The Riemann-Roch Theorem for Compact Riemann Surfaces

Jyosmita Lagachu

MS12010

A dissertation submitted for the partial fulfilment of BS-MS dual degree in Science

Indian Institute of Science Education and Research Mohali

April 2017

Certificate of Examination

This is to certify that the dissertation titled **”Riemann - Roch Theorem for Compact Riemann Surfaces”** submitted by Ms. Jyosmita Lagachu (Reg. No. MS12010) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. K. Gongopadhyay	Dr. P. Sardar	Dr. C. S. Aribam (Supervisor)

Dated: April 19, 2017

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Chandrakant S. Aribam at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Jyosmita Lagachu

Dated: April 19, 2017

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Chandrakant S. Aribam

(Supervisor)

Acknowledgement

I would like to express my gratitude to my advisor Dr. Chandrakant S. Aribam for his support, patience and encouragement throughout my project. I would also like to express thanks to my project committee members Dr. Krishnendu Gongopadhyay and Dr. Pranab Sardar for reading drafts of this dissertation and providing valuable comments that improved the contents of this dissertation. I am also grateful to my senior Rahul Chaudhary for helping me throughout the project. I would be failing if I don't mention the names of my friends Sushma and Roja for backing me up whenever I felt low. My deepest gratitude to my parents for their constant support and love in my life. I would also like to thank Indian Institute of Science Education and Research Mohali for providing me the adequate facilities and Department of Science and Technology, India for providing INSPIRE scholarship throughout my academic program.

Jyosmita Lagachu

Abstract

Riemann - Roch Theorem plays a significant role in the theory of Riemann Surfaces, which gives us certain estimate about number of linearly independent meromorphic functions subject to certain restrictions on their poles. In this dissertation we will understand the prerequisites of Riemann - Roch Theorem and will use the tools of sheaf, cohomology theory to describe it. We will generalize it and try to give a generalised proof of the theorem.

Contents

Chapter 1

Introduction

Definition 1.1. *A manifold is defined to be a topological space such that every point has a neighbourhood that is homeomorphic to the Euclidean space.*

Definition 1.2. *An n- dimensional topological manifold is defined to be a Hausdorff space that locally resembles a Euclidean space of dimension n and is second countable. All the manifolds discussed herewith are topological manifolds.*

Definition 1.3. *Let A be an open subset of P, where P is a two-dimensional manifold and B be an open subset of $\mathbb{C}$. We define a complex chart on P by a homeomorphism $\gamma : A \to B$.*
We define holomorphically compatible if

$$\gamma_2 \circ {\gamma_1}^{-1} : \gamma_1(A_1 \cap A_2) \to \gamma_2(A_1 \cap A_2)$$

where $\gamma_j : A_j \to B_j, j = 1, 2$, is biholomorphic.

Definition 1.4. *A holomorphically compatible system of charts, $\mathfrak{S} = \{\gamma_j : A_j \to B_j, j \in J\}$ which cover P, i.e., $\bigcup_{j \in J} A_j = P$ is called a complex atlas on P.*

Definition 1.5. *If every chart of* $\mathfrak{S}$ *is holomorphically compatible with every chart of* $\mathfrak{S}'$*, where* $\mathfrak{S}$ *and* $\mathfrak{S}'$ *are two complex atlases on* P*, then we say that the two atlases are analytically equivalent.*

Definition 1.6. *The analytically equivalent atlases forms an equivalence class. This equivalence class is defined as the complex structure on a two-dimensional manifold* P*.*

1.1 Riemann Surfaces

Definition 1.7. *Let P be a connected two-dimensional manifold. Then the pair (P, Γ) is called a Riemann surface, where Γ is a complex structure on P.*

Examples of Riemann Surfaces

(a) *The Complex Plane* $\mathbb{C}$. We take the mapping, $identity : \mathbb{C} \to \mathbb{C}$. This map is a chart and this chart forms an atlas. This atlas further constitutes the complex structure on $\mathbb{C}$

(b) *Tori*. Take $\lambda_1, \lambda_2 \in \mathbb{C}$. Let it be linearly independent over $\mathbb{R}$. Now define a lattice traversed by λ_1 and λ_2 as

$$\Lambda := \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2 = \{g\lambda_1 + h\lambda_2 : g, h \in \mathbb{Z}\}.$$

If $z, z' \in \mathbb{C}$ are such that $z - z' \in \Lambda$, then z, z' are said to be equivalent mod Λ. Let $\mathbb{C}/\Lambda$ represent the equivalence class. Let $\mu : \mathbb{C} \to \mathbb{C}/\Lambda$ link each $z \in \mathbb{C}$ its equivalence class mod Λ.

Now, we define a topology on $\mathbb{C}/\Lambda$. We define $A \subset \mathbb{C}/\Lambda$ to be open if $\mu^{-1}(A) \subset \mathbb{C}$ is open. Given this topology $\mathbb{C}/\Lambda$ represents a Hausdorff topological space and also have $\mu : \mathbb{C} \to \mathbb{C}/\Lambda$ to be continuous. Again $\mathbb{C}$ is connected. Thus, $\mathbb{C}/\Lambda$ is also connected. Also, the compact parallelogram

$$P := \{\alpha\lambda_1 + \beta\lambda_2 : \alpha, \beta \in [0, 1]\}$$

covers $\mathbb{C}/\Lambda$ under μ. So, $\mathbb{C}/\Lambda$ is compact. Again we have $\widehat{B} := \mu^{-1}(\mu(B))$ is open as

$$\widehat{B} = \bigcup_{\lambda \in \Lambda} (\lambda + B)$$

and every set $\lambda + B$ is open and so is $\widehat{B}$. As a result $B \subset \mathbb{C}$ is open, implying that μ is open.

Now we define the complex structure on $\mathbb{C}/\Lambda$. Let $B \subset \mathbb{C}$ be open. Let B has no two points that are equivalent under Λ. Then $A := \mu(B)$ is open and $\mu|B \to A$ is a homeomorphism. The mapping $\gamma : A \to B$ represents its inverse and also forms a complex chart on $\mathbb{C}/\Lambda$. Denote by $\mathfrak{S}$ the charts acquired in this way. Now to prove the holomorphical compatibility of $\gamma_j : A_j \to B_j, j = 1, 2$, in $\mathfrak{S}$ take the map

$$\chi := \gamma_2 \circ {\gamma_1}^{-1} : \gamma_1(A_1 \cup A_2) \to \gamma_2(A_1 \cup A_2).$$

We have $\mu(\chi(z)) = {\gamma_1}^{-1}(z) = \mu(z)$ for $z \in \gamma_1(A_1 \cup A_2)$, implying that $\chi(z) - z \in \Lambda$. Thus, on every connected component of $\gamma_1(A_1 \cup A_2)$, $\chi(z) - z$ is constant as we have Λ to be discrete and χ continuous. This gives us χ to be holomorphic. In the same way, χ^{-1} is also holomorphic. Thus, the atlas $\mathfrak{S}$ defines a complex structure on $\mathbb{C}/\Lambda$.

Definition 1.8. *Let Q be an open subset of P(P is a Riemann surface). Let $g : Q \to \mathbb{C}$. Let for every chart $\chi : A \to B$,*

$$g \circ \chi^{-1} : \chi(A \cap Q) \to \mathbb{C}$$

be holomorphic on the set $\chi(A \cap Q) \subset \mathbb{C}$, open. Then g is said to be holomorphic.

1.2 Some Definitions and Properties of Riemann surfaces

Definition 1.9. *Let $g : P \to Q$ be a continuous mapping, where P and Q are Riemann surfaces. If*

$$\chi_2 \circ g \circ {\chi_1}^{-1} : B_1 \to B_2$$

is holomorphic in the usual sense, for every pair of charts $\chi_1 : A_1 \to B_1$ on P and $\chi_2 : A_2 \to B_2$ on Q with $g(A_1) \subset A_2$, then we say that g is holomorphic.

Theorem 1.10. *(Identity Theorem) Let $g_1, g_2 : P \to Q$ be two holomorphic mappings, where P and Q are Riemann surfaces. Then g_1 and g_2 are identically equal if they coincide on a set $X \subset P$ having a limit point $x \in P$.*

Definition 1.11. *Let Q be an open subset of a Riemann surface P. Let Q' be an open subset of Q and $g : Q' \to \mathbb{C}$ be a holomorphic function. Then g is called a meromorphic function on Q if the following hold:*
(i) $Q \backslash Q'$ contains only isolated points.
(ii) For every point $s \in Q \backslash Q'$ one has

$$\lim_{y \to s} |g(y)| = \infty.$$

$\mathfrak{M}(Q)$ denotes the set of all meromorphic function.

Theorem 1.12. *Every holomorphic function on a compact Riemann surface is constant.*

Theorem 1.13. *(Liouville's Theorem) Let $g : \mathbb{C} \to \mathbb{C}$ be a bounded holomorphic function. Then g is constant.*

1.3 Homotopy of Curves

Definition 1.14. *Let $x, y \in P$, where P is a topological space. Let $g, h : J \to P$ be two curves from x to y. Let $T : J \times J \to P$ be a continuous mapping such that:*
(i) $T(l, 0) = g(l)$ for every $l \in J$,
(ii) $T(l, 1) = h(l)$ for every $l \in J$,
(iii) $T(0, k) = x$ and $T(1, k) = y$ for every $k \in J$.
then $g \sim h$ are called homotopic.

Theorem 1.15. *Let there be a topological space P. And let us suppose that x, y be in P. Define curves from x to y. These curves forms an equivalence relation. Then $x \sim y$ is defined by this equivalence relation.*

Definition 1.16. *Let x be in P. Here P is a topological space. Define $a_0 : J \to P$ by $a_0(l) = x$ for every $l \in J$. Then a_0 is called the constant curve at x.*

Definition 1.17. *Let P be a topological space. Define $a : J \to P$ a curve in P such that $a(0) = a(1)$. Then a is said to be closed.*

Definition 1.18. *Let a be a closed curve such that the initial and end point of the curve a be x. Define a_0 to be the constant curve at x. Let $a \sim a_0$. Then a is called null-homotopic.*

Definition 1.19. *Let P be a topological space. Let $f, g, h \in P$. Define curves from f to g and from g to h by $x : J \to P$ and $y : J \to P$ respectively. Define the map $x \cdot y : J \to P$, from f to h as follows:*

$$(x \cdot y)(l) := \{x(2l) \quad for\ 0 \le l \le \frac{1}{2}, \quad y(2l-1) \quad for\ \frac{1}{2} \le l \le 1.$$

Then $x \cdot y$ is called the product curve.

Definition 1.20. *Let x be a point in P. Here P is a topological space. Let the homotopy classes of curves in P be denoted by $\mu_1(P, x)$. Let the curves in $\mu_1(P, x)$ be also closed such that x is the initial and end point. We define product of the curves to be an operation on $\mu_1(P, x)$. With this operation it forms a group. It is given the name the fundamental group of P having base point x.*

Definition 1.21. *Let P be a topological space. We also suppose P to be arcwise connected and $\mu_1(P) = 0$. Then P is said to be simply connected.*

1.4 Coverings

Definition 1.22. *Let P and Q be two topological spaces. Define a continuous map $g : Q \to P$. Then by a fiber of g over p, we mean the set $g^{-1}(p)$, where p is in P.*

Definition 1.23. *Let P, Q and R be topological spaces and we define continuous maps $g : Q \to P$ and $h : R \to P$. Let $k : Q \to R$ with $g = h \circ k$, then k is said to be fiber-preserving.*

Definition 1.24. *Let P and Q be Riemann surfaces. Define a holomorphic map $g : P \to Q$. The map g is not constant. Let q be a point in Q. Suppose B is a neighbourhood of q. We have B with $g|B$ (g restrict to B) not injective. Then q is called a ramification point of g.*

Definition 1.25. ***Lifting of Mappings**: Consider two continuous maps $g : Q \to P$ and $h : R \to P$. Here P, Q and R are topological spaces. Define another continuous map $k : R \to Q$. Let the map k be defined in such a way that $h = g \circ k$. Then k is called a lifting of h with respect to g.*

Definition 1.26. *Let P and Q be topological spaces. Define a mapping $g : Q \to P$. Let us suppose g be such that every point $p \in P$ has an open neighbourhood A with*

$$g^{-1}(A) = \bigcup_{i \in I} B_i,$$

where $B_i \subset Q$ are open and disjoint and $g|B_i \to A$ are homeomorphisms. Then g is said to be a covering map.

Theorem 1.27. *(Uniqueness of Lifting.) Define a local homeomorphism $g : Q \to P$ between two Hausdorff spaces P and Q. Let R be a connected topological space.*

Define $h : R \to P$. Let h be continuous. Now again define $f_1, f_2 : R \to Q$ which are two liftings of h. The map f_1 and f_2 are such that $f_1(r_0) = f_2(r_0)$ for $r_0 \in R$, where r_0 is a point in R. Then we have $f_1 = f_2$

Definition 1.28. *Let P and Q be topological spaces. Define a continuous map $g : Q \to P$. Let every curve $a : [0, 1] \to P$ defined by $g(q_0) = a(0)$, has a lifting $\widehat{a} : [0, 1] \to Q$ of a defined by $\widehat{a}(0) = q_0$. Then, g has the curve lifting property.*

Theorem 1.29. *Let P and Q be topological spaces. Let $g : Q \to P$ be a covering map. Then g has the curve lifting property.*

Proof Suppose $a : [0, 1] \to P$ is a curve and $q_0 \in Q$ with $g(q_0) = a(0)$. Because of the compactness of $[0, 1]$, $\exists$ a partition

$$0 = l_0 < l_1 < \cdots < l_m = 1$$

and open sets $A_i \subset P$, $i = 1, \cdots, m$, with the following properties:
(i) $a([l_{i-1}, l_i]) \subset A_i$,
(ii) $g^{-1}(A_i) = \bigcup_{k \in K_i} B_{ik}$,
where the $B_{ik} \subset Q$ are open sets such that $g \mid B_{ik} \to A_i$ are homeomorphisms. Now we shall prove by induction on $i = 0, 1, \cdots, m$ the existence of a lifting $\widehat{a} \mid [0, l_i] \to P$ with $\widehat{a}(0) = q_0$. For $i = 0$ this is trivial. So suppose $i \geq 1$ and $\widehat{a} \mid [0, l_{i-1}] \to P$ is already constructed and let $\widehat{a}(l_{i-1}) =: q_{i-1}$. Since $g(q_{i-1}) = a(l_{i-1}) \in A_i$, $\exists\, k \in K_i$ such that $q_{i-1} \in B_{ik}$. Let $\chi : A_i \to B_{ik}$ be the inverse of the homeomorphism $g \mid B_{ik} \to A_i$. Then if we set

$$\widehat{a} \mid [l_{i-1}, l_i] := \chi \circ (a \mid [l_{i-1}, l_i]),$$

we obtain a continuous extension of the lifting $\widehat{a}$ to the interval $[0, l_i]$. Proved.

Definition 1.30. *Let $g : Q \to P$ be a covering map, where P and Q are connected topological spaces. Let R be another connected topological space and $h : R \to P$ be another covering map. Suppose $q_0 \in Q$ and $r_0 \in R$ with $g(q_0) = h(r_0)$ be points such that there exists a continuous fiber-preserving mapping $x : Q \to R$ with $x(q_0) = r_0$. Then g is said to be the universal covering if the above is true for every covering map h and every choice of points q_0 and r_0 with $g(q_0) = h(r_0)$.*

Theorem 1.31. *Let P be a connected manifold. Then there exists a connected, simply connected manifold $\tilde{P}$ and a covering map $g : \tilde{P} \to P$.*

Definition 1.32. *Let $g : Q \to P$ be a covering map between two topological spaces P and Q. A fiber-preserving homeomorphism $x : Q \to Q$ is called a covering transformation or deck transformation of this covering. The set of all covering transformation $g : Q \to P$ forms a group, with the group operation the composition of mappings. This group is denoted by $Deck(Q/P)$.*

Example

$exp : \mathbb{C} \to \mathbb{C}^*$ is the universal covering of $\mathbb{C}^*$, since $\mathbb{C}$ is simply connected. For $m \in \mathbb{Z}$ let $\xi_m : \mathbb{C} \to \mathbb{C}$ be translation by $2\pi im$. Then $exp(\xi_m(z)) = exp(z+2\pi im) = exp(z)$ for every $z \in \mathbb{C}$ and thus ξ_n is a covering transformation. If ζ is any covering transformation, then $exp(\zeta(0)) = exp(0) = 1$ and thus there exists $m \in \mathbb{Z}$ such that

$\zeta(0) = 2\pi im$. Since $\xi_m(0) = 2\pi im$ as well, $\zeta = \xi_m$. Thus

$$Deck(\mathbb{C} \xrightarrow{exp} \mathbb{C}^*) = \{\xi_m : m \in \mathbb{Z}\}$$

1.5 Sheaves

Definition 1.33. *Let P be a topological space and $\mathcal{C}$ be the system of open sets in P. A presheaf of abelian groups on P is a pair $(\mathfrak{Y}, \eta)$ consisting of*
(i) *a family $\mathfrak{Y} = (\mathfrak{Y}(A))_{A\in\mathcal{C}}$ of abelian groups,*
(ii) *a family $\eta = (\eta_B^A)_{A,B\in\mathcal{C},B\subset A}$ of group homomorphisms*

$$\eta_B^A : \mathfrak{Y}(A) \to \mathfrak{Y}(B),\ where\ B\ is\ open\ in\ A$$

with the following properties:

$$\eta_B^A = id_{\mathfrak{Y}(A)}\ \ for\ every\ A \in \mathcal{C},$$

$$\eta_D^B \circ \eta_B^A = \eta_D^A\ \ for\ D \subset B \subset A.$$

Definition 1.34. *Let P be a topological space and $\mathfrak{Y}$ be the presheaf on P. If for every open set $A \subset P$ and every family of open subsets $A_k \subset A$, $k \in K$, with $A = \bigcup_{k\in K} A_k$ the following Sheaf Axioms are satisfied:*
(I) *If $x, y \in \mathfrak{Y}(A)$ are elements such that $x \mid A_k = y \mid A_k$ for every $k \in K$, then $x = y$.*
(II) *Given elements $x_k \in \mathfrak{Y}(A_k), k \in K$, such that*

$$x_k \mid A_k \cap A_l = x_l \mid A_k \cap A_l\ for\ all\ k,l \in K,$$

then there exists an $x \in \mathfrak{Y}(A)$ such that $x \mid A_k = x_k$ for every $k \in K$. Then, the presheaf $\mathfrak{Y}$ is called a sheaf.

Definition 1.35. ***The Stalk of a Presheaf:*** *Suppose $\mathfrak{Y}$ is a presheaf of sets on a topological space P and $p \in P$ is a point. On the disjoint union*

$$\bigsqcup_{p\in A} \mathfrak{Y}(A),$$

where the union is taken over all the open neighbourhoods A of p, introduce an equivalence relation $\underset{p}{\sim}$ as follows: Two elements $x \in \mathfrak{Y}(A)$ and $y \in \mathfrak{Y}(B)$ are related $x \underset{p}{\sim} y$ precisely if there exists an open set D with $p \in D \subset A \cap B$ such that $x \mid D = y \mid D$. The set $\mathfrak{Y}_p$ of all equivalence classes, the so- called inductive limit of $\mathfrak{Y}(A)$, is given by

$$\mathfrak{Y}_p := \varinjlim_{p \in A} \mathfrak{Y}(A) := \left(\bigsqcup_{p \in A} \mathfrak{Y}(A) \right) \Big/ \underset{p}{\sim}$$

and is called the stalk of $\mathfrak{Y}$ at the point p.
For any neighbourhood A of p, let

$$\eta_p : \mathfrak{Y}(A) \to \mathfrak{Y}_p$$

be the mapping which assigns to each element $x \in \mathfrak{Y}(A)$ its equivalence class modulo $\underset{p}{\sim}$. One calls $\eta_p(x)$ the germ of x at p.

1.6 Differential Forms

Some Notations

Let P be a Riemann surface and $Q \subset P$ be open. We define a mapping $g : Q \to \mathbb{C}$ such that for every $A \subset Q$ and $Q \subset \mathbb{C}$ the chart $h : A \to B$ has a function $\tilde{g} \in \mathcal{H}(B)$, with $g \mid A = \tilde{g} \circ h$. Here $\mathcal{H}(B)$ represents the $\mathbb{C}$- algebra of all those functions $j : B \to \mathbb{C}$ which are infinitely differentiable. Now, let $\mathcal{H}(B)$ consists all such functions g. This $\mathcal{H}$ forms a sheaf of differential functions on X along with the natural restriction mappings on it.

Now, we denote by $\mathcal{H}_p$ the stalk at a point p in P. The germs of differential functions at the point p is contained in this stalk. The function germs which vanish at the point p are denoted by $\mathfrak{n}_p$. This is a vector subspace of $\mathcal{H}_p$. And by $\mathfrak{n}_p^2 \subset \mathfrak{n}_p$ we denote the vector subspace of the function germs which vanish to second order.

Definition 1.36. *The cotangent space of P at a point p is defined to be*

$$L_p^{(1)} := \frac{\mathfrak{n}_p}{\mathfrak{n}_p^2}$$

Definition 1.37. *Let Q subset of P be open, where P is a Riemann Surface. Define*

$$\varpi : Q \to \bigcup_{p \in Q} L_p^{(1)}.$$

Here $\varpi(p) \in L_p^{(1)}$ for every $p \in Q$. Then this mapping is called a $1 - form$ on Q or a differential form of degree one.

Example

Let $p \in Q$, then for $g \in \mathcal{H}$ we define $1 - forms$ by the mappings $(dg)(p) := d_p g$, $(d'g)(p) := d'_p g$ and $(d''g)(p) := d''_p g$.

Definition 1.38. ***The Exterior Product**: We define the exterior product $\wedge^2 B$ on a vector space B over $\mathbb{C}$ to consist of finite sum of elements of the form $b_1 \wedge b_2$, where $b_1, b_2 \in B$. Again for $b_1, b_2, b_3 \in B$ and $\alpha \in \mathbb{C}$, we have the following:*

$$(b_1 + b_2) \wedge b_3 = b_1 \wedge b_3 + b_2 \wedge b_3$$

$$(\alpha b_1) \wedge b_2 = \alpha(b_1 \wedge b_2)$$

$$b_1 \wedge b_2 = -b_2 \wedge b_1.$$

Also, $e_k \wedge e_l$, $k < l$ forms a basis for $\wedge^2 B$, where $(e_k, \cdots, e_m)$ is a basis of B.
This exterior product can be applied to $L_p^{(1)}$ and we define:

$$L_p^{(2)} := \wedge^2 L_p^{(1)}.$$

Definition 1.39. *Now, just as a $1 - form$ we can also define a $2 - form$, which is a mapping*

$$\varpi : Q \to \bigcup_{p \in Q} L_p^{(2)}$$

with $\varpi(p) \in L_p^{(2)}$, for $p \in Q$.

Example

Let $\mathcal{H}^{(1)}(Q)$ be the vector space of differential $1-forms$ on Q and $\varpi_1, \varpi_2 \in \mathcal{H}^{(1)}(Q)$. We define a $2 - form$, $\mathcal{H}^{(2)}(Q)$ by

$$(\varpi_1 \wedge \varpi_2)(p) := \varpi_1(p) \wedge \varpi_2(p)$$

with $\varpi_1 \wedge \varpi_2 \in \mathcal{H}^{(2)}(Q)$ and $p \in Q$.

Definition 1.40. *Let P be a Riemann surface and $Q \subset P$ be open. Then $\varpi \in \mathcal{H}^{(1)}(Q)$ is said to be closed if $d\varpi = 0$. It is called exact if $\varpi = dg$ for a $g \in \mathcal{H}(Q)$.*

Theorem 1.41. *Let P be a Riemann surface and Q subset of P be open. Let $\Theta(Q)$ denote the vector space of holomorphic 1-forms on Q and $\mathcal{H}^{1,0}(Q)$ represents the subspace of $\mathcal{H}^{(1)}(Q)$ of differential forms of type $(1, 0)$. Then we have the following:*
(i) ϖ is closed for every $\varpi \in \Theta(Q)$.
(ii) ϖ is holomorphic for every closed $\varpi \in \mathcal{H}^{1,0}(Q)$.

Chapter 2

Compact Riemann Surfaces

2.1 Cohomology Groups

Definition 2.1. *Let $\mathfrak{Y}$ be a sheaf of abelian groups on a topological space P. Let $\mathfrak{A} = (A_k)_{k\in K}$ be a open covering on P such that $\bigcup_{k\in K} A_k = P$. Then the $yth\ cochain\ group$ of $\mathfrak{Y}$ is defined as*

$$Z^y(\mathfrak{A}, \mathfrak{Y}) := \prod_{(k_0,\cdots,k_y)\in K^{y+1}} \mathfrak{Y}(A_{k_0} \cap \cdots \cap A_{k_y})$$

with respect to $\mathfrak{A}$ and for $y = 0, 1, 2, \cdots$
The elements of $Z^y(\mathfrak{A}, \mathfrak{Y})$ are called $y - cochains$.

Definition 2.2. *Define*

$$\vartheta : Z^0(\mathfrak{A}, \mathfrak{Y}) \to Z^1(\mathfrak{A}, \mathfrak{Y})$$

$$\vartheta : Z^1(\mathfrak{A}, \mathfrak{Y}) \to Z^2(\mathfrak{A}, \mathfrak{Y})$$

the $coboundary\ operators$ as follows:

(a) *Let* $\vartheta((h_k)_{k\in K}) = (l_{ki})_{k,i\in K}$ *with* $(h_k)_{k\in K} \in Z^0(\mathfrak{A},\mathfrak{Y})$ *and*

$$l_{ki} := h_i - h_k \in \mathfrak{Y}(A_k \cap A_i).$$

(b) *Let* $\vartheta((h_{ki})) = (l_{kin})$ *with* $(h_{ki})_{k,i\in K} \in Z^1(\mathfrak{A},\mathfrak{Y})$ *and*

$$l_{kin} := h_{in} - h_{kn} + h_{ki} \in \mathfrak{Y}(A_k \cap A_i \cap A_n).$$

The coboundary operators are group homomorphisms.

Definition 2.3. *We define* $1-cocycles$ *as the elements of* $R^1(\mathfrak{A},\mathfrak{Y})$*, where*

$$R^1(\mathfrak{A},\mathfrak{Y}) := ker(Z^1(\mathfrak{A},\mathfrak{Y}) \xrightarrow{\vartheta} Z^2(\mathfrak{A},\mathfrak{Y})).$$

Definition 2.4. *A* $1-coboundaries$ *is defined to be the elements of* $G^1(\mathfrak{A},\mathfrak{Y})$*, where* $G^1(\mathfrak{A},\mathfrak{Y})$ *is defined as*

$$G^1(\mathfrak{A},\mathfrak{Y}) := Im(Z^0(\mathfrak{A},\mathfrak{Y}) \xrightarrow{\vartheta} Z^1(\mathfrak{A},\mathfrak{Y})).$$

Definition 2.5. *We define the* $1st\ cohomology\ group$ *with respect to the covering* $\mathfrak{A}$*, to be the group*

$$H^1(\mathfrak{A},\mathfrak{Y}) := R^1(\mathfrak{A},\mathfrak{Y})/G^1(\mathfrak{A},\mathfrak{Y})$$

with coefficients in $\mathfrak{Y}$*.*

Definition 2.6. *Let* $\mathfrak{B} = (B_n)_{n\in N}$ *and* $\mathfrak{A} = (A_k)_{k\in K}$ *be two open covering of* P*. Then* $\mathfrak{B}$ *is said to be finer than* $\mathfrak{A}$ *if every* B_n *is contained in at least one* A_k*. As a result, a mapping* $\varsigma : N \to K$ *can be defined with* $B_n \subset A_{\varsigma n}$ *for* $n \in N$*.*
With ς *another map can be defined as follows:*

$$s^{\mathfrak{A}}_{\mathfrak{B}} : R^1(\mathfrak{A},\mathfrak{Y}) \to R^1(\mathfrak{B},\mathfrak{Y})$$

such that $s^{\mathfrak{A}}_{\mathfrak{B}}((h_{ki})) = (l_{nm})$ *with* $(h_{ki}) \in R^1(\mathfrak{A}, \mathfrak{Y})$ *and for* $n, m \in N$

$$l_{nm} := h_{\varsigma n,\, \varsigma m} \mid B_n \cap B_m.$$

It induces a homomorphism of the cohomology groups $s^{\mathfrak{A}}_{\mathfrak{B}} : H^1(\mathfrak{A}, \mathfrak{Y}) \to H^1(\mathfrak{B}, \mathfrak{Y})$.

Definition 2.7. $H^1(P, \mathfrak{Y})$: *Define three open coverings on* P, $\mathfrak{N}, \mathfrak{B}, \mathfrak{A}$ *such that* $\mathfrak{N} < \mathfrak{B} < \mathfrak{A}$ *then*

$$s^{\mathfrak{B}}_{\mathfrak{N}} \circ s^{\mathfrak{A}}_{\mathfrak{B}} = s^{\mathfrak{A}}_{\mathfrak{N}}.$$

Here, let $\mathfrak{A}$ *run over all open coverings of* P*, then on the disjoint union of* $H^1(\mathfrak{A}, \mathfrak{Y})$*, two cohomology classes* $\varrho \in H^1(\mathfrak{A}, \mathfrak{Y})$ *and* $\upsilon \in H^1(\mathfrak{A}', \mathfrak{Y})$ *are equivalent,* $\varrho \sim \upsilon$ *if* $\exists$ *an open covering* $\mathfrak{B} < \mathfrak{A}$ *and* $\mathfrak{B} < \mathfrak{A}'$ *with* $s^{\mathfrak{A}}_{\mathfrak{B}}(\varrho) = s^{\mathfrak{A}'}_{\mathfrak{B}}(\upsilon)$*. The inductive limit,* $H^1(P, \mathfrak{Y})$ *of* $H^1(\mathfrak{A}, \mathfrak{Y})$ *is the set of equivalence classes, i.e.*

$$H^1(P, \mathfrak{Y}) = \underset{\mathfrak{A}}{\underline{lim}} H^1(\mathfrak{A}, \mathfrak{Y}) = \left(\bigsqcup_{\mathfrak{A}} H^1(\mathfrak{A}, \mathfrak{Y}) \right) \Big/ \sim$$

Theorem 2.8. *Let* $\mathcal{H}$ *be the sheaf of differential functions on a Riemann surface* P*. Then we have* $H^1(P, \mathcal{H}) = 0$*.*

Proof Consider $\mathfrak{A} = (A_k)_{k \in K}$ an arbitrary open covering of P. Then we have $(\nu_k)_{k \in K}$, with $\nu_k \in \mathcal{H}(P)$ $\big($ $\mathcal{H}(P)$ represents differential functions with compact support $\big)$, such that:

(i) Supp$(\nu_k) \subset A_k$.

(ii) Every point of P has a neighbourhood meeting only finitely many of the sets Supp(ν_k).

(iii) $\sum_{k\in K} \nu_k = 1$.

We will show that $H^1(\mathfrak{A}, \mathcal{H}) = 0$, i.e., every cocycle $(h_{ki}) \in R^1(\mathfrak{A}, \mathcal{H})$ splits.

We have the function $\nu_i h_{ki}$ defined on $A_k \cap A_i$. By taking $\nu_i h_{ki} = 0$ outside its support this function can be differentiably extended to the whole of A_k. Thus it may be considered as an element of $\mathcal{H}(A_k)$. Now let $l_k := \sum_{i\in K} \nu_i h_{ki}$. This sum has only finitely many terms which are not zero in a neighbourhood of any point in A_k by (ii). Thus, it defines an element $l_k \in \mathcal{H}(A_k)$. Now on $A_k \cap A_i$ and for $k, i \in K$, we have

$$l_k - l_i = \sum_{n\in K} \nu_n h_{kn} - \sum_{n\in K} \nu_n h_{in} = \sum_{n} \nu_n (h_{kn} - h_{in})$$

$$= \sum_{n} \nu_n (h_{kn} + h_{ni}) = \sum_{n} \nu_n h_{ki} = h_{ki}.$$

Thus, (h_{ki}) is a coboundary. Proved.

Lemma 2.9. *The mapping*

$$s^{\mathfrak{A}}_{\mathfrak{B}} : H^1(\mathfrak{A}, \mathfrak{Y}) \to H^1(\mathfrak{B}, \mathfrak{Y})$$

is injective.

Theorem 2.10. *Let P be a topological space. Let $\mathfrak{A} = (A_k)_{k\in K}$ be an open covering of P and let $\mathfrak{Y}$ be the sheaf of abelian groups on P. Let $H^1(A_k, \mathfrak{Y}) = 0$, $k \in K$, then we define the $\mathfrak{A}$ as a* ***Leray covering*** *for the sheaf $\mathfrak{Y}$ if*

$$H^1(P, \mathfrak{Y}) \cong H^1(\mathfrak{A}, \mathfrak{Y}).$$

Proof It is enough to prove that for every open covering $\mathfrak{B} = (B_\gamma)_{\gamma\in U}$, with $\mathfrak{B} < \mathfrak{A}$, the mapping $s^{\mathfrak{A}}_{\mathfrak{B}} : H^1(\mathfrak{A}, \mathfrak{Y}) \to H^1(\mathfrak{B}, \mathfrak{Y})$ is an isomorphism. From $Lemma$ 2.9 this map is injective.

Now, we define a refining map $\varepsilon : U \to K$ such that $B_\gamma \subset A_{\varepsilon\gamma}$, $\gamma \in U$. We need to show that given any cocycle $(h_{\gamma\delta}) \in R^1(\mathfrak{B}, \mathfrak{Y})$, there exists a cocycle $(T_{ki}) \in R^1(\mathfrak{A}, \mathfrak{Y})$ such that the cocycle $(T_{\varepsilon\gamma,\varepsilon\delta}) - (h_{\gamma\delta})$ is cohomologous to zero relative to the covering $\mathfrak{B}$. Denote by $A_k \cap \mathfrak{B}$ the open covering $(A_k \cap B_\gamma)_{\gamma\in U}$ of A_k. We have

$$H^1(A_k \cap \mathfrak{B}, \mathfrak{Y}) = 0 \ \ (by\ assumption).$$

Thus, $\exists\ l_{k\gamma} \in \mathfrak{Y}(A_k \cap B_\gamma)$ such that

$$h_{\gamma\delta} = l_{k\gamma} - l_{k\delta}\ on\ A_k \cap B_\gamma \cap B_\delta.$$

Also,

$$l_{i\gamma} - l_{k\gamma} = l_{i\delta} - l_{k\delta}\ on\ A_k \cap A_i \cap B_\gamma \cap B_\delta.$$

As a result, there exists

$$T_{ki} \in \mathfrak{Y}(A_k \cap A_i)$$

such that

$$T_{ki} = l_{i\gamma} - l_{k\gamma}\ on\ A_k \cap A_i \cap B_\gamma \quad (by\ sheaf\ axiom\ II).$$

Here (T_{ki}) lies in $R^1(\mathfrak{A}, \mathfrak{Y})$ as it satisfies the cocycle relation. Now, let $o_\gamma := l_{\varepsilon\gamma,\,\gamma} \mid B_\gamma \in \mathfrak{Y}(B_\gamma)$. Then

$$T_{\varepsilon\gamma,\,\varepsilon\delta} - h_{\gamma\delta} = (l_{\varepsilon\delta,\,\gamma} - l_{\varepsilon\gamma,\,\gamma}) - (l_{\varepsilon\delta,\,\gamma} - l_{\varepsilon\delta,\,\delta})$$

$$= l_{\varepsilon\delta,\,\delta} - l_{\varepsilon\gamma,\,\gamma} = o_\delta - o_\gamma\ on\ B_\gamma \cap B_\delta.$$

Thus, $(T_{\varepsilon\gamma,\,\varepsilon\delta}) - (h_{\gamma\delta})$ splits. Proved.

Definition 2.11. *Let P be a topological space and $\mathfrak{A} = (A_k)_{k\in K}$ be an open covering of P. Define $\mathfrak{Y}$ to be a sheaf of abelian groups on P. Define*

$$R^0(\mathfrak{A}, \mathfrak{Y}) := ker(Z^0(\mathfrak{A}, \mathfrak{Y}) \xrightarrow{\vartheta} Z^1(\mathfrak{A}, \mathfrak{Y})),$$

$$G^0(\mathfrak{A}, \mathfrak{Y}) := 0,$$

$$H^0(\mathfrak{A}, \mathfrak{Y}) := R^0(\mathfrak{A}, \mathfrak{Y})/G^0(\mathfrak{A}, \mathfrak{Y}) = R^0(\mathfrak{A}, \mathfrak{Y}).$$

If $h_k \mid A_k \cap A_i = h_i \mid A_k \cap A_i$, $k, i \in K$ then a $0 - cochain$ $(h_k) \in Z^0(\mathfrak{A}, \mathfrak{Y})$ belongs to $R^0(\mathfrak{A}, \mathfrak{Y})$ (ϑ definition). Again, the elements h_k together give a $h \in \mathfrak{Y}(P)$ (by sheaf axiom II). Also,

$$H^0(\mathfrak{A}, \mathfrak{Y}) = R^0(\mathfrak{A}, \mathfrak{Y}) \cong \mathfrak{Y}(P)$$

is a natural isomorphism. As a result, we can define

$$H^0(P, \mathfrak{Y}) := \mathfrak{Y}(P)$$

as $H^0(\mathfrak{A}, \mathfrak{Y})$ is entirely independent of the covering $\mathfrak{A}$.

2.2 Some Properties of Cohomology Groups

Lemma 2.12. *Let $l \in \mathcal{H}(\mathbb{C})$ has compact support. Then, we have a $h \in \mathcal{H}(\mathbb{C})$ such that*

$$\frac{dh}{d\bar{z}} = l.$$

Theorem 2.13. *Let $P := \{z \in \mathbb{C} : |z| < R\}$, $0 < R \leq \infty$, and $l \in \mathcal{H}(P)$. Then, we have $h \in \mathcal{H}(P)$ such that*

$$\frac{dh}{d\bar{z}} = l.$$

Corollary 2.14. *Suppose P is as defined in the previous theorem. Then for $l \in \mathcal{H}(P)$, $\exists\, h \in \mathcal{H}(P)$ such that $\triangle h = l$ with $\triangle$ the Laplace operator,*

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\, \partial \bar{z}}\ .$$

Theorem 2.15. *Let $P := \{z \in \mathbb{C} : |z| < R\}$, $0 < R \leq \infty$. Then $H^1(P, \mathcal{S}) = 0$.*

Theorem 2.16. *We have $H^1(\mathbb{P}^1, \mathcal{S}) = 0$, where $\mathbb{P}^1$ is the Riemann sphere and $\mathcal{S}$ is the sheaf of holomorphic functions.*

Proof Let $A_1 := \mathbb{P}^1 \setminus \infty$ and $A_2 := \mathbb{P}^1 \setminus 0$. Now, as $A_1 = \mathbb{C}$ and A_2 is biholomorphic to $\mathbb{C}$, thus

$$H^1(A_k, \mathcal{S}) = 0 \; (by \; Theorem \; 2.15).$$

As a result we get a Leray covering of $\mathbb{P}^1$, $\mathfrak{A} = (A_1, A_2)$. Also,

$$H^1(\mathbb{P}^1, \mathcal{S}) = H^1(\mathfrak{A}, \mathcal{S}) \; (by \; Theorem \; 2.10).$$

Therefore, it is enough to prove that every cocycle $(h_{ki}) \in R^1(\mathfrak{A}, \mathcal{S})$ splits, i.e., for $h_k \in \mathcal{S}(A_k)$ and $A_1 \cap A_2 = \mathbb{C}^*$ we need

$$h_{12} = h_1 - h_2.$$

Let us suppose

$$h_{12}(z) = \sum_{m=-\infty}^{\infty} \alpha_m z^m$$

be the Laurent expansion of h_{12} on $\mathbb{C}^*$. Let

$$h_1(z) := \sum_{m=0}^{\infty} \alpha_m z^m \;\; and \;\; h_2(z) := -\sum_{m=-\infty}^{-1} \alpha_m z^m,$$

which gives us $h_1 - h_2 = h_{12}$ for $h_k \in \mathcal{S}(A_k)$. Proved.

2.3 Towards the Riemann-Roch Theorem

Theorem 2.17. *Let $Q_1 \Subset Q_2 \subset P$ be open, where P is a Riemann surface and $Q_1 \Subset Q_2$ means that Q_1 compactly contained in Q_2. Then, the image of*

$$H^1(Q_2, \mathcal{S}) \to H^1(Q_1, \mathcal{S})$$

is finite dimensional.

Corollary 2.18. *We have the $dim\ H^1(P, \mathcal{S}) < \infty$, if P is a compact Riemann surface.*

Definition 2.19. *We define*

$$l := dim\ H^1(P, \mathcal{S})$$

as the genus of P, where P is a compact Riemann surface.

Theorem 2.20. *Let $Q \Subset P$ be a relatively compact open subset of the Riemann surface P. Let $p \in Q$ be a point. Then, there exists a $h \in \mathfrak{M}(Q)$ which is holomorphic on $Q \setminus \{p\}$ and having a pole at p, for every such p.*

Theorem 2.21. *Let $Q \Subset Q' \subset P$, where P is a non-compact Riemann surface. Then*

$$Im(H^1(Q', \mathcal{S}) \to H^1(Q, \mathcal{S})) = 0$$

Definition 2.22. *Let P be a topological space and we define $\mathfrak{Y}$ and $\mathfrak{G}$ as the sheaves of abelian groups on P. Then we define a sheaf homomorphism $\gamma : \mathfrak{Y} \to \mathfrak{G}$ to be a family of group homomorphisms*

$$\gamma_A : \mathfrak{Y}(A) \to \mathfrak{G}(A) \ (A \ open \ in \ P),$$

which are compatible with the restriction homomorphisms.

Definition 2.23. ***The Kernel of a Sheaf Homomorphism**: Let P be a topological space. Now define $\mathfrak{Y}$ and $\mathfrak{G}$ to be sheaves on P. Let $\gamma : \mathfrak{Y} \to \mathfrak{G}$ be a sheaf homomorphism. Now for $A \subset P$, open define*

$$\mathfrak{K}(A) := Ker(\mathfrak{Y}(A) \xrightarrow{\gamma} \mathfrak{G}(A)).$$

The family of groups $\mathfrak{K}(A)$, together with the restriction homomorphisms induced from the sheaf $\mathfrak{Y}$, is again a sheaf. It is called the kernel of γ and is denoted by $\mathfrak{K} = Ker\,\gamma$.

Examples

On any Riemann surface one has
(a) $\mathcal{S} = Ker(\mathcal{H} \xrightarrow{d''} \mathcal{H}^{(0,1)})$,
(b) $\Theta = Ker(\mathcal{H}^{(1,0)} \xrightarrow{d} \mathcal{H}^{(2)})$,
(c) $\mathbb{Z} = Ker(\mathcal{S} \xrightarrow{ex} \mathcal{S}^*)$.

Theorem 2.24. *Dolbeault's Theorem: Let P be a Riemann surface. Then there are isomorphisms*
(a) $H^1(P, \mathcal{S}) \cong \mathcal{H}^{0,1}(P)/d''\mathcal{H}(P)$
(b) $H^1(P, \Theta) \cong \mathcal{H}^{(2)}(P)/d\mathcal{H}^{1,0}(P)$.

2.4 The Riemann-Roch Theorem

Definition 2.25. *Define a mapping*

$$W : P \to \mathbb{Z}$$

such that $W(p) \neq 0$ *for finitely many* $p \in N$*, where* N *is any compact subset of the Riemann surface* P*. Then* W *is called the divisor on* P*.*
The set of all divisors on P *denoted by* $Div(P)$ *is an abelian group with addition as the group operation.*

Definition 2.26. ***Degree of a Divisor****: Let* P *be a compact Riemann surface. Then we have only finitely many* $p \in P$ *with* $W(p) \neq 0$ *for every* $W \in Div(P)$*. Hence we define a mapping*

$$deg : Div(P) \to \mathbb{Z}$$

by

$$deg\, W := \sum_{p \in P} W(p).$$

This map is called the degree.

Definition 2.27. ***The Sheaves*** $\mathcal{S}_W$*: Let* $A \subset P$ *be open. Let* P *be a Riemann surface and* W *be a divisor on* P*. Then the set of all those meromorphic functions on* A *which are multiples of the divisor* $-W$ *are defined by* $\mathcal{S}_W$*, i.e.,*

$$\mathcal{S}_W(A) := \{h \in \mathfrak{M}(A) : ord_p(h) \geq -W(p) \; for \; every \; p \in A\}.$$

$\mathcal{S}_W$ *is a sheaf.*

Theorem 2.28. *Let* $W \in Div(P)$ *be a divisor with* $deg\, W < 0$*, where* P *is a compact Riemann surface. Then* $H^0(P, \mathcal{S}_W) = 0$*.*

Proof We proof it by contradiction. Let there exists an $h \in H^0(P, \mathcal{S}_W)$ with $h \neq 0$. Then $(h) \geq -W$ and thus

$$deg(h) \geq -deg\, W > 0$$

but $deg(h) = 0$, a contradiction. Proved.

Corollary 2.29. *Let $W \leq W'$ be divisors on a compact Riemann surface P. Then the mapping $\mathcal{S}_W \to \mathcal{S}_{W'}$ induces an epimorphism*

$$H^1(P, \mathcal{S}_W) \to H^1(P, \mathcal{S}_{W'}) \to 0.$$

Theorem 2.30. ***The Riemann-Roch Theorem**: Let P be a compact Riemann surface of genus l. Suppose W is a divisor on P. Then*

$$dim\; H^0(P, \mathcal{S}_W) - dim\; H^1(P, \mathcal{S}_W) = 1 - l + deg\, W,$$

where $H^0(P, \mathcal{S}_W)$ and $H^1(P, \mathcal{S}_W)$ are finite dimensional vector spaces.

Proof (i) For $W = 0$ we have $H^0(P, \mathcal{S}) = \mathcal{S}(P)$, which consists of only constant functions thus giving $dim\; H^0(P, \mathcal{S}) = 1$ and $dim\; H^1(P, \mathcal{S}) = l$(by definition). Thus, the Riemann-Roch holds for $W = 0$.
(ii) Let $W' = W + V$, $V \in P$. V be the divisor which takes the value 1 at V and 0 otherwise. Let the theorem be true for one of the divisor W, W'. We have the exact cohomology sequence

$$0 \to H^0(P, \mathcal{S}_W) \to H^0(P, \mathcal{S}_{W+V}) \to \mathbb{C}$$

$$\to H^1(P, \mathcal{S}_W) \to H^1(P, \mathcal{S}_{W+V}) \to 0$$

which can be split into two short exact sequences. Define

$$U := Im(H^0(P, \mathcal{S}_{W'}) \to \mathbb{C})$$

and

$$Y := \mathbb{C}/U.$$

The vector spaces are finite dimensional and we have

$$dim\ H^0(P, \mathcal{S}_{W'}) = dim\ H^0(P, \mathcal{S}_W) + dim\ U$$

$$dim\ H^1(P, \mathcal{S}_W) = dim\ H^1(P, \mathcal{S}_{W'}) + dim\ Y$$

Adding

$$dim\ H^0(P, \mathcal{S}_{W'}) - dim\ H^1(P, \mathcal{S}_{W'}) - deg\ W' = dim\ H^0(P, \mathcal{S}_W) - dim\ H^1(P, \mathcal{S}_W) - deg\ W$$

Implying that, if the Riemann-Roch theorem holds for one of the two divisors, then it also holds for the other. And the Riemann-Roch theorem holds for the divisor $W = 0$. Thus, it also holds for all $W' \geq 0$.

(iii) We can write any arbitrary divisor W as

$$W = V_1 + \cdots + V_n - V_{n+1} - \cdots - V_o$$

where, $V_i \in P$ and starting with the zero divisor and using induction, we can prove that the *Riemann-Roch Theorem* holds for any divisor W. Proved.

Theorem 2.31. *Let P be a compact Riemann surface. Let l be the genus of P. Then on P, $\exists$ a meromorphic function g having pole of order $\leq l+1$ at u which is non-constant and is otherwise holomorphic. Here $u \in P$ is a point.*

Proof Define a map

$$W : P \to \mathbb{Z},$$

by

$$W(u) = l + 1$$

and for $p \neq u$,

$$W(p) = 0.$$

This is the divisor on P. Then,

$$dim H^0(P, \mathcal{S}_W) \geq 1 - l + deg\, W = 2\ (from\ \ the\ \ Riemann - Roch\ \ Theorem).$$

Thus, a function fulfilling the condition of the theorem, g exists in $H^0(P, \mathcal{S}_W)$. Also, $g \neq constant.$ Proved.

Bibliography

[1] O. Forster, *Lectures on Riemann Surfaces*. Springer-Verlag New York, 1981

[2] W. Tutschke and H. L. Vasudeva, *An Introduction to Complex Analysis: Classical and Modern Approaches*. Chapman & Hall/CRC, 2005

[3] W. M. Boothby, *An Introduction to Differential Manifolds and Riemannian Geometry*. Academic Press, 1975

[4] P. J. Higgins, *An Introduction to Topological Groups*. London Mathematical Society Lecture Note Series: Cambridge University Press, 1974