Entanglement and Non-Locality in Continuous Variable System

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A dissertation submitted for the partial fulfilment of BS-MS dual degree in Science

Indian Institute of Science Education and Research Mohali

April 2017

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. Arvind at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Gaurav Saxena
(Candidate)

Dated: April 19, 2017

In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Arvind
(Supervisor)

Certificate of Examination

This is to certify that the dissertation titled **Entanglement and Non-Locality in Continuous Variable System** submitted by **Gaurav Saxena** (Reg. No. MS12003) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Sandeep Goyal   Dr. Kavita Dorai   Prof. Arvind (Supervisor)

Dated: April 19, 2017

Acknowledgment

I would like to acknowledge Prof. Arvind for introducing me to the field of Continuous Variable Quantum Information. It was only because of his help that I was able to explore a few important areas in this field in which extensive research around the world is being conducted. I would also like to thank him for being a very supportive, always encouraging and motivating guide. He gave me an opportunity to think independently and always guided me in hard times. Also, I would like to thank him for giving me an environment and space to do my MS thesis.

I would also like to acknowledge Chandan Kumar with whom, much of my discussions happened. I thank him for patiently looking and checking my programs and correcting or modifying them wherever required.

I would also like to thank Dr. Sandeep Goyal for discussing with me a few things whenever I got stuck. I would also like to thank Dr. Arun, Jassi, Vikash, Akanksha, Devra and Bhati bhaiya for many discussions which have always been fruitful.

Last but not the least, I would like to thank my parents who always supported my decisions and gave me independence to pursue my interests.

I would end by acknowledging Tasha, Yogesh Bhaiya, Dahlia Di, Pranay Bhaiya, Samridhi Panwar, Sanjib Kumar Das, Vishnu PK, Akshay Kumar and Ashutosh Tiwari with whom I had a great time in IISER Mohali. Without them, the life at IISER Mohali would have been incomplete.

List of Figures

Abstract

Entanglement and Nonlocality play a very important role in Quantum Information. Recently, a lot of focus has shifted to Continuous Variable Quantum Information, mainly because, continuous variable systems like coherent and squeezed state of light are easier to produce and do experiments with. In my thesis, I initially looked upon the separability criterion of bipartite Gaussian States.

Then, I explored the nonlocality of a given general bipartite Gaussian state. To examine the nonlocality of a continuous variable state, a continuous variable Bell-type inequality is required. I used two such inequalitites for my study and tried to compare them.

I also introduced noise in this study of nonlocality of Gaussian states. The main reason was that in real experiments noise is very important. I introduced two types of noise in the inequality. One in the form of thermal noise and other by using a beam splitter model. Using these two models of noise, I explored how noise would affect the nonlocality of a state and obtained some results.

Contents

Chapter 1

Introduction

1.1 Background

The advent of Quantum Mechanics, about 90 years ago, has changed the perspective through which the physicists look at nature. Postulates, as strong as the universal constancy of speed of light, on which the Special Theory of Relativity has been built, have been challenged by the results or predictions of Quantum Mechanics. The physics community got much closer towards the understanding of the working of nature after the advance of Quantum Mechanics. Even engineers have also been using Quantum Physics to bring improvements to the current technology. Now, in 21st century, industries are designing chips at nanometer scale below which the quantum effects like interference would be very prominent. In 1980s, Richard P. Feynman gave the idea that, to simulate nature, we must have computers that work on the principles of Quantum Mechanics. And from then, researchers have been trying in this direction to build a quantum computer. Though, a quantum computer has not yet been built but a lot of theory and a large number of quantum algorithms have been written down and have been implemented in various ways like on NMR or ion trap or quantum dots etc. Many problems which were considered to be NP hard(Non-deterministic Polynomial time,i.e., if a given problem cannot be solved in polynomial time in any known way it is NP hard problem) have been shown to be P(Polynomial time,i.e., P is the set of all decision problems which can be solved in polynomial time by a deterministic Turing machine) or at least have been shown to be solved in time less than NP by the use of quantum algorithms.

The most central and also the most counter-intuitive postulate on which Quantum Mechanics work is the *Superposition Principle*. It is neither an 'OR' event nor an 'AND' event. For instance, in a standard Young's Double Slit Experiment, the electron when fired passes through the two slits provided, to reach the screen. Now, the results of the experiments are such that they suggest that it cannot be the case that electron passed through either 1st or 2nd slit. Also, it cannot be inferred that the electron passed through both the slits. Though, on disturbing the electron or observing it midway before it reaches the screen, changes the results. Hence, the physicists had to settle down to the superposition principle and had to work with a counter-intuitive physical reasoning.

Another central idea of Quantum Mechanics was *Quantization*. It was observed earlier before the advance of Quantum Mechanics that at microscopic level, the nature behaved in a discrete way. For instance, in 1921, Albert Einstein was awarded the Noble Prize for his discovery of the law of photoelectric effect. This discovery opened up answers to many questions and many other questions automatically surfaced. The two main ideas conveyed with this discovery were that the light which was assumed to have wave-like properties, was now known to have particle-like properties as well and secondly, the quantization of electromagnetic radiation given by Planck was confirmed. Once the idea of quantization got included in Quantum Mechanics, it was then not difficult to map the classical fields to operators acting on quantum states.

Having been equipped with these two very important tools of Superposition Principle and Quantization, many new ideas and applications were discovered. Ideas like entanglement started to come into the picture. Because of quantization, the physics community was able to talk about the particles associated to a given field. For instance, photons were the particles of light having energy equal to an integral number of $\hbar\omega$.

Gradually as more and more research in this field continued, people studied Harmonic Oscillators from this new perspective of quantization and discovered that there are discrete levels of energy(or frequency) in a quantum harmonics oscillator. That is, the photon can not go from some energy level to any other energy level as the jump from one level to another requires a certain constant energy and hence, the levels are not continuous. Also, due to quantization, researchers were able to associate operators that could create or destroy a quanta of light. Then, later these energy levels were associated with the number of photons and hence, became known as number states. Now, this was a big leap as later people defined states like coherent states which were a linear combination of all the number states starting from vacuum state and going up to infinity. The ground state was the vacuum state with no photons but unlike classical fields, the vacuum state also had a finite energy.

1.2 Applications of Quantum Mechanics

Quantum Mechanics has a vast number of applications. It has revealed a lot about the behaviour at the subatomic levels and has strongly influenced string theory.

Quantum Mechanics has strongly influenced the area of electronics. Many electronic devices are designed keeping in mind the results and predictions of Quantum Mechanics. For example, making a LASER, transistor(and hence, the micro chips that are used as processors these days), MRI(and similarly, NMR), Atomic-force microscope etc.

Quantum Tunneling, i.e., tunneling of a particle through a barrier, which is classically impossible, has found tremendous applications. Quantum tunneling is important for nuclear fusion in stars, radioactivity, scanning tunneling microscope and to some extent in quantum biology.

Even in the field of computer science, quantum mechanics has found its applications. Quantum algorithms have been written down for many problems in computer science and have been shown to be more efficient than the classical ones. This emerging field which is the fusion of Quantum Mechanics and Computer Science is known as Quantum Computation. Similarly, quantum mechanics has also found its applications in Information theory and is known as Quantum Information. Quantum Information in itself has a lot of interesting applications like Quantum Cryptography, Quantum Teleportation etc. Cryptography is the art of enabling two parties to communicate in private[NC00]. To achieve privacy, a cryptographic protocol or a cryptosystem is used. But there is no completely secure public key cryptography. Though, with the use of the principles of quantum mechanics, there are ways to show that secure quantum key distribution can be done. Also, use of quantum mechanics allows us to detect whether any Eavesdropping has happened. On the other hand, teleportation using Quantum Mechanics is not at all what teleportation sounds like. The word 'Teleportation' has been taken from science fiction where it means to teleport objects from one place to another or from one time to another time. Quantum Teleportation does not transfer objects or states. It just transfers the information from one quantum state to another. But because of the no-cloning theorem, the information will no longer prevail in the qubit, originally containing it.

Quantum Cryptography and Quantum Teleportation work because quantum mechanics allows us to form a certain form of superposition called entanglement such that by making measurements over one system, the state of the other system entangled to it can be known without actually performing a measurement over it. Entanglement has been discussed in Chap.(2)

1.3 Motivation

In my thesis, I have looked into the Entanglement and Nonlocality of General Gaussian states. I started by understanding a result in Entanglement theory in Bipartite Gaussian States given by R.Simon[Sim00]. The result stated that the Peres-Horodecki Separability Criterion is a necessary and sufficient criterion for separability of bipartite Gaussian states(Chap.(4)). Having known these results, a natural inclination was to look into the nonlocality of the bipartite Gaussian states. Hence, I looked into two Bell-type inequalities for Continuous Variable Systems and obtained a few results by comparing the two inequalities.

Chapter 2

Entanglement and Nonlocality

Since, quantum mechanics allows us to form superpositions of various states, we can form superpositions such that the each particle can not be described independently of the other particle irrespective of the fact that they are separated by a large distance. This special type of superposition is known as Entanglement. With the emergence of entanglement, it was observed that if two particles are entangled and measurements are made on one particle, the other particle collapsed into a state such that the result of the measurement on it is correlated to the other, even though the particle would have been very far. But this violated the postulate of Special Theory of Relativity which denies instantaneous transfer of information. Hence, the nonlocality also crept into the picture of Quantum Mechanics.

2.1 Entanglement

2.1.1 History

In 1935, Albert Einstein, Boris Podolsky and Nathan Rosen published a result which questioned the completeness of quantum mechanics[EPR35]. They named it as 'EPR Paradox'. But they were not the ones to coin the word 'Entanglement'. Like Einstein, Schrodinger was also dissatisfied by the concept of entanglement as it seemed to violate the famous and well accepted postulate of Special Relativity that no information can travel faster than light. Einstein called this 'spooky action at a distance'.

2.1.2 Entanglement in Pure states

The Hilbert state of a composite system made of two non-interacting systems A and B is given as a tensor product as

$$H = H_A \otimes H_B \tag{2.1}$$

If the state of the first system is $|\psi\rangle_A$ and the state of the second system is $|\phi\rangle_B$, then the state of the composite system is $|\psi\rangle_A \otimes |\phi\rangle_B$. Such states where the state of the composite system can be written as a product of the state of the first system and the state of the second system are called separable states. If we cannot write the state of a composite system as a product of the state of the first system and the state of the second system, then the state is called entangled state.

The most general way to write a pure state for a composite system made of two qubits is

$$|\psi\rangle_{AB} = \sum_{j=0}^{1}\sum_{i=0}^{1} c_{ij}|i\rangle_A|j\rangle_B \tag{2.2}$$

where $|i\rangle_A$ and $|j\rangle_B$ are the orthogonal states for the first and the second qubit, respectively. The state will be separable if we find $c_{ij} = c_i^A c_j^B$ such that $|\psi\rangle_A = \sum_i c_i^A|i\rangle_A$ and $|\phi\rangle_B = \sum_j c_j^B|j\rangle_B$. If we are unable to find such a combination for c_{ij} then the state will not be separable or in other words, the state will be entangled.

For example, the following state is entangled

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B) \tag{2.3}$$

2.1.3 Entanglement in Mixed states

Mixed states are just the density matrices on the Hilbert space $H_A \otimes H_B$. Generalizing the definition of separability of Eq.(2.2) from the pure case to mixed case, one can say that a mixed state is separable if the density matrix can be written in the following way

$$\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i \tag{2.4}$$

where $\sum_i p_i = 1$. If we are not able to write a state in this way, then it is entangled. But then, doing this decomposition for any given density matrix is hard. So, the natural question to ask is if there exist a criterion to tell whether a given mixed state is entangled or not? The answer to this question was given by Asher Peres[Per96] and is a part of Chap.(4).

2.2 Nonlocality

A possible resolution to the EPR Paradox was to assume that the results of the measurement, somehow, depended on some kind of variables which contain the information about the past interactions of the particles. These variables were termed as 'hidden variables'. This would mean that the particles have all the information with them and no information travels from one particle to the other when the measurement is made on one particle.

In 1964, Bell showed that the predictions of Quantum Mechanics are incompatible with any physical theory that satisfies a notion of locality[Bel64]. To show this, a standard "Bell Experiment" is conducted. In this experiment, the two systems are taken such that they have previously interacted and now, they are spatially separated. Now, at each of the two sites where each system is placed, two measurements per site are allowed. Also, the number of outcomes per measurement is restricted to two. Such a system is described as $(2, 2, 2)$ where the first entry describes the number of parties, the second entry describes the number of measurements per party and the third entry describes the number of outcomes per measurements. If a large number of experiments are done, and if the local realist view or hidden variable theory is correct then, then it should satisfy Bell's inequality. But a large number of experiments have been done on many quantum composite systems and it has been observed that there are cases that do not satisfy Bell's inequalities. Thus, Bell's inequalities are a witness for a state to be local or nonlocal.

2.2.1 Bell's Inequality

Bell's inequality, as stated above can be used as a nonlocality witness. The violation of the inequality is suggestive of the nonlocal character of the state. A lot of Bell's inequalities can be found in the literature. A state is said to be nonlocal even if it violates just one of the many inequalities. The most famous Bell's inequality is the CHSH inequality. The setup for this is that we have two parties say Alice(A) and Bob(B) having sharing a composite system of two particles which have interacted in the past. Now, both of them can perform two measurements on their particles. Say, A can measure the spin along x and x' direction and B can measure spin along y and

The result obtained after measurement along x is a, along x' is a', along y is b and along y' is b'. Also, a, b, a' and b' can take only two values, i.e., $a, a', b, b' \in \{+1, -1\}$. Now, the CHSH inequality reads as

$$a.b + a.b' + a'.b - a'.b' \leq 2 \tag{2.5}$$

The above inequality has been written taking into account the locality condition and the hidden variable model. If any violation to this found for any state, the state will be called a nonlocal state.

Chapter 3

Continuous Variable Quantum Information

Systems associated with infinite dimensional Hilbert spaces are Continuous Variable Systems. In other words, quantum systems that are defined by systems having a continuous spectrum, like position and momentum, are Continuous Variable Systems. A continuous-variable (CV) system [EP03, BvL05, CLP07, AI07] of N canonical bosonic modes is described by a Hilbert space $\mathcal{H} = \otimes_{k=1}^{N} \mathcal{H}_k$ where $\mathcal{H}_k$ is an infinite-dimensional fock space associated with the mode k. As an example, we can take the following Hamiltonian

$$\hat{H} = \sum_{k=1}^{N} \hbar\omega_k(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}) \tag{3.1}$$

The Hamiltonian in Eq.(3.1) describes a system of arbitrary number N of harmonic oscillators of different frequencies, the *modes* of the field. The non-interacting quantized electromagnetic field is one such case. In the above Eq.(3.1), $\hat{a}_k^\dagger$ and $\hat{a}_k$ are the creation and annihilation operators of a photon in mode k (having frequency ω_k). They satisfy the following commutation relations

$$[\hat{a}_k, \hat{a}_k^\dagger] = \delta_{kk'}, \qquad [\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0 \tag{3.2}$$

The quadrature operators (position and momentum) for each mode are defined as

$$\hat{q}_k = (\hat{a}_k + \hat{a}_k^\dagger)/\sqrt{2} \tag{3.3}$$

$$\hat{p}_k = (\hat{a}_k - \hat{a}_k^\dagger)/\iota\sqrt{2} \tag{3.4}$$

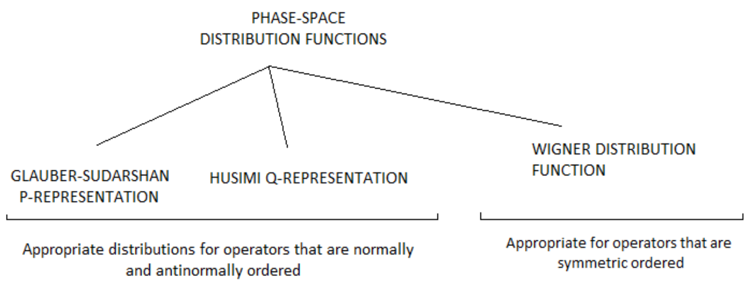

Figure 3.1: Phase-space Distributions

We can group together the canonical quadrature operators in the vector

$$\hat{R} = (\hat{q}_1, \ldots, \hat{q}_N, \hat{p}_1, \ldots, \hat{p}_N)^T \tag{3.5}$$

The above vector belongs to the real $2N$-dimensional space called *phase space*. With the help of these quadrature operators, the whole phase-space can be spanned.

We can also describe distribution functions in phase space. The geometry that is followed in the phase space is Symplectic geometry and associated to it is the Symplectic Group.

3.1 Phase Space Functions

Phase space brings out most clearly the differences in classical and quantum mechanics.[Sch05]

The complete description of any quantum state ρ of an infinite-dimensional system can be provided by one of its s-ordered characteristic functions. For every specific ordering there exists a given phase space distribution function as to obtain always the correct quantum mechanical expectation value.

Mostly, the three phase space distributions(Figure 3.1) are used:
a.) Wigner Distributions
b.) Husimi Q-Representation
c.) Glauber-Sudarshan P-Representation
Wigner distribution is proper for the operators that are symmetric ordered a and $a^\dagger$. Glauber-Sudarshan P-Representation and Husimi Q-Representation are appropriate distribution for operators that expressed in terms of normally ordered a and $a^\dagger$ and anti-normally ordered a and $a^\dagger$, respectively.

3.1.1 Wigner Distributions

Wigner phase-space distribution of a quantum state[Wig32] defined by a density operator is given by

$$W(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dq' e^{ipq'/\hbar}\langle q - \frac{q'}{2}|\hat{\rho}|q + \frac{q'}{2}\rangle \tag{3.6}$$

To represent a n^{th} statistical moment of any operator, say $\hat{A}$, Wigner distribution function can be used as:

$$\langle \hat{A}^n \rangle = tr(\hat{\rho}\hat{A}^n) = \int\int W(q,p)A^n(q,p)dqdp \tag{3.7}$$

where in the final form, $A^n(q,p)$ is a function of q and p, and not an operator as given by Weyl correspondence rule.

If an operator is given in terms of powers of $\hat{q}$ and $\hat{p}$, then

$$\langle S(\hat{q}^m,\hat{p}^n)\rangle = tr(\hat{\rho}S(\hat{q}^m,\hat{p}^n)) = \int\int W(q,p)q^m p^n dqdp \tag{3.8}$$

where S denotes symmetric ordering of the function and so, after applying Weyl corresponding rule, we get in the final form, the function as $q^m p^n$.

The Wigner distribution is not a true probability distribution function. The reason for this is that it can also acquire negative values unlike the true probability distribution functions which only acquire positive values. Therefore, it is referred as a quasi-probability distribution function.

For a function to be a Wigner distribution function, it has to satisfy a few properties, also known as Wigner qualities. The Wigner qualities are as follows:

$$\int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp W(q,p) = 1 \tag{3.9}$$

$$\int_{-\infty}^{\infty} dp W(q,p) = \langle x|\hat{\rho}|x\rangle \equiv W(q) \tag{3.10}$$

$$\int_{-\infty}^{\infty} dq W(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dq' e^{ipq'/\hbar}\langle q - \frac{q'}{2}|\hat{\rho}|q + \frac{q'}{2}\rangle \tag{3.11}$$

Also,

$$Tr(\hat{\rho}_1\hat{\rho}_2) = 2\pi\hbar\int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp' W_{\hat{\rho}_1}(q,p)W_{\hat{\rho}_2}(q,p) \tag{3.12}$$

3.1.2 Q-Representation

Here, I have given just a brief note on Q-Representation as it has not been used in any proof of the paper 'Peres-Horodecki Separability Criterion for Continuous Variable system'. For further and more detailed description, one can check the references [Hus40][Sch05][GK04].
Q-function of a pure quantum state $|\psi\rangle$ is defined as:

$$Q(\alpha_r, \alpha_i) \equiv \frac{1}{\pi}|\langle\alpha|\psi\rangle|^2 \tag{3.13}$$

where α_r and α_i are the real and imaginary parts of the complex number α that describe the coherent state. Therefore, generalizing it, we get:

$$Q(\alpha_r, \alpha_i) \equiv \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle \tag{3.14}$$

where the above is the Q-function to a mixed state with density operator $\hat{\rho}$. Hence the Q-function is the expectation value of the density operator in a coherent state.

3.1.3 P-Representation

As earlier stated, P-representation(or Glauber-Sudarshan P-representation)[Sud63][Gla63a] is appropriate distribution for operators which are expressed in normal order of a and $a^\dagger$. This representation uses the fact that the coherent states are overcomplete. Since, the coherent states form an overcomplete, normal and non-orthogonal basis, we can always represent any density matrix in a diagonal form[Gla63b][Meh67] as

$$\hat{\rho} = \int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha \tag{3.15}$$

Now, to find out the P-function, I refer to the way as done in C.L.Mehta's paper[Meh67]

$$\begin{aligned} \langle -u|\hat{\rho}|u\rangle &= \int P(\alpha)\langle -u|\alpha\rangle\langle\alpha|u\rangle d^2\alpha & (3.16) \\ &= e^{-|u|^2}\int P(\alpha)e^{-|\alpha|^2}e^{\alpha^* u-\alpha u^*}d^2\alpha & (3.17) \end{aligned}$$

where $|u\rangle$ is a coherent state and u is a complex number. Now, by inverse Fourier transform we see

$$P(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2}\int e^{|u|^2}\langle -u|\hat{\rho}|u\rangle e^{u^*\alpha-u\alpha^*}d^2u \tag{3.18}$$

To emphasize again, Glauber-Sudarshan P-representation is not a true probability distribution as it can acquire negative values. In particular, states with non-classical features such as photon number states or squeezed states have negative value of P-functions somewhere in the phase space. Thus, negativity and non-negativity is a defining condition for classicality and non-classicality, respectively. So, states that are classical in the quantum optics sense have a non-negative P-distribution function($P \geq 0$). Example of classical states include coherent states.

3.2 Symplectic Groups

The group of transformations that keep the area preserved in a phase space are Symplectic transformations. Also, symplectic groups preserve the basic kinematic relations such as the Poisson brackets in classical mechanics and commutation relations in quantum mechanics.

So, if we define a $2n$-component column vector ξ,$\hat{\xi}$ as:

$$\xi = (q_1, \ldots, q_n, p_1, \ldots, p_n) \tag{3.19}$$

$$\hat{\xi} = (\hat{q_1}, \ldots, \hat{q_n}, \hat{p_1}, \ldots, \hat{p_n}) \tag{3.20}$$

Then, classical Poisson brackets and quantum commutation relations are defined as:

$$\{\xi_a, \xi_b\} = \Omega_{ab}, \tag{3.21}$$

$$\left[\hat{\xi_a}, \hat{\xi_b}\right] = i\hbar\Omega_{ab} \tag{3.22}$$

$$and\ \Omega = \Omega_{ab} = \begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{3.23}$$

Now, we can define a transformation by a matrix S such that theses relations are preserved. The actions of this matrix are as follows:

$$S = (S_{ab}) : {\xi_a}' = S_{ab}\xi_b \tag{3.24}$$

$$\hat{\xi_a}' = S_{ab}\hat{\xi_b} \tag{3.25}$$

Thus, it leads to a condition $S\Omega S^T = \Omega$. So, the defining condition for the symplectic group in 2n dimensions:

$$Sp(2n, R) = \left\{S = real\ 2n \times 2n\ matrix \mid S\Omega S^T = \Omega\right\} \tag{3.26}$$

The matrix Ω is real, even-dimensional, anti-symmetric and non-singular. It is a "symplectic metric matrix".
Also, for each $S \in Sp(2n, R)$ it is definitely possible to construct a unitary operator $U(S)$ acting on Hilbert space H such that:

$$\hat{\xi_a}' = S_{ab}\hat{\xi_b} = U(S)^{-1}\hat{\xi_a}U(S) \tag{3.27}$$

and $U(S)^{\dagger}U(S) = 1$ on H.
This $U(S)$ is arbitrary upto an S-dependent factor.

3.2.1 Properties of Sp(2n,R) Matrices

From the defining equation of Symplectic groups(3.26), the following properties follow:
(i) $Sp(2n, R)$ is of dimension $n(2n + 1)$
(ii) $\Omega \in Sp(2n, R)$
(iii) $S \in Sp(2n, R) \Rightarrow -S, S^{-1}, S^T \in Sp(2n, R)$,
$S^T = \Omega S^{-1}\Omega^{-1}, (S^{-1})^T = \Omega S\Omega^{-1}, S^{-1} = \Omega S^T\Omega^{-1}$
(iv) $det\ S = +1$
(v) $S \in Sp(2n, R) \Rightarrow$ eigenvalue spectrum of S is invariant under reflection about the real axis, and through unit circle ($re^{i\theta} \rightarrow \frac{1}{r}e^{i\theta}$)

3.3 Covariance matrix and the uncertainty Principle

Given a bipartite density operator $\hat{\rho}$, let us define,

$$\Delta\hat{\xi} = \hat{\xi} - \langle\hat{\xi}\rangle \tag{3.28}$$

$$\Delta\hat{\xi}_\alpha = \hat{\xi}_\alpha - \langle\hat{\xi}_\alpha\rangle \tag{3.29}$$

where, $\langle\hat{\xi}_\alpha\rangle = tr(\hat{\xi}_\alpha\hat{\rho})$. As for an example, $\langle\hat{q}_1\rangle = tr(\hat{q}_1\hat{\rho})$.
The uncertainties are defined as the expectations of the Hermitian operators as

$$\{\Delta\hat{\xi}_\alpha, \Delta\hat{\xi}_\beta\} = \frac{1}{2}(\Delta\hat{\xi}_\alpha\Delta\hat{\xi}_\beta + \Delta\hat{\xi}_\alpha\Delta\hat{\xi}_\beta) \tag{3.30}$$

Now, since we know

$$V_{\alpha\beta} = \langle\{\Delta\hat{\xi}_\alpha, \Delta\hat{\xi}_\beta\}\rangle = tr(\{\Delta\hat{\xi}_\alpha, \Delta\hat{\xi}_\beta\}\hat{\rho}) = \int d^4\xi\Delta\xi_\alpha\xi_\beta W(\xi) \tag{3.31}$$

From the above eq. (3.31), we see the relation between density operator $\hat{\rho}$, Wigner distribution function W and the covariance matrix V whose elements are characterized as $V_{\alpha\beta}$.

By the eq.(3.31), we can write the elements of the covariance matrix, a few examples of which are

$$V_{11} = \langle\{\Delta\hat{q}_1, \Delta\hat{q}_1\}\rangle = \langle\hat{q}_1^2\rangle - \langle\hat{q}_1\rangle^2$$

$$V_{12} = \langle\{\Delta\hat{q}_1, \Delta\hat{q}_2\}\rangle = \langle\{\hat{q}_1, \hat{q}_2\}\rangle - \langle\hat{q}_1\rangle\langle\hat{q}_2\rangle$$

$$V_{13} = \langle\{\Delta\hat{q}_1, \Delta\hat{p}_1\}\rangle = \langle\{\hat{q}_1, \hat{p}_1\}\rangle - \langle\hat{q}_1\rangle\langle\hat{p}_1\rangle$$

and so on for the rest of the elements. But from the Cauchy-Schwarz inequality, we know,

$$\Delta_A^2\Delta_B^2 \geq \left|\frac{1}{2i}\langle[A, B]\rangle\right| \tag{3.32}$$

Taking a few specific cases

$$\Delta q_1\Delta q_2 \geq 0$$

$$\Delta q_1\Delta p_1 \geq |\frac{i\hbar}{2}|$$

$$\Rightarrow \Delta q_1\Delta p_1 + \Delta p_1\Delta q_1 \geq \frac{i\hbar}{2}$$

$$\Rightarrow V_{13} = \langle\{\Delta\hat{q}_1, \Delta\hat{p}_1\}\rangle \geq \frac{i\hbar}{2}$$

Thus, we can calculate for rest of the elements of the covariance matrix. Hence, we have the following compact form of the uncertainty principle(taking the natural system of units, $\hbar \to 1$)

$$V + i\frac{\Omega}{2} \geq 0 \tag{3.33}$$

3.4 Gaussian States

Gaussian states include, among others, coherent, squeezed and thermal states. Hence, these states are very important in quantum information, quantum optics and quantum communication with CV systems.

A state is called Gaussian if its Wigner function is a Gaussian. Since, the mean value can be changed arbitrarily by phase space translations which are local operations, so to simplify we put the mean value as zero. So, the Wigner function for a Gaussian state is given by

$$W(q,p) = \frac{exp(-\frac{1}{2}\xi^T V^{-1}\xi)}{4\pi^2\sqrt{det[V]}} \tag{3.34}$$

Gaussian states are then entirely characterized by covariance matrix,V.
Correspondingly, the characteristic function with mean($\langle\xi\rangle$) zero, becomes

$$\begin{aligned} \chi(\lambda,\eta) &= Tr(\hat{\rho}\ exp(i(\lambda_1\hat{q_1} + \lambda_2\hat{p_1} + \eta_1\hat{q_2} + \eta_2\hat{p_2}))) & (3.35)\\ &= exp(-\frac{1}{2}(\lambda_1,\lambda_2,\eta_1,\eta_2)^T V(\lambda_1,\lambda_2,\eta_1,\eta_2)) & (3.36) \end{aligned}$$

we can write (3.36) as

$$\chi(\lambda,\eta) = Tr(\hat{\rho} exp(i(\lambda^*\hat{a} + \lambda\hat{a^\dagger} + \eta^*\hat{b} + \eta\hat{b^\dagger}))) \tag{3.37}$$

where

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{q_1} + i\hat{p_1})$$

$$\hat{b} = \frac{1}{\sqrt{2}}(\hat{q_2} + i\hat{p_2})$$

$$\lambda = \frac{1}{\sqrt{2}}(\lambda_1 + i\lambda_2)$$

$$\eta = \frac{1}{\sqrt{2}}(\eta_1 + i\eta_2)$$

Now, representing the bipartite Gaussian density matrix in P-representable form

$$\hat{\rho} = \int d^2\alpha \int d^2\beta P(\alpha,\beta)|\alpha,\beta\rangle\langle\alpha,\beta| \tag{3.38}$$

where, $\hat{a}|\alpha,\beta\rangle = \alpha|\alpha,\beta\rangle$, $\hat{b}|\alpha,\beta\rangle = \beta|\alpha,\beta\rangle$ and $\langle\alpha,\beta|\alpha,\beta\rangle = 1$

On substitution of (3.38) in (3.37) and by doing some simplification, we get

$$exp\{-\frac{1}{2}(\lambda_1,\lambda_2,\eta_1,\eta_2)(V-\frac{1}{2})(\lambda_1,\lambda_2,\eta_1,\eta_2)^T\} = \int d^2\alpha \int d^2\beta\, P(\alpha,\beta) exp\{i(\lambda_1\alpha_1+\lambda_2\alpha_2+\eta_1\beta_1+\eta_2\beta_2) \tag{3.39}$$

with $\alpha = \frac{\alpha_1+i\alpha_2}{\sqrt{2}}$ and $\beta = \frac{\beta_1+i\beta_2}{\sqrt{2}}$
Since, the equation (3.39) suggests that the $P(\alpha,\beta)$ is the inverse Fourier transform of LHS, therefore, for the LHS to be a characteristic function corresponding to a non-negative probability distribution, $V-\frac{1}{2}$ has to be greater than or equal to zero,i.e., $V-\frac{1}{2}\geq 0$. Also, from this equation we can deduce that since the LHS for $V-\frac{1}{2}\geq 0$ is a positive Gaussian, the $P(\alpha,\beta)$ which is its Fourier transform will also be a non-negative Gaussian as the Fourier transform of a Gaussian is a Gaussian. Thus,

$$V-\frac{1}{2}\geq 0 \Leftrightarrow P\geq 0 \tag{3.40}$$

And we know that a non-negative $P(\alpha,\beta)$ implies classicality. Therefore, a a Gaussian state is classical if and only if $V-\frac{1}{2}\geq 0$

Chapter 4

Entanglement in Continuous Variable system

In entanglement theory, there are two central questions:

1. Is the state entangled?
2. If the state is entangled, then, how much entanglement does it have?

In my thesis work, I looked upon the first question in Continuous Variable Systems, i.e., given a state with infinite dimensional Hilbert space, whether one can tell if it is entangled or not. More precisely, I looked into bipartite Gaussian systems. For bipartite Gaussian systems, R.Simon showed that there exist a necessary and sufficient condition to check whether a state is entangled or not[Sim00]. I proving his important theorem, Simon used the results of Peres as given below in Sec.(4.1).

4.1 Peres-Horodecki Separability Criterion

Peres Separability criterion[Per96] simply states that for a given separable bipartite density matrix, if we take a partial transpose over one system, then we get a bonafide density matrix. The important point here is that this is, in general, a one way statement. So, if for any given bipartite density matrix the partial transpose is positive(i.e., it still turns out to be a positive density matrix), it does not imply that the original density matrix is separable.

Horodecki[HHH96] made an addition to this result and showed that if the bipartite system is $2 \otimes 2$ or $2 \otimes 3$ system, then this condition holds both ways, i.e., only checking if the partial transpose is positive is sufficient to tell if the original density matrix

is separable or not.
Hence, the Peres-Horodecki Separability Criterion is a necessary and sufficient condition to check if a given $2 \otimes 2$ or $2 \otimes 3$ system is separable or not.
We know that we can always form a Wigner distribution function for any given density matrix. Now, under the partial transposition, if the initial density matrix was separable then the Wigner function formed from the partial transposed density matrix would follow Wigner qualities and it would be a mirror reflection about either of the 'p' coordinates, i.e., if ρ is separable, then

$$\hat{\rho} \xrightarrow[Transpose]{Partial} \hat{\rho^T} \Rightarrow W(q_1, q_2, p_1, p_2) \xrightarrow[Transpose]{Partial} W(q_1, q_2, p_1, -p_2) \tag{4.1}$$

The proof for this is simple.

$$W(q_1, q_2, p_1, p_2) = \frac{1}{\pi^2}\int d^2q' \langle q_1 - q'|\langle q_2 - q'| \sum_j p_j \hat{\rho_{j1}} \otimes \hat{\rho_{j2}} |q_2 + q'\rangle |q_1 + q'\rangle \; exp(2i(q_1{}'p_1 + q_2{}'p_2)) \tag{4.2}$$

$$\Rightarrow W(q_1, q_2, p_1, p_2) = \frac{1}{\pi^2}\int dq_1{}'dq_2{}' \sum_j p_j \langle q_1 - q'|\hat{\rho_{j1}}|q_1 + q'\rangle \otimes \langle q_2 - q'|\hat{\rho_{j2}}|q_2 + q'\rangle \; exp(2i(q_1{}'p_1))exp(2i(q_2{}'p_2)) \tag{4.3}$$

Now, taking the partial transpose over the 2^{nd} system, we get

$$W' = \frac{1}{\pi^2}\int dq_1{}'dq_2{}' \sum_j p_j \langle q_1 - q'|\hat{\rho_{j1}}|q_1 + q'\rangle \otimes \langle q_2 - q'|\hat{\rho^T_{j2}}|q_2 + q'\rangle \; exp(2i(q_1{}'p_1))exp(2i(q_2{}'p_2)) \tag{4.4}$$

$$= \frac{1}{\pi^2}\int dq_1{}'dq_2{}' \sum_j p_j \langle q_1 - q'|\hat{\rho_{j1}}|q_1 + q'\rangle \otimes \langle q_2 + q'|\hat{\rho_{j2}}|q_2 - q'\rangle \; exp(2i(q_1{}'p_1))exp(2i(q_2{}'p_2)) \tag{4.5}$$

$$\tag{4.6}$$

Now let, $q_2' \to -q_2'$,

$$W' = -\frac{1}{\pi^2}\int_{-\infty}^{+\infty} dq_1' \int_{+\infty}^{-\infty} dq_2' \sum_j p_j \langle q_1 - q'|\hat{\rho_{j1}}|q_1 + q'\rangle \otimes \langle q_2 - q'|\hat{\rho_{j2}}|q_2 + q'\rangle \quad (4.7)$$
$$exp(2i(q_1'p_1))exp(-2i(q_2'p_2))$$
$$= \frac{1}{\pi^2}\int_{-\infty}^{+\infty} dq_1' \int_{-\infty}^{+\infty} dq_2' \sum_j p_j \langle q_1 - q'|\hat{\rho_{j1}}|q_1 + q'\rangle \otimes \langle q_2 - q'|\hat{\rho_{j2}}|q_2 + q'\rangle \quad (4.8)$$
$$exp(2i(q_1'p_1))exp(-2i(q_2'p_2))$$
$$= W(q_1, q_2, p_1, -p_2) \quad (4.9)$$

In simple terms, we can say that if initial density matrix $\hat{\rho}$ is separable then:

$$\hat{\rho} \xrightarrow[Transpose]{Partial} \hat{\rho^T} \Rightarrow W(\xi) \xrightarrow[Transpose]{Partial} W(\Lambda\xi) \quad (4.10)$$

where, $\Lambda = diag(1, 1, 1, -1)$
Similarly, the Peres-Horodecki Separability criterion can be cast with the help of Variance Matrix and the uncertainty principle we get from it. Since,

$$V_{\alpha\beta} = \langle\{\Delta\xi_\alpha, \Delta\xi_\beta\}\rangle = tr(\{\Delta\xi_\alpha, \Delta\xi_\beta\}\hat{\rho}) = \int d^4\xi \Delta\xi_\alpha \Delta\xi_\beta W(\xi) \quad (4.11)$$

$$\hat{\rho} \xrightarrow[Transpose]{Partial} \hat{\rho^T} \Rightarrow V + \frac{i}{2}\Omega \geq 0 \xrightarrow[Transpose]{Partial} \tilde{V} + \frac{i}{2}\Omega \geq 0 \quad (4.12)$$

where, $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and $\tilde{V} = \Lambda V \Lambda$.
Since, from the uncertainty principle discussed in section on Covariance Matrices, we know that $V \geq 0$ implies that the smallest eigenvalue of V should be greater than 0 and this one condition translates to the equation:

$$(detA)(detB) + (\frac{1}{4} - detC)^2 - tr(AJCJBJC^TJ) \geq \frac{1}{4}(detA + detB) \quad (4.13)$$

Since, V has the following form:

$$V = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$$

where, A,B are 2×2 real symmetric matrices and C is a real matrix. Then, by virtue of Williamson's theorem, V can be brought to a form by multiplying V by some S^T

on the left and S on the right of V such that V takes the following form:

$$\begin{pmatrix} a & 0 & c_1 & 0 \\ 0 & a & 0 & c_2 \\ c_1 & 0 & b & 0 \\ 0 & c_2 & 0 & b \end{pmatrix}$$

and where $S \in Sp(2,R) \otimes Sp(2,R) \subset Sp(4,R)$ Now, after applying the partial transposition operation on this standard form of V, it goes to $\tilde{V} = \Lambda V \Lambda$, which changes the sign of the determinant of the C matrix. Since, after applying transposition operation $A \to A$, $B \to \sigma_3 B \sigma_3$ and $C \to C\sigma_3$. Therefore, equation(4.13) reads:

$$(detA)(detB) + (\frac{1}{4} - |detC|)^2 - tr(AJCJBJC^T J) \geq \frac{1}{4}(detA + detB) \qquad (4.14)$$

This is the final form for the necessary condition of the Peres-Horodecki Separability criterion for a given bipartite system.

4.2 Peres-Horodecki-Simon Separability Criterion for Bipartite Gaussian states

The main result of Simon's paper[Sim00] was the theorem which he stated in his paper and it read as '*The Peres-Horodecki criterion (4.14) is a necessary and sufficient condition for separability, for all bipartite Gaussian states.*'

To show this, we first notice that states for which their $P(\alpha)$ in the P-representation is positive are classical in the quantum optics sense. And bipartite states which are classical in the quantum optics sense are separable. So, a Gaussian state is classical if and only if $V - \frac{1}{2} \geq 0$ as $P \geq 0 \Leftrightarrow V - \frac{1}{2} \geq 0$.
Since, we are using (4.14) as the final form of the necessary condition, so, we need to put conditions on it and eventually see that if $V - \frac{1}{2} \geq 0$ for our bipartite Gaussian states, then it is separable.
Since, the condition (3.33) implies that $A \geq 0$ and $B \geq 0$, i.e., the eigenvalues of matrices A and B are positive. So, we can conclude that their determinant is also positive. Thus, the only condition left is on the determinant if C. Simon in his paper proved a lemma that '*Gaussian states with det $C \geq 0$ are separable.*' The proof for this lemma follows as:
In the standard form of V, we can arrange $a \geq b$, $c_1 \geq c_2 > 0$. To see the matrix

operations to follow, we arrange our matrix in the (q_1, q_2, p_1, p_2) format which was originally written in (q_1, p_1, q_2, p_2) format. So,

$$\begin{pmatrix} a & 0 & c_1 & 0 \\ 0 & a & 0 & c_2 \\ c_1 & 0 & b & 0 \\ 0 & c_2 & 0 & b \end{pmatrix} \xrightarrow{rearranging} \begin{pmatrix} a & c_1 & 0 & 0 \\ c_1 & b & 0 & 0 \\ 0 & 0 & a & c_2 \\ 0 & 0 & c_2 & b \end{pmatrix}$$

I have rearranged this matrix to simplify the calculations and to get a simple picture in the block diagonal matrix form. Now, on this matrix lets apply a LOCAL canonical transformation. Local transformations means that they only act individually on one particle's phase space and since, we have Local Canonical Transformations the separability is preserved. It is only because the separability is preserved, we are using such matrices for diagonalization. The matrix to act on this rearranged matrix has a form[1]

$$S_{local} = diag(x, x^{-1}, x^{-1}, x) \tag{4.15}$$

This matrix corresponds to local reciprocal scaling. Thus,

$$S_{local} V S_{local}^T = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x^{-1} & 0 & 0 \\ 0 & 0 & x^{-1} & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \begin{pmatrix} a & c_1 & 0 & 0 \\ c_1 & b & 0 & 0 \\ 0 & 0 & a & c_2 \\ 0 & 0 & c_2 & b \end{pmatrix} \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x^{-1} & 0 & 0 \\ 0 & 0 & x^{-1} & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \tag{4.16}$$

$$= \begin{pmatrix} x^2 a & c_1 & 0 & 0 \\ c_1 & x^{-2} b & 0 & 0 \\ 0 & 0 & x^{-2} a & c_2 \\ 0 & 0 & c_2 & x^2 b \end{pmatrix} \tag{4.17}$$

[1]Since, the second and the third diagonal elements are same in this matrix, so, the same matrix acts on V in the (q_1, q_2, p_1, p_2) format

Next, we apply the matrix $S'_{local} = diag(y, y, y^{-1}, y^{-1})$ which corresponds to common LOCAL scalings at both the subsystems[2].

$$\begin{aligned}
V' &= S'_{local} S_{local} V S_{local}^T S'^T_{local} && (4.18)\\
&= \begin{pmatrix} y & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y^{-1} & 0 \\ 0 & 0 & 0 & y^{-1} \end{pmatrix} \begin{pmatrix} x^2 a & c_1 & 0 & 0 \\ c_1 & x^{-2} b & 0 & 0 \\ 0 & 0 & x^{-2} a & c_2 \\ 0 & 0 & c_2 & x^2 b \end{pmatrix} \begin{pmatrix} y & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y^{-1} & 0 \\ 0 & 0 & 0 & y^{-1} \end{pmatrix} && (4.19)\\
&= \begin{pmatrix} y^2 x^2 a & y^2 c_1 & 0 & 0 \\ y^2 c_1 & y^2 x^{-2} b & 0 & 0 \\ 0 & 0 & y^{-2} x^{-2} a & y^{-2} c_2 \\ 0 & 0 & y^{-2} c_2 & y^{-2} x^2 b \end{pmatrix} && (4.20)
\end{aligned}$$

Both the above scalings S and S' are local which means if earlier the state was separable it still would be separable and if it was entangled, it would remain entangled. But, these local squeezings can take classical to non-classical state or non-classical state to a classical state.

Here, x is chosen such that $\frac{c_1}{x^2 a - x^{-2} b} = \frac{c_2}{x^{-2} a - x^2 b}$. That is $x = [\frac{(c_1 a + c_2 b)}{c_2 a + c_1 b}]^{1/4}$. This x is chosen so as to make the state classical. The choice of this x can be seen when we take the commutation of the two block matrices obtained in (4.17) and to make them commute put the off-diagonal terms to be zero, from where we get the condition over x. We will choose y later on after making this matrix diagonal. Now, since the two block matrices commute after the choice of x,so, we can do equal rotations in $q_1 - q_2$ plane and $p_1 - p_2$ plane. This equal rotation is a compact non-local canonical transformation. This compact non-local transformation is caused by the maximal compact unitary subgroup of noncompact $SP(4, R)$. This matrix S'' is of the form $S'' = diag(X, X)$ where X is a 2×2 matrix and is the real part of $U = X - iY \in U(n)$. This compact transformation preserves the classicality of a state which means if the state was classical it would be classical and if a state was non-classical it would remain non-classical after this transformation[3]. But, it does not take care of separability which is to say, that it can take any separable state to an entangled one. So, putting in straight words, the point-wise non-negativity of the P distribution is preserved.

So, after this compact canonical equal rotations in the $q_1 - q_2$ and $p_1 - p_2$ planes, we

[2]This matrix is used in the (q_1, q_2, p_1, p_2) format which we have rearranged. If we use the (q_1, p_1, q_2, p_2) format, then the matrix would look like $S'_{local} = diag(y, y^{-1}, y, y^{-1})$

[3]Classical or non-classical in the quantum optics sense means $P \geq 0$ or $P < 0$, respectively

finally get

$$V' \to V'' = diag(\kappa_+, \kappa_-, \kappa'_+, \kappa'_-) \tag{4.21}$$

$$\kappa_\pm = \tfrac{1}{2}y^2\{x^2 a + x^{-2}b \pm [(x^2 a - x^{-2}b)^2 + 4c_1^2]^{1/2}\}$$

$$\kappa'_\pm = \tfrac{1}{2}y^{-2}\{x^{-2}a + x^2 b \pm [(x^{-2}a - x^2 b)^2 + 4c_2^2]^{1/2}\}$$

For V'', the uncertainty principle reads as $V'' + \frac{i}{2}\Omega \geq 0$ since it is still a covariance matrix as all the transformations(scalings, squeezings or compact rotations) were canonical. Recalling that earlier in the standard form of V we chose $A = diag(a, a)$ and $B = diag(b, b)$ and $a \geq b$ and it turned out from the uncertainty principle that $A \geq 1/4$ and $B \geq 1/4$. So, after the canonical transformations, in the transformed V i.e., V'', the relations are preserved. Therefore, we have $(\kappa_+, \kappa_-) \geq (\kappa'_+, \kappa'_-)$ and $\kappa'_+\kappa'_- \geq 1/4$. Now, we can choose y such that $\kappa'_+, \kappa'_- = \frac{1}{2}$. So, $y = \frac{\frac{a}{x^2}+bx^2-\sqrt{(\frac{a}{x^2}-bx^2)^2 4c_2^2}}{ax^2+\frac{b}{x^2}-\sqrt{(ax^2+\frac{b}{x^2})^2+4c_1^2}}$ So, by such a choice of y, we can write

$$V'' \geq \frac{1}{2} \Rightarrow V' \geq 0 \tag{4.22}$$

This implication is due to the fact that V'' and V' are related by a canonical rotation and therefore same relations apply to both. Hence, V' corresponds to a non-negative P-distribution implying classicality which in turn implies a separable state. And since, the transformations that took V to V' only preserved separability, hence the state corresponding to V are also separable. This can be illustrated in an easy way as:

$$V \longleftrightarrow V' \longleftrightarrow V''$$

So, to sum it up, it can said that for the final state when we got, $V'' \geq \frac{1}{2}$, it implied that the the state corresponding to V'' was classical and since V'' and V' are related by transformations that preserve classicality, therefore, for the same condition was true for V'. But, nothing can be commented about the separability of the original states by just looking at V'' as the transformation from V' to V'' do not preserve separability. So, now, we can say that $V' \geq \frac{1}{2}$ implies that the state corresponding to it is classical. And since, classicality implies separability for Gaussian states and because it came from V by transformations that preserve separability, so, the original V must be separable.

Also, we notice that for det $C > 0$ satisfies the condition (4.13) and hence it satisfies (4.14) as well. Thus, we can conclude that for $detC > 0$, the bipartite Gaussian state

is separable. This completes the proof for $detC > 0$.
For $detC = 0$, the same procedure can be applied and it is very easily seen that for $detC = 0$ as well, the Gaussian bipartite state is separable. Thus, this proves the lemma that only knowing $detC \geq 0$ tells us that the original bipartite Gaussian state is separable.
Next, we consider the case $detC < 0$. First of all, the condition (4.14) has to be satisfied for the states to be separable as it is the necessary condition. If it is not, then the states are entangled. If it is satisfied then we have to see that in this condition we have $|detC| > 0$, therefore it becomes positive and by virtue of (4.13), the condition is satisfied. And so, the state is separable.
Thus, finally, it can be said that the condition (4.14) is a necessary and sufficient condition for separability of bipartite Gaussian states. Also, if before checking this condition, we notice that $detC \geq 0$, then we can instantly say that the state is separable.

Chapter 5

Nonlocality in Continuous Variable System

To know whether a state is nonlocal or not in the continuous variable space, various Bell-type inequalities have been derived. In 1998, Arvind et.al.[AN99] used CHSH inequalities to show that the inequalities work for Continuous Variable Systems in 4-modes. They wrote the Bell-type inequalities for Multiphoton states known as Multiphoton inequalities.

In 2002, Chen et.al.[CPHZ02] proposed the idea of using pseudo-spin operators and wrote a bell type inequality based on those operators. The main idea behind these pseudo-spin operators is to divide the whole space in such a way that we only get either of the two eigenvalues and thus, the continuous variable system resemble the two dimensional Hilbert Space.

Another idea regarding testing of nonlocality in continuous variable system was proposed in 2007(popularly known as CFRD inequalities) by E. G. Cavalcanti et.al.[CFRD07] where they used the positivity of the variance in writing the inequalities.

All these and such other proposals giving the Bell-type inequalities for Continuous Variable System have been shown to work for different states but still there is no general inequality which uses the properties of the continuous variable system to tell whether a given state is nonlocal or not. Thus, it is still an open question in this field to find a set of general inequalities that work well for Continuous Variable systems.

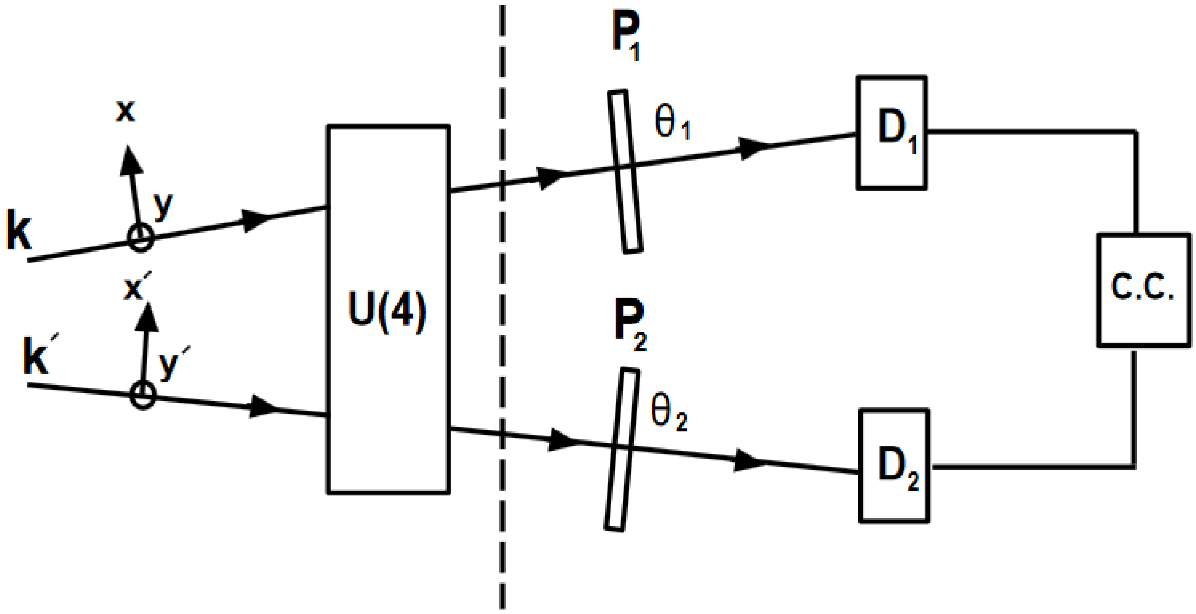

Figure 5.1: Setup to study Bell Inequality violation for states of 4-mode radiation field

5.1 Multi-photon inequalities

To study violation of Bell's inequalities, the setup used by Arvind et. al.[AN99] consisted of 4-modes of the field with propagation in two different directions, and arbitrary polarisations being allowed transverse to each direction. For photons in each propagation direction a particular polarisation is selected by a variable polariser, and finally coincidence counts are recorded using photo detector. The setup is as given in Fig.(5.1). Their basic idea in identifying the operators required for correlation function is based on the presence or absence of the photons. Using this idea, the following four Hermitian operators are defined and each of the operator having eigenvalues 0 and 1

$$\begin{aligned}\hat{A}_1 &= (I_{2\times 2} - |00\rangle\langle 00|)_k \\ \hat{A}_2 &= (I_{2\times 2} - |00\rangle\langle 00|)_{k'} \\ \hat{A}_1(\theta_1) &= (I_{\theta_1} - |0\rangle_{\theta_1\theta_1}\langle 0|)I_{\theta_1+\frac{\pi}{2}} \\ \hat{A}_2(\theta_2) &= (I_{\theta_2} - |0\rangle_{\theta_2\theta_2}\langle 0|)I_{\theta_2+\frac{\pi}{2}}. \end{aligned} \tag{5.1}$$

The subscripts θ_1 and θ_2 are the directions of the polarisers. The subscripts 1 and 2 represent the propagation direction k and k', respectively. Thus, the operators $\hat{A}_1$ and

$\hat{A}_1(\theta_1)$ are the operators belonging to the first two modes with propagation direction k and polarization along x or y (refer to Fig.(5.1)). Similarly, the operators $\hat{A}_2$ and $\hat{A}_2(\theta_2)$ are the operators belonging to the first two modes with propagation direction k' and polarization along x' or y'. Also, the expectation values of these operators are the probabilities of finding atleast one photon :
$\langle \hat{A}_1 \rangle$ = probability of detecting atleast one photon at D_1 with P_1 removed,
$\langle \hat{A}_2 \rangle$ = probability of detecting atleast one photon at D_2 with P_2 removed,
$\langle \hat{A}_1(\theta_1) \rangle$ = probability of detecting atleast one photon at D_1 with P_1 at θ_1,
$\langle \hat{A}_2(\theta_2) \rangle$ =probability of detecting atleast one photon at D_2 with P_2 at θ_2
The four types of coincidence count rates are:
(i)$P(\theta_1, \theta_2)$ = The first polariser at θ_1 and the second one at θ_2 with respect to their respective x axes.
(ii)$P(\theta_1, \;)$ = The first polariser at θ_1 and the second one removed.
(iii)$P(\;, \theta_2)$ = The first polariser removed and the second one at θ_2.
(iv)$P(\;,\;)$ = Both the polarisers removed from the setup.
And they are related to the operators in the following way:

$$\begin{aligned} P(\theta_1, \theta_2) &= \langle A_1(\theta_1) A_2(\theta_2) \rangle \\ P(\theta_1, \;) &= \langle A_1(\theta_1) A_2 \rangle \\ P(\;, \theta_2) &= \langle A_1 A_2(\theta_2) \rangle \\ P(\;,\;) &= \langle A_1 A_2 \rangle \end{aligned} \tag{5.2}$$

Now, **lemma** due to Clauser and Horne[CH74] states
if $0 \leq x, x' \leq X$ and $0 \leq y, y' \leq Y$ then,

$$-XY \leq xy - xy' + xy' + x'y' - Yx' - Xy \leq 0 \tag{5.3}$$

Using this lemma, the following inequality can be derived:

$$-P(\;,\;) \leq P(\theta_1, \theta_2) - P(\theta_1, \theta_2^{'}) + P(\theta_1^{'}, \theta_2) + P(\theta_1^{'}, \theta_2^{'}) - P(\theta_1^{'}, \;) - P(\;, \theta_2) \leq 0 \tag{5.4}$$

So, if a given quantum mechanical state does not obey this inequality then that state has non-trivial quantum properties that cannot be accommodated in realist hidden variable models based on locality.

5.2 Pseudo-Spin Inequalities

Proposed in 2002 by Chen et. al.[CPHZ02] the idea was to use such operators that divide the whole space into a set of two eigenvalues in a way that the properties of the operators are analogous with properties of the Pauli Spin Operators. Chen et.al. in their paper[CPHZ02] generalized Bell's inequalities to the CV cases for the biparty systems. They also showed that the EPR states

In two qubit case, the Bell operator reads:

$$B_{qubit} = (a.\sigma_1) \otimes (b.\sigma_2) + (a.\sigma_1) \otimes (b^{'}.\sigma_1) + (a^{'}.\sigma_1) \otimes (b.\sigma_1) - (a^{'}.\sigma_1) \otimes (b^{'}.\sigma_1) \quad (5.5)$$

where σ_j is the Pauli matrix for the $j^{th}(j = 1, 2)$ qubit; $a, a^{'}, b, b^{'}$ are four three-dimensional unit vectors.

$$a.\sigma_1 = a_x\sigma_{x_1} + a_y\sigma_{y_1} + a_z\sigma_{z_1} \quad (5.6)$$

$$b.\sigma_1 = b_x\sigma_{x_1} + b_y\sigma_{y_1} + b_z\sigma_{z_1} \quad (5.7)$$

Now it is easy to show that

$$B^2_{qubit} = 4\mathbb{I} + 4\big[(a \times a^{'})\sigma_1\big] \otimes \big[(b \times b^{'})\sigma_2\big] \quad (5.8)$$

and hence,

$$\langle B^2_{qubit}\rangle \leq 4 + 4 = 8 \quad (5.9)$$

which implies that $|\langle B_{qubit}\rangle|$ with respect to 2-qubit states is bounded by $2\sqrt{2}$ which is the famous Cirel'son bound [Cir80].

5.2.1 "Pseudospin" Operators for photons

Chen et.al. introduced the following analogous operators:

$$s_z = \sum_{n=0}^{\infty} \big[|2n+1\rangle\langle 2n+1| - |2n\rangle\langle 2n|\big] \quad (5.10)$$

$$s_- = \sum_{n=0}^{\infty} |2n\rangle\langle 2n+1| \quad (5.11)$$

$$s_+ = \sum_{n=0}^{\infty} |2n+1\rangle\langle 2n| \quad (5.12)$$

where $|n\rangle$ are the usual Fock states. And, s_- and s_+ are the parity flip operators.

It can easily be checked that

$$\begin{aligned} [s_z, s_\pm] &= \pm 2s_\pm, \qquad (5.13) \\ [s_+, s_-] &= s_z \qquad (5.14) \end{aligned}$$

From the commutation relations in Eq.(5.13, it is seen that these commutation relations are identical to those of the spin-1/2 system. Therefore, the pseudo-spin operator $\hat{s} = (s_x, s_y, s_z)$ can be regarded as a counterpart of the spin operator σ.

Now, choosing an arbitrary vector vector on the surface of a unit sphere

$$\mathbf{a} = (sin\theta_a cos\phi_a, sin\theta_a sin\phi_a, cos\theta_a)$$

θ_a being the polar angle and ϕ_a being the azimuthal angle of **a**. Also, defining

$$2s_\pm = s_x \pm \iota s_y$$

So, we have

$$\begin{aligned} a.\hat{s} &= s_x sin\theta_a cos\phi_a + s_y sin\theta_a sin\phi_a + s_z cos\theta_a \qquad (5.15) \\ &= s_z cos\theta_a + sin\theta_a(e^{\iota\phi_a} s_- + e^{-\iota\phi_a} s_+) \qquad (5.16) \end{aligned}$$

Thus, **a** may be interpreted as the direction along which the parity spin $\hat{s}$ is being measured. From the commutation relations of Eq.(5.13), we get the following

$$(a.\hat{s})^2 = I \qquad (5.17)$$

The Eq.(5.17) lead us to conclude that the outcome of the measurement of the Hermitian operator $a.\hat{s}$(with eigenvalues $+1$ or -1) is 1 or -1. This shows that exist complete analogy between Continuous-variable systems and spin-1/2 systems.

Thus, in the continuous-variable case, the Bell operator can be written as

$$B_{CHSH} = (a.\hat{s}_1) \otimes (b.\hat{s}_2) + (a.\hat{s}_1) \otimes (b^{'}.\hat{s}_1) + (a^{'}.\hat{s}_1) \otimes (b.\hat{s}_1) - (a^{'}.\hat{s}_1) \otimes (b^{'}.\hat{s}_1) \qquad (5.18)$$

Using this Bell operator, the correlation function is defined as $E(a,b) = \langle (a.\hat{s}_1) \otimes (b.\hat{s}_2) \rangle$. Hence,

$$\langle B_{CHSH} \rangle = \langle E(a,b) \rangle + \langle E(a,b') \rangle + \langle E(a',b) \rangle - \langle E(a',b') \rangle \tag{5.19}$$

Using Eq.(5.15), the above Eq.(5.19) can be written as

$$\langle B_{CHSH} \rangle = \langle E(\theta_a, \theta_b) \rangle + \langle E(\theta_a, \theta_{b'}) \rangle + \langle E(\theta_{a'}, \theta_b) \rangle - \langle E(\theta_{a'}, \theta_{b'}) \rangle \tag{5.20}$$

This is the final form of the inequality that will be used, the violation of which will tell us about the nonlocality of the state.

5.3 Comparison between Multiphoton inequalities and the Pseudo-Spin Inequalities

To compare and see which is the stronger of the two inequalities, we need to take a few states and apply the inequalities to both of them and analyze which inequality tells more about the nonlocality properties of the state. There is a very crucial point in the nonlocality of a quantum state, which is that a state does not have to violate all the inequalities. It is sufficient for a state to violate a particular inequality to be called nonlocal.

Now, to do the comparison, I took the NOPA states and 2-mode and 4-mode Gaussian states with and without noise. As described by Chen et.al.[CPHZ02], the NOPA(pulsed Nondegenerate Optical Parametric Amplifier) process The NOPA process represents a nonlinear interaction of two quantized modes (denoted by the corresponding annihilation operators a_1 and a_2) in a nonlinear medium with a strong classical pump field. In this process, the NOPA can generate the 'two-mode squeezed vacuum states', i.e., the NOPA states[RD88] [WM94]

$$|NOPA\rangle = e^{r(a_1^\dagger a_2^\dagger - a_1 a_2)}|00\rangle = \sum_{n=0}^{\infty} \frac{(tanhr)^n}{coshr}|nn\rangle \tag{5.21}$$

where $r > 0$ is the squeezing parameter.

5.3.1 Testing of Non-locality with Pseudo-Spin inequalities

1. **With NOPA states**

Using Eq.(5.20 and 5.21), Chen et. al.[CPHZ02] derived the correlation function

$$E(\theta_a, \theta_b) = \langle NOPA| s_{\theta_a}^{(1)} \otimes s_{\theta_b}^{(2)} |NOPA\rangle \quad (5.22)$$

$$= cos\theta_a cos\theta_b + K(r) sin\theta_a sin\theta_b \quad (5.23)$$

and

$$s_{\theta_a}^{(j)} = s_{jz} cos\theta_a + s_{jx} sin\theta_a \quad (5.24)$$

with $K(r) = tanh(2r) \leq 1$

Now, choosing $\theta_a = 0$, $\theta_{a'} = 0$ and $\theta_b = -\theta_{b'}$, we get

$$\langle B_{CHSH}\rangle = 2(cos\theta_b + K sin\theta_b) \quad (5.25)$$

Hence, the NOPA states always violate the Bell CHSH inequality(Eq.(5.20)) given $r \neq 0$.

2. **With 2-mode Gaussian states**

An n-mode general centered Gaussian has a Wigner distribution of the following form:

$$W(\xi) = \frac{1}{\pi^n}(DetG)^{1/2} exp(-\xi^T G \xi) \quad (5.26)$$

where $\xi^T = (q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n)$, $G = \frac{1}{2}V^{-1}$ and V is the covariance matrix and it satisfies

$$V + \imath\beta \geq 0 \quad (5.27)$$

where $\beta = \begin{pmatrix} 0_{n\times n} & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & 0_{n\times n} \end{pmatrix}$.

Rewriting the expansion of Eq.(5.19), we see,

$$\langle B_{CHSH}\rangle = \langle (a.\hat{s}_1)\otimes(b.\hat{s}_2)\rangle + \langle (a.\hat{s}_1)\otimes(b^{'}.\hat{s}_1)\rangle + \langle (a^{'}.\hat{s}_1)\otimes(b.\hat{s}_1)\rangle - \langle (a^{'}.\hat{s}_1)\otimes(b^{'}.\hat{s}_1)\rangle \quad (5.28)$$

Taking all the azimuthal angles to be zero, we can write Eq.(5.15) as

$$\langle a.\hat{s}_1 \otimes b.\hat{s}_2\rangle = \langle (s_z^{(1)} cos\theta_a + sin\theta_a(s_-^{(1)} + s_+^{(1)})) \otimes (s_z^{(2)} cos\theta_b + sin\theta_b(s_-^{(2)} + s_+^{(2)}))\rangle$$

$$= cos\theta_a cos\theta_b \langle s_z^{(1)} \otimes s_z^{(2)}\rangle + sin\theta_a sin\theta_b \langle s_x^{(1)} \otimes s_x^{(2)}\rangle \quad (5.29)$$

Since, we know that for any given two operators, $\hat{A}$ and $\hat{B}$(considering natural units so that $\hbar = 1$)

$$Tr(\hat{A}\hat{B}) = \frac{1}{2\pi} \int \int dqdp w_{\hat{A}} w_{\hat{B}} \tag{5.30}$$

where $w_{\hat{A}}$ and $w_{\hat{B}}$ are the Weyl transforms of the operators, $\hat{A}$ and $\hat{B}$, respectively. Therefore, for a given 2-mode Gaussian state,

$$\langle s_\mu^{(1)} \otimes s_\mu^{(2)} \rangle = tr(\hat{\rho}(s_\mu^{(1)} \otimes s_\mu^{(2)})) \tag{5.31}$$

$$= \int W_{\hat{\rho}}(w_{s_\mu^{(1)} \otimes s_\mu^{(2)}}) dq_1 dq_2 dp_1 dp_2 \tag{5.32}$$

where $\mu = z$ or x.

Thus, knowing the values of $\langle s_\mu^{(1)} \otimes s_\mu^{(2)} \rangle$, we know Eq.(5.29) and eventually we calculate the Bell operator, i.e., Eq.(5.19)

Now, G in Eq.(5.26) for two modes is defined as

$$G = U^{-1} S^T G_0 S U \tag{5.33}$$

$$G_0 = \kappa \mathbb{I}_{4\times4}, \qquad 0 \leq \kappa \leq 1 \tag{5.34}$$

$$\kappa = tanh\frac{\beta}{2}, \qquad \beta = \frac{\hbar\omega}{kT} \tag{5.35}$$

Here, $\kappa = 1$ implies zero temperature and $\kappa < 1$ implies certain temperature. S is a 2-mode squeezing symplectic transformation, which is a $Sp(4, R)$ matrix, and U is a passive symplectic $U(2)$ transformation whose role is to produce entanglement. Therefore, S and U can be chosen as

$$S = \begin{pmatrix} e^{-u} & 0 & 0 & 0 \\ 0 & e^{v} & 0 & 0 \\ 0 & 0 & e^{u} & 0 \\ 0 & 0 & 0 & e^{-v} \end{pmatrix}, U = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \tag{5.36}$$

Now, plotting B versus u for two different combinations of u and v,i.e., $u = v$ and $u = -v$, we can see in Fig.(5.2) that for $u = v$, we do not get any violation whereas for $u = -v$, we get a violation. Both the plots have been taken for $\theta_a = 1.13197$, $\theta_b = 0.929911$, $\theta_{a'} = 3.65681$ and $\theta_{b'} = 3.31752$. The covariance matrix for the case $u = -v$ resembles the covariance matrix for the two mode

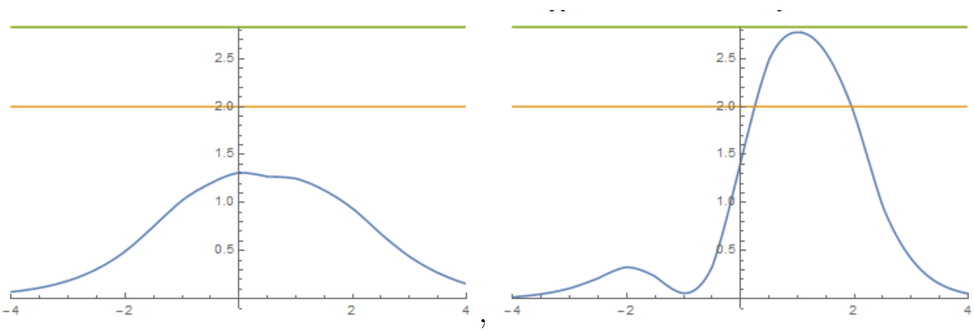

Figure 5.2: B vs u for u=v and u=-v for Pseudo Spin Inequalities

squeezed state.

5.3.2 Testing of Non-locality with Multiphoton inequalities

The given multiphoton inequality(Eq.(5.4)) require careful use of the four modes. The inequalities do not work if the state in consideration has just two modes in entangled state and two other modes are in separable states.

I analyzed the 4-mode Gaussian states with and without noise. (Please refer Appendix A for a short discussion about noise).

In the analysis of the inequality, noise is introduced in two forms. First, as a parameter which tells about the temperature. Second, by using the beam-splitter which is described below in subsection(1c).

1. **With 4-mode Gaussian states** By the definition given for an n-mode Gaussian state in the section above (Eq.(5.26)), the Wigner function of a 4-mode Gaussian state is given as

$$W(\xi) = \frac{1}{\pi^4}(DetG)^{1/2}exp(-\xi^T G\xi) \tag{5.37}$$

where $\xi^T = (q_1, q_2, q_3, q_4, p_1, p_2, p_3, p_4)$, $G = \frac{1}{2}V^{-1}$ and V is the covariance matrix. Now, defining G as

$$G = U^{-1}S^T G_0 SU \tag{5.38}$$

$$G_0 = \kappa\mathbb{I}_{8\times 8}, \qquad 0 \leq \kappa \leq 1 \tag{5.39}$$

$$\kappa = tanh\frac{\beta}{2}, \qquad \beta = \frac{\hbar\omega}{kT} \tag{5.40}$$

Here, $\kappa = 1$ implies zero temperature and $\kappa < 1$ implies certain temperature. S is a 4-mode squeezing symplectic transformation, which is a $Sp(8, R)$ matrix, and U is a passive symplectic $U(4)$ transformation whose role is to produce entanglement. As an example, S can be taken such that first and fourth mode are squeezed by equal and opposite amount, u and second and third mode are squeezed by equal and opposite amount, v. Hence,

$$S = \begin{pmatrix} e^{-u} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{v} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-v} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{u} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{u} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-v} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{v} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-u} \end{pmatrix}, U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix} \quad (5.41)$$

(a) **Without Noise**

Considering the case of no noise, we have to put $\kappa = 1$ in Eq.(5.38). Now, testing the inequality for different combinations of u and v, we notice that violations happen for some range of angles and not for all. Even a small violation(at whatever angle) is a witness for the presence of nonlocality. But it does not imply the other way round.

i. $\mathbf{u = v}$

Keeping the amount of squeezing same, G takes the form such that there is entanglement in 1-2 modes and 3-4 modes. Plotting the Bell operator as a function of squeezing yields Fig.(5.3). The two graphs in Fig.(5.3) are for two different angles and we see that there is no violation of the multiphoton inequality of Eq.(5.4).

ii. $\mathbf{u = 0}$ **or** $\mathbf{v = 0}$

Making one of parameters of squeezing vanish, the G takes a form such that there is entanglement in all the 4-modes. Hence, for some range of angles, we see that there is violation of the inequality as is shown in Fig.(5.4) where there is a small violation for a particular angle.

iii. **u=-v**

Making the amount of squeezing equal and opposite, G takes a form such that there is entanglement in 1-3 modes and 2-4 modes. This

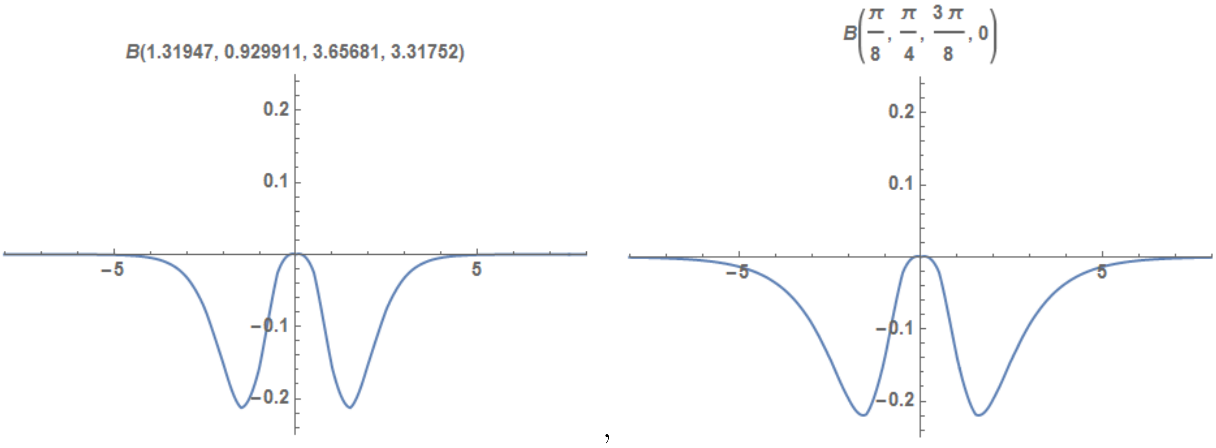

Figure 5.3: B vs u for u=v

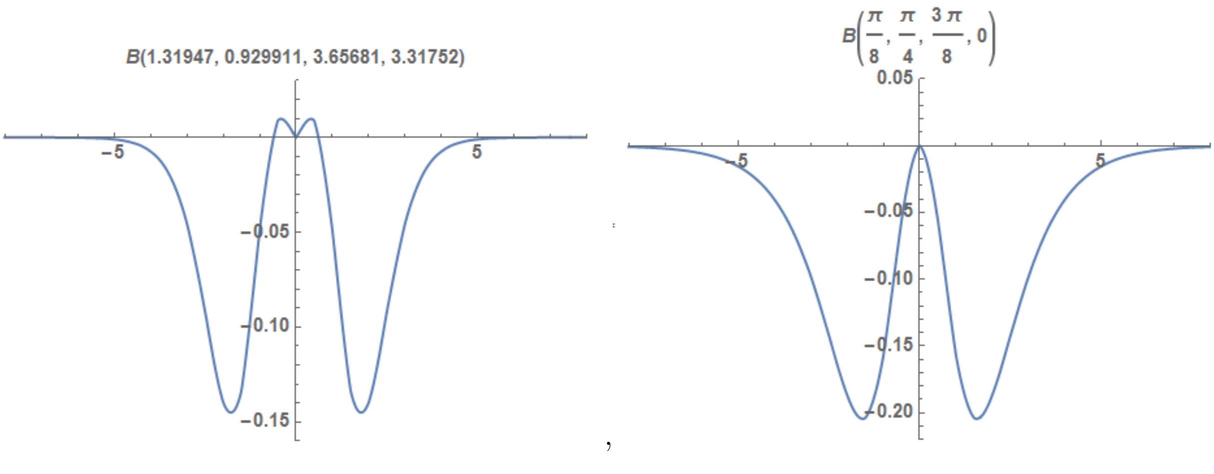

Figure 5.4: B vs u for v=0

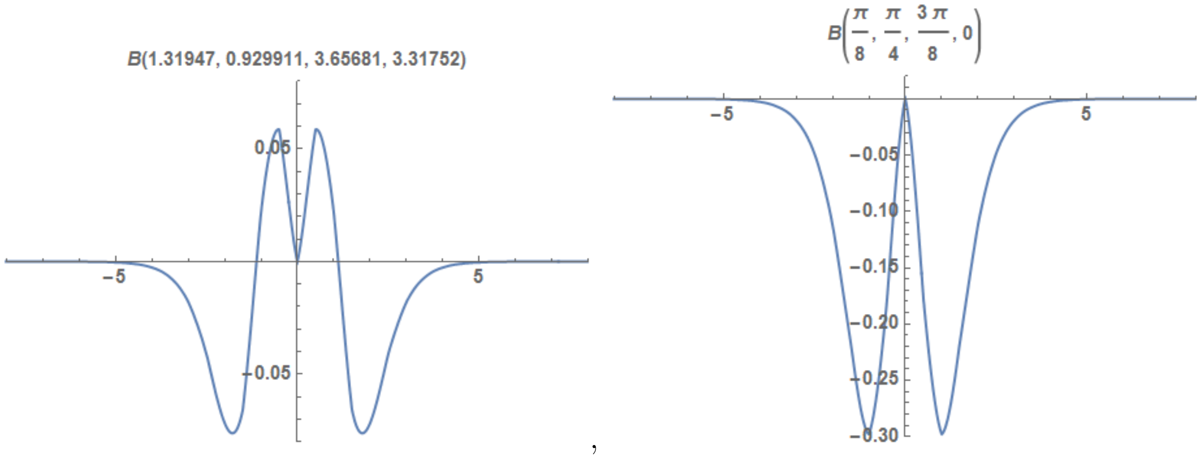

Figure 5.5: B vs u for u=- v

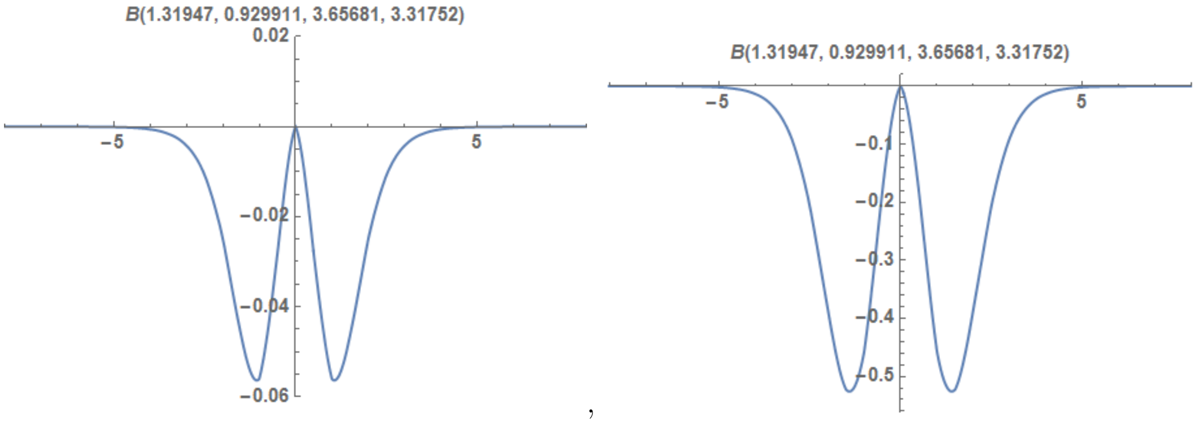

Figure 5.6: B vs u for vacuum in two modes

form of G is identical to keeping the two NOPA states, one in 1-3 mode and the other in 2-4 mode. Plotting, the Bell operator versus the squeezing parameter, we see in Fig.(5.5) that the amount of violation has increased.

iv. **A Special Case**:
Entangled states in two modes and separable states in the other two modes
If we put an entangled state in any of the two modes and separable state(be it a classical state like vacuum or coherent state or a non-classical state like a squeezed state), we do not see any violation of Eq.(5.4) for any set of angles.

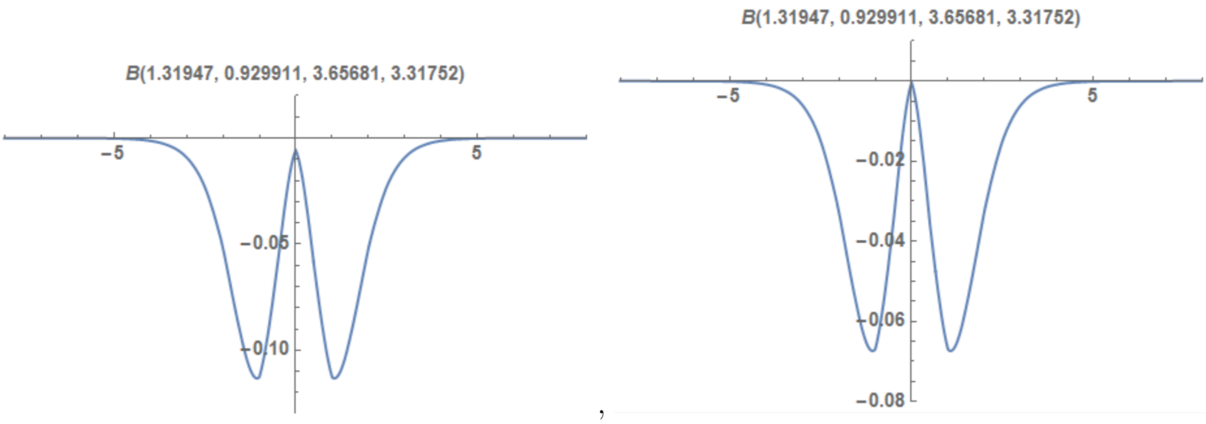

Figure 5.7: B vs u for squeezed vacuum in 2^{nd} and 4^{th} mode

In Fig.(5.6) and Fig.(5.7), I have put a two mode squeezed state in two of the four modes of the setup of the multiphoton inequality and in the remaining two modes, I have put vacuum and squeezed vacuum, respectively.
In the first graph of Fig.(5.6), I have put vacuum in 1^{st} and 3^{rd} mode and in the second graph, I have put vacuum in 2^{nd} and 4^{th} mode. In both these cases, we see that the inequality is not being violated.
Similarly, I tried putting squeezed vacuum as well but did not get any violation(See Fig.(5.7)). The first graph in Fig.(5.7) has same amount of squeezing in 2^{nd} and 4^{th} mode whereas in the second graph, I have put in different squeezing in the two modes. Hence, this proves the fact that in the setup for multiphoton inequality, if we have two modes with separable states and the remaining two have entangled states, the inequality will not be violated.

(b) **With Thermal Noise**

To introduce some finite temperature, κ should be less than 1. I have taken $\kappa = 0.8$ and then I got the following graphs(Fig.(5.8),Fig.(5.9),Fig.5.10)) of the multiphoton inequalities for various combinations of parameters, i.e., u=v, u=0 or v=0 and u=-v, respectively.

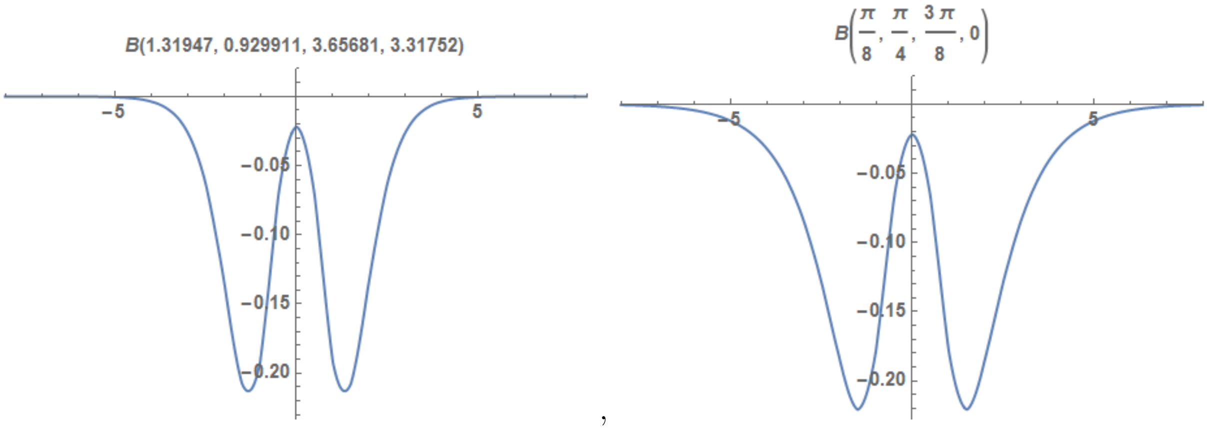

Figure 5.8: B vs u for u=v with $\kappa = 0.8$

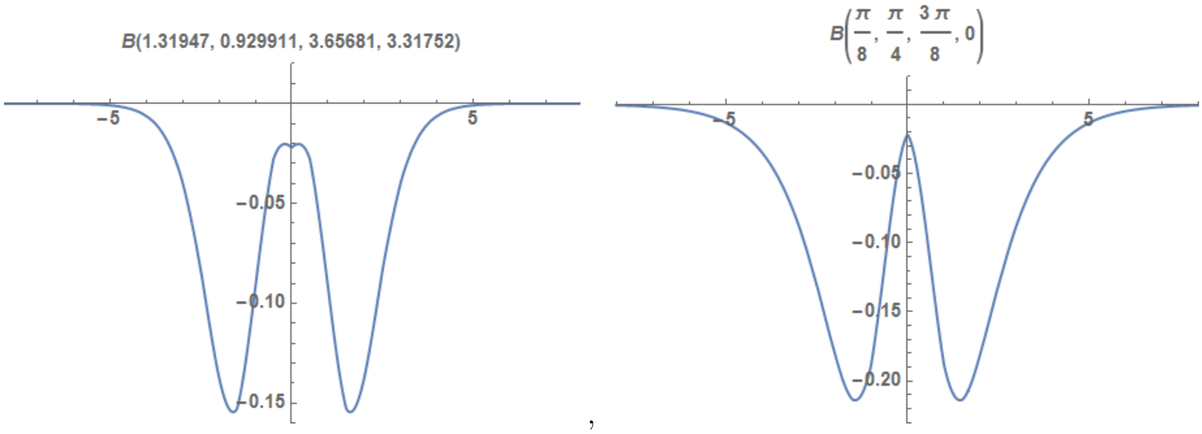

Figure 5.9: B vs u for v=0 with $\kappa = 0.8$

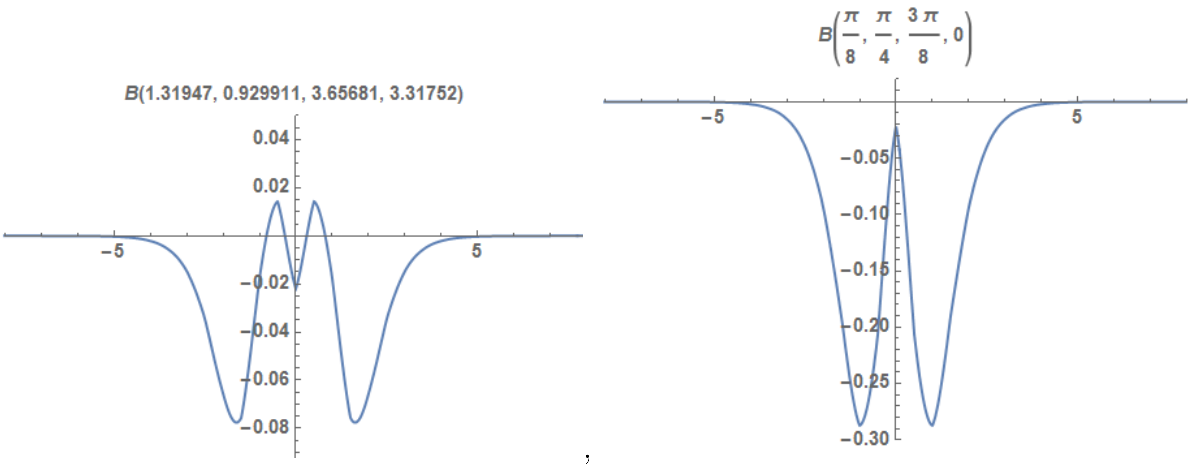

Figure 5.10: B vs u for u=- v with $\kappa = 0.8$

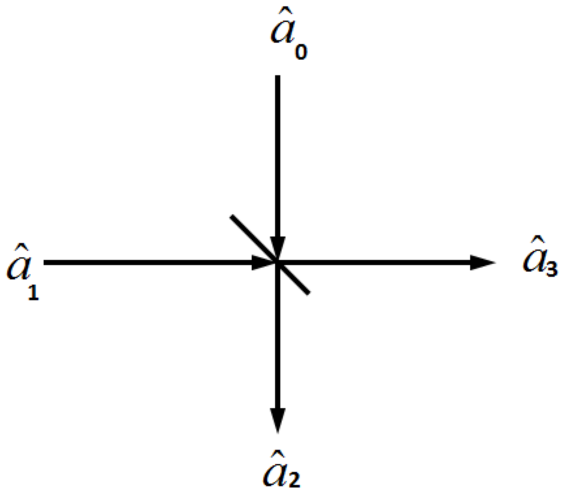

Figure 5.11: Action of a Beam Splitter

(c) **Noise introduced by using beam-splitter**

I introduce the noise by the use of a beam splitter with a certain amount of transmittance. The action of a beam splitter is shown in Fig.(5.11) Mathematically, a beam splitter's action, with a transmittivity of $\sqrt{T}$(where, $0 \leq T \leq 1$), can be written as

$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \begin{pmatrix} \sqrt{T} & \iota\sqrt{1-T} \\ \iota\sqrt{1-T} & \sqrt{T} \end{pmatrix} \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix} \tag{5.42}$$

where, $\hat{a}_0$ and $\hat{a}_1$ are the annihilation operators for the incoming modes and likewise, $\hat{a}_2$ and $\hat{a}_3$ are the annihilation operators for the outgoing modes.Modes associated to annihilation operators $\hat{a}_0$ and $\hat{a}_2$ are in a same direction and modes associated to annihilation operators $\hat{a}_1$ and $\hat{a}_3$ are in a same direction. Since, the annihilation operators can be written in terms of position and momentum operators, hence

$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \begin{pmatrix} \sqrt{T} & \iota\sqrt{1-T} \\ \iota\sqrt{1-T} & \sqrt{T} \end{pmatrix} \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix} \tag{5.43}$$

$$\begin{pmatrix} q_2 + \iota p_2 \\ q_3 + \iota p_3 \end{pmatrix} = \begin{pmatrix} \sqrt{T} & \iota\sqrt{1-T} \\ \iota\sqrt{1-T} & \sqrt{T} \end{pmatrix} \begin{pmatrix} q_0 + \iota p_0 \\ q_1 + \iota p_1 \end{pmatrix} \tag{5.44}$$

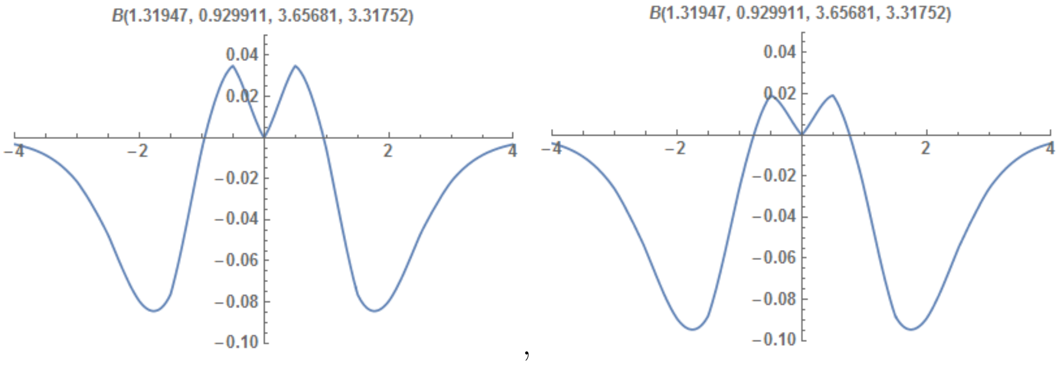

Figure 5.12: B vs u(for u=-v) with a beamsplitter with Transmitivity of 0.9 and 0.8

$$\begin{pmatrix} q_2 \\ q_3 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \sqrt{T} & 0 & 0 & -\sqrt{1-T} \\ 0 & \sqrt{T} & -\sqrt{1-T} & 0 \\ 0 & \sqrt{1-T} & \sqrt{T} & 0 \\ \sqrt{1-T} & 0 & 0 & \sqrt{T} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ p_0 \\ p_1 \end{pmatrix} \tag{5.45}$$

Now, the 4×4 matrix in Eq.(5.45) can be further extended to include two more modes and in this way, by the application of this matrix, I introduced noise in the system. By introducing the noise in the following way and then looking for the values of the Bell operator for the multiphotn inequalities versus the squeezing parameter, I got the graphs in Fig.(5.12). The first graph in Fig.(5.12) has transmittivity = 0.9 and the second graph has transmittivity = 0.8, respectively. In both the graphs, for different transmittance we see that the inequality is violated; though, the amount goes down as the transmittance is decreased but the character of the graphs remains same, ,i.e., there is no shift in the value of the Bell operator at the origin, unlike the case with thermal noise.

5.3.3 Remarks and Conclusion

We analyzed the effect on non-locality by taking in thermal noise and beam-splitter in the multiphoton inequalities.

When we introduced noise in the form of thermal noise, We notice from Fig.(5.10), Fig.(5.8) and Fig.(5.9) that when we increase the temperature, the value if the Bell operator itself goes below zero at the origin. This behaviour of the graph is such that as we increase the temperature, the violation will eventually vanish. Thus, we can infer that as the temperature increases, the nonlocality decreases for a general Gaussian state.

In the case when the noise is introduced by the beam-splitter model, we notice from Fig.(5.12), that even though the transmittance is decreasing, there is some region where the value of the Bell operator is above zero indicating a violation. Also, as mentioned above, the behaviour of the graph due to the beam splitter model noise has not changed, meaning that the value of the Bell operator is still zero at origin. Hence, only if the transmittance is completely blocked, then there will be no violation of the multiphoton inequality. The reason to this is that only the vacuum state will remain in the output modes which will not yield any violation.

By comparing the inequalities, we see that both the inequalities are violated for same set of squeezing settings. Hence, it cannot be concluded which one of the two is stronger. But since, the Multiphoton inequalities make use of 4-modes, hence, it can be said that these inequalities generalize the 2-mode Pseudo-Spin Inequalities to some extent.

Appendix A

A Note on Noise

Noise, in general, refers to some unwanted random fluctuation that hinders with the results of the ideal settings. In Quantum Mechanics, the noise(or quantum noise) is due to the uncertainty of a physical quantity. Thus, in quantum optics, the noise is because of dual nature of light as given by quantum theory of radiation.

The reason for introducing noise is that while doing real experiments, noise cannot be excluded/avoided. Hence, the analysis of the inequality to detect non-locality of a state is incomplete without taking into account the noise that can possibly creep in.

In my analysis of the multiphoton inequality, I have used thermal noise and the noise included due to the beam splitter model.

A.1 Thermal Noise

A simple way to tell how temperature might effect the quantum system is to take a an ideal closed cavity with heated walls[HK96]. These walls can emit and absorb radiation. The field which is in thermal equilibrium can be regarded as damped and generated by fluctuating currents in the walls. Thus, the ground state can be described as the state when the walls are at zero temperature. Also, Charles H. Henry and Rudolf F. Kazarinov have shown that the calculated emission from opaque walls at zero temperature mimics the flux of vacuum fluctuations from empty space[HK96].

A.2 Noise due to beam splitter

To understand how noise gets in the experiment due to the action of a beam splitter, let us first see what happens to a single photon when passed through a beam splitter.

So, to do this, let us first refer to Fig.(5.11) where we notice that there are two input modes and two output modes. Now, let us consider Eq.(5.43) and assume that the beam splitter is 50:50 beam splitter, i.e., its transmittance is 50%. Hence, we see that the electron after passing through the beam-splitter is in a superposition of reflected and transmitted state. Now, consider a continuous stream of photons thrown at regular intervals. Having kept the detectors in both the output modes, one can easily see that each detector registers a random sequence of photons. This simply implies that keeping a beam splitter in the way of continuous beam of photons introduces noise in the outgoing beams.

In regard to quantum optics, one would say that from the second input from where no beam is incident, there is vacuum state coming in. Now, this vacuum is equally divided by the beam-splitter and the noise occurring in the output modes can be explained due to the fact that the vacuum has zero-point fluctuations.

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