# The Hilbert transform

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## Certificate of Examination

This is to certify that the dissertation titled **The Hilbert Transform** submitted by **Shirina Arora** (Reg. No. MS12002) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 24, 2017

# Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. Shobha Madan at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Shirina Arora
(Candidate)

Dated: April 24, 2017

In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Shobha Madan
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Shirina Arora

# Notation

| | |
|---|---|
| $\mathbb{Z}$ | the set of integers |
| $\mathbb{R}$ | the set of real numbers |
| $\mathbb{R}^n$ | the Euclidean n-space |
| $\mathbb{C}$ | the set of complex numbers |
| $\mathcal{X}$ | the characteristic function of the set E |
| $d_f$ | the distribution function of the function $f$ |
| $< u, f >$ | the action of a distribution on a function $f$ |
| $\|T\|_{X \to Y}$ | the norm of the (bounded) operator $T : X \to Y$ |
| $\hat{f}(n) = a_n$ | $n^{th}$ Fourier coefficient |
| $S_N(f)$ | partial sum of a Fourier series of function $f$ |
| $f * g$ | convolution |
| $D_N$ | Dirichlet kernel |
| $P_n$ | Poisson kernel |
| $F_N$ | Fejér kernel |
| $A(r)$ | Abel mean |
| $\hat{f}(\xi)$ | Fourier transform |
| $L^p(X, \mu)$ | the lebesque space over the measure space $(X, \mu)$ |
| $C_0^\infty$ | the space of smooth functions with compact support |
| $S(\mathbb{R})$ | the Schwartz space |
| $S'(\mathbb{R})$ | the space of tempered distributions on $\mathbb{R}^n$ |
| $L^p$ | Lebesgue Space( p-norm for finite-dimensional vector spaces) |
| $M_p$ | the space of $L^p$ Fourier multipliers, $1 \leq p \leq \infty$ |
| $\tilde{f}$ | reflection of a function $f$ |
| $\tau_t f$ | translation of a function $f$ by t |
| $D^t f$ | dilation of a function $f$ |
| $\mathbb{T}$ | $[0, 1)$ |
| $B(\mathbb{R}^n)$ | Euclidean balls |
| $Hf$ | Hilbert transform |
| $Q_j$ | Cubes |

**NOTE : Standard mathematical notations are used.**

# Abstract

The Hilbert transform is the most important operator in analysis. There is only one singular integral in 1-D and it is Hilbert transform. The most important fact about Hilbert transform is that it is bounded on $L^p$ for $1 < p < \infty$. The aim is of this thesis is to study the basic properties of the Fourier series of a function and see whether partial sums of the Fourier series of a functions converges or not and under what constraints the series converges(uniform, pointwise and in norm convergence).
Later we will see how Hilbert transform plays a crucial role in $L^p$ norm convergence of the partial sums of the Fourier series. At the end, I will try to see how the results of 1-D works in the case of double Fourier series (that is, 2-D) and the summability methods and their convergence.

# Contents

# Chapter 1

# Introduction to Fourier Series and Integrals

## 1.1 Fourier series and Fourier coefficients

If $f : [0,1] \longrightarrow \mathbb{C}$ then the $n^{th}$ **Fourier coefficient** of f is defined by

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx, \; n \in \mathbb{Z}.$$

The **Fourier series** of f is given by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

The $N^{th}$ **partial sum** of the Fourier series of f is given by

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x}.$$

### 1.1.1 Dirichlet kernel

It is defined for $x \in \mathbb{R}$ by

$$D_n(x) = \sum_{n=-N}^{N} e^{2\pi i n x}$$

. It can expressed as follows:

$$D_N(x) = \frac{(\sin 2\pi(N+1/2)x)}{\sin 2\pi(x/2)}$$

$$\begin{aligned}
D_N(x) &= \sum_{n=0}^{N} w^n + \sum_{n=-N}^{-1} w^n \ where, \ w = e^{ix} \\
&= \frac{1-w^{N+1}}{1-w} + \frac{w^{-N}-1}{1-w} \\
&= \frac{w^{-N}-w^{N+1}}{1-w} \\
&= \frac{w^{-N-1/2}-w^{N+1/2}}{w^{-1/2}-w^{1/2}} \\
&= \frac{\sin(2\pi(N+1/2)x)}{\sin 2\pi(x/2)}
\end{aligned}$$

### 1.1.2 Convolutions

Given two functions $f$ and $g$ in $L^1(\mathbb{R})$, we define their convolution $f * g$ by

$$f * g(x) = \int_{\mathbb{R}} f(y)g(x-y)dy = \int_{\mathbb{R}} f(x-y)g(y)dy$$

Let us now see the partial sum of Fourier series in terms of convolutions.

$$\begin{aligned}
S_N(f)(x) &= \sum_{n=-N}^{N} \hat{f}(n)e^{2\pi inx} \\
&= \sum_{n=-N}^{N} \left( \int_{\mathbb{R}} f(y)e^{-2\pi iny}dy \right) e^{2\pi inx} \\
&= \int_{\mathbb{R}} f(y) \left( \sum_{-n=N}^{N} e^{2\pi in(x-y)} \right) dy \\
&= (f * D_N)(x)
\end{aligned}$$

### 1.1.3 Good kernels

A family of kernel$\{K_n(x)\}_{n=1}^{\infty}$ defined on $[0,1]$ is said to be a family of good kernels if it satisfies following properties:

1. For all $n \geq 0$,
$$\int_0^1 K_n(x)dx = 1$$
2. There exists $M > 0$ such that for all $n \geq 0$,
$$\int_0^1 |K_n(x)|dx \leq M.$$
3. For every $\eta > 0$,
$$\int_{|x|\geq\eta} |K_n(x)|dx \longrightarrow 0, \ \textit{as } n \to \infty$$

**Theorem 1.** *Let $f \in L^1(\mathbb{R})$ . Then*
$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$
*whenever $f$ is continuous at $x$.*
*Because of the above result, the family $K_n$ is sometimes referred as* ***approximation to the identity.***

**Proof** *: As $f$ is continuous at $x$, therefore for $\epsilon > 0$ choose $\eta$ such that $|y| < \eta$ implies $|f(x-y) - f(y)| < \epsilon$. Then by (1) property of good kernels, we can write*
$$\begin{aligned}
|(f * K_n)(x) - f(x)| &= |\int_0^1 K_n(y)[f(x-y) - f(x)]|dy \\
&= \int_{|y|<\eta} |K_n(y)||f(x-y) - f(x)|dy \\
&\qquad + \int_{|y|\geq\eta} |K_n(y)||f(x-y) - f(x)|dy \\
&= \epsilon \int_0^1 |K_n(y)|dy + 2B \int_{|y|\geq\eta} |K_n(y)|dy
\end{aligned}$$
*where $B$ is a bound for $f$. Using (2) property of good kernels, first term is bounded by $\epsilon M$ and by (3) property of good kernels, second term is $< \epsilon$.*
$$\therefore |(f * K_n(x)) - f(x)| \leq C\epsilon$$

**Some Important results:**We will be using the following results again and again for proving further important theorems.

- **Fubini's theorem:**

  Suppose A and B are complete metric spaces. Suppose $f(x,y)$ is $A \times B$ measurable. If $\int_{A\times B} |f(x,y)| d(x,y) < \infty$, then

$$\int_A \left( \int_B f(x,y)dy \right) dx = \int_B \left( \int_A f(x,y)dx \right) dy = \int_{A\times B} |f(x,y)| d(x,y)$$

- **Dominated Convergence theorem:**

  Let $\{f_n\}$ be a sequence of real valued measurable functions on a measure space$(X, \mu)$. Suppose that the sequence converges pointwise to a function f and is dominated by some integrable function g in the sense that

$$|f_n(x)| \leq g(x).$$

  Then f is integrable and

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n d\mu = \int_{\mathbb{R}} f d\mu$$

- **Hölder's inequality:**

  Suppose f,g are integrable functions and $p, q \in [1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^p}$$

- **Minkowski inequality:**

  Let $1 \leq p \leq \infty$. For $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^1(\mathbb{R}^n)$. We have $g * f$ exists a.e and satisfies

$$\|g * f\|_{L^p} \leq \|g\|_{L^1} \|f\|_{L^p}$$

- **Riemann-Lebesgue lemma**

  If $f \in L^1(\mathbb{R})$ then

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$$

- **Gauss-Weierstrass Summation:**

  Suppose that $\hat{f} \in L^1$. Then

$$f(0) = \lim_{\epsilon\to 0} \frac{1}{(2\pi)^n} \int \hat{f}(\xi) e^{-\epsilon|\xi|^2/2} d\xi$$

## 1.2 Convergence of Fourier Series

**Introduction:** The first question which comes to our mind is whether the partial sum of the Fourier series of f converges to f pointwise. That is

$$\lim_{N\to\infty} S_N(f)(x) = f(x) \ for\ every\ x?$$

Since we can change an integrable function at one point without changing the Fourier coefficients, so at this point it is difficult to comment on the above statement. But what if we take $f$ to be a continuous and periodic function? Answer seems to be a "yes", but it came out as a surprise when it was showed that there exists a continuous function whose Fourier series diverges at a point. What if we add more smoothness conditions on $f$: We might assume that $f$ is continuously differentiable. We will see that in that case the Fourier series of $f$ converges to $f$ uniformly. Let us state few results that will be used in proving uniform convergence of Fourier series of $f$.

**Theorem 2.** ***Uniqueness of Fourier series***
*Suppose that $f$ is an integrable function with $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f(x_0) = 0$ whenever $f$ is continuous at the point $x_0$.*

**Corollary 1.1.** *If $f \in L^1(\mathbb{R})$ and $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$.*

### 1.2.1 Uniform convergence of Fourier series

**Corollary 1.2.** *Suppose $f \in L^1(\mathbb{R})$ and that the Fourier series of $f$ is absolutely convergent , $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. Then, the Fourier series converges uniformly to $f$, that is,*

$$\lim_{N\to\infty} S_N(f)(x) = f(x) \quad uniformly\ in\ x.$$

**Proof** *: The assumption $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$ implies that the partial sums of the Fourier series of f converges absolutely and uniformly, and therefore the function $g$ defined by*

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx} = \lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}(n)e^{2\pi inx}$$

*is integrable. As a consequence of the uniform convergence of the series, the Fourier coefficients of $g$ are $\hat{f}(n)$ since we can interchange the integral with the infinite sum. Therefore, the previous corollary can be applied to the function $f - g$ yields $f = g$, as required.*

### 1.2.2 Pointwise Convergence

**Theorem 3.** *Let f be an integrable function on an interval* $[0, 1)$ *which is differentiable at a point* $x_0$*. Then* $S_N(f)(x_0) \to f(x_0)$ *as* $N \to \infty$*.*

[SS03]

**Proof** Define

$$F(t) = \begin{cases} \frac{f(x_0-t)-f(x_0)}{t} & \text{if } t \neq 0 \ and\ |t| < 1/2 \\ -\hat{f}(x_0) & \text{if } t = 0. \end{cases}$$

First F is bounded near 0 since f is differentiable there. Second, F is integrable in the interval $[-\frac{1}{2}, \frac{1}{2}]$. We know $S_N(f)(x_0) = (f * D_N)(x_0)$, where $D_N$ is the Dirichlet kernel. Since $\int_0^1 D_N = 1$, we find that

$$\begin{aligned} S_N(f)(x_0) - f(x_0) &= \int_{-1/2}^{1/2} f(x_0 - t)D_N(t)dt - f(x_0) \\ &= \int_{-1/2}^{1/2} [f(x_0 - t) - f(x_0)]D_N(t)dt \\ &= \int_{-1/2}^{1/2} F(t)tD_N(t)dt. \end{aligned}$$

We recall that

$$tD_N(t) = \frac{t}{\sin 2\pi(t/2)} sin(2\pi(N + 1/2)t),$$

where the quotient $\frac{t}{sin2\pi(t/2)}$ is continuous in the interval $[-1/2, 1/2]$, as sin function is continuous on $\mathbb{R}$. Since we can write

$$\sin(2\pi(N + 1/2)t) = sin2\pi(Nt)cos2\pi(t/2) + \cos 2\pi(Nt) \sin 2\pi(t/2),$$

Now by applying Riemann-Lebesgue lemma to Riemann integrable functions $F(t)tcos2\pi(t/2)/sin2\pi(t/2)$ and $F(t)t$ we get the desired result.

### 1.2.3 Convergence in norm

We will discuss about the norm convergence of the Fourier series of f in detail in later chapters.

## 1.3 Summability Methods

Since we have seen above that Fourier series may fail to converge at some points. Let us try to overcome this failure by taking another summability criteria.

### 1.3.1 Cesàro Summability

Suppose we are given a series of complex numbers

$$c_0 + c_1 + ....... = \sum_{k=0}^{\infty} c_k.$$

Now define the $n^{th}$ partial sum $s_n$ by

$$s_n = \sum_{k=0}^{n} c_k.$$

We define the average of the first N partial sums by

$$\sigma_N = \frac{s_0 + s_1 + ....... + s_{N-1}}{N}.$$

$\sigma_N$ is called the $N^{th}$ *Cesàro sum* of the series $\sum_{k=0}^{\infty} c_k$. If $\sigma_N$ converges to a limit $\sigma$ as N tends to $\infty$, we say that the series $\sum c_n$ is cesàro summable to $\sigma$.

**Fejér's kernel**

We know by definition $N^{th}$ Cesàro mean is

$$\sigma_N(f)(x) = \frac{S_0(f)(x) + S_1(f)(x) + ...... + S_{N-1}(f)(x)}{N}.$$

Since $S_n(f) = f * D_n$, we find that

$$\sigma_N(f)(x) = (f * F_N)(x),$$

where $F_N(x)$ is the N-th *Fejér kernel* given by

$$F_N(x) = \frac{D_0(x) + D_1(x) + ..... + D_{N-1}(x)}{N}.$$

Another expression for Fejér kernel is

$$F_N(x) = \frac{1}{N}\frac{\sin^2 2\pi(Nx/2)}{\sin^2 2\pi(x/2)}$$

which can be shown as follows:
Let $w = e^{2\pi i x}$

$$\begin{aligned}
D_n(x) &= w^{-n} + \ldots.. + w^{-1} + 1 + w^1 + \ldots\ldots. + w^n \\
&= (w^{-n} + \ldots.. + w^{-1}) + (1 + w^1 + \ldots\ldots. + w^n) \\
&= w^{-1}\left(\frac{w^{-n}-1}{w^{-1}-1}\right) + \frac{1-w^{n+1}}{1-w} \\
&= \frac{w^{-n}-1}{1-w} + \frac{1-w^{n+1}}{1-w} \\
&= \frac{w^{-n}-w^{n+1}}{1-w}
\end{aligned}$$

So,

$$\begin{aligned}
NF_N(x) &= \sum_{n=0}^{N-1} \frac{w^{-n}-w^{n+1}}{1-w} \\
&= \frac{1}{1-w}\left(\sum_{n=0}^{N-1} w^{-n} - \sum_{n=0}^{N-1} w^{n+1}\right) \\
&= \frac{1}{1-w}\left(\frac{w^{-N}-1}{w^{-1}-1} - w\frac{1-w^N}{1-w}\right) \\
&= \frac{1}{1-w}\left(\frac{w^{-N+1}-w}{1-w} - w\frac{1-w^N}{1-w}\right) \\
&= w\left(\frac{w^{-N}-2+w^N}{(1-w)^2}\right) \\
&= \frac{1}{(w^{-1/2})^2}\frac{(w^{N/2}-w^{-N/2})^2}{(1-w)^2} \\
&= \frac{(w^{N/2}-w^{-N/2})^2}{(w^{1/2}-w^{-1/2})^2} \\
&= \frac{-4\sin^2 2\pi(Nx/2)}{-4\sin^2 2\pi(x/2)} \\
F_N(x) &= \frac{1}{N}\frac{\sin^2 2\pi(Nx/2)}{\sin^2 2\pi(x/2)}
\end{aligned}$$

**Lemma 1.1.** *The Fejér kernels, $F_N$ are good kernel.*

**Proof** Since $\int_0^1 e^{2\pi int} = 1$, if $n = 0$ and otherwise 0, we can clearly see that $\int_0^1 F_N(t)dt = 1$ and from the expression of $F_N$ we can see that $F_N > 0$ and thus $\int_0^1 |F_N(t)|dt = 1$ for all $N$. Since $F_N(t) \leq \frac{c}{N} min(N^2, t^{-2})$ using the property that $|\sin nt| \leq n|\sin t|$. So now we have

$$\int_{|t|\geq\eta} F_N(t)dt \leq \int_{|t|\geq\eta} CN^{-1}t^{-2}dt \leq C(\eta N)^{-1}.$$

As $N \to \infty$, integral tends to zero. Thus Fejér kernel is a good kernel.

**Theorem 4.** *If $f \in L^p, 1 \leq p < \infty$, or if $f$ is continuous and $p = \infty$, then*

$$\lim_{N\to\infty} \|\sigma_N f - f\|_p = 0.$$

**Proof** $F_N$ are good kernel, so $\int F_N = 1$, by *Minkowski's inequality* we see that

$$\begin{aligned}
\|\sigma_N f - f\|_p &= \int_{-1/2}^{1/2} \|f(\cdot - t) - f(\cdot)\|_p F_N(t)dt \\
&\leq \int_{|t|<\delta} \|f(\cdot - t) - f(\cdot)\|_p F_N(t)dt + 2\|f\|_p \int_{\delta<|t|<1/2} F_N(t)dt.
\end{aligned}$$

Since for $1 \leq p < \infty$,

$$\lim_{t\to 0} \|f(\cdot - t) - f(\cdot)\|_p = 0,$$

and the same limit holds for the case $p = \infty$ and f is continuous, the first term can be made as small as possible by choosing a suitable $\delta$. And by the (3) property of $F_N$ being a good kernel i.e,

$$\lim_{N\to\infty} \int_{\delta<|t|<1/2} F_N(t)dt = 0 \ if \ \delta > 0,$$

therefore for fixed $\delta$, the second term tends to 0.

### 1.3.2 Abel Summability

A series of complex numbers $\sum_{k=0}^{\infty} c_k$ is said to be *Abel summable* to s if for every $0 \leq r < 1$, the series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k$$

converges, and

$$\lim_{r \to 1} A(r) = s.$$

The quantity A(r) are known as *Abel means* of the series.

**The Poisson kernel**

We define Abel mean of the function $f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$ by

$$A_r(f)(x) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{2\pi i n x}.$$

Just like Cesàro means, Abel means can be written as convolutions

$$A_r(f)(x) = (f * P_r)(x),$$

where $P_r(x)$ is the *Poisson kernel* given by

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{2\pi i n x}.$$

In fact,

$$\begin{aligned} A_r(f)(x) &= \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{2\pi i n x} \\ &= \sum_{n=-\infty}^{\infty} r^{|n|} \left( \int_0^1 f(\varphi) e^{-2\pi i n \varphi} d\varphi \right) e^{2\pi i n x} \\ &= \int_0^1 f(\varphi) \left( \sum_{n=-\infty}^{\infty} r^{|n|} e^{-2\pi i n (\varphi - x)} \right) d\varphi, \end{aligned}$$

Poisson kernel can be expressed in another form as below:

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos 2\pi\theta + r^2}.$$

As

$$\begin{aligned}
P_r(\theta) &= \sum_{n=0}^{\infty} w^n + \sum_{n=1}^{\infty} \bar{w}^n \ with\ w = re^{2\pi i\theta}, \\
&= \frac{1}{1-w} + \frac{\bar{w}}{1-\bar{w}} \\
&= \frac{1-\bar{w}+(1-w)\bar{w}}{(1-w)(1-\bar{w})} \\
&= \frac{1-|w|^2}{|1-w|^2} \\
&= \frac{1-r^2}{1-2r\cos 2\pi\theta + r^2}
\end{aligned}$$

## 1.4 Distribution Function

**Definition 1.1.** *For a measurable function $f$ on $X$, the distribution function of $f$ is the function $d_f$ defined on $[0,\infty)$ as follows:*

$$d_f(\alpha) = \mu(\{x \in X : |f(x)| > \alpha\}).$$

*The distribution function provides us with the information about the size of $f$ but not about the behaviour of $f$ itself.*

**Proposition 1.1.** *Let $(X,\mu)$ be a $\sigma$-finite measure space. Then for $f$ in $L^p(X,\mu), 0 < p < \infty$, we have*

$$\|f\|_{L^p}^p = p\int_0^\infty \alpha^{p-1} d_f(\alpha)d\alpha.$$

**Proof** *: Let $E_\alpha = \{x : |f(x)| > \alpha\}$*

$$\begin{aligned}
p\int_0^\infty \alpha^{p-1} d_f(\alpha)d\alpha &= p\int_0^\infty \alpha^{p-1} \int_X \mathcal{X}_{E_\alpha} d\mu(x)d\alpha \\
&= \int_X \int_0^{|f(x)|} p\alpha^{p-1} d\alpha d\mu(x) \\
&= \int_X |f(x)|^p d\mu(x) \\
&= \|f\|_{L^p}^p
\end{aligned}$$

**Definition 1.2.** *For $0 < p < \infty$, the space weak $L^p(X, \mu)$ is defined as the set of all $\mu$-measurable functions $f$ such that*

$$\begin{aligned}\|f\|_{L^{p,\infty}} &= \inf\{C > 0 : d_f(\alpha) \leq \frac{C^p}{\alpha^p} \text{ for all } \alpha > 0\} \\ &= \sup\{\gamma d_f(\gamma)^{1/p} : \gamma > 0\}\end{aligned}$$

*is finite.*

# Chapter 2

# Fourier Transform and Schwartz Space

## 2.1 $L^1$ theory of Fourier transform

**Introduction:** In the previous chapter we have seen that the theory of Fourier series applies to a periodic functions on $\mathbb{R}$, or equivalently to functions on the circle. Here we will develop an analogous theory for the study of functions which are non periodic on the entire real line.Let us recall the Fourier coefficients $a_n$ of a function f defined on the circle which is given as follows

$$a_n = \int_0^1 f(x)e^{-2\pi inx}dx.$$

Roughly speaking, continuous version of Fourier coefficients is Fourier Transform.
**Definition:** Given a function $f \in L^1(\mathbb{R})$, we define the Fourier transform as

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx$$

Fourier transform has some nice properties. Let $f, g \in L^1(\mathbb{R})$.

1. If $h(x) = f * g(x)$, then

$$\begin{aligned}
\hat{h}(\xi) &= \int e^{-2\pi ix\xi} \int f(y)g(x-y)dydx \\
&= \int e^{-2\pi iy\xi} f(y)dy \int e^{-2\pi iz\xi} g(z)dz \\
\hat{h}(\xi) &= \hat{f}(\xi)\hat{g}(\xi)
\end{aligned}$$

2. If $\tau_h f(x) = f(x+h)$, then $\widehat{\tau_h f}(\xi) = \hat{f}(\xi)e^{2\pi ih\xi}$

3. If $h(x) = f'(x)$, then $\hat{h}(\xi) = 2\pi i\xi \hat{f}(\xi)$

   **Proof** :

$$\begin{aligned}
\hat{h}(\xi) &= \int_{-N}^{N} f'(x)e^{-2\pi ix\xi}dx \\
&= [f(x)e^{-2\pi ix\xi}]_{-N}^{N} + 2\pi i\xi \int_{-N}^{N} f(x)e^{-2\pi ix\xi}dx
\end{aligned}$$

   As $N \to \infty$ we get the desired result.

4. If $f \in L^1(\mathbb{R})$, then $\hat{f}$ is continuous

   **Proof** : For $f \in L^1$, using Dominated Convergence Theorem,

$$\begin{aligned}
|\hat{f}(\xi+h) - \hat{f}(\xi)| &= \int f(x)(e^{-2\pi ix(\xi+h)} - e^{-2\pi ix\xi})dx \\
&\leq \int |f(x)||e^{2\pi ixh} - 1|dx
\end{aligned}$$

   which tends to zero as $h \to 0$ as $|e^{-2\pi ix\xi}| = 1$

5. $\widehat{rf}(\xi) = r\hat{f}(\xi)$, if $r \in \mathbb{R}$; $\widehat{f+g}(\xi) = \hat{f}(\xi) + \hat{g}(\xi)$, so $\hat{f}$ is a linear operator.

6. $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$

   **Proof** : $|\hat{f}(\xi)| = |\int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}d\xi| \leq \int_{-\infty}^{\infty} |f(x)e^{-2\pi ix\xi}|d\xi = \|f\|_1$
   Therefore,

$$\sup|\hat{f}| \leq \|f\|_1$$

$$\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$$

## 2.2 Schwartz functions and Fourier transform on $L^p$

Roughly speaking, a function is **Schwartz** if it is smooth and all its derivatives decay faster than the reciprocal of any polynomial at infinity.

**Definition:** A $C^\infty$ complex valued function f on $\mathbb{R}^n$ is called a Schwartz function if for every pair of multi indices $\alpha, \beta$ there exists a positive constant $C_{\alpha,\beta}$ such that

$$\rho_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| = C_{\alpha,\beta} < \infty,$$

where $x^\alpha = (x_1^{\alpha_1}, x_2^{\alpha_2}, x_3^{\alpha_3}, \cdots x_n^{\alpha_n})$ and $\partial^\beta = \frac{\partial^\beta}{\partial x^\beta}$.

The quantities $\rho_{\alpha,\beta}(f)$ are called the Schwartz seminorms of f. The set of all Schwartz functions on $\mathbb{R}^n$ is denoted by $S(\mathbb{R}^n)$

**Key Point:** The Fourier transform is often introduced as an operator on $L^1$. Since the Schwartz functions are $C^\infty$ functions whose derivatives decay faster than any polynomials. Since D is sense in $L^1$ space, Fourier transform and its properties remains same for Schwartz functions.

### 2.2.1 The Schwartz Topology

The topology on $S(\mathbb{R})$ is generated by the family of semi-norms $\{\rho_{\alpha,\beta}\}$. . The functions $\{\rho_{\alpha,\beta}\}$ are semi-norms on the vector space $S(\mathbb{R})$, in the sense that

$$\rho_{\alpha,\beta}(f+g) \leq \rho_{\alpha,\beta}(f) + \rho_{\alpha,\beta}(g)$$

and

$$\rho_{\alpha,\beta}(zf) = z\rho_{\alpha,\beta}(f)$$

for all $f, g \in S(\mathbb{R})$, and $z \in \mathbb{C}$. For this semi-norm, an open ball of radius $r$ centered at some $f \in S(\mathbb{R})$ is given by

$$B\rho_{\alpha,\beta}(f; r) = \{g \in S(\mathbb{R}) : \rho_{\alpha,\beta}(gf) < r\}$$

Thus each $\rho_{\alpha,\beta}$ specifies a topology $\tau(\alpha, \beta)$ on $S(\mathbb{R})$. A set is open according to $\tau(\alpha, \beta)$ if it is a union of open balls. The topologies $\tau(\alpha, \beta)$ put all together, generate the standard Schwartz topology $\tau$ on $S(\mathbb{R})$. This is the smallest topology containing all the sets of $\tau(\alpha, \beta)$ for all $\alpha, \beta \in \mathbb{Z}$.

### 2.2.2 The Fourier transform on $S$

The Fourier transform of a function $f \in S$ is defined by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$$

We use the following notation

$$f(x) \longrightarrow \hat{f}(\xi)$$

which means that $\hat{f}$ is the Fourier transform of $f$. Some properties of the Fourier transform are stated in the following proposition.

**Proposition 2.1.** *If $f \in S(\mathbb{R})$, then:*

*(i)* $f(x+h) \longrightarrow \hat{f}(\xi) e^{2\pi i h \xi}$ *whenever* $h \in \mathbb{R}$.

*(ii)* $f(x) e^{-2\pi i x h} \longrightarrow \hat{f}(\xi + h)$ *whenever* $h \in \mathbb{R}$.

*(iii)* $f'(x) \longrightarrow 2\pi i \xi \hat{f}(\xi)$.

*(iv)* $-2\pi i x f(x) \longrightarrow \frac{d}{d\xi} \hat{f}(\xi)$.

*(v)* $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$

*(vi)* $f(tx) \longrightarrow t^{-1} \hat{f}(t^{-1}\xi)$ *whenever* $t > 0$.

***Example:*** *If* $f(x) = e^{-\pi x^2}$, *then* $\hat{f}(\xi) = f(\xi)$.
***Proof:*** *Let us first check*

$$\int_{-\infty}^{\infty} e^{-2\pi x^2} dx = 1.$$

*To see this let us proceed as follows:*

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-2\pi x^2}dx\right)^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} \\ &= \int_0^{2\pi}\int_0^{\infty} e^{-\pi r^2} r dr d\theta \\ &= \int_0^{\infty} 2\pi r e^{-\pi r^2} dr \\ &= \big[-e^{-\pi r^2}\big]_0^{\infty} \\ &= 1.\end{aligned}$$

*Now we will prove the example.*

$$F(\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx,$$

*and using the above calculation we see that $F(0) = 1$. Using result (iv) of previous proposition and the fact that $f'(x) = -2\pi x f(x)$, we see that*

$$F'(\xi) = \int_{-\infty}^{\infty} f(x)(-2\pi i x)e^{-2\pi i x\xi}dx = i\int_{-\infty}^{\infty} f'(x)e^{-2\pi i x\xi}dx$$

*And again by using the(iii)property of the above proposition, we find that*

$$F'(\xi) = i(2\pi i\xi)\hat{f}(\xi) = -2\pi\xi F(\xi).$$

*Let us define $G(\xi) = F(\xi)e^{\pi\xi^2}$, and we see that $G'(\xi) = 0$, which shows that $G$ is a constant. Since $F(0) = 1$, we get the constant equals to 1, therefore $F(\xi) = e^{-\pi\xi^2}$.*

### 2.2.3 Some Important Results

(i) Multiplication Formula: If $f, g \in S(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)dx = \int_{-\infty}^{\infty} \hat{f}(y)g(y)dy$$

(ii) Fourier Inversion Formula: If $f \in S(\mathbb{R})$, then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi$$

### 2.2.4 The Fourier transform on $L^p, 1 < p \leq 2$

**Theorem 5.** ***Plancherel theorem***
*The Fourier transform is an isometry on $L^2$; i.e, for $f \in S$*

$$\|\hat{f}\|_2 = \|f\|_2$$

**Proof** For $f \in S(\mathbb{R})$, define $g(x) = \overline{f(-x)}$. Then $\hat{g}(\xi) = \overline{\hat{f}(\xi)}$. Let $h = f * g$. Then we have

$$\hat{h}(\xi) = \hat{f}(\xi)\hat{g}(\xi) = |\hat{f}(\xi)|^2 \textit{ and } h(0) = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

Now using Fourier Inversion Formula for $x = 0$,

$$h(0) = \int_{-\infty}^{\infty} \hat{h}(\xi) d\xi = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi.$$

This implies

$$\|\hat{f}\|_2 = \|f\|_2$$

**Parceval's Identity:** For $f, h$ in $S(\mathbb{R})$

$$\int_{S(\mathbb{R})} f(x)\overline{h(x)}dx = \int_{S(\mathbb{R})} \hat{f}(\xi)\overline{\hat{h}(\xi)}d\xi$$

## 2.3 The Class of Tempered Distributions[SW71]

The dual space(i.e the space of continuous linear functionals on the sets of test functions) we introduced is denoted by

$$(S(\mathbb{R}^n))' = S'(\mathbb{R}^n)$$

Elements of $S'(\mathbb{R}^n)$ are called **Tempered Distributions**.That is, it is a set of all functions

$$f : S(\mathbb{R}) \to \mathbb{C}$$

which are linear and continuous.

- If $< f, a\varphi + b\psi > = \ a < f, \varphi > +b < f, \psi >$ for all $\varphi, \psi \in S(\mathbb{R}) \textit{ and } a, b \in \mathbb{C}$, then $f$ is linear.

- If $\varphi = \lim_{n\to\infty} \varphi_n \textit{ in } S(\mathbb{R})$, then $< f, \varphi >= \lim_{n\to\infty} < f, \varphi_n >$, then $f$ is continuous.

**Definition:** A linear functional u on $S(\mathbb{R}^n)$ is a *tempered distribution* iff there exists $C > 0$ and k,m integers such that

$$| < u, f > | \leq C \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq k}} \rho_{\alpha,\beta}(f), \ \ for \ f \in S(\mathbb{R}^n)$$

where action of distribution u on f is represented as $< u, f >= u(f)$.

**Definition 2.1.** *Let $u \in S'$ and $\alpha$ is a multi-index. Define*

$$< \partial^\alpha u, f > = \ (-1)^{|\alpha|} < u, \partial^\alpha f > .$$

*The derivative in the sense of distributions is known as distributional derivatives.*

**Definition 2.2.** *For a tempered distribution u, we define the Fourier transform $\hat{u}$ and Inverse Fourier transform $u^\vee$ by*

$$< \hat{u}, f > = < u, \hat{f} > \ \ and \ \ < u^\vee, f > = < u, f^\vee >$$

**Definition 2.3.** *The dilation $D^t u$, the translation $\tau^t u$ and the reflection $\tilde{u}$ of a tempered distribution u is defined as follows:*

$$\begin{aligned} < \tau^t u, f > \ &= < u, \tau^{-t} f >, \\ < D^t u, f > \ &= < u, t^{-n} D^{\frac{1}{t}} f >, \\ < \tilde{u}, f > \ &= < u, \tilde{f} >, \end{aligned}$$

*for $t \in \mathbb{R}^n$*

**Definition 2.4.** *Let $u \in S'$ and $h \in S$.Define the convolution $h * u$ by*

$$< h * u, f > = < u, \tilde{h} * f >, \ f \in S$$

Next proposition will extend the properties of Fourier transform to tempered distributions.

**Proposition 2.2.** *Given $u, v$ in $S'(\mathbb{R}^n)$, $b$ a complex scalar, $\alpha$ a multi-index and $a > 0$, we have*

*(i)* $\widehat{u+v} = \hat{u} + \hat{v}$,

*(ii)* $\widehat{bu} = b\hat{u}$,

*(iii)* $(\tilde{u})^\wedge = (\hat{u})^\sim$,

*(iv)* $(\hat{u})^\vee = u$,

*(v)* $\widehat{f * u} = \hat{f}\hat{u}$,

*(vi)* $\widehat{fu} = \hat{f} * \hat{u}$,

*(vii)* $(\tau^y u)^\wedge = e^{-2\pi i y\xi}\hat{u}$,

*(viii)* $(e^{2\pi i x y}u)^\wedge = \tau^y \hat{u}$,

*(ix)* $(D^t u)^\wedge = (\hat{u})_t = t^{-n} D^{t^{-1}}\hat{u}$,

*(x)* $(\partial^\alpha u)^\wedge = (2\pi i\xi)^\alpha \hat{u}$,

*(xi)* $\partial^\alpha \hat{u} = ((-2\pi i x)^\alpha u)^\wedge$.

# Chapter 3

# Hardy-Littlewood Maximal Function

**Introduction:** The Hardy-Littlewood maximal function $Mf : \mathbb{R}^n \longrightarrow [o, \infty)$ of a locally integrable function $f : \mathbb{R}^n \to (-\infty, \infty)$ is defined by

$$Mf(x) = sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(y)| dy,$$

where sup is over all $r > 0$. Here $|B_r| = |B(0, r)|$ denotes the volume of the ball $B(0, r)$.
We may also define maximal functions over cubes centered at $x$ and cubes containing $x$: If $Q_r = [-r, r]^n$ is a cube, we define

$$M'f(x) = sup_{r>0} \frac{1}{(2r)^n} \int_{Q_r} |f(y)| dy$$

For $n = 1, M \ and \ M'$ coincide. Furthermore, since the n-dimensional volumes of the unit cube and unit ball are equal upto a multiplicative constant depending only on n, it is immediate that Mf and $M'f$ are comparable in the sense that

$$c_n M'f(x) \leq Mf(x) \leq C_n M'f(x).$$

for constants $c_n \ and \ C_n$ depending upon n. In fact, we can define a more general maximal function

$$M''f(x) = sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

One sometimes distinguishes between $M'$ and $M''$ by referring to the former as the centered and latter as the non-centered maximal operator.

## 3.1 Hardy Littlewood Maximal Theorem

**Lemma 3.1.** ***Vitali Covering Lemma***
*Let $B_1$, $B_2$, $B_3$,....,$B_M$ be balls in $\mathbb{R}^n$ then there exists a sub collection of pairwise disjoint balls, $B_{\alpha_j}$, such that*

$$\bigcup_{\alpha=1}^{M} B_\alpha \subset \bigcup_{j=1}^{m} B^*_{\alpha_j}$$

*where $B^*$ is the ball with same centre as B, but three times the radius of B.*

**Proof** We have a finite collection of balls $B = \{B_\alpha\}_{\alpha=1}^{M}$. Now select $B_{\alpha_1}$ to has a radius as large as possible(chosen from the given M balls). Having selected $B_{\alpha_1}$, we select $B_{\alpha_2}$ subject to two conditions that (1) it be disjoint from $B_{\alpha_1}$ and that (2) it has a radius as large as possible.
Continue in the preceding fashion. If $B_{\alpha_1}$, $B_{\alpha_2}$, $B_{\alpha_3}$,......,$B_{\alpha_p}$ have been selected, then we select $B_{\alpha_{p+1}}$ such that (1) it is disjoint from $B_{\alpha_1}$, $B_{\alpha_2}$,.....$B_{\alpha_p}$ and (2) it has a radius as large as possible i.e

$$B_{\alpha_j} \cap \big(\bigcup_{i=1}^{j-1} B_{\alpha_i}\big) = \emptyset$$

Clearly, this process must stop because we have finite balls.
Let $\acute{B} = \{B_{\alpha_j}\}_{j=1}^{m}$ be the sub collection of balls from the above selection process. First, by design, the sub collection $\acute{B}$ is pairwise disjoint.
Second, whenever a ball $B_\alpha$ from the original collection is not chosen, then it must intersect some chosen ball $B_{\alpha_j}$ and be of smaller radius.Let $B^*_{\alpha_j}$ be one of the dilated $\{3B_{\alpha_j}\}$ balls.
**Claim:** These dilated balls cover each of the original $B_{\alpha_j}$ balls
**Proof of the claim:** If $B_\alpha$ is equal to one of the selected balls $B_{\alpha_j}$, then of course it is covered by the dilated balls $\{3B_{\alpha_j}\}$. If instead it is not one of the selected balls, then let $B_{\alpha_q} = \mathbb{B}(c_{\alpha_q}, r_{\alpha_q})$ be the first selected ball that intersect $B_\alpha = \mathbb{B}(c_\alpha, r_\alpha)$. Then the radius of $r_{\alpha_q}$ of $B_{\alpha_q}$ must be at least as great as the radius $r_\alpha$ of $B_\alpha$, otherwise we would have selected $B_\alpha$ instead of $B_{\alpha_q}$ at the $q^{th}$ step. Now it follows from triangle

inequality $3\mathbb{B}(c_{\alpha_q}, r_{\alpha_q}) \equiv \mathbb{B}(c_{\alpha_q}, 3r_{\alpha_q})$ covers $\mathbb{B}(c_\alpha, r_\alpha)$ Therefore,

$$\bigcup_{\alpha=1}^{M} B_\alpha \subset \bigcup_{j=1}^{m} B^*_{\alpha_j}$$

**Lemma 3.2.** ***Covering Lemma***

*Let $\{I_\alpha\}_{\alpha\epsilon A}$ be a collection of intervals in $\mathbb{R}$ and let $K$ be a compact set contained in their union. Then there exists a finite sub collection $\{I_j\}$ such that*

$$K \subset \bigcup_j I_j \textit{ and } \sum \mathcal{X}_{I_j}(x) \leq 2$$

**Proof** $\{I_\alpha\}_{\alpha\epsilon A}$ is a collection of open intervals. Using compactness of K, we can say that $\{I_\alpha\}$ will be an open cover of K and therefore, there is a finite subcover $I_{\alpha_j}; j = 1, 2, ..., n$

$$\Rightarrow K \subset \bigcup_{j=1}^{n} I_{\alpha_j}$$

**To prove :** $\sum \mathcal{X}_{I_j}(x) \leq 2$

Denote $\{I_{\alpha_j}\} = \{I_\alpha\}$

Let us take an interval $\{I_1\} = (a, b)$ with maximum length possible. Now, define one interval $\{I_p\}$ whose right bound is equal to the right most extreme point possible (greater than b) and covering some part of $\{I_1\}$. Similarly, define another interval $\{I_q\}$ whose left bound is equal to the left most extreme point possible(less than a) and covering some part of $\{I_1\}$. Intervals are selected in such a way that

$$I_{\alpha_1} \cap I_{\alpha_2} = \emptyset$$

and those intervals which are completely inside one of the intervals can be removed completely.

Repeat the above process and as there are finite sub collection of intervals, process will end. Above selection of intervals shows that any point $x$ can be in maximum of two intervals. Thus,

$$\sum \mathcal{X}_{I_j}(x) \leq 2$$

**Theorem 6.** *The operator $M$ is weak $(1,1)$ and strong $(p,p), 1 < p \leq \infty, i.e$*
*(1)There exists a constant $C > 0$ depending only on $n$, such that for every $f \in L^1(\mathbb{R}^n)$,*

$$|\{x \in \mathbb{R}^n : Mf(x) > t\}| \leq \frac{C}{t}\|f\|_1 \ \forall t > 0$$

*(2)For $1 < p \leq \infty$, there exists a constant $C_p > 0$ depending only on $p$ and $n$, such that for every $f \in L^p(\mathbb{R}^n)$*

$$\|Mf\|_p \leq C_p\|f\|_p.$$

We will now see that maximal function has importance in the study of approximation of the identities which can be shown by the following result.

**Proposition 3.1.** *Let $\phi$ be a function which is positive, radial, decreasing $\big($as a function on $(0,\infty)$ $\big)$ and integrable. Then*

$$\sup_{t>0} |\phi_t * f(x)| \leq \|\phi\|_1 Mf(x).$$

**Proof** Let us assume that $\phi$ is a simple function, that is, it can be written as

$$\phi(x) = \sum_j a_j \mathcal{X}_{B_{r_j}}(x)$$

with $a_j > 0$. Let us check that $\phi$ satisfies the hypothesis.
Since $a_j$ are positive, therefore $\phi(x) > 0$. Radial property of $\phi(x)$ can be seen as follows:
For $x \neq x'$ we have $d(x,0) = d(x',0)$. But $\mathcal{X}_{B_{r_j}}(x) = \mathcal{X}_{B_{r_j}}(x') \ \forall \ r_j$, thus $\phi(x) = \phi(x')$.
Also to show decreasing property we need to show that for $y \geq x$ we have $\varphi(y) \leq \varphi(x)$
This is easy to see because if $x \in B_{r_m}$ then for all $m' > m$, $x \in B_{r_{m'}}$ Let us take $y > x$ and $y \in B_{r_k}$ then $k > m$ and for all $k > k'$, $y \in B_{r_{k'}}$ Thus,

$$\phi(x) = \sum_{j=m}^{\infty} a_j$$

and

$$\phi(y) = \sum_{j=k}^{\infty} a_j$$

since $a_j > 0$,

$$\sum_{j=k}^{\infty} a_j < \sum_{j=m}^{\infty} a_j$$

$$\implies \phi(x) < \phi(y)$$

Now,

$$\phi * f(x) = \sum_j a_j |B_{r_j}| \frac{1}{|B_{r_j}|} \mathcal{X}_{B_{r_j}} * f(x) \leq \|\phi\|_1 Mf(x)$$

since $\|\phi\|_1 = \sum a_j |B_{r_j}|$ as $a_j > 0$

An arbitrary function $\phi$ satisfying the hypothesis can be approximated by sequence of simple functions and any further dilation $\phi_t$ will be another function with same properties and same integral and thus satisfying the same inequality.

## 3.2 Marcinkiewicz Interpolation Theorem

**Theorem 7.** *Let $(X, \mu)$ and $(X, \nu)$ be measure spaces and let $1 \leq p_0 < p_1 \leq \infty$. Let $T$ be a sub-linear operator defined on the space $L^{p_o}(X) + L^{p_1}(X)$ and taking values in the space of measurable functions on $Y$. Assume that there exist two positive constants $A_0$ and $A_1$ such that*

$$\|T(f)\|_{L^{p_0,\infty}}(Y) \leq A_0 \|f\|_{L^{p_0}}(X) \ for\ all\ f \in L^{p_0}(X),$$

$$\|T(f)\|_{L^{p_1,\infty}}(Y) \leq A_1 \|f\|_{L^{p_1}}(X) \ for\ all\ f \in L^{p_1}(X).$$

*Then for all $p_0 < p < p_1$ and for all f in $L^p(X)$ we have the estimate*

$$\|T(f)\|_{L^p}(Y) \leq A \|f\|_{L^p}(X),$$

*where*

$$A = 2\left(\frac{p}{p - p_0} + \frac{p}{p_1 - p}\right)^{\frac{1}{p}} A_0^{\frac{\frac{1}{p} - \frac{1}{p_1}}{\frac{1}{p_0} - \frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0} - \frac{1}{p}}{\frac{1}{p_0} - \frac{1}{p_1}}}.$$

*[Gra14]*

**Proof** : Given $f \in L^p$, for each $\lambda > 0$ decompose f as $f_0 + f_1$, where

$$f_0 = f\chi_{\{x:|f(x)| > c\lambda\}},$$

$$f_1 = f\chi_{\{x:|f(x)| \leq c\lambda\}};$$

the constant c will be fixed later. Then $f_0 \in L^{p_0}(\mu)$ and $f_1 \in L^{p_1}(\mu)$. Furthermore,

$$|T(f)| \leq |T(f_0)| + |T(f_1)|,$$

which implies

$$\{y \in Y : |T(f)(y)| > \lambda\} \subseteqq \{y \in Y : |T(f_0)(y)| > \lambda/2\} \cup \{y \in Y : |T(f_1)(y)| > \lambda/2\},$$

and therefore

$$d_{T(f)}(\lambda) \leq d_{T(f_0)}(\lambda/2) + d_{T(f_1)}(\lambda/2)$$

Let us consider 2 cases.

*Case 1*: $p_1 = \infty$. Choose $c = \frac{1}{2A_1}$, where $A_1$ is such that $\|Tg\|_\infty \leq A_1\|g\|_\infty$. Then $a_{Tf_1}(\lambda/2) = 0$. Using weak $(p_0, p_1)$ inequality,

$$a_{Tf_0}(\lambda/2) \leq \left(\frac{2A_0}{\lambda}\|f_0\|_{p_0}\right)^{p_0};$$

hence,

$$\begin{aligned}
\|Tf\|_p^p &\leq \int_0^\infty \lambda^{p-1-p_0}(2A_0)^{p_0} \int_{x:|f(x)|>c\lambda} |f(x)|^{p_o} d\mu d\lambda \\
&= p(2A_0)^{p_0} \int_X |f(x)|^{p_0} \int_0^{|f(x)|/c} \lambda^{p-1-p_0} d\lambda d\mu \\
&= \frac{p}{p-p_0}(2A_0)^{p_0}(2A_1)^{p-p_0}\|f\|_p^p.
\end{aligned}$$

*Case 2*: $p_1 < \infty$ Using given hypothesis and proposition 1.9 we obtain that $d_{T(f)}(\lambda) \leq \frac{A_0^{p_0}}{(\lambda/2)^{p_0}} \int_{|f|>c\lambda} |f(x)|^{p_0} d\mu(x) + \frac{A_1^{p_1}}{(\lambda/2)^{p_1}} \int_{|f|\leq c\lambda} |f(x)|^{p_1} d\mu(x)$.

$$
\begin{aligned}
\|T(f)\|_{L^p}^p &\leq p(2A_0)^{p_0} \int_0^\infty \lambda^{p-1}\lambda^{-p_0} \int_{|f|>c\lambda} |f(x)|^{p_0} d\mu(x) d\lambda \\
&\qquad + p(2A_1)^{p_1} \int_0^\infty \lambda^{p-1}\lambda^{-p_1} \int_{|f|\leq c\lambda} |f(x)|^{p_1} d\mu(x) d\lambda \\
&= p(2A_0)^{p_0} \int_X |f(x)|^{p_0} \int_0^{\frac{1}{c}|f(x)|} \lambda^{p-1-p_0} d\lambda d\mu(x) \\
&\qquad + p(2A_1)^{p_1} \int_X |f(x)|^{p_1} \int_{\frac{1}{c}|f(x)|}^\infty \lambda^{p-1-p_1} d\lambda d\mu(x) \\
&= \frac{p(2A_0)^{p_0}}{p-p_0} \frac{1}{c^{p-p_0}} \int_X |f(x)|^{p_0} |f(x)|^{p-p_0} d\mu(x) \\
&\qquad + \frac{p(2A_1)^{p_1}}{p_1-p} \frac{1}{c^{p-p_1}} \int_X |f(x)|^{p_1} |f(x)|^{p-p_1} d\mu(x) \\
&= p\left( \frac{p(2A_0)^{p_0}}{p-p_0} \frac{1}{c^{p-p_0}} + \frac{p(2A_1)^{p_1}}{p_1-p} \frac{1}{c^{p-p_1}} \right) \|f\|_{L^p}^p,
\end{aligned}
$$

Interchange of integrals can be done using Fubini's theorem which uses the hypothesis that $(X, \mu)$ is a $\sigma$-finite measure space. We pick c such that

$$
(2A_0)^{p_0} \frac{1}{c^{p-p_0}} = (2A_1)^{p_1} c^{p_1-p},
$$

Thus proving the result for $p_1 < \infty$.

# Chapter 4

# Hilbert Transform

## 4.1 The conjugate Poisson kernel

Let us define a Poisson kernel on the upper half plane;

$$F(x,y) = P_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}.$$

Poisson kernel on the upper half plane is a harmonic function. Harmonic function is a twice continuously differentiable function which satisfies Laplace equation, i.e. $\triangle F = 0$.

Let us check that Poisson kernel is a harmonic function:

**To prove:**

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$$

$$\begin{aligned}
\frac{\partial F}{\partial x} &= -\frac{1}{\pi}\frac{2xy}{(x^2+y^2)^2} \\
\frac{\partial^2 F}{\partial x^2} &= -\frac{2}{\pi}\left[\frac{(x^2+y^2)^2 y - 4x^2 y(x^2+y^2)}{(x^2+y^2)^4}\right] \\
&= -\frac{2}{\pi}\left[\frac{y^5 - 3x^4 y - 2x^2 y^3}{(x^2+y^2)^4}\right]
\end{aligned}$$

Similarly,

$$\frac{\partial F}{\partial y} = \frac{x^2 - y^2}{\pi(x^2+y^2)^2}$$
$$\frac{\partial^2 F}{\partial x^2} = -\frac{2}{\pi}\left[\frac{3x^4y + 2x^2y^3 - y^5}{(x^2+y^2)^4}\right]$$

$$\Rightarrow \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$$

We know that $f \in S$, $\hat{f} \in S$, then

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

and

$$\left(\frac{c}{1+x^2}\right)^{\wedge}(\xi) = ce^{-|\xi|},\ \textit{where c is such that} \int \frac{c}{1+x^2}dx = 1.$$

Similarly,

$$P_y(x) = \frac{1}{y}P\left(\frac{x}{y}\right)$$
$$= \frac{cy}{y^2+x^2}$$
$$\Rightarrow \hat{P}_y(\xi) = ce^{-2\pi y|\xi|}.$$

where, $P(x) = \frac{c}{1+x^2}$ Given a function f in $S(\mathbb{R})$, $u(x,y) = P_y * f(x)$ is also an harmonic function which can be proved using the following convolution property

$$D(f*g) = Df*g = f*Dg,$$

where $f \in L^1(\mathbb{R})$ and g is a differentiable function with bounded derivative.

**Proof of the property:**

$$\lim_{h\to 0} \frac{f*g(x+h) - f*g(x)}{h} = \lim_{h\to 0}\int f(y)\frac{g(x+h-y) - g(x-y)}{h}dy$$

We can take the limit inside the integral using DCT provided $f$ is in $L^1$ and $D_g$ is bounded and differentiable.Thus,

$$\begin{aligned}
\lim_{h\to 0}\frac{f*g(x+h)-f*g(x)}{h} &= \int \lim_{h\to 0} f(y)\frac{g(x+h-y)-g(x-y)}{h}dy \\
&= \int f(y)\lim_{h\to 0}\frac{g(x+h-y)-g(x-y)}{h}dy \\
\Rightarrow D(f*g) = f*Dg
\end{aligned}$$

We can also write:

$$\begin{aligned}
u(z) &= \int_0^\infty \hat{f}(\xi)e^{2\pi i z\xi}d\xi + \int_{-\infty}^0 \hat{f}(\xi)e^{2\pi i\bar{z}\xi}d\xi, \\
&= \int_0^\infty \hat{f}(\xi)e^{2\pi i x\xi}e^{-2\pi y\xi}d\xi + \int_{-\infty}^0 \hat{f}(\xi)e^{2\pi i x\xi}e^{2\pi y\xi}d\xi, \\
&= \int_{-\infty}^\infty \hat{f}(\xi)e^{2\pi i x\xi}e^{-2\pi y|\xi|}d\xi
\end{aligned}$$

where z= x+iy. If we now define,

$$\begin{aligned}
iv(z) &= \int_0^\infty \hat{f}(\xi)e^{2\pi i z\xi}d\xi - \int_{-\infty}^0 \hat{f}(\xi)e^{2\pi i\bar{z}\xi}d\xi, \\
v(z) &= \int_0^\infty (-i)\hat{f}(\xi)e^{2\pi i z\xi}d\xi - \int_{-\infty}^0 (-i)\hat{f}(\xi)e^{2\pi i\bar{z}\xi}d\xi,\ (multiply\ by\ -i\ on\ both\ sides) \\
v(z) &= \int_0^\infty (-i)\hat{f}(\xi)e^{2\pi i x\xi}e^{-2\pi y\xi}d\xi - \int_{-\infty}^0 (-i)\hat{f}(\xi)e^{2\pi i x\xi}e^{2\pi y\xi}d\xi, \\
v(z) &= \int_{\mathbb{R}} -isgn(\xi)e^{-2\pi y|\xi|}\hat{f}(\xi)e^{2\pi i x\xi}d(\xi),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
v(x,y) &= Q_y * f(x) \\
\hat{Q}_y(\xi) &= -isgn(\xi)e^{-2\pi y|\xi|} \qquad \text{(i)}
\end{aligned}$$

as

$$
\begin{aligned}
v(z) &= \int_{\mathbb{R}} -isgn(\xi)e^{-2\pi y|\xi|}\hat{f}(\xi)e^{2\pi ix\xi}d(\xi),\\
&= \int_{\mathbb{R}} \hat{Q}_y(\xi)\hat{f}(\xi)e^{2\pi ix\xi}d(\xi)\\
&= \int_{\mathbb{R}}\int_t \hat{Q}_y(\xi)f(t)e^{-2\pi i\xi t}e^{2\pi ix\xi}d(\xi)\\
&= \int_{\mathbb{R}}\int_t \hat{Q}_y(\xi)e^{2\pi i(x-t)\xi}f(t)d(\xi)\\
&= \int_{\mathbb{R}} Q_y(x-t)f(t)dt
\end{aligned}
$$

and Fourier transform of $Q_y$ is defined as $Q_y \in L^2(\mathbb{R})$ which can be proved as follows: If we prove $P_x = \frac{1}{1+x^2}$ is in $L^2(\mathbb{R})$, then we know that after dilation also function belongs to $L^2(\mathbb{R})$ and

$$Q_y(x) = \frac{1}{x}P(\frac{y}{x}).$$

$$
\begin{aligned}
\|P_x\|_2 &= \int_{\mathbb{R}} \frac{1}{(1+x^2)^2}dx\\
&= \int_{\mathbb{R}} \frac{\sec^2\theta}{\sec^4\theta}d\theta \qquad \textit{substituting } x = \tan\theta\\
&= \int_{\mathbb{R}} \cos^2\theta d\theta
\end{aligned}
$$

which is finite and thus $P_x \in L^2(\mathbb{R})$
Inverse Fourier Transform is:

$$\begin{aligned}
Q_y(x) &= \int\limits_{-\infty}^{+\infty} -isgn(\xi)e^{-2\pi y\xi}e^{2\pi ix\xi}d\xi \\
&= \int\limits_{0}^{+\infty} -ie^{-2\pi y\xi}e^{2\pi ix\xi}d\xi + \int\limits_{-\infty}^{0} ie^{-2\pi y\xi}e^{2\pi ix\xi}d\xi \\
&= \int\limits_{0}^{+\infty} -ie^{-2\pi\xi(y-ix)}d\xi + \int\limits_{-\infty}^{0} ie^{-2\pi\xi(y+ix)}d\xi \\
&= \frac{-ie^{-2\pi\xi(y-ix)}}{-2\pi(y-ix)}\Bigg|_0^\infty + \frac{ie^{2\pi\xi(y+ix)}}{2\pi(y+ix)}\Bigg|_{-\infty}^0 \\
&= \frac{-i}{2\pi(y-ix)} + \frac{i}{2\pi(y+ix)} \\
&= \frac{i}{2\pi}\left(\frac{-y+-ix+t-ix}{y^2+x^2}\right) \\
&= \frac{-2i^2x}{2\pi(y^2+x^2)} \\
&= \frac{x}{\pi(y^2+x^2)}
\end{aligned}$$

$Q_y(x) = \frac{1}{\pi}\frac{x}{x^2+y^2}$, is also known as the conjugate Poisson kernel. Now similar to $u$, $v$ is also harmonic in $\mathbb{R}^2$ and both $u$ and $v$ are real if $f$ is. Furthermore, $u+iv$ is analytic, so $v$ is the harmonic conjugate of $u$ and $u+iv = 2\int_0^\infty \hat{f}(\xi)e^{2\pi iz\xi}d(\xi)$ is analytic.$u+iv$ is analytic can be proved using Morera's Theorem:

**Morera's Theorem:**

Continuous complex valued function $f$ defined on a connected open set $D$ in the complex plane that satisfies

$$\oint_\gamma f(z)dz = 0,$$

for every closed pointwise curve $\gamma$ in D must be holomorphic on D.

**Proof of Analyticity:**

$$\oint \int_0^\infty \hat{f}(\xi)e^{2\pi iz\xi}d(\xi) = \int \oint_0^\infty \hat{f}(\xi)e^{2\pi iz\xi}d(\xi) \tag{1}$$

$$= 0 \tag{2}$$

First step is using Fubini's Theorem and Second step is using Cauchy Integral Theorem which states that:

If f(z) is analytic in some simply connected region $\mathbb{R}$, then

$$\oint_\gamma f(z)dz = 0$$

for any closed contour $\gamma$ completely contained in $\mathbb{R}$.

## 4.2 Principal value of 1/x

We define a tempered distribution called the principal value of 1/x, abbreviated p.v.1/x, by

$$p.v.\frac{1}{x}(\phi) = \lim_{\epsilon\to 0}\int_{\frac{1}{\epsilon}>|x|>\epsilon}\frac{\phi(x)}{x}dx, \phi \in S.$$

To see that this expression defines a tempered distribution, we rewrite it as

$$p.v.\frac{1}{x}(\phi) = \int_{|x|<1}\frac{\phi(x)-\phi(0)}{x}dx + \int_{|x|>1}\frac{\phi(x)}{x}dx;$$

this holds since the integral of 1/x on $\epsilon < |x| < 1$ is zero(odd function).Further

$$\begin{aligned}
\Big|p.v.\frac{1}{x}(\phi)\Big| &= \Big|\int_{|x|<1}\frac{\phi(x)-\phi(0)}{x}dx + \int_{|x|>1}\frac{\phi(x)}{x}dx\Big| \\
&\leq \Big|\int_{|x|<1}\frac{\phi(x)-\phi(0)}{x}dx\Big| + \Big|\int_{|x|>1}\frac{\phi(x)}{x}dx\Big| \\
&\leq \Big|\int_{|x|<1}\frac{\phi(x)-\phi(0)}{x}dx\Big| + \Big|\int_{|x|>1}\frac{x\phi(x)}{x^2}dx\Big| \\
\Big|p.v.\frac{1}{x}(\phi)\Big| &\leq C(\|\phi'\|_\infty + \|x\phi\|_\infty);
\end{aligned}$$

where we get first term by Mean Value Theorem and second expression is bounded as $\frac{1}{x^2}$ is integrable.

**Proposition 4.1.** *In* $S^{'}, \lim\limits_{y\to 0} Q_y = \frac{1}{\pi} p.v.\frac{1}{x}$.

**Proof** *For each $\epsilon > 0$, the functions $\psi_\epsilon(x) = x^{-1}\mathcal{X}_{\{|x|>\epsilon\}}$ are bounded and define tempered distributions. It follows at once from the definition that in $S^{'}$,*

$$\lim_{\epsilon\to 0} \psi_\epsilon = p.v.\frac{1}{x}.$$

*Therefore, it will suffice to prove that in $S^{'}$*

$$\lim_{y\to 0}\left(Q_y - \frac{1}{\pi}\psi_y\right) = 0.$$

*Fix $\phi \in S$; then*

$$\begin{aligned}
(\pi Q_y - \psi_y)(\phi) &= \int_{\mathbb{R}} \frac{x\phi(x)}{y^2+x^2}dx - \int_{|x|>y} \frac{\phi(x)}{x}dx \\
&= \int_{|x|<y} \frac{x\phi(x)}{y^2+x^2}dx + \int_{|x|>y} \left(\frac{x}{y^2+x^2} - \frac{1}{x}\right)\phi(x)dx \\
&= \int_{|x|<1} \frac{x\phi(yx)}{1+x^2}dx - \int_{|x|>1} \frac{\phi(yx)}{x(1+x^2)dx.}
\end{aligned}$$

*If we take limit as $y \to 0$ and apply DCT, we get two integrals of odd functions on symmetric domains. Hence, the limit equals 0.*

## 4.3 Different equivalent expressions for Hilbert Transform

Suppose,

$$T : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

is an operator which satisfies following conditions:

(1) commutes with translation(is a multiplier)

(2) commutes with dilation and

(3) anti-commutes with reflection

$(3')$ commutes with reflection

An operator satisfying condition 1,2,$3'$ is an Identity Operator,

$$T \;=\; CI$$

$m(L^2(\mathbb{R}) \approx L^\infty(\mathbb{R})$ and

$$(Tf)^\wedge(\xi) = m(\xi)\hat{f}(\xi).$$

Now,

$$TD_\delta f = D_\delta Tf, \qquad where\ D_\delta f = f(\delta x)$$

$$\begin{aligned}(D_\delta f)^\wedge(\xi) &= \int f(\delta x)e^{-2\pi i x\xi}dx \\ &= \frac{1}{\delta}\int f(y)e^{-2\pi i\xi\frac{y}{\delta}}dy \\ &= \frac{1}{\delta}\hat{f}(\frac{\xi}{\delta})\end{aligned}$$

Then,

$$\begin{aligned}T(D_\delta f)^\wedge(\xi) &= m(\xi)(D_\delta f)^\wedge(\xi) \\ &= m(\xi)\frac{1}{\delta}\hat{f}(\frac{\xi}{\delta})\end{aligned}$$

and

$$\begin{aligned}(D_{\deg}Tf)^\wedge(\xi) &= \frac{1}{\delta}(Tf)^\wedge(\frac{\xi}{\delta}) \\ &= \frac{1}{\delta}m(\frac{\xi}{\delta})\hat{f}(\frac{\xi}{\delta})\end{aligned}$$

$$\Rightarrow m(\xi) = m(\frac{\xi}{\delta}),\ \forall\delta > 0$$

Now by Anti-Reflection property,

$$Rf(x) = f(-x),$$

$$RTf(x) = -TRf(x)$$

and

$$(Rf)^\wedge(\xi) = \hat{f}(-\xi).$$

The above equation will imply

$$m(\xi) = -m(-\xi)$$

If all the three conditions 1,2 and 3 are satisfied, then the multiplier is given by

$$m(\xi) = csgn(\xi).$$

We take $c = -i$ and $m(\xi) = -isgn(\xi)$.
Given a function $f \in S$, we define its Hilbert transform such that

$$(Hf)^\wedge(\xi) = -isgn(\xi)\hat{f}(\xi). \tag{4.1}$$

Other equivalent expressions for Hilbert Transform are:

$$Hf = \lim_{y\to 0} Q_y * f, \tag{4.2}$$

$$Hf = \frac{1}{\pi}p.v.\frac{1}{x} * f, \tag{4.3}$$

where as a result of above proposition we get that

$$\lim_{y\to 0} Q_y * f(x) = \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|t|>\epsilon}\frac{f(x-t)}{t}dt,$$

and using equation (i) we get

$$\Big(\widehat{\frac{1}{\pi}p.v.\frac{1}{x}}\Big)(\xi) = -isgn(\xi)$$

The first expression also lets us define the Hilbert Transform of functions in $L^2\mathbb{R}$; it satisfies

$$\|Hf\|_2 = \|f\|_2, \tag{4.4}$$

$$H(Hf) = -f, \tag{4.5}$$

$$\int Hf.g = -\int f.Hg \tag{4.6}$$

**Proof** : Equation (4.4): Using Plancherel's Theorem,

$$\begin{aligned}\|Hf\|_2 &= \|\widehat{Hf}\|_2 \\ &= \|\hat{f}\|_2 \qquad (from\ equation\ (4.1)) \\ &= \|f\|_2\end{aligned}$$

Equation (4.5):

$$\begin{aligned}(\widehat{H(Hf)})(\xi) &= -\iota sgn(\xi)(\widehat{Hf})(\xi) \\ &= (-\iota sgn(\xi))(-\iota sgn(\xi))\hat{f}(\xi) \\ &= -\iota\hat{f}(\xi)\end{aligned}$$

Now, $(H^2 f) = (\widehat{H(Hf)})^{\vee} = (-\hat{f}(\xi))^{\vee} = -f$, for $f \in S$

Equation (4.6):

$Hf \in L^2$ and $g \in L^2$. Now,

$$\begin{aligned}\langle Hf, g\rangle &= \langle \widehat{Hf}, \hat{g}\rangle \\ &= \langle -\iota sgn(.)f\hat{(.)}, \hat{g}\rangle \\ &= \langle \hat{f}, \iota sgn(.)g\hat{(.)}\rangle \\ &= -\langle \hat{f}, \widehat{Hg}\rangle\end{aligned}$$

## 4.4 Theorem of Kolmogorov

[Duo01]

**Theorem 8.** *For $f \in S(\mathbb{R})$, the following assertion is true:*
*H is weak (1,1):*

$$|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}| \leq \frac{C}{\lambda}\|f\|_1.$$

Before proceeding further, we need the following lemma:

**Lemma 4.1.** ***Calderón Zygmund Lemma***
*Let $f \geq 0$, $f \in L^1(\mathbb{R})$ & $\lambda > 0$ There exists a countable collections of cube with sides parallel to the axis,$Q_j$ with disjoint interior such that for each j,*

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j} f(x)dx \leq 2^n\lambda$$

**Proof of theorem 8**:

Fix $\lambda > 0$ and $f$ is non negative and integrable function. Now, partition the space into cubes of equal size.Notation,

$$f_Q = \frac{1}{Q}\int_Q f(x)dx$$

. If $f_Q > \lambda$, choose the cube and if $f_Q < \lambda$ divide it further unto 4 sub-cubes and repeat the process as we have countable cubes.

So, $\Omega = \cup Q_j$, where picked up Q's are enumerated as $Q_j$. $Q_j$ are mutually disjoint a.e. Now, let us define $\widetilde{Q}_j$ as a bigger cube of which $Q_j$ is one of the part out of four. Form the Calderón- Zygmund decomposition of $f$; this yields a sequence of disjoint cubes $\{\widetilde{Q}_j\}$ such that

$$\begin{aligned} &\frac{1}{|\widetilde{Q}_j|}\int_{\widetilde{Q}_j} f(x)dx \leq \lambda \\ =&\frac{1}{2^n|Q_j|}\int_{\widetilde{Q}_j} f(x)dx \leq \lambda \end{aligned}$$

Therefore,

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j} f(x)dx \leq \frac{1}{|Q_j|}\int_{\widetilde{Q}_j} f(x)dx \leq 2^n\lambda.$$

Let $F = \mathbb{R}^2 \backslash \Omega$ and

$$|\Omega| \leq \sum |Q_j| \leq \frac{1}{\lambda}\int_{Q_j} f(x)dx \leq \frac{1}{\lambda}\|f\|_1 \tag{4.7}$$

Now we decompose $f$ as the sum of two functions, $g$ and $b$, defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathsf{F}, \\ \sum f_{Q_j}\mathcal{X}_{Q_j}(x) & \text{if } x \in \Omega; f_{Q_j} = \frac{1}{|Q_j|}\int_{Q_j} f(x)dx, \end{cases}$$

and

$$\begin{aligned} b(x) &= f(x) - g(x) \\ &= \sum_{j=1} \big(f(x) - f_{Q_j}\big)\mathcal{X}_{Q_j}(x) \\ &= \sum_j b_j. \end{aligned}$$

Then $g(x) \leq 2^n\lambda$ almost everywhere, and $b_j$ is supported on $Q_j$. Since $Hf = Hg+Hb$,

$$|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}| \leq |\big\{x \in \mathbb{R} : |Hg(x)| > \frac{\lambda}{2}\big\}|+|\big\{x \in \mathbb{R} : |Hb(x)| > \frac{\lambda}{2}\big\}|. \quad (4.8)$$

We estimate the first term of the equation (4.8):

$$\|g\|_2^2 = \int_F |g(x)|^2 + \int_\Omega |g(x)|^2 dx$$

First term of the above expression can b written as :

$$\begin{aligned}\int_F g(x)^2 &= \int_F f(x)^2 \\ &\leq \lambda \int_F f(x)dx \\ &\leq \lambda \|f\|_1\end{aligned}$$

Second term can be written as :

$$\begin{aligned}\int_\Omega |g(x)|^2 &= \sum_j \int_{Q_j} (f_{Q_j})^2 dx \\ &\leq 2^n\lambda \sum_j \int_{Q_j} f_{Q_j} dx \\ &\leq C\|f\|_1\end{aligned}$$

Therefore,

$$\int_\mathbb{R} |g(x)|^2 dx \leq C^{'}\|f\|_1$$

Now

$$\begin{aligned}|\big\{x \in \mathbb{R} : |Hg(x)| > \frac{\lambda}{2}\big\}| &\leq \big(\frac{2}{\lambda}\big)^2 \int_\mathbb{R} |Hg(x)|^2 dx \\ &= \frac{4}{\lambda^2} \int_\mathbb{R} g(x)^2 dx \\ &\leq \frac{C''}{\lambda}\|f\|_1. \qquad (4.9)\end{aligned}$$

Let us now estimate the second term of the equation (4.8):

$$b_j(x) = \big(f(x) - f_{Q_j}\big)\mathcal{X}_{Q_j}$$

and

$$\int b_j(x)dx = \int_{Q_j} (f(x) - f_{Q_j})$$
$$= 0$$

Now,

$$Hb = H\big(\sum_j b_j\big)$$
$$= \sum_j Hb_j (in\ the\ sense\ of\ L^2)$$

As,

$$|\{x \in \mathbb{R} : |Hb_j(x)| > \lambda\}| \leq |\Omega| + |\{x \notin \Omega : |Hb_j(x)| > \lambda\}|$$

Replace $\Omega$ by $\Omega^*$ and $\Omega \subset \Omega^* = \cup Q_j^*$, where $Q_j^*$ has same center as $Q_j$ and of length twice.
Now,

$$|Q_j^*| \leq \sum_{j=1}^{\infty} |Q_j^*|$$
$$= 2\sum_{j=1}^{\infty} |Q_j|$$
$$= 2|\Omega| \tag{4.10}$$

Let us take $x \notin \Omega^*$ $(x \notin Q_j^*)$.

$$Hb_j(x) = \int_{y \in Q_j} \frac{b_j(x)}{x - y}$$
$$= \int_{y \in Q_j} \left[\frac{b(y)}{x - y} - \frac{b(y)}{x - c_j}\right] dy \quad (where\ c_j\ is\ the\ center)$$
$$= \int_{y \in Q_j} b(y) \frac{y - c_j}{(x - y)(x - c_j)} dy$$

Since,

$$|c_j - y| \leq \frac{1}{2}|x - c_j| \tag{4.11}$$

and

$$
\begin{aligned}
|x-y| &\leq |x-c_j| - |c_j - y| \\
&\geq \frac{1}{2}|x-c_j| \qquad (Using\ equation\ 4.10)
\end{aligned}
$$

Therefore,

$$|Hb_j(x)| \leq c\int_{y\in Q_j} \frac{|b_j(y)||l(Q_j)|}{|x-c_j|^2}dy$$

$$
\begin{aligned}
\int_{\mathbb{R}\setminus\Omega^*} |Hb_j(x)| &\leq cl(Q_j)\int_{|x-c_j|>2l(Q_j)} \frac{1}{|x-c_j|^2}\int_{Q_j}|b_j(y)|dy \\
&\leq c'l(Q_j)\frac{1}{l(Q_j)}\|b_j\|_1 \\
&\leq c'\|b_j\|_1 && (4.12)
\end{aligned}
$$

Now,

$$
\begin{aligned}
|\{x \notin Q_j^* : |Hb_j(x)| > \lambda\}| &\leq \frac{1}{\lambda}\int_{\mathbb{R}\setminus\Omega^*}|Hb_j(x)|dx \\
&\leq \frac{c}{\lambda}\|b_j\|_1 && (4.13)
\end{aligned}
$$

As b is support on $Q_j$,

$$\|b\|_1 = \sum \|b_j\|_1$$

So now,

$$
\begin{aligned}
|\{x \notin \Omega^* : Hb(x) > \lambda\}| &\leq \frac{1}{\lambda}\|Hb\|_1 \\
&\leq \frac{1}{\lambda}\int_{\mathbb{R}\setminus\Omega^*}|Hb(x)|dx \\
&\leq \frac{1}{\lambda}\int_{\mathbb{R}\setminus\Omega^*}\sum_j |Hb_j(x)|dx \\
&\leq \frac{1}{\lambda}\sum_j\int_{\mathbb{R}\setminus\Omega^*}|Hb_j(x)|dx \\
&\leq \frac{c}{\lambda}\sum_j \|b_j\|_1 \qquad (Using\ equation\ (4.13))
\end{aligned}
$$

As,

$$\begin{aligned}\|b_j\|_1 &\leq \int_{Q_j} |f(x) - f_{Q_j}|dx \\ \sum \|b_j\| &\leq \sum \int_{Q_j} |f(x) - f_{Q_j}|dx \\ &\leq 2\sum_j \int_{Q_j} |f(x)|dx \\ &\leq 2\|f\|_1\end{aligned}$$

Therefore,

$$|\{x \notin \Omega^* : Hb(x) > \lambda\}| \leq \frac{C}{\lambda}\|f\|_1. \tag{4.14}$$

From equation (4.9) and (4.14) we can say that:

$$|\{x \in \mathbb{R} : |Hb(x)| > \lambda\}| \leq \frac{C'}{\lambda}\|f\|_1. \tag{4.15}$$

This proves the weak(1,1)inequality:

$$|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}| \leq \frac{C}{\lambda}\|f\|_1.$$

## 4.5 Norm convergence of Fourier Series

**Theorem 9.** *For $f \in L^p(\mathbb{R}^n), 1 < p < \infty$,*
*Let*

$$\begin{aligned}S_N f(x) &= \int_{-N}^{N} \hat{f}(\xi)e^{2\pi i\xi x}d\xi \\ &= \int_{\mathbb{R}^n} \mathcal{X}_{\prod[-N,N]}\hat{f}(\xi)e^{2\pi i\xi x}d\xi\end{aligned}$$

*Then,*

$$S_N f \longrightarrow f \text{ in } L^p \text{ norm as } N \to \infty \text{ i.e}$$

*If f is such that supp $\hat{f}$ is compact then for large N, $S_N f = f$*

**To Prove:**

$$\lim_{N\to\infty} \|S_N f - f\|_p = 0$$

One of the key idea is that if we can show that the operators $S_N$ are uniformly bounded in $L^p(\mathbb{R})$.
Let us assume

$$\|S_N\| < C$$

for $C$ be some constant independent of $N$. Start with $F \in L^p$, approximate by a $g \in S$ with compact support $\hat{g}$. Then,

$$\begin{aligned}
\|f - S_N\|_p &\leq \|f-g\|_p + \|g - S_N g\|_p + \|S_N g - S_N f\|_p \\
&\leq \|S_N\|\|f-g\| + \frac{\epsilon}{2+2C} \\
&\leq C * \frac{\epsilon}{2+2C} + \frac{\epsilon}{2+2C} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&\leq \epsilon
\end{aligned}$$

second term gets zero for large N. Now let us prove that $S_N$ are uniformly bounded operators on $L^p$.
We know

$$S_N f(x) = \int\limits_R \mathcal{X}_{[-N,N]} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$$

and

$$H\hat{f}(\xi) = -isgn(\xi) f\hat{(}\xi)$$

Let us take,

$$(S_1 f(\xi))^\wedge = \mathcal{X}_{|x\leq 1|}(\xi)\hat{f}(\xi).$$

We see that

$$\mathcal{X}_{(0,\infty)}(\xi) = \frac{I + iH}{2}$$

and

$$\mathcal{X}_{(-\infty,0)}(\xi) = \frac{I - iH}{2}.$$

Therefore, $\mathcal{X}_{(0,\infty)}(\xi)$ and $\mathcal{X}_{(-\infty,0)}(\xi)$ are multipliers too. Product of the two will be multiplier too.

$$\mathcal{X}_{[-1,1]}(\xi) = \mathcal{X}_{(0,\infty)}(1-\xi) \cdot \mathcal{X}_{(-\infty,0)}(\xi - 1).$$

So we can write $S_1 = H_1 H_2$
Now,

$$\|S_1\|_{L^p \to L^p} = \|H_1\|_{L^p \to L^p} \|H_2\|_{L^p \to L^p}$$

As we know that Hilbert transform is bounded on $L^p \ for \ 1 < p < \infty$, therefore we can see that $S_1$ is bounded on $L^p \ for \ 1 < p < \infty$. Due to scale invariance each $S_N$ has same norm as $\mathcal{X}$ is scale invariant (value is 1 irrespective of interval length) thus we can say that $S_N$ is uniformly bounded in $L^p$. Now using Uniform Boundedness Principle which states that:
Let $X, Y$ be Banach spaces.Let $\{T_\alpha\}$ be a family of bounded linear maps from $X$ into $Y$ . If

$$\sup_\alpha \|T_\alpha x\| < \infty$$

for all $x \in X$, then

$$\sup_\alpha \|T_\alpha\| < \infty$$

Now we can say that $S_N$ are uniformly bounded operators on $L^p(\mathbb{R})$

# Chapter 5

# Double Fourier Series

[Kra99] **Definition:** The Fourier series of $f(x, y)$ can be written in the following form:

$$f(x, y) \sim \sum_{m,n=-\infty}^{\infty} C_{m,n} e^{2i(mx+ny)}$$

where,

$$C_{m,n} = \frac{1}{4\pi^2} \int \int_{\mathbb{R}} f(x, y) e^{-i(mx+ny)} dx dy$$

**Method of Partial Summation: Square Summability**

$$\hat{f}(\xi, \eta) = \int \int f(x, y) e^{-2\pi i \xi x} e^{-2\pi i \eta y} dx dy$$

Now we define,

$$S_N^{Sq} f(x, y) \equiv \sum_{\xi, \eta \leq N} \hat{f}(\xi, \eta) e^{2\pi i(\xi x + \eta y)}$$

So basic question will be whether $S_N^{Sq} f(x, y) \longrightarrow f(x, y)$ *as* $N \to \infty$. So before answering this question let us see how Fourier multiplier can be used to get the answer of the above question.

## 5.1 Application of Fourier Multiplier to summation of double Fourier series

**Lemma 5.1.** *Fix $1 < p < \infty$. Let $Q_R$ be the $R$-fold dilate of the unit cube $Q$. If $M_Q$ is bounded on $L^p(\mathbb{R}^n)$, then the operator*

$$M_{Q_R} : f \longmapsto [\mathcal{X}_{Q_R}\hat{f}]^\vee$$

*is bounded on $L^p(\mathbb{R}^n)$, independent of $R$.*

**Corollary 5.1.** *If $M_Q$ is bounded on $L^p(\mathbb{R}^n)$ for some $1 < p < \infty$, then the operator*

$$M_{Q_R} : f \longrightarrow \sum_{j\in\mathbb{Z}} \mathcal{X}_{Q_R}(-j)\hat{f}(j)e^{ijx}$$

*is bounded on $L^p(\mathbb{R}^n)$, is independent of $R$.*

**Proof** : We take $f$ to be in $S(\mathbb{R}^n)$ which is certainly dense in $L^p$, so we need not worry about the convergence of the integrals. Now

$$\begin{aligned}
[\mathcal{X}_{Q_R}\hat{f}]^\vee(x) &= (2\pi)^{-N}\int \mathcal{X}_{Q_R}(\xi)\hat{f}(\xi)e^{-ix.\xi}d\xi \\
&= R^N(2\pi)^{-N}\int \mathcal{X}_{Q_R}(R\xi)\hat{f}(R\xi)e^{-ix.R\xi}d\xi \\
&= R^N(2\pi)^{-N}\int \mathcal{X}_{Q}(\xi)\widehat{\alpha^R f}(\xi)(\xi)e^{-ix.R\xi}d\xi \\
&= R^N[M_Q(\alpha^R f)](Rx).
\end{aligned}$$

We have substituted $\xi = R\xi$ in $2^{nd}$ step of the above calculation and in the third step $f_R(x) = Rf(Rx) = (T_r f)x$ so here we have denoted $T_r = \alpha^R$ that is, $\alpha^R$ is an operator.Now let us calculate the $L^p$ norm:

$$
\begin{aligned}
\|M_{Q_R} f\|_{L^p} &= R^N \left[ \int |M_Q[\alpha^R f](Rx)|^p dx \right]^{1/p} \\
&= R^N \cdot R^{-N/p} \left[ \int |M_Q[\alpha^R f](x)|^p dx \right]^{1/p} \\
&\leq C \cdot R^N \cdot R^{-N/p} \|\alpha^R f\|_{L^p} \\
&= C \cdot R^N \cdot R^{-N/p} \left[ \int |R^{-N} f(R^{-1}x)|^p dx \right]^{1/p} \\
&= C \cdot R^N \cdot R^{-N/p} \left[ \int R^N \cdot R^{-Np} |f(x)|^p dx \right]^{1/p} \\
&= C \left[ \int |f(x)|^p dx \right]^{1/p}
\end{aligned}
$$

So we have proved the bound on the $L^p$ norm of $M_{Q_R} f$, depending on $f$ and independent of $R$.

**Theorem 10.** *Let $P$ be a point of $\mathbb{R}^2, v \in \mathbb{R}^2$ be a unit vector, and set*

$$E_v = \{x \in \mathbb{R}^2 : (x - P) \cdot v \geq 0\}.$$

*Then the operator*

$$f \longmapsto (\mathcal{X}_{E_v} \cdot \hat{f})^\vee$$

*is bounded on $L^p, 1 < p < \infty$.*

**Proof** We have proved in the previous chapter that Hilbert transform is bounded on $L^p(\mathbb{R})$ and we have used the operator, $M = \frac{1}{2}(I + iH)$ because of its simple multiplier $m = \mathcal{X}_{[0,\infty)}$. So now we will express the multiplier for a half space as an amalgam of multipliers for the half-line using Fubini's theorem. After the composition of translation and rotation, we bring point P to the origin and v is the vector $(0, 1)$ which is shown as below:

$$M_a f(x) = e^{2\pi i a x} f(x)$$

$$(M_a f)^\wedge(\xi) = \hat{f}(\xi - a)$$

$$
\begin{aligned}
(H M_a f)^\wedge(\xi) &= (M_a f)^\wedge(\xi) \mathcal{X}_{[0,\infty)}(\xi) - (M_a f)^\wedge(\xi) \mathcal{X}_{(-\infty,0]}(\xi), \\
&= \hat{f}(\xi - a) \mathcal{X}_{[0,\infty)}(\xi) - \hat{f}(\xi - a) \mathcal{X}_{(-\infty,0]}(\xi)
\end{aligned}
$$

$$
\begin{aligned}
[M_{-a}(HM_af)]^\wedge(\xi) &= (HM_af)^\wedge(\xi+a) \\
&= (M_af)^\wedge(\xi+a)\mathcal{X}_{[0,\infty)}(\xi+a) - (M_af)^\wedge(\xi+a)\mathcal{X}_{(-\infty,0]}(\xi+a) \\
&= \hat{f}(\xi)\mathcal{X}_{[0,\infty)}(\xi+a) - \hat{f}(\xi)\mathcal{X}_{(-\infty,0]}(\xi+a)
\end{aligned}
$$

and an anti-clockwise rotation of a function by an angle  implies that its Fourier transform is also rotated anti-clockwise by the same angle.

Fix $1 < p < \infty$. Take $f$ to be in $S(\mathbb{R})$ since Schwartz functions are dense in $L^p(\mathbb{R}^2)$. Notation $f_{x_1}(x_2) \equiv f(x_1, x_2)$. Now we calculate
$(\mathcal{X}_E\hat{f})^\vee(x_1, x_2) = (2\pi)^{-2}\int_0^\infty \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(t_1, t_1) \times e^{i\xi_1 t_1}e^{i\xi_2 t_2}dt_1 dt_2 e^{-i\xi_1 x_1}e^{-i\xi_2 x_2}d\xi_1 d\xi_2$.
The two integrals give rise to a Schwartz function, so all integrals converge absolutely. By Fubini's theroem, the last line equals

$$
\begin{aligned}
\tfrac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}&\left[\tfrac{1}{2\pi}\int_0^\infty \left[\int_{\mathbb{R}} f(t_1, t_1)e^{i\xi_2 t_2}dt_2\right]e^{-i\xi_2 x_2}d\xi_2\right]e^{-i\xi_1 x_1}d\xi_1 e^{i\xi_1 t_1}dt_1 \\
&= \tfrac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} M(f_1)(x_2)e^{it_1\xi_1}dt_1 e^{-i\xi_1 x_1}d\xi_1.
\end{aligned} \tag{5.1}
$$

Also,

$$
\begin{aligned}
\int_{t_1\in\mathbb{R}} \|Mf_{t_1}(\cdot)\|^p_{L^p(\mathbb{R})}dt_1 &\le C\int_{t_1\in\mathbb{R}} \|f_{t_1}(\cdot)\|^p_{L^p(\mathbb{R})}dt_1 \\
&= C\|f\|^p_{L^p(\mathbb{R}^2)}.
\end{aligned}
$$

In particular, for almost every $x_2$, the function

$$
t_1 \longmapsto F_{x_2}(t_1) \equiv M(f_{t_1})(x_2)
$$

lies in $L^p(\mathbb{R})$. We can write the right hand side of equation (5.1) as

$$
\lim_{\epsilon\to 0}\int_{\mathbb{R}}\frac{1}{2\pi}\int_{\mathbb{R}} F_{x_2}(t_1)e^{it_1\xi_1}dt_1 e^{-i\xi_1 x_1}e^{-\epsilon|\xi|^2}d\xi_1
$$

using Gauss-Weierstrass summation. But this equals

$$
\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\mathbb{R}} F^\vee_{x_2}(-\xi_1)e^{-i\xi_1 x_1}e^{-\epsilon|\xi|^2}d\xi_1 = F_{x_2}(x_1)
$$

for almost every $x_1$ and we have proved that the latter function is norm dominated by $C\|f\|_{L^p(\mathbb{R}^2)}$

**Theorem 11.** *The method of square summation is valid for double trigonometric series in $L^p$ norm,* $1 < p < \infty$.

**Proof** : Let

$$E_1 \equiv \{(x,y) \in \mathbb{R}^2 : (-1,0) \cdot [(x,y) - (1,0)] \geq 0\},$$

$$E_2 \equiv \{(x,y) \in \mathbb{R}^2 : (1,0) \cdot [(x,y) - (-1,0)] \geq 0\},$$

$$E_3 \equiv \{(x,y) \in \mathbb{R}^2 : (0,-1) \cdot [(x,y) - (0,1)] \geq 0\},$$

$$E_4 \equiv \{(x,y) \in \mathbb{R}^2 : (0,1) \cdot [(x,y) - (0,-1)] \geq 0\}.$$

So the common intersection of the above 4 half planes is a unit square $Q = \{(x,y) : |x| \leq 1, |y| \leq 1\}$ in $\mathbb{R}^2$. Let $T_j$ be the multiplier operator associated to $\mathcal{X}_{E_j}$,that is,

$$T_j : f \longrightarrow (\mathcal{X}_{E_j} \cdot \hat{f})^{\vee}.$$

Then the multiplier operator associated with this closed square is $T_1 \circ T_2 \circ T_3 \circ T_4$. As we have proved earlier that each $T_j$ is bounded on $L^p, 1 < p < \infty$. As a result $T_1 \circ T_2 \circ T_3 \circ T_4$ is certainly bounded on $L^p$. Since the multiplier operator associated with the unit square is bounded, we can say that square summation is valid for double Fourier series, $1 < p < \infty$.

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