BIAS IN THE DISTRIBUTION OF PRIMES

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Certificate of Examination

This is to certify that the dissertation titled **Bias In The Distribution Of Primes** submitted by **Mr. Nishant Malik** (Reg. No. MS10083) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Kapil H. Paranjape at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Kapil H. Paranjape

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Abstract

Any model based on the randomness of primes would strongly suggest that every residual class $a(mod\,q)$ must contain roughly the same number of primes for $(a, q) = 1$. But despite the obviously seeming flow of logic, the above is inaccurate as a bias is obsereved in the distribution of primes when taken from different residual classes. A bias also exists in the disribution of prime pairs of form $(p, p + 2k)$ where $k \in \mathbb{N}$. This report is a humble attempt to discover these biases and provide conjuctural explaination of such phenomenons.

Introduction.

Any prime number divided by 4 gives us a remainder of 1 or 3. But do they have any preference?

Let $c_1\,(p)= \#$ { primes less than or equal to p with remainder 1 when divided by 4 } and $c_3= \#$ { primes less than or equal to p with remainder 3 when divided by 4 } upto, $p = 101$; $c_1\,(p) = 12$ & $c_3\,(p) = 13$, at $p = 1009$; $c_1\,(p) = 81$ & $c_3\,(p) = 87$, at $p = 10007$;$c_1\,(p) = 609$ & $c_3\,(p) = 620$.

In this race between $c_1\,(p)$ & $c_3\,(p)$, for the majority of times $c_3\,(p) > c_1\,(p)$ and $c_3\,(p)$ always seems to maintain a narrow lead. This phenomenon was first observed in a letter written by Chebyshev to M. Fuss in 1853. It is also commonly known as CHEBYSHEV's BIAS.

The bias is violated for the first time at $p = 26861$ where $c_1\,(p) > c_3\,(p)$, but the real zone of violation is 11 primes from $p = 616877$ to 617011. $c_1\,(p)$ holds the lead at only 1939 of the first 5.8 million primes & don't hold it once in last 4988472 of them.

Also, if you divide primes by 3 and count that give remainder 1 or 2, the bias goes towards 2 and this bias doesn't get violated until $p = 608981813029$. It was found by Bays and Hunson in 1978.

So our attempts here are to understand these biases and further estimate them using tools from Analytic Number Theory.

Primes in Arithmatic Progression.

Theorem 1 :- There are infinitely many primes of form $4n+1$.

Proof : Let N be any integer > 1. We will show that there is a prime $p > N$ s.t. $p \equiv 1 (mod\, 4)$.

Let $m = (N!)^2 + 1$, Note that m is odd, $m > 1$.

Let p be the smallest prime factor of m. None of the numbers $2, 3, \ldots.., N$ divides m, so $p > N$.

Also we have

$$(N!)^2 \equiv -1 (mod\, p),$$

raising both members by power of $\frac{p-1}{2}$,

$$(N!)^{p-1} \equiv (-1)^{\frac{p-1}{2}} (mod\, p)$$

But $(N!)^{p-1} \equiv 1 (mod\, p)$ by Fermat's Theorem,

So,

$$(-1)^{\frac{p-1}{2}} \equiv 1 (mod\, p)$$

Now the difference $(-1)^{\frac{p-1}{2}} - 1$ is either 0 or -2, and it can't be -2 because it's divisible by p, so it must be 0.

That means, $\frac{p-1}{2}$ is even

$\Rightarrow p \equiv 1 (mod\, 4)$.

In other words, we have shown that for each integer $N > 1$ there is a prime $p > N$ s.t. $p \equiv 1 (mod\, 4)$.

Therefore, there are infinitely many primes of form $4n+1$.

Theorem 2 :- (Dirichlet's theorem on primes in arithmatic progression) ([1]: Theorem 7.9, page 154) If a, q are relatively prime positive integers, then the arithmatic progression "a, $q+a$, $2q+a, \ldots..$" contains infinitely many primes.

Theorem 3 :- (Prime Number Theorem for arithmetic progression) ([1]: Section 7.9, page 154) Let $\pi(x)$ be the number of primes less than or equal to x & $\pi(x; q, a)$ be the number of primes not exceeding x and congruent to $a (mod\, q)$.

If $q > 0$ & $(a, q) = 1$, then

$$\pi(x; q, a) \sim \frac{\pi(x)}{\varphi(q)} \sim \frac{x}{\ln(x)\varphi(q)}$$

as $x \to \infty$.

Theorem 4 :- ([1]: Theorem 7.10, page 155) If the relation

$$\pi(x;q,a) \sim \frac{\pi(x)}{\varphi(q)}$$

as $x \to \infty$ holds for every integer a relatively prime to q then,
$\pi(x;q,a) \sim \pi(x;q,b)$ as $x \to \infty$ whenever $(a,q) = (b,q) = 1$.
Converse also holds.
Proof: $\pi(x;q,a) \sim \frac{\pi(x)}{\varphi(q)} \sim \pi(x;q,b)$ is obvious.
To prove the converse we assume $\pi(x;q,a) \sim \pi(x;q,b)$ as $x \to \infty$ to be true, whenever $(a,q) = (b,q) = 1$.
and let $A(q)$ denote the number of primes that divide q.
If $x > q$ we have

$$\pi(x) = \sum_{p \le x} 1 = A(q) + \sum_{p \le x,\, p \nmid q} 1$$

$$\Rightarrow A(q) + \sum_{a=1,\,(a,q)=1}^{q} \; \sum_{p \le x,\, p \equiv a (mod\, q)} 1$$

$$\Rightarrow A(q) + \sum_{a=1,\,(a,q)=1}^{q} \pi(x;q,a)$$

Therefore,

$$\frac{\pi(x) - A(q)}{\pi(x;q,b)} = \sum_{a=1,\,(a,q)=1}^{q} \frac{\pi(x;q,a)}{\pi(x;q,b)}$$

But $\frac{\pi(x;q,a)}{\pi(x;q,b)} \to 1$, as $x \to \infty$, so the sum tends to $\varphi(q)$.
Hence,

$$\frac{\pi(x)}{\pi(x;q,b)} - \frac{A(q)}{\pi(x;q,b)} \to \varphi(q)$$

as $x \to \infty$.
But $\frac{A(q)}{\pi(x;q,b)} \to 0$. So,

$$\frac{\pi(x)}{\pi(x;q,b)} \to \varphi(q)$$

$$\Rightarrow \pi(x;q,b) \sim \frac{\pi(x)}{\varphi(q)}$$

for every integer b relatively prime to q.

Since the above result is independent of a, one might expect to find the same number of primes in each residue class $a(mod\, q)$, if $(a, q) = 1$.

But the Chebyshev Bias is the observation that contrary to expectations, $\pi(x; q, N) > \pi(x; q, R)$ majority of time, when N is a non-quadratic modulo q and R is from quadratic modulo class. i.e.

$$\delta(x, q) = \pi(x; q, N) - \pi(x; q, R)$$

Where, $\delta(x, q) > 0$ for the majority of time.

Chebyshev's Conjecture.

In a letter to M. Fuss in 1853, Chebyshev conjectured that

$$\lim_{x\to\infty} \sum (-1)^{\frac{(p-1)}{2}} \exp \frac{-p}{x} = -\infty$$

and gave the bias a mathematical form.

Later it was shown by Hardy, Littlewood & Landau that the above conjecture holds if and only if the function,

$$L\left(s, \chi'\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$

does not vanish for $Re(s) > \frac{1}{2}$.

"$Li(x) - \pi\left(x\right)$"

From the Prime Number Theorem, we have

$$Li(x) \sim \pi(x) \sim \frac{x}{ln(x)}$$

where $Li(x) = \int_2^x \frac{dt}{ln(t)}$.

And similarly to Chebyshev Bias, there is also a race between $Li(x)$ and $\pi(x)$ where $Li(x)$ leads the race for majority of time.

In essence, Chebyshev Bias $\delta\left(x, 4\right) = \pi(x; 4, 3) - \pi(x; 4, 1)$ is similar to $\delta(x) = Li(x) - \pi(x)$ as $\delta(x)$ is for $L(s, 1)$ (Dirichlet L-function for $\zeta(s)$) whereas, $\delta\left(x, 4\right)$ is corresponding to $L(s, \chi_4)$ i.e the Dirichlet L-function with modulo 4 character.

Theorem 5 :- (Littlewood) ([4]) The difference $\pi(x) - Li\,(x)$ changes sign infinitely often.

Littlewood proved the above in 1914. His proof is very innovative and worth mentioning because he used a technique that later became a very useful tool in Analytic Number Theory.

First he assumed that the Riemann Hypothesis is true and showed that $\pi(x) - Li\,(x)$ changes sign infinitely often. Then he showed that the same is true if Riemann Hypothesis is assumed to be false.

$Li(x)$ was Gauss' original approximation to $\pi(x)$. Riemann attempted to improve upon this in the following manner.

The probability of choosing a prime randomly less than x, would be $\frac{1}{ln(x)}$ if one counted not only primes but also the weighted powers of the primes. That will give

$$Li(x) \approxeq \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) \cdots$$

upon inverting it

$$\pi(x) \approxeq Li(x) - \frac{1}{2}Li(x^{\frac{1}{2}}) - \frac{1}{3}Li(x^{\frac{1}{3}}) \cdots$$

Theorem 6a :- (Littlewood's oscillation theorem) ([4])

$$\psi(x) - x = \Omega_{\pm}\left(\sqrt{x}\, \ln\ln\ln(x)\right)$$

where

$$\psi(x) = \sum_{p^k \le x} \ln(p).$$

Theorem 6b :- ([4])

$$\pi(x) - Li(x) = \Omega_{\pm}\left(\frac{\sqrt{x}}{\ln(x)}\, \ln\ln\ln(x)\right)$$

i.e. for +ve constants c_1& c_2

$$\pi(x) - Li(x) > c_1 \frac{\sqrt{x}}{\ln(x)}\, \ln\ln\ln(x)$$

&

$$\pi(x) - Li(x) < -c_2 \frac{\sqrt{x}}{\ln(x)}\, \ln\ln\ln(x)$$

Proof:
From $Li(x) \approxeq \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) \cdots$
we can develop an explicit formula for $\pi(x)$ i.e.

$$F(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) \cdots$$

Therefore after connecting $F(x)$ with $\psi(x)$ by a partial summation.

$$F(x) = \sum_{2}^{x} \frac{\psi(n) - \psi(n-1)}{\ln(n)}$$

$$\Rightarrow \sum_{2}^{x} \frac{1}{\ln(n)} + \sum_{2}^{x} \frac{(\psi(n) - n) - (\psi(n-1) - (n-1))}{\ln(n)}$$

$$F(x) = Li(x) + O(1) + \sum_{2}^{x} \frac{(\psi(n) - n) - (\psi(n-1) - (n-1))}{\ln(n)}$$

$$\Rightarrow Li(x) + O(1) + \sum_{2}^{x-1} \{\psi(n) - n\} \left(\frac{1}{\ln(n)} - \frac{1}{\ln(n+1)}\right) + \frac{\psi(x) - [x]}{\ln([x]+1)}$$

$$F(x) - Li(x) - \frac{\psi(x) - x}{\ln(x)} = \sum_{2}^{x} \frac{\psi(n) - n}{n(\ln(n))^2} + O\left(\sum_{2}^{x} \frac{|\psi(n) - n|}{n^2(\ln(n))^2}\right) + O(1)$$

$$\Rightarrow F(x) - Li(x) - \frac{\psi(x) - x}{\ln(x)} = \sum_{2}^{x} \frac{\psi(n) - n}{n(\ln(n))^2} + O(1)$$

Let $\chi(x) = \sum_{2}^{x} \{\psi(n) - n\}$,
Then, $\chi(n) = O\left(n^{\frac{3}{2}}\right)$,
Hence,

$$\sum_{2}^{x} \frac{\psi(n) - n}{n(\ln(n))^2} = \sum_{2}^{x} \frac{\chi(n) - \chi(n-1)}{n(\ln(n))^2}$$

$$= \sum_{2}^{x} \chi(n) \left[\frac{1}{n(\ln(n))^2} - \frac{1}{(n+1)(\ln(n+1))^2}\right] + \frac{\chi[x]}{([x]+1)(\ln([x]+1))^2}$$

$$= O\left(\frac{1}{\sqrt{n}(\ln(n))^2}\right) + O\left(\frac{\sqrt{x}}{(\ln(x))^2}\right)$$

$$= O\left(\frac{\sqrt{x}}{(\ln(x))^2}\right).$$

Combining the two equations we get,

$$F(x) - Li(x) - \frac{\psi(x) - x}{\ln(x)} = O\left(\frac{\sqrt{x}}{(\ln(x))^2}\right)$$

But from theorem 6a we have,

$$\psi(x) - x = \Omega_{\pm}\left(\sqrt{x}\,\ln\ln\ln(x)\right)$$

which upon substituting into the above equation gives us the desired result.

The Logarithmic Density.

The modern way to study this problem of Bias is to look at the set of integers for which the bias exists and define the 'logarithmic density' on that set.

Let us define the logarithmic density of a set $P \subset \mathbb{N}$ by

$$\delta(P) := \lim_{x\to\infty} \frac{1}{\ln(x)} \sum_{t\in P, t\leq x} \frac{1}{t}$$

if the limit exists.

In general, we define,

$$\bar{\delta}(P) := \limsup_{x\to\infty} \frac{1}{\ln(x)} \sum_{t\in P, t\leq x} \frac{1}{t}$$

$$\underline{\delta}(P) := \liminf_{x\to\infty} \frac{1}{\ln(x)} \sum_{t\in P, t\leq x} \frac{1}{t}$$

& set $\delta(P) = \bar{\delta}(P) = \underline{\delta}(P)$ if the latter two limits exist & are equal.

In 1994, Rubinstein and Sarnak developed a framework to calculate the above and showed that

For any r-tuple $(a_1, a_2, \ldots\ldots a_r)$ of admissible residue classes mod q (where, $(a_i, q) = 1$), the logarithmic density of the set

$$P_{q;a_1,a_2\ldots.a_r} := \{\, x : \, \pi(x;q,a_1) > \pi(x;q,a_2) > \pi(x;q,a_3) \cdots > \pi(x;q,a_r) \,\}$$

which we denote by $\delta(q; a_1, a_2 \ldots. a_r)$ exists, and is not equal to 0 or 1.

Assuming the following hypothesis –

1. *Generalized Riemann Hypothesis*: For every primitive character $\chi(mod\, q)$, all non-trivial zeros of $L(s,\chi)$ lie on the line $Re(s) = \frac{1}{2}$.

2. *Grand Simplicity Hypothesis*: For every fixed modulus q, the set

$$\bigcup_{\chi(mod\, q)} \{\, Im(\rho_\chi) \,:\, L(\rho_\chi, \chi) = 0,\ 0 < Re(\rho_\chi) < 1,\ Im(\rho_\chi) \geq 0\}$$

is linearly independent over $\mathbb{Q}$.

Introduce the vector valued functions

$$E_{q;a_1,a_2\ldots.a_r}(x) = \frac{\ln(x)}{\sqrt{x}} \left(\varphi(q)\,\pi(x;q,a_1) - \pi(x), \cdots\cdots, \varphi(q)\,\pi(x;q,a_r) - \pi(x)\right)$$

Theorem 7 :- (Rubinstein and Sarnak) ([7]: Theorem 1.1, page 174) Assume GRH. Then $E_{q;a_1,a_2....a_r}(x)$ has a limiting distribution $\mu_{q;a_1,a_2....a_r}(x)$ on $\mathbb{R}^r$, i.e.

$$\lim_{X\to\infty}\frac{1}{\ln(X)}\int_2^X f(E_{q;a_1,a_2....a_r}(x))\frac{dx}{x} = \int_{\mathbb{R}^r} f(x)\,d\mu_{q;a_1,a_2....a_r}(x)$$

for all bounded functions f on $\mathbb{R}^r$.

The measures μ are very localized but not compactly supported.

Set

$$B'_R = \left\{\, x\in\mathbb{R}^r \,|\, \left|x\right| \geq R \,\right\},$$

$$B^+_R = \{\, x\in B'_R \,|\, \varepsilon(a_j)x_j > 0\,\},$$

$$B^-_R = -B^+_R,$$

where $\varepsilon(a) = 1$ if $a\equiv 1(mod\, q)$ and $\varepsilon(a) = -1$ otherwise.

Theorem 8 :- (Rubinstein and Sarnak) ([7]: Theorem 1.2, page 175) Assume GRH. There are positive constants c_1, c_2, c_3 and c_4 depending only on q, such that

$$\mu_{q;a_1,a_2....a_r}(B'_R) \leq c_1\exp(-c_2\sqrt{R}),$$

$$\mu_{q;a_1,a_2....a_r}(B^{\pm}_R) \geq c_3\exp(-\exp(c_4 R)).$$

To better understand the above two theorems we have the following examples.

The case $q=1$, concerning the density of

$$P_1 = \{\, x\geq 2 \,|\, \pi(x) > Li(x)\,\}$$

Denote by μ_1 the limiting distribution of

$$E_1 = (\pi(x)\; - \; Li(x))\,\frac{\ln(x)}{\sqrt{x}}.$$

Then assuming Riemann Hypothesis, we have for $\lambda \gg 1$,

$$c_7\exp(-\exp(c_8\lambda)) \leq \; \mu_1\big[\lambda,\infty\big) \;\leq c_5\exp(-c_6\sqrt{\lambda}),$$

$$c_7\exp(-\exp(c_8\lambda)) \leq \; \mu_1(-\infty,\lambda] \;\leq c_5\exp(-c_6\sqrt{\lambda})$$

for absolute positive constants c_5, c_6, c_7 and c_8.

Similarly for the case concerning the excess of primes from quadratic modulo class than primes from the non-quadratic modulo class.

$$P_{q;N,R} = \{\, x \geq 2 \,|\, \pi(x;q,N) - \pi(x;q,R),$$

$$P_{q;R,N} = \{\, x \geq 2 \,|\, \pi(x;q,R) - \pi(x;q,N)$$

One can give lower bounds for the tails of the limiting distribution $\mu_{q;N,R}$ of

$$E_{q;N,R} = (\pi(x;q,N) \, - \, \pi(x;q,R)) \, \frac{\ln(x)}{\sqrt{x}}$$

Consequently, we have

$$\underline{\delta}(P_{q;N,R}) \, \underline{\delta}(P_{q;R,N}) > 0.$$

Assuming GRH & GSH, Rubinstein and Sarnak found an explicit formula for the Fourier transform of $\mu_{q;a_1,a_2....a_r}$.

i.e.

$$\hat{\mu}_{q;a_1,a_2....a_r}(\xi_1,\xi_2,.....,\xi_r) = \exp\left(i \sum_{j=1}^{r} c(q,a_j)\xi_j \right) \times \prod_{\substack{\chi \neq \chi_0 \\ \chi (mod\, q)}} \prod_{\gamma_\chi > 0} J_0 \left(\frac{2 \,|\, \sum_{j=1}^{r} \chi(a_j)\xi_j |}{\sqrt{\frac{1}{4} + \gamma_\chi^2}} \right),$$

where χ_0 is the principal character,

$$c(q,a) = -1 + \sum_{\substack{b^2 \equiv a(q) \\ 0 \leq b \leq q-1}} 1,$$

and $J_0(z)$ is the Bessel function

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}z)^{2m}}{(m!)^2}.$$

The formula implies that, for $r < \varphi(q)$, $\mu_{q;a_1,a_2....a_r} = f(x)\,dx$ with a rapidly decreasing entire function f.

As a consequence, under GRH & GSH, each $\delta(P_{q;a_1,a_2....a_r})$ does exist and is non-zero.

Theorem 9 :- (Rubinstein and Sarnak) ([7]: Theorem 1.5, page 177) Assume GRH & GSH. Then, for fixed r,

$$\max_{1 \leq a_1,\ldots,a_r \leq q,\, (a_i,q)=1} \left| \delta(q;a_1,a_2....,a_r) - \frac{1}{r!} \right| \to 0$$

as $q \to \infty$.

Which means that the bias dissolves as $q \to \infty$. i.e. $\delta(P_{q;N,R}) \to \frac{1}{2}$ as $q \to \infty$.

The race $(q; a_1, a_2...., a_r)$ is said to be unbiased, if the density function of $\mu_{q;a_1,a_2....a_r}$ is invariant under permutations of $(x_1, x_2....x_r)$.

In this case we have, $\delta(q; a_1, a_2...., a_r) = \frac{1}{r!}$.

Theorem 10 :- (Rubinstein and Sarnak) ([7]: Theorem 1.4, page 177) Under GRH & GSH, $(q; a_1, a_2...., a_r)$ is unbiased if and only if either $r = 2$ and $c(q, a_1) = c(q, a_2)$, where $c(q, a) = -1 + \sum_{\substack{b^2 \equiv a(q) \\ 0 \leq b \leq q-1}} 1$, or $r = 3$ and there exists $\rho \neq 1$ such that $\rho^3 \equiv 1(mod\, q)$, $a_2 \equiv a_1\rho(mod\, q)$ and $a_3 \equiv a_1\rho^2(mod\, q)$.

Theorem 11 :- (Rubinstein and Sarnak) ([7]: Theorem 1.6, page 177) Assume GRH & GSH. Let $\tilde{\mu}_{q;N,R}$ be the limiting distribution of

$$\frac{E_{q;N,R}(x)}{\sqrt{\ln(q)}}.$$

Then $\tilde{\mu}_{q;N,R}$ converges in measure to the Gaussian $(2\pi)^{-\frac{1}{2}} \exp(-\frac{x^2}{2})dx$ as $q \to \infty$.

Numerical investigations of the above framework:

([7]: Section 4, page 188)

$\delta(P_1^{comp})$	$0.99999973\cdots$
$\delta(P_{3;N,R})$	$0.9990\cdots$
$\delta(P_{4;N,R})$	$0.9959\cdots$
$\delta(P_{5;N,R})$	$0.9954\cdots$
$\delta(P_{7;N,R})$	$0.9782\cdots$
$\delta(P_{11;N,R})$	$0.9167\cdots$
$\delta(P_{13;N,R})$	$0.9443\cdots$

Regularized Chebyshev's Bias.

In 1984, Guy Robin reformulated the the uncondition bias of $\delta(x) = Li(x) - \pi(x)$ as a conditional one involving the second chebyshev's function

$$\psi(x) = \sum_{p^k \leq x} \ln(p),$$

The inequality $\delta'(x) := Li[\psi(x)] - \pi(x) > 0$ is equivalent to Riemann Hypothesis.

In attempt to regularize the $\delta(x)$, Robin introduced the function

$$B(x; q, a) = Li[\varphi(q)\psi(x; q, a)] - \varphi(q)\pi(x; q, a)$$

Proposition :- (G. Robin) ([9]) Let $B(x; q, a)$ be the Robin B-function and R & N be quadratic & non-quadratic residue modulo q respectively. Then, the statement

$$\delta'(x, q) := B(x; q, N) - B(x; q, R) > 0$$

is equivalent to Genralized Riemann Hypothesis for the modulus q.

For a prime modulud q, we define the bias so as to obtain the average over all differences $\pi(x; q, N) - \pi(x; q, R)$ as

$$\delta(x, q) = -\sum \left(\frac{a}{q}\right) \pi(x; q, a),$$

where $\left(\frac{a}{q}\right)$ is the Legendre symbol.

Correspondingly, we define the regularized bias as

$$\delta'(x, q) = \frac{1}{\lfloor q/2 \rfloor} \sum \left(\frac{a}{q}\right) B(x; q, a).$$

Bias in the Distribution of Consecutive Primes.

Polignac's Conjecture:

For every positive natural number k, there are infinetely many consecutive prime pairs p & p' such that $p - p' = 2k$.

The case $k = 1$ is the Twin Prime conjecture.

Hardy Littlewood k-tuple Conjecture:

Let $0 < m_1 < m_2 < \cdots < m_k$, then the k-tuple conjecture predicts that the number of primes $p \leq x$ such that $p + 2m_1, p + 2m_2, \cdots\cdots, p + 2m_k$ are all prime is

$$\pi_{m_1,m_2,\cdots,m_k}(x) \sim C(m_1, m_2, \cdots\cdots m_k) \int_2^x \frac{dt}{\ln^{(k+1)} t},$$

Where,

$$C(m_1, m_2, \cdots\cdots m_k) = 2^k \prod_q \frac{1 - \frac{w(q;m_1,m_2,\cdots\cdots m_k)}{q}}{(1 - \frac{1}{q})^{k+1}},$$

the product is over odd primes q, and $w(q; m_1, m_2, \cdots\cdots m_k)$ denotes the number of distinct residues of $0, m_1, m_2, \cdots\cdots m_k (mod\, q)$.

If $k = 1$, then this becomes

$$C(m) = 2 \prod_q \frac{q(q-2)}{(q-1)^2} \prod_{q|m} \frac{q-1}{q-2}.$$

Also the value of the product $\prod_q \frac{q(q-2)}{(q-1)^2} \approx 0.6601618158\ldots$

On further calculating the product appearing in $C(m)$.

$2m$	2	4	6	8	10	12	14	16	18	20
$\prod_{q\|m} \frac{q-1}{q-2}$	1	1	2	1	4/3	2	6/5	1	2	4/3

According to this conjecture the density of twin prime pairs is equivalent to the density of prime pairs with gap 4 or 8 and so on.

Also the density of primes pairs with difference 6 turns out to be twice the density of twin primes, which means that the prime pairs $(p, p + 6)$ occur twice as much as the twin primes when counted for a large number x. (see the numeric table below for evidence)

([17]: Table 8, page 31)

x	$\pi_2(x)$	$\pi_4(x)$	$\pi_6(x)$	$\pi_8(x)$	$\pi_{10}(x)$
10^3	35	41	74	38	51
10^4	205	203	411	208	270
10^5	1,224	1,216	2,447	1,260	1,624
10^6	8,169	8,144	16,386	8,242	10,934
10^7	58,980	58,622	117,207	58,595	78,211
10^8	440,312	440,258	879,908	439,908	586,811
10^9	3,424,506	3,424,680	6,849,047	3,426,124	4,567,691
10^{10}	27,412,679	27,409,999	54,818,296	27,411,508	36,548,839
10^{11}	224,376,048	224,373,161	448,725,003	224,365,334	299,140,330
10^{12}	1,870,585,220	1,872,585,459	3,741,217,498	1,870,580,394	2,494,056,601

References

[1] T. M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, 1976.

[2] H. Davenport, Multiplicative number theory, Second edition, Springer Verlag, New York (1980).

[3] P. Chebyshev, Lettre de M. le professeur Tche´bychev a M. Fuss, sur un nouveau the´ore`me re´latif aux nombres premiers contenus dans la formes 4n+1 et 4n+3. (French), Bull. de la Classe phys. math. de l'Acad. Imp. des Sciences St. Petersburg 11 (1853), 208.

[4] J. E. Littlewood, Sur la distribution des nombres premiers (French), Comptes Rendus (22 June 1914).

[5] G. H. Hardy and J. E. Littlewood, Some problems of Partitio Numerorum III: On the expression of a number as a sum of primes, Acta Math. 44 (1922) 1–70.

[6] G. H. Hardy and J. E. Littlewood, Contributions to the theory of the Riemann Zeta-function and the theory of the distribution of primes. Acta Math. 41 (1917), 119–196.

[7] M. Rubinstein, P. Sarnak, "Chebyshev's bias", Experiment. Math. 3 (1994), no.3, 173–197.

[8] A. E. Ingham," A note on distribution of primes", Acta Arith. 1 (1936), 201–211.

[9] G. Robin, Sur la difference $\mathrm{Li}(\vartheta(x)) - \pi(x)$, Ann. Fac. Sc. Toulouse 6 (1984) 257-268.

[10] C. Bays and R. H. Hudson, Numerical and graphical description of all axis crossing regions for the moduli 4 and 8 which occurs before 1012, Intern. J. Math. & Math. Sci., 2, 111-119 (1979).

[11] C. Bays and R. H. Hudson, A new bound for the smallest x with $\pi(x) > \mathrm{li}(x)$, Math. Comp., 69(231), 1285-1296 (2000).

[12] E. Landau, U¨ber einen Satz von Tschebyschef (German), Mathematische Annalen 61 (1905), 527-550.

[13] E. Landau, Über einige ältere Vermutungen und Behauptungen in der Primzahltheorie. I. Math. Z., 1 (1918), 1–24.

[14] S. Skewes, On the difference $\pi(x) - \mathrm{li}(x)$ (II), Math. Tables and other aids to computataion 13 (1955), 272-284.

[15] S. Knaposwki and P. Tura´n, Comparative Prime-Number Theory I, Acta Math. Acad. Sci. Hung. 13 (1962), 299-314.

[16] S. Knaposwki, On sign changes of $\pi(x) - li(x)$. II, Monatsh. Math. 82 (1976), 163-175.

[17] A. Granville, G. Martin, Prime number races. Amer. Math. Monthly 113 (2006), no. 1, 1–33.

[18] H. M. Stark, A problem in comparative prime number theory, Acta Arith. 18 (1971), 311–320.