag{2.2}$$

Clearly ${}^{C}E^1_{1,2} = 0$. Recall that the differential d^0 at E^0 level has bidegree (-1,-1). We obtain ${}^{C}E^1_{3,3}$ as the homology of ${}^{C}E^0_{3,3}$ with respect to d^0. That is ${}^{C}E^1_{3,3} = ker(d^0_{3,3})/im(d^0_{4,4})$. We illustrate these maps more clearly below

$${}^{C}E^0_{4,4} \xrightarrow{d^0_{4,4}} {}^{C}E^0_{3,3} \xrightarrow{d^2_{3,3}} {}^{C}E^0_{2,2}.$$

But the terms on either side are zero. So

$$^{C}E^1_{3,3} = {}^{C}E^0_{3,3} = coker(\gamma).$$

Moreover observe that our double complex have non-zero entries only in bottom two rows. So $H_{q-p}(B_{p,*}, d_v)$ is non-zero only when $q = p$ or $q = p+1$. Thus we may invoke Proposition 2.3 to obtain the following long exact sequence

$$\to {}^{C}H_3(\mathrm{Tot}\mathbf{B}) \to {}^{C}E^1_{3,3} \to {}^{C}E^1_{1,2} \to .$$

The terms on either side of $E^1_{3,3}$ is 0. So

$$coker(\gamma) = E^1_{3,3} = 0.$$

One may prove (i) similarly. □

Our next example is of Snake Lemma. We don't intend to give the entire proof here. Usually the construction of connecting homomorphism involves a choice. Using spectral sequence we eliminate the need for choice.

Example 2.2 (Snake Lemma)**.** *Snake lemma says, given the following commutative diagram with exact rows;*

$$\begin{array}{ccccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C & & \end{array}$$

we can obtain the following long exact sequence

$$0 \longrightarrow ker(\alpha) \longrightarrow ker(\beta) \longrightarrow ker(\gamma) \xrightarrow{\delta} coker(\alpha) \longrightarrow coker(\beta) \longrightarrow coker(\gamma) \longrightarrow 0.$$

Proof. As mentioned already we only intend to show how to construct δ. Consider the following positive double complex

$$\begin{array}{cccccccccccccc} \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longleftarrow & 0 & \longleftarrow & C & \underset{g}{\longleftarrow} & B & \underset{f}{\longleftarrow} & A & \longleftarrow & ker(f) & \longleftarrow & 0 & \longleftarrow \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow & \\ 0 & \longleftarrow & coker(g') & \longleftarrow & C' & \underset{g'}{\longleftarrow} & B' & \underset{f'}{\longleftarrow} & A' & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow . \end{array}$$

The rows are exact, so the horizontal homologies vanish. Thus ${}^{R}E^{\infty}_{p,q} ={}^{R} E^{0}_{p,q} = 0$. So we have both spectral sequences converging to zero. If we perform our computations as in the case of five lemma we shall see that the spectral sequence at ${}^{C}E^{0}$ level looks as follows. We will show only the non-zero terms.

$$ker(\gamma) \xleftarrow[g*]{} ker(\beta) \xleftarrow[f*]{} ker(\alpha) \longleftarrow ker(f) \longleftarrow 0$$

$$0 \longleftarrow coker(g') \longleftarrow coker(\gamma) \xleftarrow[g'*]{} coker(\beta) \xleftarrow[f'*]{} coker(\alpha) \qquad .$$

It can be verified that both the rows are exact. We shall skip this verification. Since the rows are exact, so ${}^{C}E^{1}$ shall have only two non-zero entries, $coker(g*)$ and $ker(f'*)$. Clearly $coker(g*) = {}^{C}E^{1}_{2,3}$ and $ker(f'*) = {}^{C}E^{1}_{4,4}$. We have a map

$$d^{1}_{4,4} : {}^{C}E^{1}_{4,4} \to {}^{C}E^{1}_{2,3}.$$

Our double complex have non-zero entries only in bottom two rows. So $H_{q-p}(B_{p,*}, d_v)$ is zero if $q \neq p$ or $q \neq p + 1$. Thus in view of Corollary 2.4 ${}^{C}E^{2}_{p,q} ={}^{C} E^{\infty}_{p,q} = 0$. This (${}^{C}E^{2}_{p,q} = 0 \quad \forall \quad p, q$) is possible only if $d^{1}_{4,4}$ is an isomorphism. Let us denote the inverse of $d^{1}_{4,4}$ by ψ. Now we define δ such that the square below is commutative

$$\begin{array}{ccccccccc} \longrightarrow & ker(\beta) & \xrightarrow{g*} & ker(\gamma) & \overset{\delta}{\dashrightarrow} & coker(\alpha) & \xrightarrow{f'*} & coker(\beta) & \longrightarrow \\ & & & \downarrow & & \uparrow & & & \\ & & & coker(g*) & \xrightarrow{\psi} & ker f'* & & & . \end{array}$$

Thus we have constructed δ such that the row is exact. □

Example 2.3 (Balancing Tor). *Let A be a right R-module and C be a left R-module. Now $A \otimes_R -$ and $-_R \otimes C$ are functors from Category of R-modules to Category of R-modules. Consider the derived functors $Tor_n(A, -)$ and $Tor_n(-, C)$. Using spectral sequences we prove that*

$$Tor_n(A, -)(C) = Tor_n(-, C)(A).$$

Proof. Let

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

be a projective resolution of A. Similarly we choose a projective resolution

$$\cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow B \longrightarrow 0$$

of C. Define

$$B_{p,q} := P_p \otimes Q_q.$$

Let $\mathbf{B} = \{B_{p,q}\}_{p,q}$. Following is a diagrammatic representation of $\mathbf{B}$

$$
\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \\
P_0 \otimes Q_2 & \longleftarrow & P_1 \otimes Q_2 & \longleftarrow & P_2 \otimes Q_2 & \longleftarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
P_0 \otimes Q_1 & \longleftarrow & P_1 \otimes Q_1 & \longleftarrow & P_2 \otimes Q_1 & \longleftarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
P_0 \otimes Q_0 & \longleftarrow & P_1 \otimes Q_0 & \longleftarrow & P_2 \otimes Q_0 & \longleftarrow & .
\end{array}
$$

Since $P_p \otimes -$ and $- \otimes Q_p$ are functors, so $\mathbf{B}$ is a double complex. Projective modules are flat. Hence rows and columns of $\mathbf{B}$ are exact.
Notice that $\mathbf{B}$ is positive. So both the spectral sequences associated with $\mathbf{B}$ converges. We compute ${}^C E$ first. Recall that

$$ {}^C E^0_{p,q} = H_{q-p}(B_{p,*}, d^v). $$

The columns are exact, so we have non-zero entries coming from only the bottom row of $\mathbf{B}$. Now $H_0(B_{p,*}, d^v) = H_0(P_p \otimes Q_*)$. So

$$ H_0(P_p \otimes Q_*) = coker(P_p \otimes Q_1 \to P_p \otimes Q_0). $$

Since P_p is flat, so we have

$$ coker(P_p \otimes Q_1 \to P_p \otimes Q_0) = P_p \otimes C. $$

Hence we have

$$ {}^C E^0_{p,q} = \begin{cases} P_p \otimes C & \text{if} \quad p = q; \\ 0 & \text{if} \quad p \neq q. \end{cases} $$

We illustrate ${}^C E^0$ below. We have the sequence

$$ \cdots \leftarrow E^0_{p-1,p-1} \leftarrow E^0_{p,p} \leftarrow \cdots . $$

In other words, we have the sequence

$$ P_0 \otimes C \longleftrightarrow P_1 \otimes C \longleftarrow P_2 \otimes C \longleftarrow \cdots . $$

Now we compute ${}^C E^1_{p,p}$ only, since the other terms are all zero. We know that

$$ {}^C E^1_{p,p} = H_p(H_0(\mathbf{B}, d^v), d^h). $$

Hence

$$ {}^C E^1_{p,p} = H_p(P_* \otimes C, d^h). $$

By definition $H_p(P_p \otimes C)$ is $Tor_p(-, C)(A)$. Our spectral sequence clearly satisfies Corollary 2.4. Thus we have ${}^C E^1_{p,p} = H_p(\mathrm{Tot}\mathbf{B})$. Or in other words $H_p(\mathrm{Tot}\mathbf{B})$ is precisely $Tor_p(-, C)(A)$.

Now we turn to the spectral sequence associated with row filtration. Again the rows in $\mathbf{B}$ are exact. So only the last term from each row will survive after applying homology. Thus we have

$$ {}^R E^0_{p,q} = \begin{cases} H_0(B_{*,p}, d^h) & \text{if} \quad q = p; \\ 0 & \text{if} \quad q \neq p. \end{cases} $$

Clearly $H_0(B_{*,p}, d^h) = A \otimes Q_q$. The following is a representation of ${}^R E^0$

$$ \begin{array}{c} \vdots \\ \downarrow \\ A \otimes Q_2 \\ \downarrow \\ A \otimes Q_1 \\ \downarrow \\ A \otimes Q_0. \end{array} $$

Notice that ${}^R E^1_{p,q}$ is non-zero only when $p = q$. So we compute ${}^R E^1_{p,p}$ only. Now ${}^R E^1_{p,p}$ is given by

$$ H_p(H_0(\mathbf{B}, d^h), d^v) = H_p(A \otimes Q_p). $$

Thus by definition ${}^R E^1_{p,q}$ is $Tor_p(A, -)(C)$. Clearly ${}^R E$ also satisfies the Corollary 2.4. Thus we have that $H_p(\mathrm{Tot}\mathbf{B}) = Tor_p(A, -)(C)$. So

$$ Tor_p(A, -)C = Tor_p(-, C)(A). $$

□

Example 2.4 (Base Change for Tor)**.** *Let R, S be two rings. Let $f : R \to S$ be a ring homomorphism. Now consider an R-module A and an S-module C. Via f we can also consider C as an R-module. Our aim is to understand how $Tor^R_n(-, C)$ and $Tor^S_n(-, C)$ are related. We employ spectral sequences for that.*

As in previous example let

$$ \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0 $$

be a projective resolution of A. Let

$$ \cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow C \longrightarrow 0 $$

be a projective resolution of C as an S-module. We construct a positive double complex $\mathbf{B}$

$$\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \\
P_0 \otimes Q_2 & \longleftarrow & P_1 \otimes Q_2 & \longleftarrow & P_2 \otimes Q_2 & \longleftarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
P_0 \otimes Q_1 & \longleftarrow & P_1 \otimes Q_1 & \longleftarrow & P_2 \otimes Q_1 & \longleftarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
P_0 \otimes Q_0 & \longleftarrow & P_1 \otimes Q_0 & \longleftarrow & P_2 \otimes Q_0 & \longleftarrow & .
\end{array}$$

Notice that Q_p need not be projective as R-module. Hence only the columns will be exact this time. So as in previous example we obtain

$$H_q(\mathrm{Tot}\mathbf{B}) = Tor_q(A, -)(C).$$

So in view of Proposition 1.5 the row spectral sequence ${}^R E^r$ converges to graded associated with $Tor_q(A, -)(C) = Tor_q(A, C)$ filtered row-wise.

We use $P_p \otimes_R Q_q = (P_p \otimes_R S) \otimes_S Q_q$ to rewrite $\mathbf{B}$ as follows

$$\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \\
(P_0 \otimes_R S) \otimes_S Q_2 & \longleftarrow & (P_1 \otimes_R S) \otimes_S Q_2 & \longleftarrow & (P_2 \otimes_R S) \otimes_S Q_2 & \longleftarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
(P_0 \otimes_R S) \otimes_S Q_1 & \longleftarrow & (P_1 \otimes_R S) \otimes_S Q_1 & \longleftarrow & (P_2 \otimes_R S) \otimes_S Q_1 & \longleftarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
(P_0 \otimes_R S) \otimes_S Q_0 & \longleftarrow & (P_1 \otimes_R S) \otimes_S Q_0 & \longleftarrow & (P_2 \otimes_R S) \otimes_S Q_0 & \longleftarrow & .
\end{array}$$

Since Q_p is a projective S-module, so $- \otimes_S Q_p$ is an exact functor. By definition $H_p(P_* \otimes_R S) = Tor_p(-, S)(A)$. Thus we have the following as our row spectral sequence at E^0 level

$$\begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \\
Tor_0^R(A, S) \otimes_S Q_2 & Tor_1^R(A, S) \otimes_S Q_2 & Tor_2^R(A, S) \otimes_S Q_2 & \cdots \\
\downarrow & \downarrow & \downarrow & \\
Tor_0^R(A, S) \otimes_S Q_1 & Tor_1^R(A, S) \otimes_S Q_1 & Tor_2^R(A, S) \otimes_S Q_1 & \cdots \\
\downarrow & \downarrow & \downarrow & \\
Tor_0^R(A, S) \otimes_S Q_0 & Tor_1^R(A, S) \otimes_S Q_0 & Tor_2^R(A, S) \otimes_S Q_0 & \cdots
\end{array} .$$

Here $Tor^R_{q-p}(A,S)\otimes_S Q_p$ is actually the ${}^RE^0_{p,q}$ term. Now the complexes in these columns are the ones used to calculate the derived functors of $-\otimes_S C$. So the pq-th entry at the E^1 level is

$$Tor^S_p(Tor^R_{q-p}(A,S),C).$$

Thus far we have proved the following proposition.

Proposition 2.5. *Let R, S be two rings. Let $f: R\to S$ be a ring homomorphism. Now consider an R-module A and an S-module C. Then there exist a spectral sequence converging to the graded module associated with $Tor^R_q(A,C)$ such that*

$$E^1_{p,q}=Tor^S_p(Tor^R_{q-p}(A,S),C).$$

Corollary 2.6. *Suppose S is flat as an $R-$ module. Then $Tor^R_n(A,S)=0$ for $n>0$. Thus at E^1 level only the bottom row of spectral sequence is non-zero. Hence by Corollary 2.2 we have*

$$Tor^S_n(A\otimes_R S,C)=Tor^R_n(A,C).$$

Example 2.5 (The Universal Coefficient Theorem)**.** *Let $C=(C_*,d_*)$ be a a chain complex consisting of free abelian groups. Let A be any abelian group. Then the universal coefficient theorem predicts the existence of the following short exact sequence*

$$0\longrightarrow H_n(C_*)\otimes A\longrightarrow H_n(C_*\otimes A)\longrightarrow Tor_1(H_{n-1}(C),A)\longrightarrow 0.$$

Proof. Let

$$\cdots\longrightarrow P_2\longrightarrow P_1\longrightarrow P_0\longrightarrow A\longrightarrow 0$$

be a projective resolution of A. We obtain a double complex $\mathbf{B}$ by setting $B_{p,q}=P_p\otimes C_q$

$$\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \\
P_0\otimes C_2 & \longleftarrow & P_1\otimes C_2 & \longleftarrow & P_2\otimes C_2 & \longleftarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
P_0\otimes C_1 & \longleftarrow & P_1\otimes C_1 & \longleftarrow & P_2\otimes C_1 & \longleftarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
P_0\otimes C_0 & \longleftarrow & P_1\otimes C_0 & \longleftarrow & P_2\otimes C_0 & \longleftarrow & .
\end{array}$$

We first look at the spectral sequence associated with filtration by columns. Since $P_p\otimes -$ is exact, so

$$H_q(P_p\otimes C_*)=P_p\otimes H_q(C_*).$$

Thus our the spectral sequence at E^0 level looks as follows.

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ P_0 \otimes H_2(C) & \longleftarrow & P_1 \otimes H_2(C) & \longleftarrow & P_2 \otimes H_2(C) & \longleftarrow & \\ P_0 \otimes H_1(C) & \longleftarrow & P_1 \otimes H_1(C) & \longleftarrow & P_2 \otimes H_1(C) & \longleftarrow & \\ P_0 \otimes H_0(C) & \longleftarrow & P_1 \otimes H_0(C) & \longleftarrow & P_2 \otimes H_0(C) & \longleftarrow & \end{array}$$

Here $P_p \otimes H_{q-p}(C)$ is the ${}^C E^0_{p,q}$ term of spectral sequence.
These complexes along rows are the ones used for computing derived functors of $A\otimes -$. Thus

$${}^C E^1_{p,q} = Tor_p^{\mathbb{Z}}(A, H_{q-p}(C)).$$

Let us look at this more closely. Given a C_n we have the following short exact sequence

$$0 \longrightarrow im(d_{n+1}) \longrightarrow ker(d_n) \longrightarrow H_n(C) \longrightarrow 0 \quad .$$

Let us apply $A\otimes -$ to this short exact sequence. Then we shall obtain the following long exact sequence

$$\longrightarrow Tor_2(A, im(d_n+1)) \longrightarrow Tor_2(A, ker(d_n)) \longrightarrow Tor_2(A, H_n(C)) \longrightarrow$$
$$\longrightarrow Tor_1(A, im(d_n+1)) \longrightarrow Tor_1(A, ker(d_n)) \longrightarrow Tor_1(A, H_n(C)) \longrightarrow$$
$$\longrightarrow Tor_0(A, im(d_n+1)) \longrightarrow Tor_0(A, ker(d_n)) \longrightarrow Tor_0(A, H_n(C)).$$

We may obtain such a long exact sequence for each n. Here $ker(d_n)$ and $im(d_{n+1})$ are subgroups of C_n. Since C_n is free, so it's subgroups are also free. Free groups are flat. Hence higher Tor groups of $ker(d_n)$ and $im(d_{n+1})$ vanishes. Thus from the long exact sequence we deduce that

$$Tor_i(A, H_n(C)) = 0 \qquad \text{for} \quad i \geq 2.$$

Recall that

$${}^C E^1_{p,q} = Tor_p^{\mathbb{Z}}(A, H_{q-p}(C)).$$

So we have ${}^C E^1_{p,q} = 0$ if $p \neq 0$ or $p \neq 1$. Recall that by Proposition 2.1 we have the following short exact sequence

$$0 \longrightarrow {}^C E^1_{0,q} \longrightarrow H_q(\mathrm{Tot}\mathbf{B}) \longrightarrow {}^C E^1_{1,q} \longrightarrow 0. \tag{2.3}$$

Substituting values of ${}^{C}E^{1}_{p,q}$ calculated above, we get

$$0 \longrightarrow A \otimes H_q(C) \longrightarrow H_q(\mathrm{Tot}\mathbf{B}) \longrightarrow Tor_1^{\mathbb{Z}}(A, H_{q-1}(C)) \longrightarrow 0 \quad . \tag{2.4}$$

We now compute $H_q(\mathrm{Tot}\mathbf{B})$. For that we turn to the spectral sequence associated with the row filtration. Since C_n are free, so the rows of $\mathbf{B}$ are exact. Hence at E^0 level the only non-zero terms will be ${}^{R}E^{0}_{p,p}$. To make it clear recall that

$$ {}^{R}E^{0}_{p,q} = H_{q-p}(B_{*,p}). $$

In our case

$$ H_{q-p}(B_{*,p}) = H_{q-p}(P_* \otimes C_p). $$

Since the rows are exact so the above groups are non-zero only when $q - p = 0$. Let us explicitly compute ${}^{R}E^{0}_{p,p}$:

$$\begin{aligned} {}^{R}E^{0}_{p,p} &= H_{q-p}(P_* \otimes C_p) \\ &= H_0(P_* \otimes C_p) \\ &= Tor_0(A \otimes C_p) \\ &= A \otimes C_p. \end{aligned}$$

Now ${}^{R}E^{1}_{p,p} = H_p(A \otimes C_p)$. Notice that our spectral sequence satisfies Corollary 2.4, so $H_q(\mathrm{Tot}\mathbf{B}) ={}^{R} E^{1}_{p,p}$. Hence

$$ H_q(\mathrm{Tot}\mathbf{B}) = H_q(A \otimes C_q). $$

Now we shall rewrite the short exact sequence given by (2.4) to obtain

$$0 \longrightarrow A \otimes H_q(C) \longrightarrow H_q(A \otimes C_*) \longrightarrow Tor_1^{\mathbb{Z}}(A, H_{q-1}(C)) \longrightarrow 0 \quad .$$

□

Chapter VII

The Grothendieck Spectral Sequence

Alexander Grothendieck introduced a special kind of spectral sequence in his famous Tohoku paper. Now it is named after him. Let $F\colon \mathfrak{A} \to \mathfrak{B}$ and $G\colon \mathfrak{B} \to \mathfrak{C}$ be two additive functors between abelian categories. Under some assumptions, this spectral sequence relates the composition of derived functors of G and F to derived functors of $G \circ F$ as follows

$$(R^pG)(R^{q-p}F)(A) \Rightarrow R^q(GF)(A).$$

Familiarity with Chapter VI is essential to understand this Chapter. We would like to remind the reader that in this Chapter we will be working with co-homologicaly graded spectral sequences (cf. II.4.2).

1 Cartan - Eilenberg Resolution

Let $\mathfrak{A}$ be an abelain category with enough injectives. Consider the category $\mathfrak{C}$ of co-chain complexes in $\mathfrak{A}$. Then $\mathfrak{C}$ is an abelian category. Given an object, $A \in \mathfrak{A}$, we can regard it as a complex concentrated in degree zero as follows:

$$\cdots \to 0 \to A \to 0 \to \cdots .$$

Further, given an injective resolution of I^* of A we can regard it as a quasi-isomorphism as follows:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array} \tag{1.1}$$

More generally, one can define injective objects in $\mathfrak{C}$. Moreover, we can also take injective resolutions in $\mathfrak{C}$ by the following procedure. Given a bounded-below object say

$$\mathbf{C} = \cdots \to 0 \to C^0 \to C^1 \to C^2 \to \cdots \tag{1.2}$$

in $\mathfrak{C}$ we will explain below how to construct a quasi-isomorphism

$$\mathbf{C} \to \mathbf{I}$$

where $\mathbf{I}$ is an injective object in the category $\mathfrak{C}$. In fact, the complex $\mathbf{I}$ will be constructed in terms of injective resolutions of $\{C^n\}_{n\in\mathbb{Z}}$. This is possible by the following technique which goes by the name of Cartan-Eilenberg resolution. This is an *injective replacement of complexes.* As a consequence of this procedure, we see that the category $\mathfrak{C}$ of complexes of objects in $\mathfrak{A}$ also has enough injectives whenever $\mathfrak{A}$ has enough of them.

Let us denote the co-cycles of $\mathbf{C}$ by Z^r and co-boundaries by B^r. Then we have two sets of obvious short exact sequences:

$$0 \to Z^r \to C^r \to B^{r+1} \to 0; \tag{1.3}$$

and

$$0 \to B^r \to Z^r \to Z^r/B^r = H^r(\mathbf{C}) \to 0. \tag{1.4}$$

We make use of these short exact sequences to construct the desired injective replacement of $\mathbf{C}$.

Lemma 1.1. *For each value of r we may resolve (1.3) and (1.4) as*

$$\begin{array}{ccccccccc} & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & L^{r,1} & \longrightarrow & K^{r,1} & \longrightarrow & H^{r,1} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & L^{r,0} & \longrightarrow & K^{r,0} & \longrightarrow & H^{r,0} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & B^{r} & \longrightarrow & Z^{r} & \longrightarrow & H^{r} & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccccccc} & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K^{r,1} & \longrightarrow & J^{r,1} & \longrightarrow & L^{r+1,1} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K^{r,0} & \longrightarrow & J^{r,0} & \longrightarrow & L^{r+1,0} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & Z^{r} & \longrightarrow & C^{r} & \longrightarrow & B^{r+1} & \longrightarrow & 0 \end{array} \tag{1.5}$$

such that each column is an augmented injective resolution of the object appearing at it's head. Furthermore for fixed values of x and y the sequences

(i) $0 \to K^{x,y} \to J^{x,y} \to L^{x+1,y} \to 0$

(ii) $0 \to L^{x,y} \to K^{x,y} \to H^{x,y} \to 0$

are split-exact.

Proof. We shall prove the proposition by induction on r. For $r = 0$, we have $B^0 = 0$. We choose an arbitrary injective resolution

$$0 \to B^0 \to L^{0,0} \to L^{0,1} \to L^{0,2} \to \cdots$$

of B^0 by injective modules.

Similarly, we also choose

$$H^0(\mathbf{C}) \longrightarrow H^{0,0} \longrightarrow H^{0,1} \longrightarrow \cdots$$

an arbitrary injective resolution of $H^0(\mathbf{C}) = Z^0/B^0$. Applying Horse shoe Lemma we can obtain an injective resolution

$$Z^0 \longrightarrow K^{0,0} \longrightarrow K^{0,0} \longrightarrow \cdots$$

of Z^0 such that each row of the following diagram is exact

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & L^{0,1} & \longrightarrow & K^{0,1} & \longrightarrow & H^{0,1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & L^{0,0} & \longrightarrow & K^{0,0} & \longrightarrow & H^{0,0} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B^0 & \longrightarrow & Z^0 & \longrightarrow & H^0 & \longrightarrow & 0.
\end{array}$$

Thus we have obtained an injective resolution of Z^0 compatible with those of B^0 and H^0.

Similarly let

$$B^1 \longrightarrow L^{1,0} \longrightarrow L^{1,1} \longrightarrow \cdots$$

be an injective resolution of B_1. By the Horse shoe Lemma(cf. Lemma [1, III.5.4]) we can obtain an injective resolution of C^0

$$C^0 \longrightarrow J^{0,0} \longrightarrow J^{0,1} \longrightarrow \cdots$$

such that each row in the following diagram is exact in every row and column

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K^{0,1} & \longrightarrow & J^{0,1} & \longrightarrow & L^{1,1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K^{0,0} & \longrightarrow & J^{0,0} & \longrightarrow & L^{1,0} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Z^0 & \longrightarrow & C^0 & \longrightarrow & B^1 & \longrightarrow & 0.
\end{array}$$

Thus we have shown the diagram (1.5) for $r = 0$. We may suppose by induction that diagram (1.5) exists for values from 0 to $r-1$. So one has an injective resolution of B^r at one's disposal. Now, we let

$$H^r(\mathbf{C}) \longrightarrow H^{r,0} \longrightarrow H^{r,1} \longrightarrow \cdots$$

be an arbitrary injective resolution of $H^r(\mathbf{C}) = Z^r/B^r$. As above, we can obtain an injective resolution of Z^r

$$Z^r \longrightarrow K^{r,0} \longrightarrow K^{r,1} \longrightarrow \cdots$$

such that each row of the following diagram is exact

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & L^{r,1} & \longrightarrow & K^{r,1} & \longrightarrow & H^{r,1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & L^{r,0} & \longrightarrow & K^{r,0} & \longrightarrow & H^{r,0} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B^r & \longrightarrow & Z^r & \longrightarrow & H^r & \longrightarrow & 0.
\end{array}$$

Thus we have obtained a resolution of Z^r compatible with B^r and H^r. We then use an arbitrary resolution of B^{r+1} to construct an injective resolution of

$$0 \to Z^r \to C^r \to B^{r+1} \to 0$$

such that each row is exact. This shows the diagram (1.5) for the value r. We have thus proved the induction hypothesis. The assertion about split-exactness follows immediately, by basic homological algebra, because the sequences are short exact by construction and each of the modules involved is injective. □

Notice that by construction

$$J^{0,y} \to J^{1,y} \to J^{2,y} \to^{\cdots}$$

is a co-chain complex for each y. By combining diagrams of the form

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K^{r,1} & \longrightarrow & J^{r,1} & \longrightarrow & L^{r+1,1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K^{r,0} & \longrightarrow & J^{r,0} & \longrightarrow & L^{r+1,0} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Z^r & \longrightarrow & C^r & \longrightarrow & B^{r+1} & \longrightarrow & 0
\end{array}$$

for two successive values of r, we obtain the *Cartan-Eilenberg resolution of* $\mathbf{C}$

$$\begin{array}{ccccccccccccc}
\uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & J^{0,1} & \longrightarrow & \cdots & \longrightarrow & J^{r,1} & \longrightarrow & J^{r+1,1} & \longrightarrow & J^{r+2,1} & \longrightarrow & \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & J^{0,0} & \longrightarrow & \cdots & \longrightarrow & J^{r,0} & \longrightarrow & J^{r+1,0} & \longrightarrow & J^{r+2,0} & \longrightarrow & \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C^{0} & \longrightarrow & \cdots & \longrightarrow & C^{r} & \longrightarrow & C^{r+1} & \longrightarrow & C^{r+2} & \longrightarrow & .
\end{array} \tag{1.6}$$

Now we can set $\mathbf{I} = Tot(J^{*,*})$. Since $J^{*,*}$ consists only of injective objects in each bi-degree, $\mathbf{I}$ also consists only of injective objects in each degree. It is a standard fact in homological algebra that $\mathbf{I}$ is an injective object in the category of complexes. This can be proven by arguing on the degree of the given complexes starting from degree 0.

Proposition 1.2. *Let* $\mathbf{I}$ *be as defined above. Then* $\mathbf{I}$ *is an injective object in Category of co-chain complexes in the Abelian category* $\mathfrak{A}$*.*

Proof. Let $\mathbf{A}$ be and $\mathbf{D}$ be two co-chain complexes. Suppose we have the following diagram

$$\begin{array}{ccc}
\mathbf{A} & \xrightarrow{i} & \mathbf{D} \\
{\scriptstyle \alpha}\downarrow & & \\
\mathbf{I} & &
\end{array}$$

where i is a monomorphism.

We need to show that there exist a map $\beta : \mathbf{D} \to \mathbf{I}$ such that the diagram is commutative. Now each I^n is given by direct sum of injective objects $J^{p,q}$ such that $p+q=n$. Since direct sum of injective objects is injective, so each I^n is injective.

Consider $\alpha_n : A^n \to I^n$ and $i_n : A^n \to D^n$. Since for each n, i_n is a monomorphism and I^n is injective, so we have $\beta_n : D^n \to I^n$ such that the following diagram is commutative

$$\begin{array}{ccc}
A^n & \xrightarrow{i_n} & D^n \\
{\scriptstyle \alpha_n}\downarrow & \swarrow {\scriptstyle \beta_n} & \\
I^n. & &
\end{array}$$

Set $\beta = \{\beta_n\}$. Thus we have produced the following commutative in category of complexes

$$\begin{array}{ccc}
\mathbf{A} & \xrightarrow{i} & \mathbf{D} \\
{\scriptstyle \alpha}\downarrow & \swarrow {\scriptstyle \beta} & \\
\mathbf{I}. & &
\end{array}$$

□

Further we have a natural map of complexes

$$\mathbf{C} \to \mathbf{I}.$$

We claim that this map is a quasi-isomorphism. We may augment the complex **C** into a *positive* double co-chain complex simply by adding the modules 0. Thus we have a map of double co-chain complexes

$$\mathbf{C}^{*,*} \to \mathbf{J}^{*,*}. \tag{1.7}$$

Associating a double complex to it's total complex filtered either by column or row is clearly functorial. By dual of equations 1.8 and 1.9 of Chapter IV we have a functor which associates a filtered chain complex to an exact couple. Now recall by Proposition III.1.3 that the process of associating a spectral sequence to an exact couple defines a functor. Thus, by functoriality of the construction associating a spectral sequence to a double complex , we obtain morphism of the column spectral sequences

$$^{C}E(Tot\mathbf{C}) \to^{C} E(Tot\mathbf{J}). \tag{1.8}$$

Since both these complexes are *positive*, therefore by Proposition 1.5 of Chapter VI , these spectral sequences converge to $H_n(Tot\mathbf{C})$ and $H_n(Tot\mathbf{J})$ respectively. Let us denote by $\mathbf{B}^{*,*}$ a double-complex when we want to make assertions on either of $\mathbf{C}^{*,*}$ and $\mathbf{I}^{*,*}$. Dualizing Proposition 1.3 of Chapter VI , for the column spectral sequence, we have

$$^{C}E_0^{p,q} = H_{q-p}(\mathbf{B}^{p,*}, d^v).$$

It is (possibly) non-zero only when $q - p = 0$. In fact, we have

$$H_0(\mathbf{B}^{*,*}, d^v) = \mathbf{C}^*$$

for both double complexes. Since by construction each column $\mathbf{J}^{p,*}$ resolves C^p, so via (1.7), we see that the morphism of spectral sequences, at level 0, in (1.8) is actually the identity morphism. Thus afortiori, we have the identity morphism in (1.8) at level 1 also between the spectral sequences. Now since the degree for cohomological spectral sequences at level r is $(r + 1, 1)$, thus we see that both spectral sequences collapse at level 1. This has two consequences. Firstly, $E_1^{p,q} = E_\infty^{p,q}$ for all values of p and q. Secondly, $H_p(TotB)$ is filtered by a filtration of length 1. So it is its own graded. Therefore it is isomorphic to $E_\infty^{p,p}$ which is isomorphic to $E_1^{p,p}$. Since, the map in (1.8) at level 1 is identity therefore we have

$$H_p(\mathbf{C}^*) = H_p(Tot\mathbf{C}^{*,*}) =^C E_1^{p,p}(\mathbf{C}^{*,*}) \xrightarrow{id} {}^C E_1^{p,p}(\mathbf{J}^{*,*}) = H_p(Tot\mathbf{J}).$$

This proves the quasi-isomorphism.

2 Grothendieck spectral sequence

We shall need the following definition.

Definition 2.1. *We say an object B in $\mathfrak{B}$ is right G-acyclic if;*

$$R^q G(B) = \begin{cases} G(B), & q = 0; \\ 0, & q \geq 1. \end{cases} \tag{2.1}$$

Theorem 2.1 (Grothendieck spectral sequence)**.** *Let $F\colon \mathfrak{A} \to \mathfrak{B}$, $G\colon \mathfrak{B} \to \mathfrak{C}$ be additive functors between abelian categories. Assume that $\mathfrak{A}$ and $\mathfrak{B}$ have enough injectives and for every injective object I of $\mathfrak{A}$ the object $F(I)$ is G-acyclic. Then there is a spectral sequence $E_n(A)$ corresponding to each object A in $\mathfrak{A}$, such that*

$$E_1^{p,q} = (R^p G)(R^{q-p} F)(A) \Rightarrow R^q(GF)(A). \tag{2.2}$$

This spectral sequence converges finitely to the graded object associated with $R^q(GF)(A)$, suitably filtered.

Proof. Let A be an object in $\mathfrak{A}$. Let

$$\mathbf{I} = I^0 \to I^2 \to \cdots$$

be an injective resolution of A. Then

$$F(\mathbf{I}) = FI^0 \to FI^1 \to FI^2 \to \tag{2.3}$$

is a cochain complex of objects in $\mathfrak{B}$. Let,

$$\begin{array}{ccccccccc}
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & J^{0,1} & \longrightarrow & J^{1,1} & \longrightarrow & J^{2,1} & \longrightarrow & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & J^{0,0} & \longrightarrow & J^{1,0} & \longrightarrow & J^{2,0} & \longrightarrow & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & FI^0 & \longrightarrow & FI^1 & \longrightarrow & FI^2 & \longrightarrow &
\end{array} \tag{2.4}$$

be a Cartan-Eilenberg resolution of (2.3) obtained by replacing the role of $\mathbf{C}$ in Lemma 1.1 with $F(\mathbf{I})$. By applying G to (2.4) we obtain a double co-chain complex

$$\begin{array}{ccccccccc}
 & & d^v\uparrow & & d^v\uparrow & & d^v\uparrow & & \\
0 & \longrightarrow & GJ^{0,2} & \xrightarrow{d^h} & GJ^{1,2} & \xrightarrow{d^h} & GJ^{2,2} & \xrightarrow{d^h} & \cdots \\
 & & d^v\uparrow & & d^v\uparrow & & d^v\uparrow & & \\
0 & \longrightarrow & GJ^{0,1} & \xrightarrow{d^h} & GJ^{1,1} & \xrightarrow{d^h} & GJ^{2,1} & \xrightarrow{d^h} & \cdots \\
 & & d^v\uparrow & & d^v\uparrow & & d^v\uparrow & & \\
0 & \longrightarrow & GJ^{0,0} & \xrightarrow{d^h} & GJ^{1,0} & \xrightarrow{d^h} & GJ^{2,0} & \xrightarrow{d^h} & \cdots .
\end{array} \tag{2.5}$$

Let us denote this double co-chain complex by $\mathbf{B}$.

Lemma 2.2. *For the diagram (2.4), in the notation of Lemma 1.1, recall the complex* $\{H^{x,y}\}_{x\in\mathbb{Z}}$. *We have*

$$H_p(\mathbf{B}^{*,q}, d^h) = GH^{p,q}. \tag{2.6}$$

Proof. We view the double complex $\mathbf{B}$ as a complex of vertical complexes $\{\mathbf{B}^{p,*}\}_{p\in\mathbb{Z}}$. For all values of x and y we have the following split short exact sequence

$$0 \to K^{p,q} \to J^{p,q} \to L^{p+1,q} \to 0.$$

Since G is an additive functor, so it preserves split exactness(cf. I.2) . Thus

$$0 \to GK^{p,q} \to GJ^{p,q} \to GL^{p+1,q} \to 0$$

is also split exact. It follows that $GJ^{p,q} \to GJ^{p+1,q}$ the kernel identifies with $GK^{p,q}$ and cokernel identifies with $GL^{p+1,q}$. Thus

1. the complex $\tilde{Z}^p$ of p co-cycles identifies with $G(K^{p,*})$.
2. Further, the complex $\tilde{B}^p$ of p co-boundaries identifies with $G(L^{p,*})$.

Thus the complex of p cohomologies identifies with the complex $G(K^{p,*})/G(L^{p,*})$.

For all values of x and y we have the following split short exact sequence

$$0 \to L^{p,q} \to K^{p,q} \to H^{p,q} \to 0.$$

Since G is an additive functor, so it preserves split exactness(cf. I.2) . Thus

$$0 \to GL^{p,q} \to GK^{p,q} \to GH^{p,q} \to 0$$

is also split exact. Hence we have $GK^{p,q}/GL^{p,q} \cong GH^{p,q}$. Thus we have shown $H_p(\mathbf{B}^{*,q}, d^h) = GH^{p,q}$. □

Let us denote the r co-cycles and the r co-boundaries of the co-chain complex $F(\mathbf{I})$ by Z^r and B^r respectively. So by definition

$$R^rF(A) = H^r(F(\mathbf{I})) = Z^r/B^r. \tag{2.7}$$

We compute ${}^RE_1^{p,q}$ by taking homology with respect to the differential induced by vertical differential on ${}^RE_0$. Recall by equation (1.17) of Chapter VI that

$${}^RE_1^{p,q} = H^p(H^{q-p}(\mathbf{B}, d^h), d^v).$$

By equation 2.6, taking homology of the complex $G(H^{q-*,*})$ we obtain

$${}^RE_1^{p,q} = H^p(G(H^{q-p,p})).$$

By Lemma (1.1), fixing $q-p$, we have an injective resolution of $H^{q-p}(F(I))$ given by the complex

$$\cdots \to H^{q-p,s} \to H^{q-p,s+1} \to \cdots .$$

Thus we have ${}^{R}E_1^{p,q} = (R^pG)(H^{q-p}(F\mathbf{I}))$ which by definition is

$$(R^pG)(R^{q-p}F(A)). \tag{2.8}$$

Since the double complex (2.5) is positive, so Proposition VI.1.5 assures finite convergence.

Now we consider the spectral sequence arizing from column-wise filtration of Tot$\mathbf{B}$. Thus ${}^{C}E_0^{p,q}$ is computed by filtering $\mathbf{B}$ vertically. Recall by equation (1.16) of Chapter VI that

$${}^{C}E_0^{p,q} = H^{q-p}(B^{p,*}, d^v).$$

In the complex $\mathbf{B}$ above the term $B^{p,q}$ is given by $GJ^{p,q}$. Hence

$${}^{C}E_0^{p,q} = H^{q-p}(GJ^{p,*}, d^v).$$

Since by our hypothesis FI^p are G-acyclic so

$$\begin{aligned} {}^{C}E_0^{p,q} &= G(FI^p), \quad q-p=0 \\ &= 0, \qquad\quad q-p \neq 0. \end{aligned}$$

Recall that equation (1.16) of Chapter VI says

$${}^{C}E_1^{p,q} = H^p(H^{q-p}(\mathbf{B}, d^v), d^h).$$

Thus we have ${}^{C}E_1^{p,q} = H^q(GFI^*)$ if $q-p=0$ and the module 0 otherwise. Further, by definition $H^q(GFI^*) = R^q(GF)(A)$. By dualizing the formula (1.15) of chapter IV it follows that the bi-degree of d_r is $(r+1,1)$. Thus the spectral sequence collapses at level 1. Consequently,

1. ${}^{C}E_1^{p,q}$ is possibly non-zero only when $q-p=0$ and
2. ${}^{C}E_1^{q,q} = {}^{C}E_\infty^{q,q}$
3. ${}^{C}E_\infty^{q,q} = R^q(GF)A$
4. Since $\mathbf{B}$ is positively graded, so by Proposition 1.5 of Chapter VI, the column spectral sequence of $\mathbf{B}$ does indeed converge to $H^q(\text{Tot}\mathbf{B})$.
5. Applying Corollary 2.4 to $Tot(\mathbf{B})$ we have ${}^{C}E_1^{q,q} = H^q(\text{Tot}\mathbf{B})$.

So we have

$$H^q(\text{Tot}\mathbf{B}) = R^q(GF)A. \tag{2.9}$$

The theorem now follows by applying Proposition 1.5 of Chapter VI to (2.8) and (2.9) . □

2.1 Lyndon-Hochschild-Serre Spectral Sequence

In the following we give an application of Grothendieck spectral sequence in group co-homology. The spectral sequence we will obtain is known as Lyndon-Hochschild-Serre spectral sequence. In this discussion we will assume familiarity with group co-homology. Readers lacking knowledge in Group cohomology are requested to refer [3, 6].

Let H be a group and N be a normal subgroup of H. We shall denote the quotient H/N by Q. Thus we have a short exact sequence of groups

$$0 \to N \xrightarrow{i} H \xrightarrow{p} Q \to 0.$$

Let $\mathfrak{A}$ be the category of H-modules and $\mathfrak{B}$ be the category of Q-modules. Let A^N denote the subgroup of A fixed by N. Now we have the functors

$$F : \mathfrak{A} \to \mathfrak{B} \tag{2.10}$$

and

$$G : \mathfrak{B} \to \mathfrak{C}, \tag{2.11}$$

where $F(A) = A^N$ and $G(B) = B^Q$. Notice that for $a \in A$ and $h \in H$ the assignment

$$(ph)(a) = ha$$

defines a Q-action on A. Hence we may regard A^N as a Q-module.

Claim 2.3. *The functors F and G are additive. Further $GF(A) = A^H$.*

Proof. The fact that these functors are additive follows from construction.

We now prove the second assertion. Given a group H and a H-module A, it is well known fact that $A^H = Hom_H(\mathbb{Z}, A)$, where the H-action on $\mathbb{Z}$ is trivial (cf. [3, 6.1.1]).

Similarly we know that given a ring homomorphism $f : R \to S$ and a fixed S-module N, the functors $Hom_R(N, -) : \mathfrak{R} - \mathfrak{mod} \to \mathfrak{Ab}$ and $- \otimes_S N : \mathfrak{S} - \mathfrak{mod} \to \mathfrak{Ab}$ forms an adjoint pair (cf. [3, 2.6.3]). Thus for an S-module M and R-module Q, we have

$$Hom_R(M \otimes_S N, Q) \cong Hom_S(M, Hom_R(N, Q)).$$

There is natural morphism from the group ring $\mathbb{Z}H$ to $\mathbb{Z}Q$. Thus we may use the above to obtain

$$GF(A) = Hom_Q(\mathbb{Z}, Hom_H(\mathbb{Z}, A) = Hom_H(\mathbb{Z} \otimes_Q \mathbb{Z}, A) = Hom_H(\mathbb{Z}, A) = A^H.$$

□

Lemma 2.4. *Let $\rho : \mathfrak{B} \to \mathfrak{A}$ be the functor such that $\rho(B)$ is a H-module with H-action given by*

$$hb = (ph)b.$$

Then F is right adjoint to ρ.

Proof. Let A be an object in $\mathfrak{A}$ and B be an object in $\mathfrak{B}$.

We need to show that

$$Hom_H(\rho(B), A) \cong Hom_Q(B, F(A)).$$

Let $f : B \to F(A) = A^N \in Hom_Q(B, F(A))$. Thus by abuse of notation we have a morphism $f : B \to A^N$ of abelian groups. Consider B as a H-module via ρ. Let us denote the H-linear map given by f from $\rho(B)$ to $A^N \subset A$ by f^*. Thus given a $f \in Hom_Q(B, F(A))$ we have produced a $f^* \in Hom_H(\rho(B), A)$.

Now let $g : \rho(B) \to A \in Hom_H(\rho(B), A)$. Consider $b \in \rho(B)$. Now $g(b)$ belongs to A. Let $n \in N$. Now $ng(b) = g(nb)$. By definition of H-action on $\rho(B)$, $g(nb) = g(p(n)b) = g(b)$ (because $p(n) = 1$) . Thus $g(b) \in A^N$. Notice that $\rho(B)$ is essentially B equipped with an H-action. Thus the abelian groups $\rho(B)$ and B are same. Consider the morphism of ableian groups $g : B \to A^N$. Clearly this shall give us a morphism $g^* : B \to A^N = F(A)$ of Q-modules. Thus given a $g \in Hom_H(\rho(B), A)$ we have produced $g^* \in Hom_Q(B, F(A))$. □

Now we are in a position prove the following theorem.

Theorem 2.5. *Let H be a group and N be a normal subgroup of H. Consider an H-module A, then there is a natural action of $Q = H/N$ on the cohomology groups $H^n(N, A)$. Further there is a spectral sequence $\{E_n(A)\}$ such that*

$$E_1^{p,q} = H^p(Q, H^{q-p}(N, A)) \Rightarrow H^q(H, A).$$

Proof. We shall first verify hypothesis of Theorem 2.1 for the functors F and G given by equations (2.10) and (2.11). We have already showed that F and G are additive. Now it remains to verify that if I is an injective H-module then $F(I) = I^N$ is G-acyclic. By Lemma 2.4 we have that F is right adjoint to the forgetful functor $\rho : \mathfrak{B} \to \mathfrak{A}$. Clearly ρ preserves monomorphisms, so F preserves injectives (cf. Lemma I.0.1). Thus for I injective in $\mathfrak{A}$, $F(I) = I^N$ is injective in $\mathfrak{B}$. Hence I^N is plainly G-acyclic.

Now we shall apply Theorem 2.1 to the functors F and G.

Since $\mathbb{Z}H$ is a free $\mathbb{Z}N$ module, it follows that H-injective resolution of A is also an N-injective resolution. Further given any H-injective resolution of A,

$$I_0 \to I_1 \to I_2 \to \cdots$$

we have the following Q-complex

$$Hom_N(\mathbb{Z}, A) \to Hom_N(\mathbb{Z}, I_0) \to Hom_N(\mathbb{Z}, I_1) \to \cdots .$$

Thus the cohomology groups $H^m(N, A)$ also acquires the structure of Q-modules and

$$R^m F(A) = H^m(N, A). \tag{2.12}$$

By definition

$$R^m G(B) = H^m(Q, B)$$

and

$$R^m GF(A) = H^m(H, A).$$

Thus by Theorem 2.1 and equation (2.12) the result follows. □

Chapter VIII

Simplicial Sets

In this chapter we define the notion of Simplicial Sets and Simplicial Homotopy. We introduce minimum machinery needed for the subsequent two Chapters, where Spectral Sequences arise from Simplicial situations.

1 The definitions

For $n \geq 0$, let $[n]$ denote the ordered set $\{0, 1, \cdots, n\}$. By ordinal maps $[m] \to [n]$ we shall mean non-decreasing maps of ordered sets. The category of ordinals has $[n]$ as objects and ordinal maps as morphisms. We shall denote this category by *Ord*.

Let $\mathfrak{C}$ be a category.

Definition 1.1. *A simplicial object in $\mathfrak{C}$ is simply a contravariant functor $Ord \to \mathfrak{C}$.*

This definition is good for defining simplicial objects and morphisms. The following equivalent definition is better for computations.

Definition 1.2. *A simplicial object in $\mathfrak{C}$ is given by a sequence $\{K_n\}_{n\in\mathbb{N}}$ of objects in $\mathfrak{C}$, along with face maps*

$$d_i : K_n \to K_{n-1},$$

and degeneracy maps

$$s_i : K_n \to K_{n+1},$$

satisfying the following simplicial identities:

$$d_i d_j = d_{j-1} d_i, \qquad if \quad i < j \tag{1.1}$$
$$s_i s_j = s_{j+1} s_i, \qquad if \quad i \leq j \tag{1.2}$$
$$d_i s_j = s_{j-1} d_i, \qquad if \quad i < j \tag{1.3}$$
$$d_i s_i = id = d_{i+1} s_i, \tag{1.4}$$
$$d_i s_j = s_j d_{i-1}, \qquad if \quad i > j + 1. \tag{1.5}$$

Definition 1.3. *A simplicial map $f : K \to L$ between two simplicial objects K and L in a category $\mathfrak{C}$ consists of $\{f_n\}$ where $f_n : K_n \to L_n$ is such that*

$$f_n d_i = d_i f_{n+1} \tag{1.6}$$
$$f_n s_i = s_i f_{n-1}. \tag{1.7}$$

Definition 1.4. *A semi-simplicial object in a category $\mathfrak{C}$ is given by a sequence of objects in $\mathfrak{C}$, $\{K_n\}_{n\in\mathbb{N}}$ along with face maps*

$$d_i : K_n \to K_{n-1},$$

satisfying (1.1) which are known as semi-simplicial identities.

Definition 1.5. *A semi-simplicial map $f : K \to L$ between two semi-simplicial objects K and L consists of $\{f_n\}$ where $f_n : K_n \to L_n$ is such that*

$$f_n d_i = d_i f_{n+1} \tag{1.8}$$

Definition 1.6. *For $n \geq 0$, we shall denote by $\Delta[n]$ the simplicial set whose m-simplices are*

$$\Delta[n]_m = \{f : [m] \to [n]\}.$$

The face and degeneracy maps are the obvious maps induced from ordinal maps $[m] \to [m-1]$ or $[m] \to [m+1]$ respectively.

We shall denote by $\overset{\circ}{\Delta}[n]$ the simplicial set whose m-simplices are

$$\{f : [m] \to [n] | f \quad \text{is not surjective} \}.$$

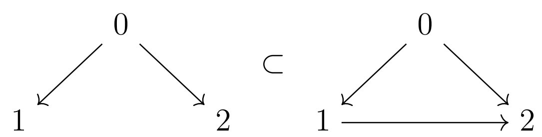
The face and degeneracy maps are those induced by the inclusion $\overset{\circ}{\Delta}[n] \hookrightarrow \Delta[n]$.

We shall denote by Λ^n_k the simplicial set called the k-th horn of $\Delta[n]$. Its m-simplices are

$$\{f : [m] \to [n] | k \notin Img(f)\}.$$

The face and degeneracy maps are those induced by the inclusion $\Lambda^n_k \hookrightarrow \Delta[n]$.

Geometrically the k-th horn corresponds to the skeleton of the n-simplex with the face opposite the k-th vertex removed. One could represent Λ^2_0 , for example, by the picture



Definition 1.7. *Let K be a simplicial set, and let $k_0 \in K_0$ be a 0-simplex. Then we have a sub-simplicial set of K generated by k_0 . For each $n \geq 0$, there is exactly one simplex in degree n*

$$s_{n-1}s_{n-2}...s_0k_0.$$

We will use k_0 to denote both the sub-simplicial set it generates as well as any of its simplices. We shall call k_0 a base point of the simplicial set K.

Simplicial sets with a distinguished base point is called a pointed simplicial space.

2 Examples

Example 2.1 (Nerve of a Category)**.** *Let $\mathfrak{C}$ be a small category. We define the simplicial set called called nerve of the category $\mathfrak{C}$ as follows:*

$N\mathfrak{C}_0 =$*objects in* $\mathfrak{C}$

$N\mathfrak{C}_1 =$*morphisms in* $\mathfrak{C}$

$N\mathfrak{C}_2 =$*the collection of composable morphisms in* $\mathfrak{C}$

$\vdots$

$N\mathfrak{C}_n =$*the collection of n-times composable morphisms in* $\mathfrak{C}$

The degeneracy map $s_i : N\mathfrak{C}_n \to N\mathfrak{C}_{n+1}$ is defined as a map which takes a collection of n-composable morphisms and inserts identity at i-th position. That is given a n-composable collection of morphisms $\{f_i\}$ in $\mathfrak{C}$ we have

$$s_i(f_0 f_1 ... f_i f_{i+1} ... f_n) = f_0 f_1 ...(id) f_i f_{i+1} ... f_n.$$

The face map $d_i : N\mathfrak{C}_n \to N\mathfrak{C}_{n-1}$ composes the i-th and $i+1$-th morphims if $0 < i < n$, and leaves out the first or last morphism for $i = 0$ or n respectively. That is for $0 < i < n$

$$d_i(f_0 f_1 ... f_i f_{i+1} ... f_n) = f_0 f_1 ...(f_i \circ f_{i+1}) ... f_n,$$

for $i = 0$

$$d_0(f_0 f_1 ... f_i f_{i+1} ... f_n) = f_1 ... f_i f_{i+1} ... f_n,$$

for $i = n$

$$d_0(f_0 f_1 ... f_i f_{i+1} ... f_n) = f_1 ... f_i f_{i+1} ... f_{n-1}.$$

One may easily see that Nerve of Category forms a simplicial set.

Example 2.2. *We know that a group G can be considered as a category with one object G and morphisms $g : G \to G$ for each element $g \in G$. Now we may define nerve of a group. This way we can associate a simplicial set to a group.*

Definition 2.1. *A geometric braid on n strands(strings) is a subset $\beta \subset \mathbb{R}^2 \times [0,1]$ such that it is composed of n disjoint topological intervals (maps from the unit interval into a space). Furthermore, β must satisfy the following conditions:*

1. $\beta \cap (\mathbb{R}^2 \times \{0\}) = \{(1,0,0), (2,0,0), ..., (n,0,0)\}$
2. $\beta \cap (\mathbb{R}^2 \times \{1\}) = \{(1,0,1), (2,0,1), ..., (n,0,1)\}$
3. $\beta \cap (\mathbb{R}^2 \times \{t\})$ *consists of n points for all* $t \in [0,1]$
4. *For any string in β, there exists a projection $p : \mathbb{R}^2 \times [0,1] \to [0,1]$ taking that string homoeomorphically to the unit interval.*

   Taking a base of n distinct points in R^2, geometric braids forms a group. Composition of braids is simply given by stacking one braid atop another.

Example 2.3 (Simplicial structure on braids)**.** *Let $K_n = B_{n+1}$ be the braid group of $(n+1)$-strands with faces and degeneracies given by:*

the braid $d_i\beta$ is obtained by removing the $(i+1)$-th strand of β and $s_i\beta$ is the braid obtained by doubling the $(i+1)-th$ strand of β (that is the $(i+1)$-th strand is replaced by two untwisted strands in its small neighborhood).

Then it can be easily verified that $K = \{K_n\}$ is a simplicial set.

Example 2.4. *From a simplicial set K, one may construct a simplicial abelian group $\mathbb{Z}K$, with $(\mathbb{Z}K)_n$ equal to the free abelian group on K_n . The face and degeneracy operators are the ones induced from K. We may associate a chain complex to $\mathbb{Z}K$, called its Moore complex also written $\mathbb{Z}K$, with*

$$(\mathbb{Z}K)_0 \overset{\delta}{\leftarrow} (\mathbb{Z}K)_1 \overset{\delta}{\leftarrow} (\mathbb{Z}K)_2 \overset{\delta}{\leftarrow} \cdots$$

and

$$\delta = \sum_{i=0}^{n} (-1)^i d_i$$

in degree n.

2.1 Basic Constructions

Let K and L be two simplicial sets. We define their product $K \times L$ as follows:

$$(K \times L)_n = K_n \times L_n,$$

$$d_i^{K\times L} = (d_i^K, d_i^L) \quad \text{and} \quad s_i^{K\times L} = (s_i^K, s_i^L).$$

Claim 2.1. *The set $K \times L$ is a simplicial set.*

Proof. We need to show that $\{d_i^{K\times L}\}$ and $\{s_i^{K\times L}\}$ satisfy simplicial identities.

i) Now for $i < j$ we have $d_i^{K\times L} \circ d_j^{K\times L} = (d_i^K, d_i^L) \circ (d_j^K, d_j^L) = (d_i^K d_j^K, d_i^L d_j^L)$. By the first simplicial identity this equals $(d_{j-1}^K d_i^K, d_{j-1}^L d_i^L) = (d_{j-1}^K, d_{j-1}^L) \circ (d_i^K, d_i^L) = d_{j-1}^{K\times L} \circ d_i^{K\times L}$.

ii) For $i \le j$ we have $s_i^{K\times L} \circ s_j^{K\times L} = (s_i^K, s_i^L) \circ (s_j^K, s_j^L) = (s_i^K s_j^K, s_i^L s_j^L)$. By the second simplicial identity this equals $(s_{j+1}^K s_i^K, s_{j+1}^L s_i^L) = (s_{j+1}^K, s_{j+1}^L) \circ (s_i^K, s_i^L) = s_{j+1}^{K\times L} \circ s_i^{K\times L}$.

iii) For $i < j$ we have $d_i^{K\times L} \circ s_j^{K\times L} = (d_i^K, d_i^L) \circ (s_j^K, s_j^L) = (d_i^K s_j^K, d_i^L s_j^L)$. By the third simplicial identity this equals $(s_{j-1}^K d_i^K, s_{j-1}^L d_i^L) = (s_{j-1}^K, s_{j-1}^L) \circ (d_i^K, d_i^L) = s_{j-1}^{K\times L} \circ d_i^{K\times L}$.

iv) Now we have $d_i^{K\times L} \circ s_i^{K\times L} = (d_i^K, d_i^L) \circ (s_i^K, s_i^L) = (d_i^K s_i^K, d_i^L s_i^L)$. By the fourth simplicial identity this equals $(id^K, Id^L) = id^{K\times L}$. Similarly one can show that $d_{i+1}^{K\times L} \circ s_i^{K\times L} = id$.

v) For $i > j+1$ we have $d_i^{K\times L} \circ s_j^{K\times L} = (d_i^K, d_i^L) \circ (s_j^K, s_j^L) = (d_i^K s_j^K, d_i^L s_j^L)$. By the fifth simplicial identity this equals $(s_j^K d_{i-1}^K, s_j^L d_{i-1}^L) = (s_j^K, s_j^L) \circ (d_{i-1}^K, d_{i-1}^L) = s_j^{K\times L} \circ d_{i-1}^{K\times L}$. □

Definition 2.2. *Let K and L be pointed simplicial sets. The wedge $K \vee L$ of K and L is the simplicial set obtained by identifying the basepoint of K with the basepoint of Y. The smash product of K and L is defined to be the simplicial quotient $X \times Y/K \vee L$*

Definition 2.3. *Let K be a pointed Simplicial set. We define the reduced cone of K by setting*

$$(CK)_n = \{(x,q)|x \in K_{n-q}, \quad 0 \le q \le n\} \quad \textit{with } (*,q) \textit{ all identified to } *,$$

$$d_i(x,q) = \begin{cases} (x, q-1) & \textit{for} \quad 0 \le i < q \\ (d_{i-q}x, q) & \textit{for} \quad q \le i \le n \end{cases} \tag{2.1}$$

$$s_i(x,q) = \begin{cases} (x, q+1) & \textit{for} \quad 0 \le i < q \\ (s_{i-q}x, q) & \textit{for} \quad q \le i \le n \end{cases} \tag{2.2}$$

where for $x \in K_0$, $d_1(x,1) = *$. By identifying x with $(x,0)$, we may see that K is a simplicial subset of CK.

The reduced suspension ΣK of K is defined as the simplicial quotient

$$\Sigma K = CK/K.$$

We now give an alternate description of the cone CK. Let K be a simplicial set with no chosen base point. Let $\tilde{x}_0$ be a new point not in K. Let

$$(\tilde{C}K)_n = K_n \sqcup s_{-1}(K_{n-1}) \sqcup ... \sqcup s^n_{-1}(K_0) \sqcup \{s^n_0\tilde{x}_0\} = \{s^n_0\tilde{x}_0\} \coprod_{k=0}^{n} s^k_{-1}(K_{n-k})$$

be a disjoint union as a set, where $s_{-1}K_j = K_j$. Consider s_{-1} as a (-1)-st degenracy by setting

$$d_i s_{-1} = \begin{cases} id & \text{if} \quad i = 0 \\ s_{-1}d_{i-1} & \text{if} \quad i \ge 0 \end{cases} \tag{2.3}$$

$$s_i s_{-1} = \begin{cases} s_{-1}s_{-1} & \text{if} \quad i = 0 \\ s_{-1}s_{i-1} & \text{if} \quad i \ge 0 \end{cases} \tag{2.4}$$

We shall identify $s_{-1}x$ with (x,q). For $x \in K_0$ set $d_1(s_{-1}x) = \tilde{x}_0$. Now d_i and s_i induces operations on $\tilde{C}K$ which gives relations as in equations (2.1) and (2.2). We shall call this simplicial set $\tilde{C}K$ the *unreduced* cone of K. The reduced cone CK is then the quotient given by the relations $x_0 \sim \tilde{x}_0$ and $(x_0, 1) = s_{-1}x_0 \sim (s_0x_0, 0) = s_0x_0 \sim s_0\tilde{x}_0$ for the basepoint $x_0 \in K_0$.

3 Kan Complex

We now introduce the all-important Kan condition.

Definition 3.1. *We say that a collection of n many $(n-1)$-simplices*

$$x_0, x_1, ..., x_{k-1}, -, x_{k+1}, ..., x_n$$

satisfy the compatibility condition if

$$d_i x_j = d_{j-1} x_i, \forall i < j, k \notin \{i, j\}. \tag{3.1}$$

A simplicial set is a Kan complex if for every collection of n, $(n-1)$-simplices

$$x_0, x_1, ..., x_{k-1}, -, x_{k+1}, ..., x_n$$

satisfying the compatibility condition, there exists an n-simplex y such that

$$d_i y = x_i, \forall i \neq k.$$

Example 3.1. *The standard n-simplex, is a simplicial set defined as the functor $Hom_{Ord}(-, [n])$ where $[n]$ denotes the ordered set $0, 1, ..., n$.*

Given a topological space X, let us denote the standard topological n-simplex by Δ_n. We define a singular n-simplex of X to be a continuous map from Δ_n to X,

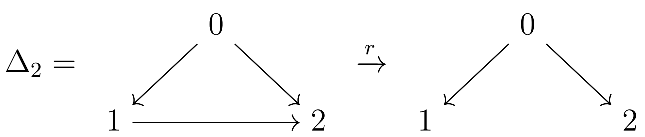
$$f : \Delta_n \to X.$$

We denote this simplicial set by $S(X)$.

The union of any $n+1$ faces of Δ_{n+1} is a strong deformation retract of Δ_{n+1}. So any continuous function defined on these faces can be extended to Δ_{n+1}. Hence $S(X)$ is a Kan complex.

In other words Horn of an n-simplex is a strong deformation retract of that simplex. Therefore any continuous function defined on the horn of an n-simplex can be extended to the n-simplex. For this reason Kan condition is also known as the 'Horn filler' condition.

The following is an illustration of this fact for the case $n = 1$



Definition 3.2. *A map of simplicial sets $f : K \to L$ is a Kan fibration if for every collection of n many $(n-1)$-simplices of K,*

$$x_0, x_{k-1}, -, x_{k+1}, ..., x_n$$

which satisfy the compatibility condition of Definition 3.1, and for every n-simplex $y \in L_n$ such that

$$d_i y = f(x_i), i \neq k,$$

there exists an n-simplex $x \in K_n$ such that $d_i x = x_i$, for all $i \neq k$ and $f(x) = y$.

Geometrically this condition is nicely summarized by the following diagram

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{(x_0,\cdots,x_{k-1},-,x_{k+1},\cdots,x_n)} & K \\ \downarrow & \overset{x}{\nearrow} & \downarrow f \\ \Delta[n] & \xrightarrow{\quad y \quad} & L \end{array} \tag{3.2}$$

which is also known as the Horn-filler diagram. The solid arrows represent the given data and the existence of the dotted arrow is the Kan condition.

The following lemma gives a natural example of a Kan complex.

Lemma 3.1. *If G is a simplicial group, then the underlying simplicial set is a Kan complex.*

Proof. Suppose we are given $(n+1)$ many n-simplices. That is we have

$$x_0, x_1, ..., x_{k-1}, -, x_{k+1}, ..., x_{n+1}$$

elements in G_n such that

$$d_i x_j = d_{j-1} x_i \quad \forall i < j \quad k \notin \{i, j\}. \tag{3.3}$$

We wish to find a g in G_{n+1} such that

$$d_i g = x_i \quad \forall i \neq k. \tag{3.4}$$

For $n+1 \geq r \geq -1$, it suffices to construct $g_r \in G_{n+1}$ such that

$$d_i g_r = x_i \quad \forall i \neq k \quad i \leq r. \tag{3.5}$$

Then the g sought will be g_{n+1}. The g_r will be constructed inductively.

Set $g_{-1} = 1$. Then the condition (3.5) is vacuously true. Suppose we have constructed g_{r-1} such that

$$d_i g_{r-1} = x_i \quad \forall\, i \leq r-1, \quad i \neq k. \tag{3.6}$$

If $k = r$ then we take $g_r := g_{r-1}$. Now condition (3.5) holds for $i \leq r = k$. Now we assume $r \neq k$. We first consider the element

$$y = x_r^{-1}(d_r g_{r-1}) \in G_n.$$

Claim 3.2. *We have $d_i(y) = 1 \quad \forall \quad i < r$ and $i \neq k$*

Proof. We have by definition

$$d_i(y) = d_i(x_r^{-1}) d_i d_r g_{r-1} \tag{3.7}$$

$$\overset{(1.1)}{=} d_i(x_r)^{-1} d_{r-1} d_i g_{r-1} \tag{3.8}$$

$$\overset{(3.6)}{=} d_i(x_r^{-1}) d_{r-1}(x_i) \tag{3.9}$$

$$\overset{(3.3)}{=} d_i(x_r)^{-1} d_i(x_r) = 1. \tag{3.10}$$

□

Further by simplicial identity (1.3) we have

$$d_i s_r y = s_{r-1} d_i y \quad \forall \quad i < r, i \neq k. \tag{3.11}$$

We set

$$g_r := g_{r-1}(s_r y)^{-1}. \tag{3.12}$$

Let us check (3.5):
Case $i < r$ we have by (3.12) that $d_i g_r = d_i(g_{r-1} s_r(y^{-1})) = d_i(g_{r-1}) d_i s_r(y^{-1})$

$$\overset{(3.11)}{=} d_i(g_{r-1}) s_{r-1} d_i(y^{-1}) \overset{3.2}{=} d_i g_{r-1} = x_i$$

Case $i = r$ we have $d_r g_r = d_r(g_{r-1} s_r(y^{-1})) = d_r g_{r-1} d_r s_r(y^{-1}) \overset{(1.4)}{=} d_r g_{r-1} y^{-1} = x_r$. This verifies (3.5) for r.

This proves the Lemma.

□

4 Group structures

Definition 4.1. *Let K be a simplicial set. Let $x, x' \in K_n$ be two n-simplices satisfying*

$$d_j x = d_j x', \quad \forall \quad 0 \leq j \leq n \tag{4.1}$$

or equivalently, denoting $\overset{\circ}{\Delta}[n+1]$ the skeleton of the simplex $\Delta[n+1]$

$$\overset{\circ}{\Delta}[n+1] \xrightarrow{(s_{n-1}d_0 x, \cdots, s_{n-1}d_{n-1}x, x, x')} K \tag{4.2}$$

We shall say that they are homotopic if there exists $y \in K_{n+1}$ such that

$$d_n y = x \tag{4.3}$$

$$d_{n+1} y = x' \tag{4.4}$$

$$d_i y = s_{n-1} d_i x = s_{n-1} d_i x' \quad \text{for} \quad 0 \leq i < n. \tag{4.5}$$

We call the $(n+1)$-simplex y a homotopy from x to x' and we write it as

$$x \overset{y}{\sim} x'.$$

More geometrically $x \overset{y}{\sim} x'$ means

$$\begin{array}{ccc} \overset{\circ}{\Delta}[n+1] & \xrightarrow{(s_{n-1}d_0 x, \cdots, s_{n-1}d_{n-1}x, x, x')} & K \\ \downarrow & \nearrow_{y} & \\ \Delta[n+1] & & \end{array} \tag{4.6}$$

Lemma 4.1. *If K is a simplicial set satisfying the Kan condition then the homotopy relation $\sim$ is an equivalence relation on K_n for each $n \geq 0$.*

Proof. Reflexivity: Let x be an n-simplex in K. We verify the existence of the morphism

$$\overset{\circ}{\Delta}[n+1] \xrightarrow{(s_{n-1}d_0x,\cdots,s_{n-1}d_{n-1}x,x,x)} K. \tag{4.7}$$

Clearly $d_j(x) = d_j(x)$ for $0 \leq j \leq n$. Set $y := s_n x$. Then $d_n s_n x \overset{1.4}{=} x$ and $d_{n+1} s_n x \overset{1.4}{=} x$. Now by the simplicial identity (1.3) $d_i y = d_i s_n x = s_{n-1} d_i x$ for $0 \leq i \leq n$.

Claim 4.2. *To show that the relation is symmetric and transitive it suffices to prove that if $x' \sim x$ and $x'' \sim x$ then $x' \sim x''$.*

Proof. For symmetry, we set $x'' := x$. It follows by hypothesis that $x \sim x'$. For transitivity, suppose we are given $x' \sim x$ and $x \sim x''$, then we apply symmetry to $x \sim x''$ to get $x'' \sim x$. Then by hypothesis, in the claim, it follows that $x' \sim x''$. □

Now we check the claim. Let $x, x', x'' \in K_n$ be n-simplices such that $x' \overset{y'}{\sim} x$ and $x'' \overset{y''}{\sim} x$. In words, $y' \in K_{n+1}$ is a homotopy from x' to x and $y'' \in K_{n+1}$ is a homotopy from x'' to x. Geometrically this means that we are given

$$\begin{array}{ccc} \overset{\circ}{\Delta}[n+1] & \xrightarrow{(s_{n-1}d_0x,\cdots,s_{n-1}d_{n-1}x,x',x)} & K \\ \downarrow & \nearrow_{y'} & \\ \Delta[n+1] & & \end{array} \tag{4.8}$$

and

$$\begin{array}{ccc} \overset{\circ}{\Delta}[n+1] & \xrightarrow{(s_{n-1}d_0x,\cdots,s_{n-1}d_{n-1}x,x'',x)} & K \\ \downarrow & \nearrow_{y''} & \\ \Delta[n+1] & & \end{array} \tag{4.9}$$

In long hand, this means we have the following relations

$$d_i x' = d_i = d_i x'' \quad \text{for} \quad 0 \leq i \leq n \tag{4.10}$$

$$d_i y' = \begin{cases} s_{n-1} d_i x' & 0 \leq i < n \\ x' & i = n \\ x & i = n+1 \end{cases} \tag{4.11}$$

$$d_i y'' = \begin{cases} s_{n-1} d_i x' & 0 \leq i < n \\ x'' & i = n \\ x & i = n+1 \end{cases} \tag{4.12}$$

For $0 \leq j < n$ we set

$$z_j := s_{n-1} s_{n-1} d_j x'.$$

Claim 4.3. *The $(n+2)$ many $n+1$-simplices*

$$z_0, z_1 \cdots z_{n-1}, -, z_{n+1} := y', z_{n+2} := y''$$

satisfy the Kan condition. In other words, we have a morphism

$$\Lambda_n^{n+2} \xrightarrow{(z_0, z_1 \cdots z_{n-1}, -, y', y'')} K. \tag{4.13}$$

Proof. We should check that for $0 \leq i < j \leq \mathbf{n+2}$ such that $n \notin \{i, j\}$ we have

$$d_i z_j = d_{j-1} z_i. \tag{4.14}$$

Case $j < n$: We remark at the outset that

$$s_{n-1} s_{n-1} d_j \overset{1.3}{=} s_{n-1} d_j s_n = d_j s_n s_n. \tag{4.15}$$

Thus for $0 \leq i \leq j < n$ we have

$$\begin{aligned} d_i z_j = d_i(s_{n-1} s_{n-1} d_j x') &\overset{(4.15)}{=} d_i(d_j s_n s_n x') \\ &\overset{1.1}{=} d_{j-1} d_i s_n s_n x' = d_{j-1}(d_i s_n s_n x') \\ &\overset{4.15}{=} d_{j-1}(s_{n-1} s_{n-1} d_i x') \\ &= d_{j-1} z_i. \end{aligned}$$

Case $j = n+1$: So for $0 \leq i < n$, we should check that

$$d_{j-1} z_i = d_i z_j = d_i z_{n+1}.$$

Now

$$d_{j-1} z_i = d_n z_i = d_n s_{n-1} s_{n-1} d_i x' \overset{1.3}{=} s_{n-1} d_i x' = d_i y' = d_i z_{n+1}.$$

Case $j = n+2$: So for $0 \leq i < n+2$ and $i \neq n$, we should check that

$$d_{j-1} z_i = d_i z_j = d_i z_{n+2}.$$

Sub case $0 \leq i < n$: we have

$$\begin{aligned} d_{j-1} z_i = d_{n+1} z_i = d_{n+1}(s_{n-1} s_{n-1} d_i x') &= (d_{n+1} s_{n-1}) s_{n-1} d_i x') \\ &\overset{1.5}{=} (s_{n-1} d_n) s_{n-1} d_i x' \\ &= s_{n-1}(d_n s_{n-1}) d_i x' \\ &\overset{1.3}{=} s_{n-1} d_i x' \\ &= d_i y'' = d_i z_{n+2} \end{aligned}$$

Sub case $i = n+1$: so we should check

$$d_{j-1} z_i = d_i z_j.$$

Now, $d_{j-1} z_i = d_{n+2-1} z_{n+1} = d_{n+1} y' = x$. On the other hand, $d_i z_j = d_{n+1} z_{n+2} = d_{n+1} y'' = x$.

□

Therefore by the Kan property of K, there exist an $(n+2)$-simplex z such that

$$\begin{aligned} d_{n+2}z &= y'' && (4.16)\\ d_{n+1}z &= y' && (4.17)\\ d_i z &= z_i \quad 0 \leq i < n. && (4.18) \end{aligned}$$

In other words, we have

$$\begin{array}{ccc} \Lambda^{n+2}_n & \xrightarrow{(z_0, z_1 \cdots z_{n-1}, -, y', y'')} & K \\ \downarrow & \nearrow_{z} & \\ \Delta[n+2] & & \end{array} \qquad (4.19)$$

Claim 4.4. *A homotopy from x' to x'' is given by $d_n z$.*

Proof. Let us check

$$\begin{array}{ccc} \overset{\circ}{\Delta}[n+1] & \xrightarrow{(s_{n-1}d_0x', \cdots, s_{n-1}d_{n-1}x', x', x'')} & K \\ \downarrow & \nearrow_{d_n z} & \\ \Delta[n+1] & & \end{array} \qquad (4.20)$$

We have

1. $d_j(x'') = d_j(x')$ for $0 \leq j \leq n$ clearly.
2. $d_n d_{n+1} z \overset{1.1}{=} d_n d_n z = d_n(y') = x'$ and
$$d_{n+1} d_n z = d_n d_{n+2} z = d_n y'' = x''.$$
3. for $0 \leq i < n$, $d_i(d_n z) = d_{n-1} d_i z = d_{n-1} z_i = d_{n-1}(s_{n-1}s_{n-1}d_i)x' = (d_{n-1}s_{n-1})s_{n-1}d_i x' = s_{n-1}d_i x' = s_{n-1}(d_i x') = s_{n-1}(d_i x'')$.

□

We obtain *symmetry* by setting $x'' = x$. And *transitivity* follows. □

Let k_0 be a basepoint (cf. Definition (1.7)) of a simplicial set K. If K is a Kan complex, then we shall say that (K, k_0) is a *Kan pair*.

Definition 4.2. *Let (K, k_0) be a Kan pair. Then we define*

$$\pi_n(K, k_0) = \frac{\{x \in K_n | d_i x = k_0\}}{\sim}$$

where $\sim$ is the equivalence relation described above.

Let (K, k_0) be a Kan pair. Let $\alpha, \beta \in \pi_n(K, k_0)$. We choose a representatives x for α and y for β.

Claim 4.5. *Then the* $(n+1)$ *many* n*-simplices*

$$z_0 = k_0, z_1 = k_0, ..., z_{n-2} = k_0, z_{n-1} = x, -, z_{n+1} = y$$

satisfy the hypothesis of the Kan condition.

Proof. By definition, we should be able to define a morphism

$$\Lambda_n^{n+1} \xrightarrow{(k_0,k_0,\cdots,k_0,x,-,y)} K. \tag{4.21}$$

In other words, we should check that for $0 \leq i < j \leq n+1$ and $n \notin \{i, j\}$ we have

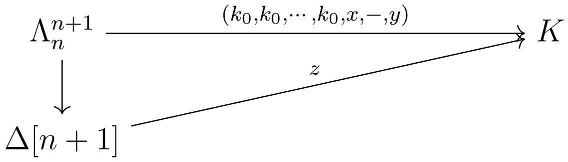
$$d_i z_j = d_{j-1} z_i.$$

Thus $0 \leq i \leq n-1$, so by hypothesis on x, y and k_0, we have

$$d_i z_j = k_0 \in K_{n-1},$$

and $d_{j-1} z_i$ is also $k_0 \in K_{n-1}$. □

Therefore there exists an $(n+1)$ simplex z such that



(4.22)

or equivalently,

$$d_{n+1} z = y, \tag{4.23}$$

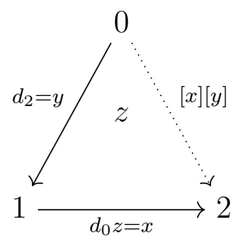
$$d_{n-1} z = x, \tag{4.24}$$

$$d_i z = k_0 \qquad 0 \leq i < n-1. \tag{4.25}$$

We define $\alpha\beta$ to be the homotopy equivalence class in K_n of $d_n z$ namely

$$\alpha\beta := [d_n z].$$

For the case $n = 1$ the following diagram encodes *multiplication geometrically.* When $n = 1$ we shall have no k_0



The relation $d_0 z = x$ can be read from the diagram as: the face opposite to 0-th vertex of z. Other relations can be read similarly. We obtain the product of $[x]$ and $[y]$ as the face opposite vertex 1.

Lemma 4.6. *The multiplication $\alpha\beta$ is well defined.*

Proof. We first check that the multiplication is well defined *with respect to the horn filler z*. Let z' be another $(n+1)$-simplex. Suppose z' also satisfies $d_{n+1}z' = y$, $d_{n-1}z' = x$ and $d_i(z') = k_0$ for $0 \leq i < n-1$. In other words, z' also fits in

$$\begin{array}{ccc} \Lambda^{n+1}_n & \xrightarrow{(k_0,k_0,\cdots,k_0,x,-,y)} & K \\ \big\downarrow & \overset{z'}{\nearrow} & \\ \Delta[n+1] & & \end{array} \tag{4.26}$$

Claim 4.7. *Then the $n+2$ many $(n+1)$-simplices*

$$a_0 = k_0, a_1 = k_0, \cdots, a_{n-1} = s_n d_{n-1} z, -, a_{n+1} = z, a_{n+2} = z'$$

satisfy the hypothesis of the Kan condition i.e we have a morphism

$$\Lambda^{n+2}_n \xrightarrow{(k_0,k_0,\cdots,k_0,s_n d_{n-1}z,-,z,z')} K. \tag{4.27}$$

Proof. We should check that for $0 \leq i < j \leq n+2$ and $n \notin \{i,j\}$, we have

$$d_i a_j = d_{j-1} a_i.$$

Notice that for $i < n$:

1. we have $d_i k_0 = k_0$,
2. now

$$\begin{aligned} d_i a_{n-1} &= d_i s_n d_{n-1} z \\ &= d_i s_n x \\ &= s_{n-1} d_i x \\ &= s_{n-1} k_0 = k_0, \end{aligned}$$

3. $d_i z \overset{(4.25)}{=} k_0$,
4. $d_i z' = k_0$.

For $i = n+1$, we have $j = n+2$.

So, $d_i a_j = d_{n+1} a_{n+2} = d_{n+1} z' = y \overset{(4.23)}{=} d_{n+1} z = d_{n+2-1} a_{n+1} = d_{j-1} a_i$.

□

Therefore by the Kan condition on K, there exists a $(n+2)$-simplex w commuting

$$\begin{array}{ccc} \Lambda^{n+2}_n & \xrightarrow{(k_0,k_0,\cdots,k_0,s_n d_{n-1}z,-,z,z')} & K \\ \downarrow & \nearrow_{w} & \\ \Delta[n+2] & & \end{array} \tag{4.28}$$

In long hand, we have

$$d_{n+2}w = z', \tag{4.29}$$

$$d_{n+1}w = z, \tag{4.30}$$

$$d_{n-1}w = s_n d_{n-1}z, \tag{4.31}$$

and

$$d_i w = k_0 \qquad 0 \leq i < n-1. \tag{4.32}$$

Claim 4.8. *The* $(n+1)$*-simplex* $d_n w$ *defines a homotopy from* $d_n z$ *to* $d_n z'$*.*

Proof. We should check the commutativity of

$$\begin{array}{ccc} \overset{\circ}{\Delta}[n+1] & \xrightarrow{(s_{n-1}d_0 d_n z,\cdots,s_{n-1}d_{n-1}d_n z,d_n z,d_n z')} & K \\ \downarrow & \nearrow_{d_n w} & \\ \Delta[n+1] & & \end{array} \tag{4.33}$$

In other words, we should check that

$$\begin{aligned} d_n d_n w &= d_n z \\ d_{n+1} d_n w &= d_n z' \\ d_i d_n w &= s_{n-1} d_i d_n z \\ &= s_{n-1} d_i d_n z' \qquad 0 \leq i \leq n-1 \end{aligned}$$

Now

$$d_n d_n w \overset{(1.1)}{=} d_n(d_{n+1}w) = d_n z$$

and

$$d_{n+1} d_n w \overset{1.1}{=} d_n(d_{n+2}w) = d_n z'.$$

For $i \leq n-1$ we have $d_i d_n w = d_{n-1} d_i w = d_{n-1} k_0 = k_0$. On the other hand, $d_i d_n z = d_{n-1} d_i z = d_{n-1}$ of k_0, x or y. Thus it equals k_0 so it's s_{n-1} is also k_0. So

$$[d_n z] = [d'_z]$$

□

This completes the proof of the independence with respect to the Horn-filler.

Now we check *independence with respect to the lift of β:* Suppose we had chosen another representative y' of β. Then by considering

$$k_0, ..., k_0, x, -, y$$

as in Claim 4.5, we should have found a homotopy z' from x to y' commuting

$$\begin{array}{ccc} \Lambda^{n+1}_n & \xrightarrow{(k_0,k_0,\cdots,k_0,x,-,y')} & K \\ \downarrow & \nearrow_{z'} & \\ \Delta[n+1] & & \end{array} \tag{4.34}$$

Then we would define

$$\alpha\beta := [d_n z'].$$

Then since $x \overset{z'}{\sim} y'$,so in long-hand we have

$$d_{n+1}z' = y, \tag{4.35}$$

$$d_{n-1}z' = x, \tag{4.36}$$

$$d_i z' = k_0 \qquad 0 \leq i < n-1. \tag{4.37}$$

Since y and y' are in the same homotopy class (defined by β), so let $y \overset{w}{\sim} y'$ be a homotopy between them. Or in other words we have the following commutative diagram

$$\begin{array}{ccc} \overset{\circ}{\Delta}[n+1] & \xrightarrow{(s_{n-1}d_0y,\cdots,s_{n-1}d_{n-1}y,y,y')} & K \\ \downarrow & \nearrow_{w} & \\ \Delta[n+1]. & & \end{array} \tag{4.38}$$

Claim 4.9. *: The $n+2$-many $(n+1)$-simplices*

$$b_0 = k_0, b_1 = k_0, ..., b_{n-2} = k_0, b_{n-1} := z, b_n := z', -, b_{n+2} = w$$

satisfy the hypothesis of Kan condition. In other words we have a morphism

$$\Lambda^{n+2}_{n+1} \xrightarrow{(k_0,k_0,\cdots,k_0,z,z',-,w)} K. \tag{4.39}$$

Proof. We should check that for $0 \leq i < j \leq n+2$ and $n+1 \notin \{i,j\}$ we have

$$d_i b_j = d_{j-1} b_i.$$

So $i \leq n$.

Case $i \leq n-2$: For any j we have $d_i b_j = k_0$ and since $b_i = k_0$ so

$$d_{j-1} b_i = k_0.$$

Case $i = n-1$:
Sub-case $j = n$:

$$d_i b_j = d_{n-1} b_n = d_{n-1} z' = x$$

On the other hand

$$d_{j-1} b_i = d_{n-1} z = x.$$

Sub-case $j = n+2$:

$$d_i b_j = d_{n-1} b_{n+2} = d_{n-1} w = y$$

and

$$d_{j-1} b_i = d_{n+1} b_{n-1} = d_{n+1} z = y$$

Case $i = n$**:**
We should check that $d_n b_{n+2} = d_{n+2-1} b_n$.
Now $d_n b_{n+2} = d_n w = y$ and also $d_{n+1} z = y$. □

Therefore by the Kan condition on K there exist an $n+2$-simplex u commuting

$$\begin{array}{ccc} \Lambda^{n+2}_{n+1} & \xrightarrow{(k_0,k_0,\cdots,k_0,z,z',-,w)} & K \\ \downarrow & \nearrow_{u} & \\ \Delta[n+2] & & \end{array} \tag{4.40}$$

In other words, we have

$$d_i u = k \qquad 0 \leq i < n-1, \tag{4.41}$$

$$d_{n-1} u = z, \tag{4.42}$$

$$d_n u = z', \tag{4.43}$$

$$d_{n+2} u = w. \tag{4.44}$$

Consider the $n+1$-simplex $v := d_{n+1} u$. We have $d_i v = k_0$ for $0 \leq i < n-1$, because $d_i v = d_i d_{n+1} u = d_n d_i u = d_n$ of k_0 or z which are all k_0.

Further

$$\begin{aligned} d_{n-1} v = d_{n-1} d_{n+1} u &= d_n d_{n-1} u \\ &= d_n z = x, \end{aligned} \tag{4.45}$$

$$\begin{aligned} d_n v = d_n d_{n+1} u &= d_n d_n u \\ &= d_n z' \end{aligned} \tag{4.46}$$

$$\begin{aligned} d_{n+1} v = d_{n+1} d_{n+1} u &= d_{n+1} d_{n+2} u \\ &= d_{n+1} w = y \end{aligned} \tag{4.47}$$

By (4.47) and (4.45), v along with z is also a horn filler of

$$k_0, k_0, ..., x, -, y.$$

Thus

$$[d_n z] = [d_n v] \stackrel{(4.46)}{=} [d_n z']$$

where the first equality follows from our first check that the group law does not depend on the choice of the Horn filler z between x and y. □

Proposition 4.10. *Let (K, k_0) be a Kan pair. Then with the above multiplication, $\pi_n(K, k_0)$ is a group for $n \geq 1$.*

Proof. (i). *Neutral element* Let us check that $[k_0]$ is the identity element. Let $\alpha \in \pi_n(K, k_0)$ and let x be a representative. So

$$d_i x = k_0 \,\forall\, i.$$

The set of $n+1$ many n-simplices

$$k_0, k_0, ..., x, -, k_0$$

clearly satisfy the hypothesis of the Kan condition. In fact we can take $s_{n-2}x$ as the Horn filler. Then for $i < n-2$ we have $d_i s_{n-2}x = s_{n-3}d_i x = k_0$. Further $d_{n-2}s_{n-2}x = x$ and $d_n s_{n-2}x = s_{n-2}d_{n-1}x = s_{n-2}k_0 = k_0$. Now

$$[x][k_0] = [d_n(s_{n-1}x)] = [x].$$

So we have showed that $\alpha[k_0] = \alpha$.

Similarly we check that k_0 is the left identity as follows. Consider

$$k_0, ..., k_0, -, x.$$

These satisfy the Kan filler condition because any face of any element of the above collection is k_0. Now as a horn filler we may take $s_n x$. Indeed for $i \leq n-1$

$$d_i s_n x = s_{n-1}d_i x = s_{n-1}k_0 = k_0$$

and

$$d_{n+1}s_n x = x.$$

Thus $[k_0][x] = [d_n s_n x] = [x] = \alpha$ itself. Similarly one checks that k_0 is the right identity.

(iii). *Divisibility:* Let $\alpha, \beta \in \pi_n(K, k_0)$. We choose a representative x for α and y for β. The $n+1$ n-simplices

$$k_0, ..., k_0, -, y, x$$

satisfy the hypothesis of the Kan condition because the face of any element in the above collection is k_0. So there exists an $n+1$ simplex z such that $d_i z = k_0$ for $0 \leq i \leq n-1$, $d_n z = y$ and $d_{n+1}z = x$. Then by definition of group law we have

$$[d_{n-1}z]\alpha = [d_n z] = [y] = \beta.$$

This proves left divisibility. Similarly the $n+1$ many n-simplices

$$k_0, ..., k_0, y, x, -$$

also satisfy the hypothesis of the Kan condition. So we have an $n+1$-simplex z such that $d_i z = k_0$ for $0 \leq i \leq n-1$, $d_{n-1} z = y$ and $d_n z = x$. Now by definition of group law we have

$$\beta[d_{n+1}z] = [d_n z] = [x] = \alpha.$$

(ii). *Associativity:* Let x, y, z be representatives of $\alpha, \beta, \gamma \in \pi_n(K, k_0)$ respectively. Using the extension conditions choose w_{n-1}, w_{n+1} and w_{n+2} such that

$$\begin{aligned} d_i w_j &= k_0 & 0 \leq i < n-1, && (4.48) \\ d_{n-1} w_{n-1} &= x & d_{n+1} w_{n-1} = y, && (4.49) \\ d_{n-1} w_{n+1} &= d_n w_{n-1} & d_{n+1} w_{n+1} = z, && (4.50) \\ d_{n-1} w_{n+2} &= y & d_{n+1} w_{n+2} = z. && (4.51) \end{aligned}$$

Geometrically, we have

$$\begin{array}{ccc} \Lambda^{n+1}_n & \xrightarrow{(k_0, k_0, \cdots, k_0, x, -, y)} & K \\ \downarrow & \nearrow_{w_{n-1}} & \\ \Delta[n+1] & & \end{array} \qquad (4.52)$$

$$\begin{array}{ccc} \Lambda^{n+1}_n & \xrightarrow{(k_0, k_0, \cdots, k_0, d_n w_{n-1}, -, z)} & K \\ \downarrow & \nearrow_{w_{n+1}} & \\ \Delta[n+1] & & \end{array} \qquad (4.53)$$

$$\begin{array}{ccc} \Lambda^{n+1}_n & \xrightarrow{(k_0, k_0, \cdots, k_0, y, -, z)} & K \\ \downarrow & \nearrow_{w_{n+2}} & \\ \Delta[n+1] & & \end{array} \qquad (4.54)$$

In other words, we have homotopies $x \overset{w_{n-1}}{\sim} y$, $d_n w_{n-1} \overset{w_{n+1}}{\sim} z$ and $y \overset{w_{n+1}}{\sim} z$. Further, the product $[x][y]$ is represented by the class of $d_n w_{n-1}$

By compatibility relations in (4.48), it follows that we have a morphism

$$\Lambda^{n+2}_n \xrightarrow{(k_0, k_0, \cdots, k_0, w_{n-1}, -, w_{n+1}, w_{n+2})} K \qquad (4.55)$$

Thus the hypothesis of the Kan condition is satisfied. So we may choose a $u \in K_{n+2}$ commuting

$$\begin{array}{ccc} \Lambda^{n+2}_n & \xrightarrow{(k_0, k_0, \cdots, k_0, w_{n-1}, -, w_{n+1}, w_{n+2})} & K \\ \downarrow & \nearrow_{u} & \\ \Delta[n+2] & & \end{array} \qquad (4.56)$$

In long-hand, we may choose $u \in K_{n+2}$ such that $d_i u = k_0$ for $0 \leq i < n-1$ and $d_i u = w_i$ for $i = n-1, n+1, n+2$. Then

$$d_{n-1} d_n u = x,$$

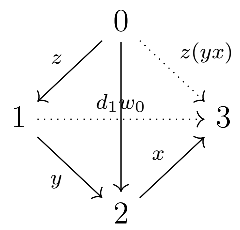
$$d_{n+1} d_n u = d_n w_{n+2}$$

and

$$d_i u = k_0 \qquad 0 \leq i \leq n-1.$$

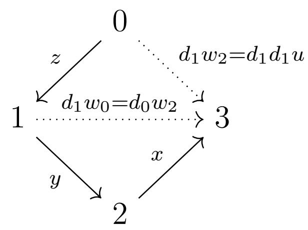
We now explain a diagrammatic convention for a 3-simplex to reveal the meaning of calculations proving associativity.

We will denote u being the 3-simplex which is a tetrahedron. We will denote by $w_j = d_j u$ the faces opposite the vertex j. Since there is a natural order on the vertices, so this will allow us to relabel the vertices of w_j also. When we write $d_k w_j$ we shall mean the face opposite the k-th largest vertex of w_j. Setting $n = 1$ has the advantage that k_0 terms are not there. Reader may verify that for $n = 1$ this information can be encoded in the following diagram. As usual the solid arrows represent the given data and the dotted arrows are constructed



We remark that from the arrows x,y and z there is only one way using the dotted arrows to complete to a tetrahedron.

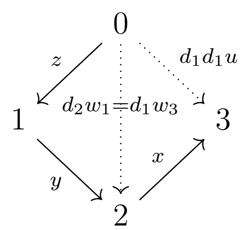
Therefore, for $n = 1$ we shall obtain the following diagram



which for an arbitrary n will read as follows:

$$(\alpha\beta)\gamma = [d_n w_{n-1}]\gamma = [d_{n-1} w_{n+1}]\gamma = [d_n w_{n+1}]$$

Similarly we have



And now for arbitrary n the above equality extends to

$$(\alpha\beta)\gamma = [d_n w_{n-1}]\gamma = [d_{n-1}w_{n+1}]\gamma = [d_n w_{n+1}] = [d_n d_n u] = \alpha[d_n w_{n+2}] = \alpha(\beta\gamma).$$

□

We would like to remark here that the interior of the tetrahedron (u) gives 2-isomorphisms between $[\alpha]([\beta][\gamma])$ and $([\alpha][\beta])[\gamma]$.

Proposition 4.11. *The homotopy groups $\pi_n(K, k_0)$ is Abelian if $n \geq 2$.*

Proof. Let $w, x, y, z \in K_n$ respectively be the representatives of $[w], [x], [y], [z] \in \pi_n(K_n, k_0)$.

The proof will be divided into three preparatory steps and one final step combining the first step with the third. We remark that the second step is only an ingredient to the third.

By the symbol $[w, x, y, z]$ the reader may want to imagine a tetrahedron "T such that $d_i T$ is these elements in this order. We invite the reader to put $n = 2$ in the following proof.

(i) "$[w, x, y, k_0] \Rightarrow [y][w] = [x]$" Suppose there exists a $v_{n+1} \in K_{n+1}$ satisfying $d_i v_{n+1} = k_0$ for $0 \leq i < n-2$, $d_{n-2}v_{n+1} = w$, $d_{n-1}v_{n+1} = x$, $d_n v_{n+1} = y$ and $d_{n+1}v_{n+1} = k_0$, then

Claim 4.12. *We have $[y][w] = [x]$.*

Proof. Choose $v_{n-1} \in K_{n+1}$ with faces $d_i v_{n-1} = k_0$, for $0 \leq i \leq n-2$,

$$d_n v_{n-1} = x$$

and

$$d_{n+1}v_{n-1} = w.$$

Let $t = d_{n-1}v_{n-1}$. Let $v_i = k_0$, for $0 \leq i < n-2$. Finally let $v_{n-2} = s_n w$ and $v_{n+2} = s_{n-2}w$

Claim 4.13. *The $n+2$ many $(n+1)$-simplices*

$$v_0, v_1, ..., v_{n-1}, -, v_{n+1}, v_{n+2}$$

satisfies the compatibility conditions. In other words we claim that the following map exists

$$\Lambda^{n+2}_n \xrightarrow{(v_0, v_1, ..., v_{n-1}, -, v_{n+1}, v_{n+2})} K.$$

Proof. It is enough to check the compatibility condition for $j = n-1$, $j = n+1$ and $j = n+2$. The other relations are easily obtained because $v_i = k_0$ for $0 \leq i \leq n-2$.

Case $j = n-1$**:** We have $d_i v_{n-1} = k_0 = d_n v_i$ for $0 \leq i < n-2$. Now $d_{n-2}v_{n-1} = k_0$. On the other hand $d_{n-2}v_{n-2} = d_{n-2}s_n w = s_{n-1}d_{n-2}w = s_{n-1}d_{n-2}d_{n+1}v_{n-1} = s_{n-1}d_n d_{n-2}v_{n-1} = k_0$.

Case $j = n+1$**:** We have $d_{n-2}v_{n+1} = w$. On the other hand $d_n v_{n-2} = d_n s_n w = w$. Now $d_{n-1}v_{n+1} = x = d_n v_{n-1}$. When $i \leq n-2$ we clearly have both LHS and RHS of compatibility relations equalling to k_0.

Case $j = n+2$**:**

Subcase $i = n-2$*:* We have

$$d_{n-2}v_{n+2} = d_{n-2}s_{n-2}w = w = d_{n+1}s_{n+1}v_{n-2} = d_{n+1}v_{n-2}.$$

Subcase $i = n-1$*:* We have $d_{n-1}v_{n+2} = d_{n-1}s_{n-2}w = w = d_{n+1}v_{n-1}$.

Subcase $i = n+1$*:* We have

$$d_{n+1}v_{n+2} = d_{n+1}s_{n-2}w = s_{n-2}d_n w =$$

$$s_{n-2}d_n d_{n-2}v_{n+1} = s_{n-2}d_{n-2}d_{n+1}v_{n+1} = k_0 = d_{n+1}v_{n+1}.$$

□

Therefore we have an $n+2$-simplex r satisfying

$$d_i r = v_r \quad i \neq n.$$

In other words the following diagram is commutative

$$\begin{array}{ccc} \Lambda^{n+2}_n & \xrightarrow{(v_0,v_1,\ldots,v_{n-1},-,v_{n+1},v_{n+2})} & K \\ \downarrow & \nearrow_{r} & \\ \Delta[n+2]. & & \end{array}$$

Set $v_n := d_n r$. Now

$$d_i v_n = k_0 \quad \text{for} \quad 0 \leq i \leq n-2, \quad d_{n-1}v_n = t, \quad d_n v_n = y,$$

and $d_{n+1}v_n = k_0$. Therefore $[t][k_0] = [y]$; but by the choice of v_{n-1}, $[t][w] = [x]$, hence

$$[y][w] = [x].$$

□

(ii) "$[w, k_0, y, z] \Rightarrow [w][y] = [z]$*"* Suppose $v_n \in K_{n+1}$ satisfies $d_i v_n = k_0$ for $0 \leq i < n-2$, $d_{n-2}v_n = w$, $d_{n-1}v_n = k_0$, $d_n v_n = y$ and $d_{n+1}v_n = z$.

Claim 4.14. *We have* $[w][y] = [z]$.

Proof. Choose $v_{n-1} \in K_{n+1}$ with faces $d_i v_{n-1} = k_0$, for $0 \leq i \leq n-2$,

$$d_{n-2}v_{n-1} = w,$$

and

$$d_{n-1}v_{n-1} = k_0 = d_{n+1}v_{n-1}.$$

Let $t = d_n v_{n-1}$. Let $v_i = k_0$, for $0 \leq i < n-2$. Finally let

$$v_{n-2} = s_{n-2}w$$

and

$$v_{n+2} = s_n z.$$

Claim 4.15. *We have the following map*

$$\Lambda^{n+2}_{n+1} \xrightarrow{(v_0,v_1,\ldots,v_{n-1},v_n,-,v_{n+2})} K.$$

In long hand, the $n+2$ many $(n+1)$-simplices

$$v_0, v_1, \ldots, v_{n-1}, v_n, -, v_{n+2}$$

satisfies the compatibility conditions.

Proof. It is enough to check the compatibility condition for $j = n-1$, $j = n$ and $j = n+2$. The other relations are easily obtained because $v_i = k_0$ for $0 \leq i \leq n-2$.

Case $j = n-1$**:** We have $d_i v_{n-1} = k_0 = d_n v_i$ for $0 \leq i < n-2$. Now $d_{n-2}v_{n-1} = w$. On the other hand $d_{n-2}v_{n-2} = d_{n-2}s_{n-2}w = w$.

Case $j = n$**:** We have $d_{n-2}v_n = w$. On the other hand $d_{n-1}v_{n-2} = d_{n-1}s_{n-2}w = w$. Now $d_{n-1}v_n = k_0 = d_{n-1}v_{n-1}$. When $i \leq n-2$ we clearly have both LHS and RHS of compatibility relation equalling to k_0.

Case $j = n+2$**:**

Subcase $i = n-2$: We have $d_{n-2}v_{n+2} = d_{n-2}s_n z = s_{n-1}d_{n-2}z$
$= s_{n-1}d_{n-2}d_{n+1}v_n = s_{n-1}d_n d_{n-2}v_n = s_{n-1}d_n w = d_{n+1}s_{n-2}w = d_{n+1}v_{n-2}$

Subcase $i = n-1$: We have $d_{n-1}v_{n+2} = d_{n-1}s_n z = s_{n-1}d_{n-1}d_{n+1}v_n = s_{n-1}d_n d_{n-1}v_n = k_0 = d_{n+1}v_{n-1}$.

Subcase $i = n$: We have

$$d_n v_{n+2} = d_n s_n z = z = d_{n+1}v_n.$$

□

Therefore we have an $n+2$-simplex r such that the following diagram is commutative

$$\begin{array}{ccc} \Lambda^{n+2}_{n+1} & \xrightarrow{(v_0,v_1,\ldots,v_{n-1},v_n,-,v_{n+2})} & K \\ \downarrow & \overset{r}{\nearrow} & \\ \Delta[n+2]. & & \end{array}$$

That is we have an $n+2$-simplex r satisfying

$$d_i r = v_i \quad i \neq n.$$

Let $v_{n+1} = d_{n+1}r$. Now

$$d_i v_n + 1 = k_0 \quad \text{for} \quad 0 \leq i \leq n-2, \quad d_{n-1}v_{n+1} = t, \quad d_n v_{n+1} = y,$$

and $d_{n+1}v_{n+1} = z$. Therefore $[t][z] = [y]$; but by the choice of v_{n-1}, and (i) $[t][w] = [k_0]$. Hence

$$[w][y] = [z].$$

□

(iii) "$[w, x, y, z] \Rightarrow [w]^{-1}[x][z] = [y]$" Suppose $v_{n+2} \in K_{n+1}$ satisfies $d_i v_{n+2} = k_0$ for $0 \leq i < n-2$, $d_{n-2}v_{n+2} = w$, $d_{n-1}v_{n+2} = x$, $d_n v_{n+2} = y$ and $d_{n+1}v_{n+2} = z$.

Claim 4.16. *We have $[w]^{-1}[x][z] = [y]$.*

Proof. Choose $v_{n-2} \in K_{n+1}$ with faces $d_i v_{n-2} = k_0$, for $i \neq n-2, n+1$,

$$d_{n+1}v_{n-2} = w$$

and let $t = d_{n-2}v_{n-2}$. Choose $v_{n-1} \in K_{n+1}$ with faces $d_i v_{n-1} = k_0$, for $i \neq \{n-2, n+1\}$,

$$d_{n+1}v_{n-1} = x,$$

$$t = d_{n-2}v_{n-1}.$$

Let $u = d_{n-2}v_{n-1}$. Let $v_i = k_0$, for $0 \leq i < n-2$. Finally let

$$v_n = s_n.y$$

Claim 4.17. *The $n+2$ many $(n+1)$-simplices*

$$v_0, v_1, ..., v_{n-1}, v_n, -, v_{n+2}$$

satisfies the compatibility conditions. In terms of diagram this means that we have the following map

$$\Lambda^{n+2}_{n+1} \xrightarrow{(v_0,v_1,...,v_{n-1},v_n,-,v_{n+2})} K.$$

Proof. It is enough to check the compatibility condition for $j = n-1$, $j = n$ and $j = n+2$. The other relations are easily obtained because $v_i = k_0$ for $0 \leq i \leq n-2$.

Case $j = n-1$: We have $d_i v_{n-1} = k_0 = d_n v_i$ for $0 \leq i < n-2$. Now $d_{n-2}v_{n-1} = t$. On the other hand $d_{n-2}v_{n-2} = t$.

Case $j = n$: We have $d_{n-2}v_n = d_{n-2}s_n y = s_{n-1}d_{n-2}d_n v_{n+2} = s_{n-1}d_{n-1}w = s_{n-1}d_{n-1}d_{n+1}v_{n-2} = s_{n-1}d_n d_{n-1}v_{n-2} = k_0$. On the other hand $d_{n-1}v_{n-2} = k_0$.

Now $d_{n-1}v_n = d_{n-1}s_n y = s_{n-1}d_{n-1}d_n v_{n+2} = s_{n-1}d_{n-1}d_{n-1}v_{n+2}$ $= s_{n-1}d_{n-1}x = s_{n-1}d_{n-1}d_{n+1}v_{n-1} = s_{n-1}d_n d_{n-1}v_{n-1} = k_0 = d_{n-1}v_{n-1}$. When $i \leq n-2$ we clearly have both LHS and RHS of compatibility relation equalling to k_0.

Case $j = n+2$:

Subcase $i = n-2$: We have $d_{n-2}v_{n+2} = w = d_{n+1}v_{n-2}$.

Subcase $i = n-1$: We have $d_{n-1}v_{n+2} = x = d_{n+1}v_{n-1}$.

Subcase $i = n$: We have

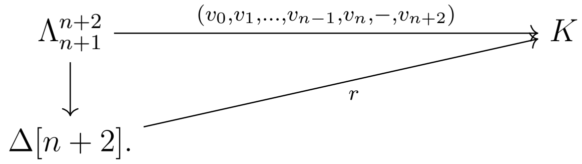
$$d_n v_{n+2} = y = d_{n+1}s_n y = d_{n+1}v_n.$$

□

Therefore we have an $n+2$-simplex r satisfying

$$d_i r = v_i \qquad i \neq n+1.$$

Set $v_{n+1} := d_{n+1} r$. In the language of diagrams this means that we have an $n+2$-simplex r such that the following diagram is commutative



Now

$$d_i v_{n+1} = k_0 \quad \text{for} \quad 0 \leq i \leq n-2, \quad d_n v_{n+1} = y,$$

and $d_{n+1} v_{n+1} = z$. Set $d_{n-1} v_{n+1} =: u$. Therefore $[u][z] = [y]$; but by the choice of v_{n-1}, and (i) $[t][w] = [k_0]$. Now by (ii) we have $[t] = [w]$ and $[t][u] = [x]$. Combining we get

$$[w]^{-1}[x][z] = [y].$$

□

Combining (iv) Set $z = k_0$ in (iii). Then $[w]^{-1}[x] = [y]$. By applying (i) to v_{n+2} of (iii), we find $[y] = [x][w]^{-1}$. Therefore for any $[x]$ and $[w]$,

$$[x][w]^{-1} = [w]^{-1}[x].$$

□

4.1 Relative situation

Definition 4.3. *Let K be a simplicial set. Let L be a sub-simplicial set of K. We say that two n-simplices x, x' of K are homotopic relative to L if*

1. *we have $d_j x = d_j x' \; \forall \; 1 \leq j \leq n$,*

2. *$d_0 x \overset{y}{\sim} d_0 x'$ in for a $y \in L_n$,*

3. *there exists $w \in K_{n+1}$ such that*

$$d_0 w = y, d_n w = x, d_{n+1} w = x',$$

$$d_i w = s_{n-1} d_i x = s_{n-1} d_i x' \quad \textit{for} \quad 1 \leq i < n.$$

We write $x \overset{w}{\sim}_L x'$ (or simply $x \sim_L x'$) and we say that w is a homotopy from x to x' relative to L.

Definition 4.4. *Let K be a Kan complex with a "base-point" k_0. We call (K, L, k_0) a Kan triple if $k_0 \in L_0$ and L is a sub Kan complex of K.*

Definition 4.5. *Let* (K, L, k_0) *be a Kan triple, then we define the relative n-th homotopy group*

$$\pi_n(K, L, k_0) = \frac{\{x \in K_n | d_0 x \in L_{n-1}, d_i x = k_0 \, 1 \leq i \leq n\}}{\sim_L}.$$

For $n \geq 2$*, we define multiplication in* $\pi_n(K, L, k_0)$ *in a way analogous to the absolute situation as follows. Choose* $\alpha, \beta \in \pi_n(K, L, k_0)$*. Let* x *and* y *be representatives of* α *and* β *respectively. Then* $d_0 x$*,* $d_0 y \in L_{n-1}$ *and, we have that* $[d_0 x][d_0 y] = [d_{n-1} z]$ *for* $z \in L_n$ *satisfying* $d_i z = k_0$ *,* $i \leq n-3$ *and* $d_{n-2} z = d_0 x, d_n z = d_0 y$*. The* $n+1$*-many* n*-simplices*

$$z, k_0, ..., k_0, x, -, y$$

satisfy the hypothesis of the Kan condition. Thus there exists $w \in K_{n+1}$ *such that* $d_i w = k_0$ *for* $1 \leq i \leq n-2$ *and* $d_0 w = z$*,* $d_{n-1} z = x$*,* $d_{n+1} z = y$*. We define*

$$\alpha\beta = [d_n w].$$

As before this group law is well-defined for $n \geq 2$. For $n \geq 3$, we shall prove that these relative groups are abelian.

Let $[x]$ be a relative homotopy class in $\pi_{n+1}(K, L, k_0)$. We define a map $\delta : \pi_{n+1}(K, L, k_0) \to \pi_n(L, k_0)$ by setting

$$\delta[x] := [d_0 x].$$

By definition of relative homotopy classes, it follows immediately that this assignment is well-defined.

By exactness of a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of pointed sets at B we shall mean that $g^{-1}(*) = Img(f)$.

Proposition 4.18. *Let* (K, L, k_0) *be a Kan triple. Then the sequence*

$$\cdots \xrightarrow{j} \pi_{n+1}(K, L, k_0) \xrightarrow{\delta} \pi_n(L, k_0) \xrightarrow{i} \pi_n(K, k_0) \xrightarrow{j} \pi_n(K, L, k_0) \to \cdots$$

of sets with distinguished element k_0 *is exact, where the maps* i *and* j *are induced by inclusion.*

Proof. i) Consider the following part of the sequence

$$\cdots \pi_{n+1}(K, L, k_0) \xrightarrow{\delta} \pi_n(L, k_0) \xrightarrow{i} \pi_n(K, k_0) \cdots$$

we shall prove that $i\delta = k_0$:

Let $[x] \in \pi_{n+1}(K, L, k_0)$. Let $x \in K_{n+1}$ be a representative of $[x]$. By definition $i\delta[x] = i[\delta_0 x]$.

Claim 4.19. *The* $(n+2)$ *many* $n+1$ *simplices*

$$-, a_1 = k_0, a_2 = k_0, ..., a_{n+1} = k_0, a_{n+2} = x$$

satisfy compatibility condition. In language of diagrams, we claim that we have the following map

$$\Lambda_0^{n+2} \xrightarrow{(-, k_0, \cdots, k_0, x)} K.$$

Proof. We should check that for $i < j$ and $0 \notin \{i, j\}$, we have

$$d_i a_j = d_{j-1} a_i.$$

So $i \geq 1$ and therefore $j > 1$.

Case $1 < j < n+2$ **:** Clearly $d_i a_j = k_0 = d_{j-1} a_i$.

Case $j = n+2$ **:** Let us consider $d_i x$. Since $i \geq 1$, this is equal to k_0 by definition of relative homotopy. Thus the claim is verified. □

Therefore we have an $n+2$ simplex z such that the following diagram is commutative

$$\begin{array}{ccc} \Lambda_0^{n+2} & \xrightarrow{(-,k_0,\cdots,k_0,x)} & K \\ \downarrow & \nearrow_{z} & \\ \Delta[n+2]. & & \end{array} \tag{4.57}$$

In long hand this means $d_i z = k_0$ for $1 \leq i \leq n+1$ and $d_{n+2} z = x$. Now we show that $d_0 z$ is a homotopy between $d_0 x$ and k_0. By the first simplicial identity, for $0 \leq i < n+1$ we have

$$d_i d_0 z = d_0 d_{i+1} z = k_0.$$

Further $d_{n+1} d_0 z = d_0 d_{n+2} z = d_0 x$. Furthermore since $d_i x = k_0$, so we see that for $0 \leq i < n$ we have

$$d_i(d_0 z) = k_0 = s_{n-1} d_i x.$$

Thus $d_0 z$ is a homotopy between $d_0 x$ and k_0. In the language of diagram this means that the following diagram commutes

$$\begin{array}{ccc} \overset{\circ}{\Delta}[n] & \xrightarrow{(k_0,k_0\cdots,k_0,k_0,d_0 x)} & K \\ \downarrow & \nearrow_{d_0 z} & \\ \Delta[n] & & \end{array}$$

So we shown that

$$i\delta[x] = [d_0 x] = [k_0].$$

ii) Consider the following part of the sequence

$$\cdots \pi_{n+1}(K, L, k_0) \xrightarrow{\delta} \pi_n(L, k_0) \xrightarrow{i} \pi_n(K, k_0) \to \cdots$$

we shall prove that Image δ = Kernel i:

Let $[y]$ be an element of $\pi_n(L, k_0)$ such that $i[y] = k_0$. We choose a representative $y \in L_n$ of $[y]$. Clearly $y \sim k_0$ in K. Let $z \in K_{n+1}$ be the homotopy between k_0 and y. Then we have $d_i z = k_0$ for $0 \leq i \leq n$ and

$$d_{n+1} z = y.$$

Claim 4.20. *The following map exists*

$$\Lambda^{n+2}_{n+2} \xrightarrow{(z,k_0,k_0,\ldots,k_0,-)} K.$$

In long hand, the $n+2$ many $n+1$ simplices

$$b_0 = z, b_1 = k_0, b_2 = k_0, ..., b_{n+1} = k_0, -$$

satisfy the compatibility condition.

Proof. We should check that for $i < j$ and $n+2 \notin \{i, j\}$, we have

$$d_i b_j = d_{j-1} b_i.$$

So $j \geq 1$. When $j = 1$, we have $d_0 b_1$ equals $d_0 k_0 = k_0$ just by substituting. On the other hand $d_0 b_0 = d_0 z_0 = k_0$ by substituting again. Now let us assume that $j > 1$. Now $j - 1 \leq n$. Thus, dividing in two cases $i = 0$ and $i > 0$, we see immediately from hypothesis that $d_i b_j = d_{j-1} b_i = k_0$. Thus the claim is verified. □

Therefore we have an $n+2$ simplex w such that $d_0 w = z$ and $d_i w = k_0$ for $1 \leq i \leq n+1$. Or in other words the following diagram commutes

$$\begin{array}{ccc} \Lambda^{n+2}_{n+2} & \xrightarrow{(z,k_0,k_0,\ldots,k_0,-)} & K \\ \downarrow & \nearrow_{w} & \\ \Delta[n+2] & & \end{array} \tag{4.58}$$

Let us consider $d_{n+2} w$ more closely. Now we have

$$d_i d_{n+2} w \overset{1.1}{=} d_{n+1} d_i w = d_{n+1} k_0 = k_0 \quad \text{for} \quad 1 \leq i \leq n+1$$

$$d_0 d_{n+2} w = d_{n+1} d_0 w = d_{n+1} z = y.$$

Thus $\delta[d_{n+2} w] = [d_0 d_{n+2} w] = [y]$. So for an arbitary element $[y] \in \pi_n(L, k_0)$ such that $i[y] = k_0$, we have produced an element $t = [d_{n+2} w]$ in $\pi_{n+1}(K, L, k_0)$ such that $\delta t = [y]$.

iii)Consider the following part of the sequence

$$\cdots \pi_n(L, k_0) \xrightarrow{i} \pi_n(K, k_0) \xrightarrow{j} \pi_n(K, L, k_0) \to \cdots$$

we shall prove that $ji = k_0$:

Let $y \in L_n$ be a representative of $[y] \in \pi_n(L, k_0)$. we need to show that k_0 is relative homotopic to y.

Claim 4.21. *The $n+1$ many n-simplices*

$$-, k_0, k_0, ..., k_0, y$$

satisfy the hypothesis of the Kan condition.

Proof. The statement is clearly true for $0 < i < j \leq n$. Indeed, by definition all faces of y are k_0. □

Therefore there exists $z \in L_{n+1}$ which satisfies $d_i z = k_0$ for $1 \leq i \leq n$,

$$d_{n+1} z = y$$

and further all faces of $d_0 z$ are k_0. Thus we see that z is a relative homotopy between k_0 and y.

iv) Consider the following part of the sequence

$$\cdots \pi_n(L, k_0) \xrightarrow{i} \pi_n(K, k_0) \xrightarrow{j} \pi_n(K, L, k_0) \to \cdots$$

we shall prove that Image i = Kernel j:

Let $[x] \in \pi_n(K, k_0)$ be such that $j[x] = k_0$. We choose $x \in K_n$ to be a representative of $[x]$. We need to find a representative of $[x]$ in L_n. To this end, we shall construct an element $d_{n+1}v \in L_n$ and further, we have to show that $d_{n+1}v$ and x are homotopic in K.

Since $j[x] = k_0$ in $\pi_n(K, L, k_0)$ so we may choose a $n+1$-simplex w such that

$$d_i w = k_0 \quad \forall \quad 1 \leq i \leq n \tag{4.59}$$

$$d_{n+1} w = x \tag{4.60}$$

$$d_0 w = z \in L_n. \tag{4.61}$$

Thus $x \overset{w}{\sim}_L k_0$.

Claim 4.22. *The $n+1$ many n-simplices*

$$s_0 := z, s_1 := k_0, ..., s_n := k_0, -$$

are compatible in L. In the language of diagrams this means we claim the existence of following map

$$\Lambda^{n+1}_{n+1} \xrightarrow{(z, k_0, k_0, \ldots, k_0, -)} L$$

Proof. We need to check that for $i < j$ and $n+1 \notin \{i, j\}$ we have

$$d_i s_j = d_{j-1} s_i.$$

For $i \geq 1$, we have $j \geq 1$. In this case, all s_i, and s_j are k_0 and so are their faces. Now let us take $i = 0$. We have $d_{j-1} s_0 = d_{j-1} z = d_{j-1} d_0 w \overset{1.1}{=} d_0 d_j w$. Now $j \geq 1$, so we have $d_j w = k_0$ and thus so is $d_0 d_j w$. □

Therefore there exists a $n+1$-simplex v such that the following diagram is commutative

$$\begin{array}{ccc} \Lambda^{n+1}_{n+1} & \xrightarrow{(z, k_0, k_0, \ldots, k_0, -)} & L \\ \downarrow & \overset{v}{\nearrow} & \\ \Delta[n+1] & & \end{array}$$

Moreover $d_0 v = z$ and $d_i v = k_0$ for $1 \leq i \leq n$. We now relate v and w.

Claim 4.23. *The following map exists*

$$\Lambda^{n+2}_{n+2} \xrightarrow{(s_{n-1}z, k_0, k_0, \dots, v, w, -)} K.$$

Or in other words the $n+2$ many $n+1$-simplices

$$a_0 = s_{n-1}z, a_1 = k_0, a_2 = k_0, \dots, a_n = v, a_{n+1} = w, -$$

satisfy the hypothesis of Kan condition.

Proof. We need to check that for $i < j$ and $n+2 \notin \{i, j\}$, we have

$$d_i a_j = d_{j-1} a_i.$$

So $j \geq 1$. **Case** $j = 1$**:** We have $d_0 k_0 = d_0 s_{n-1} z = s_{n-1}(d_0 z)$. In the proof of claim 4.22, setting $j = 1$, we obtain $d_0 z = k_0$. Thus so is $s_{n-1}(d_0 z)$.

Case $1 < j < n$**:** In this case for all $i < j$ we clearly have $d_i a_j = d_{j-1} a_i = k_0$.

Case $j = n$**:** We have $d_i a_j = d_i a_n = d_i v = k_0$ for $1 \leq i < j$ by the construction of v. On the other hand, $d_{j-1} a_i = d_{n-1} a_i$ is clearly k_0 for $1 \leq i < j$. Now we consider the case $(i, j) = (0, n)$. Here

$$d_0 a_n = d_0 v = z = d_{n-1} s_{n-1} z = d_{n-1} a_0.$$

Case $j = n+1$**:** Thus $i \leq n$. Let us treat the case $1 \leq i \leq n$. We have $d_i a_{n+1} = d_i w = k_0 = d_{n+1-1} a_i$. When $i < n$, this is clearly k_0. When $i = n$, we also have $d_n a_n = d_n v = k_0$. Now let us check the case $(i, j) = (0, n+1)$. Here

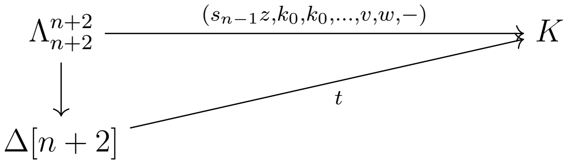
$$d_0 a_{n+1} = d_0 w = z \overset{1.4}{=} d_n s_{n-1} z = d_n a_0 = d_{n+1-1} a_0.$$

□

Therefore we have a $n+2$-simplex t such that $d_i t = k_0$ for $1 \leq i \leq n-1$,

$$d_0 t = s_{n-1} z, \qquad d_n t = v, d_{n+1} t = w.$$

In other words the following diagram is commutative



Now we show that $d_{n+2} t$ is a homotopy between $d_{n+1} v$ and x. We have

$$d_n d_{n+2} t \overset{1.1}{=} d_{n+1} d_n t = d_{n+1} v$$

and

$$d_{n+1}d_{n+2}t = d_{n+1}d_{n+1}t = d_{n+1}w = x.$$

Further

$$d_i d_{n+1}v = d_i x = k_0 \quad \text{for} \quad 1 \leq i \leq n.$$

Furthermore we have for $0 \leq i < n$,

$$d_i d_{n+2}t = d_{n+1}d_i t = k_0 = s_{n-1}d_i x = s_{n-1}d_i t.$$

Thus we have shown

$$x \overset{d_{n+2}t}{\sim} d_{n+1}v.$$

v) Consider the following part of the sequence

$$\cdots \pi_n(K, k_0) \xrightarrow{j} \pi_n(K, L, k_0) \xrightarrow{\delta} \cdots$$

we shall prove that $\delta j = k_0$:

Let $[x] \in \pi_n(K, k_0)$. Let x be a representative . Then by definition $\delta j[x] = [\delta_0 x]$. Since $\delta_0 x = k_0$ and j so we have $[\delta_0 jx] = [k_0]$.

vi) Consider the following part of the sequence

$$\cdots \pi_n(K, k_0) \xrightarrow{j} \pi_n(K, L, k_0) \xrightarrow{\delta} \cdots$$

we shall prove that Image $j =$ Kernel δ:

Let $[x] \in \pi_n(K, L, k_0)$ be such that $\delta[x] = [d_0 x] = k_0$. Let $x \in K_{n+1}$ be a representative of $[x]$. Since $[d_0 x] = k_0$, so there exists an element $z \in L_n$ such that $d_i z = k_0$, for $0 \leq i < n$ and $d_n z = d_0 x$.

Claim 4.24. *The $n+1$ many n-simplices*

$$z, k_0, k_0, ..., -, x$$

are compatible.

Proof. Since the only non-k_0 entries are at 0-th and $n+1$-th position, so the claim follows from $d_{n+1-1}z = d_n z = d_0 x$. □

Therefore we have a $n+1$-simplex y such that $d_0 y = z$, $d_i y = k_0$, for $0 \leq i < n$ and $d_{n+1}y = x$. Thus x is homotopic to $d_n y$ relative to L i.e

$$x \overset{y}{\sim}_L d_n y.$$

Now since $d_i d_n y = k_0$, $0 \leq i \leq n$ we have

$$[x] = j[d_n y].$$

□

5 Simplicial homotopy

Definition 5.1. *Let K and L be simplicial objects in a category $\mathfrak{C}$. Then two simplicial maps $f, g : K \to L$ are simplicially homotopic (f is homotopic to g) if there exist morphisms $h_i : K_n \to L_{n+1}$, for $0 \leq i \leq n$ such that*

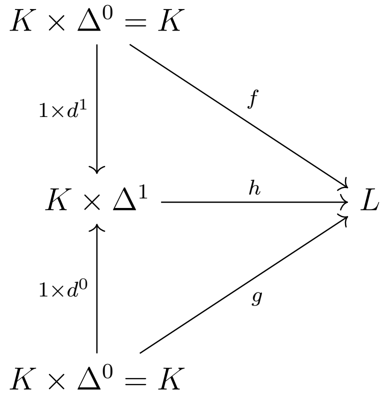
$$d_0 h_0 = f, d_{n+1} h_n = g, \tag{5.1}$$

$$d_i h_j = \begin{cases} h_{j-1} d_i & \text{if} \quad i < j \\ d_i h_{i-1} & \text{if} \quad i = j = 0, \\ h_j d_{i-1} & \text{if} \quad i > j+1 \end{cases} \tag{5.2}$$

$$s_i h_j = \begin{cases} h_{j+1} s_i & \text{if} \quad i \leq j \\ h_j s_{i-1} & \text{if} \quad i > j. \end{cases} \tag{5.3}$$

We say $h = \{h_i\}$ is a simplicial homotopy from f to g and we write $f \simeq g$.

We may encode the definition of simplicial homotopy from f to g in the following commutative diagram



Definition 5.2. *Let K and L be two semi-simplicial objects in a category $\mathfrak{C}$. Let $f, g : K \to L$ be two semi-simplicial maps. We say $h = \{h_i\}$ is a semi-simplicial homotopy from f to g if it satisfies only conditions (5.1) and (5.2) above.*

Definition 5.3. *Let K and L be two simplicial sets, and $f : K \to L$ a simplicial map. We say that*

1. *f is a homotopy equivalence if there exists a simplicial map $g : L \to K$ such that*

$$g \circ f \simeq id_K$$
$$f \circ g \simeq id_L.$$

2. *f is a weak homotopy equvialence if it induces isomoprhisms*

$$\pi_n(K, k_0) \tilde{\to} \pi_n(L, f(k_0))$$

for all $n \leq 0$ and for all $k_0 \in K_0$.

We say that two simplicial sets K and L are homotopy equivalent if there exists a homotopy equivalence $f : K \to L$. Homotopy equivalence implies weak homotopy equivalence. Thus if $K \simeq L$ we have that $\pi_i(K, k_0) \overset{\sim}{=} \pi_i(L, f(k_0))$.

Definition 5.4. *Let K be a simplicial object in $\mathfrak{A}$. Then the path space of K is the simplicial object PK. Thus PK is a simplicial object with $(PK)_n = K_{n+1}$, and the i-th face and degeneracy operators of PK, d_i^P and s_i^P are the d_{i+1} and s_{i+1} operators of K. We have a simplicial map*

$$p : PK \to K$$

coming from the maps $d_0 : K_{n+1} \to K_n$.

Lemma 5.1. *Let K be a simplicial object. Then PK, the path space of K, viewed as a semi-simplicial set is homotopy equivalent to the constant simplicial object at K_0 .*

Proof. We begin by defining simplicial maps

$$s = \{s_n\} : C(K_0) \to PK$$

which level-wise is given by

$$s_n = s_0^{n+1} : C(K_0)_n = K_0 \to K_{n+1} = (PK)_n$$

and

$$d : PK \to CK_0$$

which level-wise is given by

$$d_n : (PK)_n \to (CK_0)_n$$

as d_0^P composed with itself $n + 1$ times, that is

$$d_1 \circ \cdots \circ d_1 : K_{n+1} \to K_0.$$

By the simplicial identities, we have clearly

$$d_s = Id_{K_0}.$$

We show a semi-simplicial homotopy as follows . Define semi-simplicial maps

$$h_i : (PK)_n \to (PK)_{n+1} \quad \text{for} \quad 0 \leq i \leq n$$

by setting

$$h_j = s_0^{j+1}(d_0^P)^j,$$

where we follow the convention that raising index means so many fold composition of the map with itself.

Notice that $s_0^{j+1}(d_0^P)^j = s_0^{j+1}d_1^j$. Let us check that $\{h_j\}_{0 \leq j \leq n}$ define semi-simplicial homotopies id_{PK}.

By the simplicial identities we have

$$d_0^P h_0 = d_1 s_0 = id \tag{5.4}$$

and

$$d_{n+1}^P h_n = d_{n+2}(s_0^{n+1} d_1^n) = s_0^{n+1} d_1^{n+1} = s_n d_n, \tag{5.5}$$

where the last equality holds by definition.

Now let us check the remaining semi-simplicial identities. Let us consider $d_i^P h_j^P$ which by definition is $d_{i+1}(s_0^{j+1} d_1^j)$.

Case $i < j$: This should be

$$h_{j-1}^P d_i^P = (s_0^j d_1^{j-1}) d_{i+1}.$$

Now, $d_{i+1} s_0^{j+1} \overset{(1.5)}{=} s_0 d_i s_0^j \overset{(1.5)}{=} \cdots \overset{(1.5)}{=} s_0^{i-1} d_i^2 s_0^{j-i+2} \overset{(1.5)}{=} s_0^i d_1 s_0^{j-i+1} \overset{(1.4)}{=} s_0^j s_0^{j-i} = s_0^j$. Thus we have established

$$d_{i+1} s_0^{j+1} d_1^j = s_0^j d_1^j.$$

On the other hand,

$$d_1^{j-1} d_{i+1} = \overset{(1.1)}{=} d_1^{j-2} d_i d_1 \overset{(1.1)}{=} d_1^{j-3} d_{i-1} d_1^2 \cdots \overset{(1.1)}{=} d_1^{j-1-i} d_1^{i+1-i} d_1^i = d_1^j.$$

Thus, $(s_0^j d_1^{j-1}) d_{i+1}$ is also equal to $s_0^j d_1^j$.

Case $i = j \neq 0$: we have $d_i^P h_i^P = d_{i+1} s_0^{i+1} d_1^i$ by definition. This should be equal to

$$d_i^P h_{i-1}^P$$

which is $d_{i+1} s_0^i d_1^{i-1} \overset{(1.5)}{=} d_1 s_0^i d_1 d_1^{i-1} = d_1 s_0^i d_1^i$. On the other hand, we have

$$d_{i+1} s_0^{i+1} d_1^i = d_1 s_0^i d_1 s_0 d_1^i = d_1 s_0^i (d_1 s_0) d_1^i \overset{(1.4)}{=} d_1 s_0^i d_1^i.$$

Case $i > j + 1$: we have $d_i^P h_j^P = d_{i+1} s_0^{j+1} d_1^j$. This should be equal to

$$h_j^P d_{i-1}^P = s_0^{j+1} d_1^j d_i.$$

Now $d_{i+1} s_0^{j+1} \overset{(1.5)}{=} s_0^{j+1} d_{i-j}$ and $d_{i-j} d_1^j \overset{(1.1)}{=} d_1^j d_i$. This proves this case. □

We remark that one cannot upgrade this result to a simplicial homotopy. A counter example to the condition (5.3) for small values is not hard to construct.

We define simplicial retraction in the spirit of usual retraction.

Definition 5.5. *Let K be a simplicial set and let $j : L \hookrightarrow K$ be a simplicial subset of K. We say K retracts to L if there exists a simplicial map*

$$r : K \to L$$

such that the composite $r \circ j = id_K$. In other words r restricted to K is identity. If this happens we may also say that j admits a retraction.

Proposition 5.2 (Contractibility). *Let K be a pointed simplicial space. Let CK denote it's simplicial cone (cf. Definition (2.3)) . Then the inclusion $j : K \hookrightarrow CK$ admits a simplicial retraction if and only if there exists a function $s_{-1} : K_n \to K_{n+1}$ for each $n \geq 0$ such that $s_{-1}(*) = *$ and Identities (2.1) and (2.2) hold.*

Proof. Suppose j admits a retraction. Then we have a map $r : CK \to K$ such that $r \circ j = id_K$. Define $s_1(x) = r(x, 1)$ for $x \in K_n$. Then we check that identities (2.3) and (2.4) holds:

$$d_i s_{-1}(x) = d_i r(x, 1) = r d_i(x, 1) = r(d_i s_{-1}(x, 0)),$$

$$s_i s_{-1}(x) = s_i r(x, 1) = r s_i(x, 1) = r(s_i s_{-1}(x, 0)).$$

Conversely, suppose s_{-1} is defined as such in the hypothesis. Then we define $r : CK \to K$ by setting $r(x, 0) = x$ and $r(x, q) = s_{q-1}x$ for $q > 0$. By identities (2.3) and (2.4), r is a simplicial map. And the restriction of r to K is clearly the identity map on X. □

Chapter IX

D'après Daniel Quillen

1 Introduction

Let X be a topological space with an open covering $\{U_i\}_{i\in I}$. We begin by mentioning two well known results.

The van Kampen theorem in algebraic topology describes the fundamental group $\pi_1(X)$ as a push-out of those of U_i and their finite intersections.

Let us set $U := \sqcup_{i\in I} U_i$ as the disjoint union of U_i. Thus we have a surjective morphism $U \to X$ in the category of topological spaces. Let U^l denote the l-th fibered product

$$U \times_X \times \cdots \times_X U,$$

of U with itself over X. Then we know well how to associate a simplicial topological space U_* whose $l-1$-term is U^l. Given any homology theory $\{H_q\}_{n\geq 0}$, we may consider the simplicial object

$$H_q(U_*)$$

in the category of abelian groups. Taking the p-th homology of the associated Moore complex, we get

$$H_p(H_q(U_*)).$$

There is a well-known homological spectral sequence,

$$E^2_{p,q} = H_p(H_q(U_*)) \Rightarrow H_{p+q}(X).$$

We omit the details here because in the next chapter, we shall explain the construction of a spectral sequence associated with any simplicial topological space.

M.Artin and B.Mazur [8] gave generalization of the van Kampen theorem, in the spirit of spectral sequence of homology, to higher homotopy groups. Varying l, for each $q \geq 1$, the q-th homotopy groups $\pi_q(U^l)$ form a simplicial group

$$\pi_q(U_*).$$

Now taking the p-th homotopy of $\pi_q(U_*)$, one can define

$$\pi_p(\pi_q(U_*)) \quad \text{for} \quad p \geq 0, q \geq 1.$$

The theorem of Artin-Mazur may be stated as follows.

Theorem 1.1. *There is a spectral sequence of homological type whose terms* $E^2_{p,q}$ *is* $\pi_p(\pi_q(U_*))$ *and whose abutment is the associated graded group of a certain filtration of* $\pi_*(X)$ *i.e*

$$E^2_{p,q} = \pi_p(\pi_q(U_*)) \Rightarrow \pi_{p+q}(X).$$

We remind the reader that abutment and convergence are two different names for the same concept of spectral sequences. This simply means here that $\pi_{p+q}(X)$ has a filtration whose associated graded has p-th piece

$$F^p(\pi_{p+q}(X))/F^{p-1}(\pi_{p+q}(X))$$

is isomorphic to $E^\infty_{p,q}$.

The original proof of this theorem was very complicated. Daniel Quillen provided a simplified proof which is the object of this chapter. Given a double simplicial group $G_{*,*}$, he showed the existence of two spectral sequences converging to the homotopy groups of the simplicial group $\Delta G_{*,*}$.

2 Double simplicial groups

We shall denote by e the final and initial object in the category of groups. Simplicial groups are simplicial objects in Category of groups. We denote a simplicial group by $G_* = \{G_q, d_j, s_j\}$: here d_j and s_j are group homomorphism. Given a simplicial group G_* we define the following associated simplicial groups:

1. The simplicial group EG: The q-simplices of EG are given by

$$(EG)_q = \{x \in G_{q+1} | d_0^{q+1} = e\}$$

and the degeneracy maps $d_j : (EG)_q \to (EG)_{q-1}$ for $j \leq q$ is induced by $d_j : G_{q+1} \to G_q$. Similarly, the face maps $s_j : (EG)_q \to (EG)_{q+1}$ is induced by $s_j : G_{q+1} \to G_{q+2}$;

2. The constant simplicial group $C\pi_0(G)$ with $C\pi_0(G)_q = \pi_0(G)$ for all p, q. We set all s_j and d_j to be identity;

3. The morphism $\theta_q : (EG)_q \to G_q$, as induced by $d_{q+1} : G_{q+1} \to G_q$;

4. The morphism $j_q : G_q \to \pi_0(G)$ as the composition $G_q \xrightarrow{d_0^q} G_0 \to \pi_0(G)$.

Now we have an exact sequence of simplicial groups

$$0 \to \Omega G \xrightarrow{i} EG \xrightarrow{\theta} G \xrightarrow{j} C\pi_0(G) \to 0 \tag{2.1}$$

where $i : \Omega G \to EG$ is the kernel of θ.

Lemma 2.1. *If* K *is a simplicial Kan set with base point then* EK *is contractible. In fact there is a canonical homotopy* $h : EK \times \Delta[1] \to EK$ *functorial in* K *such that* $h_0 = id$ *and* $h_1 = *$.

Proof. □

Actually we will only use the following corollary to the above lemma. We refer the reader to Chapter 1 Section 7 of [7].

Corollary 2.2. *We have $\pi_k(EK) = e$ for $k \geq 1$.*

Let IG be the image of θ. Then from the sequence (2.1) we can obtain the following long exact homotopy sequences (cf. Chapter VIII, Proposition 4.18)

$$\cdots \to \pi_p(\Omega G \to \pi_p(EG) \to \pi_p(IG) \to \pi_{p-1}(\Omega G) \to \cdots$$

$$\cdots \to \pi_p(IG) \to \pi_p(G) \to \pi_p C(\pi_0(G)) \to \pi_{p-1}(IG) \to \cdots .$$

Now by Corollary (2.2) for $k \geq 1$ we have

$$\pi_k(EG) = e.$$

So we have isomorphisms

$$\pi_{p-1}(\Omega G) \simeq \pi_p(G), \qquad p \geq 1. \tag{2.2}$$

Now we turn to the case of double simplicial groups. Let

$$G_{*,*} = \{G_{p,q} : d_j^h : G_{p,q} \to G_{p-1,q}, s_j^h : G_{p,q} \to G_{p+1,q}, d_j^v, s_j^v\}$$

be a double simplicial group. By the p-th vertical simplicial group, we shall mean

$$G_{p,*} = \{G_{p,q} : d_j^v : G_{p,q} \to G_{p,q-1}, s_j^v : G_{p,q} \to G_{p,q+1}\}.$$

Applying π_0 to vertical simplicial groups $G_{p,*}$ for each p, we obtain groups which we denote

$$\pi_0^v G_{p,*}.$$

Note that $\{\pi_0^v G_{p,*}\}_{p\geq 0}$ actually form a simplicial group indexed by p and with face and degeneracy maps induced by $G_{*,*}$. Now we apply the functor C to each $\pi_0^v G_{p,*}$. We get thus a double simplicial group, which we denote as

$$C_v \pi_0^v G.$$

We have a natural map of double simplicial groups

$$G \to C_v \pi_0^v G.$$

We apply the construction of equation (2.1) to each of the vertical simplicial groups $G_{p,*} \to \pi_0^v G_{p,*}$ and induce them with the horizontal face and degeneracy maps to get now an exact sequence of *double* simplicial groups

$$0 \to \Omega_v G \xrightarrow{i^v} E^v G \xrightarrow{\theta^v} G \xrightarrow{j^v} C^v \pi_0^v G \to 0 \tag{2.3}$$

where $i^v : \Omega_v G \to E^v G$ is the kernel of θ^v.

We define the diagonal simplicial group ΔG of a double simplicial group $G_{*,*}$ by

$$(\Delta G)_n = G_{n,n} \quad d_j = d_j^h d_j^v \quad s_j = s_j^h s_j^v. \tag{2.4}$$

Letting $I^v G$ be the image of θ^v we obtain the long exact sequences

$$\to \pi_p(\Delta\Omega_v G) \to \pi_p(\Delta E^v G) \to \pi_p(\Delta I^v G) \to \pi_{p-1}(\Delta\Omega_v G) \to \tag{2.5}$$

$$\to \pi_p(\Delta I^v G) \to \pi_p(\Delta G) \to \pi_p(\Delta C^v \pi_0^v G) \to \pi_{p-1}(\Delta I^v G) \to \tag{2.6}$$

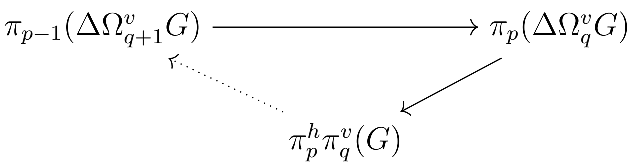
Lemma 2.3. *We have $\pi_p(\Delta E^v G) = 0$ for all p.*

Proof. We have $\Delta E^v G = E\Delta G$. Thus the result follows from Corollary 2.2. □

As $\pi_p(\Delta C^v \pi_0^v(G) = \pi_p^h \pi_0^v G$, in view of Lemma 2.3 we shall obtain the following long exact sequence from the sequences (2.5) and (2.6)

$$\to \pi_{p-1}(\Delta\Omega_v G) \to \pi_p(\Delta G) \to \pi_p^h \pi_0^v(G) \to \pi_{p-1}(\Delta\Omega_v G) \to \tag{2.7}$$

Using $\pi_0^v(\Omega_q^v G) = \pi_q^v(G)$ we set $D_{p,q} = \pi_p(\Delta\Omega_q^v G)$ and $E_{p,q} = \pi_p^h \pi_q^v(G)$. Now applying (2.7) to the groups $\Omega_q^v G$, we obtain the following exact couple



We now compute the r for which $E_{p,q}^r = \pi_p^h \pi_{q-p}^v(G)$. We see that d^r is the composition

$$\pi_p^h \pi_q^v(G) \to \pi_{p-1}(\Delta\Omega_v^{q+1} G) \to \pi_{p-1}^h \pi_{q+1}^v(G)$$

and so $r = 1$.

The *abutment* of spectral sequence of this couple is $\pi_q(\Delta G)$ with the filtration

$$F_p \pi_{q-p}(\Delta G) = Im(\pi_p(\Delta\Omega_{q-p}^v G) \to \pi_q(\Delta G)).$$

In other words since

$$F_{-1}\pi_{q-p}(\Delta G) = 0,$$

so the E^∞ term (cf. Section 2 Chapter III) is given by Graded associated with $\pi_q(\Delta G)$ filtered by it's sub-objects

$$F_p \pi_{q-p}(\Delta G) = Im(\pi_p(\Delta\Omega_{q-p}^v G) \to \pi_q(\Delta G)).$$

So we have established the following

Theorem 2.4. *If G is a double simplicial group, then there are two spectral sequences*

$$E_{p,q}^1 = \pi_p^h \pi_{q-p}^v G \Rightarrow \pi_q(\Delta G)$$

$$E_{p,q}^1 = \pi_p^v \pi_{q-p}^h G \Rightarrow \pi_q(\Delta G)$$

Chapter X

D'après Graeme Segal

1 Introduction

In this chapter we show how one may construct spectral sequences from a simplicial topological space together with a (co)homology theory suitable for topological spaces. The key observation is the geometric realization is naturally filtered. This construction generalizes vastly the second result stated in the last chapter about existence of spectral sequence for Cech covers whenever we have a homology theory. For simplicity we shall assume that our cohomology groups are vector spaces over a field.

2 Simplicial Topological spaces

By a simplicial space we shall mean a simplicial object in Category of Topological Spaces. Let Δ^p denote the standard p-simplex. It is defined as

$$\Delta^p = \{r = (r_0, \cdots, r_p) \in \mathbb{R}^{p+1} | \sum r_i = 1 \quad r_i \geq 0\}.$$

Let Δ^p_d denote the $(p-1)$ skeleton of Δ^p. Similarly, for a simplicial set A let A^d_p denote the degenerate part of A_p. It is defined as the union of the images of all face maps $A_{p-1} \to A_p$.

We recall the *geometric realization* ΔA of a simplicial set A. One takes the disjoint sum

$$\sqcup_p \Delta^p \times A_p,$$

equipped with discrete topology on A_p and then takes the quotient topological space by the following equivalence relation. Given $\theta : [p] \to [r]$, for $x \in \Delta^p$ and $a \in A_r$, one identifies

$$\Delta^p \times A_p \ni (x, A(\theta)(a)) = (\Delta(\theta)(x), a) \in \Delta^r \times A_r. \tag{2.1}$$

If A is a simplicial space its realization ΔA has a natural filtration

$$\Delta^0 A \subset \Delta^1 A \subset ... \subset \Delta A, \tag{2.2}$$

where $\Delta^p \times A_p$ is the image of $\Delta^p \times A_p$ in A. (In fact $\Delta^p A$ is a quotient space of $\Delta^p \times A$.)

Proposition 2.1. *We have a relative homeomorphism*

$$(\Delta^p \times A_p, (\Delta^p \times A_p^d) \cup (\Delta_d^p \times A_p)) \to (\Delta^p A, \Delta^{p-1} A).$$

Proof. By the relation used to make geometric realization of a simplicial set, we see that any element of $\Delta^p \times A_p^d \subset \Delta^p \times A_p$ is related to an element of $\Delta^{p-1} \times A_{p-1}$. Similarly we see that any element of $\Delta_d^p \times A_p$ is also related to $\Delta^{p-1} \times A_{p-1}$. Consider the natural composition

$$\Delta^p \times A_p \to \Delta^p A \to \Delta^p A/\Delta^{p-1} A.$$

We see that the image of $\Delta_d^p \times A_p \cup \Delta^p \times A_p^d \subset \Delta^p \times A_p$ becomes the distinguished point of $\Delta^p A/\Delta^{p-1}A$. So we have a factorization

$$\frac{\Delta^p \times A_p}{\Delta_d^p \times A_p \cup \Delta^p \times A_p^d \subset \Delta^p \times A_p} \to \frac{\Delta^p A}{\Delta^{p-1} A}.$$

By definition, this map is surjective. Let us check injectivity. So let us take a point $(q, a) \in \Delta^p \times A_p$ such that $x \notin \Delta_d^p$ and $a \notin A_p^d$. Let us consider the equivalence class of (x, a). So let us pick another element (x', a') from it. With notation as in (2.1), we have $p = r$. Further let us assume that for $\theta : [p] \to [p]$ we have

$$(x, a) = (x, A(\theta)(a')) = (\Delta(\theta)(x), a') = (x', a').$$

Since a is non-degenerate, so a' is non-degenerate. This forces θ to be $id : [p] \to [p]$. This means that $x' = x$. So we have checked injectivity.

□

Recall that the usual suspension of a pair (X, A) is defined as

$$(X, A) \wedge (S^1, 1) = (X \times S^1, A \times S^1 \cup X \times 1).$$

Note that the pair $(S^1, 1)$ is relatively homeomorphic to (Δ^1, Δ_d^1). Further, the p-fold suspension of (Δ^1, Δ_d^1) identifies with (Δ^p, Δ_d^p).

Proposition 2.2. *The pair $(\Delta^p A, \Delta^{p-1} A)$ can be identified with the p-fold suspension of (A_p, A_p^d).*

Proof. This follows immediately from Proposition 2.1. □

3 The construction

Let X be a topological space and A be a closed sub-space. Let $k^* = \{k^q\}_{q \in F}$ denote the co-homology theory defined on a category of pairs (X, A). We shall assume k^* has the following properties:

1. It is a contravariant δ-functor.

2. If $f_0 \simeq f_1 : (X, A) \to (Y, B)$, then $k^*(f_0) = k^*(f_1)$.

3. If $f : (X, A) \to (Y, B)$ is a relative homeomorphism, in the sense that it induces a homeomorphism $X/A \to Y/B$, then $k^*(f)$ is an isomorphism.

4. Let $\sqcup$ denote Topological sum. Then

$$k^*(\sqcup_\alpha X_\alpha) \stackrel{\simeq}{\to} \Pi_\alpha k^*(X_\alpha)$$

for any family of spaces $\{X_\alpha\}$

The filtration (2.2) and the cohomology theory k^* lead to a spectral sequence as follows.

Proposition 3.1. *To a simplicial space A is associated a spectral sequence whose termination is $k^*(\Delta A)$, with $E_1^{p,q} = H^p(k^q(A))$, the p-th co-homology group of the simplicial co-chain complex $k^q(A)$*

In the proof below we have preferred to use complementary degree instead of total degree because of an immediate relation with Cartan-Eilenberg systems as we shall soon see.

Proof. Setting $H(p, q) = k^q(\Delta^p A, \Delta^{p-1} A)$ we get a Cartan-Eilenberg system (cf Chapter II Subsection 3.2). Thus we have a spectral sequence.

Recall by Prop 2.2 that the pair $(\Delta^p A, \Delta^{p-1} A)$ can be identified with the p-fold suspension of (A_p, A_p^d). Thus,

$$E_1^{p,q} \simeq k^{p+q}(\Delta^p A, \Delta^{p-1} A) \simeq k^q(A_p, A_p^d)$$

because suspension shifts degrees.

We will show below that the natural map $E_1^{p,q} \to k^q(A_p)$ is compatible with the differential of the co-chain complex $k^q(A)$. Thus, the group $E_2^{p,q}$ can be calculated from $k^q(A)$ as follows. The group $k^q(A, A_p^d)$ is a direct summand in $k^q(A_p)$ complementary to the subgroup of degenerate co-chains: indeed denoting by S and T ordinal sets we have

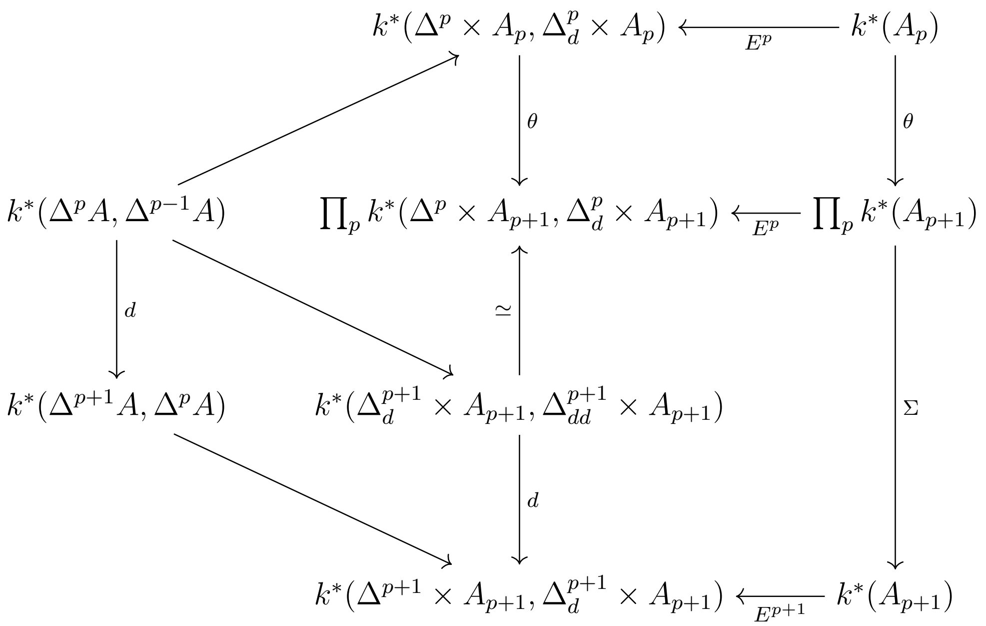
$$k^q(A_S) \simeq \bigoplus_T k^q(A_T, A_T^d), \tag{3.1}$$

where T runs through the quotient ordinal sets of S because our cohomology groups are vector spaces.

Fixing q, we now show that for every p, the differential $d_1^{p,q}$ and that of the co-chain complex $k^q(A)$ are compatible. This means we will show that the following diagram is commutative

$$\begin{array}{ccc} k^*(\Delta^p \times A_p, \Delta_d^p \times A_p) & \longrightarrow & k^*(A_p) \\ \downarrow & & \downarrow \\ k^*(\Delta^{p+1} A, \Delta^p A) & \longrightarrow & k^*(A_{p+1}). \end{array}$$

This follows from the commutativity of the following diagram



where: Δ^p_d denotes the $p-1$-skeleton of Δ^p. Similarly Δ^p_{dd} denotes the $p-2$-skeleton of Δ^p,

the maps θ are induced by the $p+2$ injections $[p] \to [p+1]$

E^p denotes the p-fold suspension, and

Σ denotes the summation with alternating signs, so that the composition of the right-most vertical arrows is the differential of the simplicial co-chain complex $k^q(A_p)$. The horizontal arrows labeled by E^p and E^{p+1} are all isomorphisms. The left-most vertical vertical arrows are the differentials of $E_1^{p,q}$. One checks that the diagram commutes. So we have a spectral sequence

□

An astute reader would have observed that the spectral sequence in this chapter starts at $r = 1$ while the one in the last chapter starts only at $r = 2$. This is because working with vector spaces and cohomological functors, one is able to get a decomposition as in (3.1) at level $r = 1$. This is not available in the set-up of double simplicial groups.

Bibliography

[1] Hilton, P.J., Stammbach, U.: *A course in Homological Algebra*, Springer, 2nd edition, 1996.

[2] Cartan, H., Eilenberg, S. : *Homological Algebra*, Princeton University Press, 1956.

[3] Weibel, C.A. : *An Introduction to Homological Algebra*, Cambridge University Press, 1994.

[4] May, J.P. : *Simplicial Objects In Algebraic Topology*, The University of Chicago Press,1967.

[5] Quillen, D.G. : *Spectral Sequence of Double Semi-Simplicial Group*, Topology Vol. 5 (1966), 155-157.

[6] Segal, G. : *Classifying Spaces and Spectral Sequences*,Manuscrit refu le 14 avril (1967).

[7] Jardine, J.F., Goerss, P.G., : *Simplicial Homotopy Theory*, Birkhäuser.

[8] Artin, M., Mazur, B., : *On the Van Kampen Theorem*, Topology Vol. 5 (1966), 179-189.