Some Topics in Riemannian Geometry

Nitesh Kumawat

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Certificate of Examination

This is to certify that the dissertation titled "Some Topics in Riemannian Geometry" submitted by Mr. Nitesh Kumawat (Reg. No. MS12034) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. K. Gongopadhyay   Dr. Mahender Singh   Dr. Pranab Sardar
(Supervisor)

Dated: April 21, 2017

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr.Pranab Sardar at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Nitesh Kumawat

(Candidate)

Dated: April 21, 2017

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr.Pranab Sardar

(Supervisor)

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IISER MOHALI

Dedicated to my family

For their endless love,support and encouragement

Contents

Introduction

Riemannian Geometry is the study of Riemannian manifolds which are, roughly speaking, smooth manifolds where we can measure the lengths of the tangent vectors. This helps us to compute lengths of curves in these spaces and talk about shortest paths etc. Hence, we can do geometry on these spaces.

In this expository thesis after introducing some basic notions of differentiable manifolds (Chapter 1) we define Riemannian metrics and show the existence of metrics on arbitrary differentiable manifolds (Chapter 2). Then we introduce connections (Chapter 3) and parallel transport. We incorporate a complete proof of the Levi-Civita's theorem on the existence and uniqueness of symmetric connections compatible with the metric. Then using the connections we define geodesics on Riemannian manifolds (Chapter 4). We discuss exponential maps after that and prove Gauss lemma. The thesis ends with introducing curvature of Riemannian manifolds. We show that the sphere S^2 and the hyperbolic plane $\mathbb{H}^2$ have constant sectional curvature. An important feature of the thesis is that we discuss many examples to illustrate the concepts.

However, no originality is claimed on the part of the author. The results and concepts dealt with in this thesis are quite standard. We have closely followed do Carmo's *Riemannian Geometry* and Barrett O'neill's S*semi-Riemannian Geometry* all the time.

Chapter 1

Some prerquisites

1.1 Definition

A topological space is *second countable* if it has a countable basis. A neighborhood of a point p in a topological space M is any open set containing p. An open cover of M is a collection $\{U_a\}_{a \in A}$ of open sets in M whose union $\cup_{a \in A} U_a$ is M.

1.2 Topological manifold

A topological space M is said to be *locally Euclidean* of dimension n if every point p in M has a neighborhood U such that there is a homeomorphism ϕ from an open subset V of $\mathbb{R}^n$ onto U . We call the pair (U, ϕ) a chart, U a *coordinate neighborhood* or a *coordinate open set*, and ϕ a *coordinate map* or a *coordinate system* on U. We say that a chart (U,ϕ) is *centered* at $p \in U$ if $\phi(p) = 0$.For any $p \in U$ the coordinates of $\phi^{-1}(p) = \ (x_1, .., x_n)$ are said to be the coordinates of p.Note that x_i's are functions $U \to \mathbb{R}$. The pair $(U, (x_1, .., x_n))$ is also referred to as a coordinate system.

1.3 Definition

A *topological manifold* is a Hausdorff, second countable, locally Euclidean space. It is said to be of *dimension n* if it is locally Euclidean of *dimension n*.

1.4 Differentiable manifold

We define a smooth atlas on a locally Euclidean space M of dimension n as follows : A collection $\{(U_a, \phi_a)\}$ of coordinate charts is called an atlas if it satisfies the following two properties:-

1. $\cup_a \phi_a(U_a) = M$;

2. for any pair α, β with $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) \neq \emptyset$,the map

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is a smooth map.

An atlas is called *maximal* if it is not contained in any other atlas. A locally Euclidean space M of dimension n with a smooth atlas $\{(U_\alpha, \phi_\alpha)\}$ is called a differentiable or smooth manifold of *dimension n.*

Remark : property 1 together with 2 is called *differentiable structure* on M.

1.5 Example

Differentiable structure on spheres. Let

$$S^n := \{x := (x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$$

A smooth atlas is given by stereographic projections from the north pole and the south pole. Let $N = (0, 0, ..., 1)$ be the north pole of S^n. We define $(\phi : \mathbb{R}^n \to S^n - \{N\})$ by

$$\phi(x) = (1 + \|x\|^2)^{-1}(x_1, x_2, ..., x_n, \|x\|^2 - 1)$$

where $x := (x_1, x_2, ...x_n, 0) \in \mathbb{R}^n$. Similarly let $S := (0, 0, ..., -1)$.Define $(\psi : \mathbb{R}^n \to S^n - \{S\})$ by

$$\psi(x) = (1 + \|x\|^2)^{-1}(x_1, x_2, ..x_n, 1 - \|x\|^2).$$

Note that $(S^n - N) \cup (S^n - S) = S^n$ and the transition map $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ given by

$$\phi \circ \psi^{-1}(x) = \|x\|^{-2}(x_1, x_2, ..x_n, 0)$$

which is smooth.So (ϕ, U) and (ψ, V) define a *differentiable structure* on S^n.

Smooth maps between Manifolds: Let M and N be smooth manifolds of dimension m and n, respectively. A continuous map $F : M \to N$ is smooth at a point p in M if there are charts (V, ψ) about F(p) in N and (U, ϕ) about p in M such that the composition $\psi \circ F \circ \phi^{-1}$, a map from the open subset $\phi(F^{-1}(V) \cap U)$ of $\mathbb{R}^m$ to $\mathbb{R}^n$, is smooth at $\phi(p)$. The continuous map $F : M \to N$ is said to be smooth if it is smooth at every point of M.

Diffeomorphism: Let M and N be smooth manifolds.A mapping $F : M \to N$ is a *diffeomorphism* if it is differentiable,bijective,and its inverse F^{-1} is differentiable.If $F : M \to N$ is diffeomorphism then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism for all $p \in M$.

REMARK: We $C^\infty(M)$ as the set of all real-valued smooth function on M.

1.6 Tangent space and tangent bundle

Let M be a smooth manifold. For all $p \in M$ let θ_P be the set of all real valued functions defined on some neighbourhood of p and differentiable there clearly θ_p has an $\mathbb{R}$-algebra structure. A tangent vector to M at the point P is an $\mathbb{R}$-linear function $D : \theta_P \to \mathbb{R}$ which satisfy the Leibniz rule :

$$D(fg)(p) = f(p)(Dg) + g(p)(Df).$$

The set of all tangent vectors at $p \in M$ is called tangent space of M at p. We denote it by T_PM.This has a natural vector space structure. Mention $\dim T_pM = \mathbb{R}^n$.

A smooth map $F : M \to N$ induces a linear map $dF_p : T_pM \to T_{F(p)}N$ such that $d(id)_p = id_M$ and it satisfies chain rule.

Vector Field : A *vector field* X on a differentiable manifold M is an assignment of tangent vectors to each point $q \in M$.A *vector field* is calld smooth or differentiable if Xf is smooth for $f \in C^\infty(M)$.

Theorem: If $(U, (x_1, .., x_n))$ is a coordinate system at $p \in M$ then

$$\{\frac{\partial}{\partial x_1} |_p, ..., \frac{\partial}{\partial x_n} |_p\}.$$

forms a basis of T_pM for all p in U.And $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$. are called coordinate vector fields. So we can write any *vector field* X as $X = \sum_i X_i \frac{\partial}{\partial x_i}$ on U.It is easy to check that

$X_1, ..., X_n$ are smooth function on U if and only if X is smooth.

Tangent bundle: Let M be a smooth manifold then we define the tangent bundle of M as follows.As a set

$$TM = \{(p, v); p \in M, v \in T_P M\}.$$

Topology on TM : Let $\pi : TM \to M$ be the canonical projection.Let $(U, (x_1, .., x_n))$ be a coordinate system on M. Every $v \in T_pM, p \in M$ can be written as $v = \sum_i a_i \frac{\partial}{\partial x_i} |_p$ where a_i's are real numbers.Define a map

$$\varphi : \pi^{-1}(U) \to \mathbb{R}^{2n}$$

by

$$\varphi(\sum_i a_i \frac{\partial}{\partial x_i}) = (x_1(\pi(v)), .., x_n(\pi(v)), a_1, .., a_n).$$

We define the weakest topology on TM such that φ are homeomorphism onto their image.

Now we will give differential structure on TM. Let M be a smooth manifold .Let $\{(x_\alpha, U_\alpha)\}$ be smooth atlas on M .Coordinates on U_α are denoted by $({x_1}^\alpha, .., {x_n}^\alpha)$and , basis of tangent space at $x_\alpha(U_\alpha)$ is given by

$$\{\frac{\partial}{\partial {x_1}^\alpha}, ..., \frac{\partial}{\partial {x_n}^\alpha}\}.$$

Define the map $F_\alpha : U\alpha \times \mathbb{R}^n \to TM$ for each α by

$$F_\alpha(x_1^\alpha, .., x_n^\alpha, a_1, .., a_n) = (x_\alpha(x_1^\alpha, .., x_n^\alpha), \sum_{i=1}^n a_i \frac{\partial}{\partial {x_i}^\alpha})$$

where $(a_1, ..., a_n) \in \mathbb{R}^n$.

Now we are going to prove that $\{(U_\alpha \times \mathbb{R}^n), F_\alpha)\}$ is smooth structure on M. Since $x_\alpha(U_\alpha)$ cover M and $((dx_\alpha)_q(\mathbb{R}^n)) = T_{x_\alpha(q)}M$ where $q \in U_\alpha$. we have that

$$\cup_\alpha F_\alpha(U_\alpha \times \mathbb{R}^n) = TM.$$

Where $x_\alpha : \mathbb{R}^n \to U_\alpha$ and $(dx_\alpha)_q : T_q(\mathbb{R}^n) \to T_{x_\alpha(q)}M$.

Now we have to check that transition map should be smooth:
so let $(p, v) \in F_\alpha(U_\alpha \times \mathbb{R}^n) \cap F_\beta(U_\beta \times \mathbb{R}^n)$ then

$$(p, v) = (x_\alpha(q_\alpha), dx_\alpha(u_\alpha)) = (x_\beta(q_\beta), dx_\beta(u_\beta))$$

where $q_\alpha \in U_\alpha$, $q_\beta \in U_\beta$ and $v_\alpha, v_\beta \in \mathbb{R}^n$.

Therefore

$$F_\beta{}^{-1} \circ F_\alpha(q_\alpha, v_\alpha) = F_\beta{}^{-1}(x_\alpha(q_\alpha), dx_\alpha(v_\alpha)) = ((x_\beta{}^{-1} \circ x_\alpha)(q_\alpha), d(x_\beta{}^{-1} \circ x_\alpha)(v_\alpha)).$$

so clearly $F_\beta{}^{-1} \circ F_\alpha$ is differential . Hence TM is smooth manifold with smooth structure $\{(U_\alpha \times \mathbb{R}^n), F_\alpha)\}$.

1.6.1 Partition of unity

Let M be a manifold and $U = \{U_\alpha\}_{i \in A}$ be an open cover of M.A collection of smooth functions $\{f_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ is called partition of unity subordinate to U if it satisfies the following property:

(a). $0 \leq f_\alpha(x) \leq 1$ $\forall \alpha \in A$ and all $x \in M$.

(b). Supp$f_\alpha \subset U_\alpha$.

(c). The collection of supports, $\{Suppf_\alpha\}_{\alpha \in A}$, is locally finite.

(d). $\sum_{\alpha \in A} f_\alpha(x) = 1$ for all $x \in M$.

Theorem: Let M be a smooth manifold and $U = \{U_\alpha\}_{i \in A}$ be an open cover of M.Then there exist a smooth partition of unity subordinate to U.

We omit the *proof*.

Chapter 2

Riemannian metric on manifolds

2.1 Definition

Let M be a smooth manifold of dimension n.A Riemannian metric on M is a family of inner products

$$\langle , \rangle_q : T_qM \times T_qM \to \mathbb{R},\ q \in M$$

such that

$$q \mapsto \langle (X(q), Y(q)) \rangle_q$$

defines a smooth function $M \to \mathbb{R}$ for all smooth vector field X ,Y on M.

2.2 Examples

1. Let $(U, (x_1, .., x_n))$ be a coordinate system on all of $\mathbb{R}^n$ then Riemannian metric on $\mathbb{R}^n$ is given by $\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle = \delta_{ij}$.

Immersed manifolds : Let $F : N \to M$ be an immersion , which means :

(1). F is differentiable .

(2). $dF_p : T_pN \to T_{F(p)}M$ is injective $\forall p \in M$.

If M has a Riemannian metric then N also has a Riemannian metric induced from F defined by $\langle u, v \rangle_p = \ \langle dF_p(u), dF_p(v) \rangle_{F(p)}\ \forall u, v \in T_pN$.

2. **Induced Riemannian metric on S^2 from $\mathbb{R}^3$.**

Let $S^2 = \{x \in \mathbb{R}^3 : \sum_{i=1}{}^3 x_i{}^2 = 1\}$.

Let $\phi : \mathbb{R}^2 \to S^2 - \{N\}$ be the inverse of the stereographic projection from the north pole and it is given by

$$\phi(p) = (x, y, z)$$

where

$$x = \frac{2u}{u^2 + v^2 + 1}, y = \frac{2v}{u^2 + v^2 + 1}$$

and

$$z = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}$$

where $p = (u, v) \in S^2$.

So the metric on S^2 induced from R^3 is given by $\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \rangle := \langle d\phi(\frac{\partial}{\partial u}), d\phi(\frac{\partial}{\partial v} \rangle$.

Now,

$$d\phi(\frac{\partial}{\partial u}) = \frac{\partial x}{\partial u}.\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}.\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}.\frac{\partial}{\partial z}$$

and

$$d\phi(\frac{\partial}{\partial v}) = \frac{\partial x}{\partial v}.\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}.\frac{\partial}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial}{\partial z}.$$

Now a simple computation shows that

$$\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \rangle = \frac{4}{(u^2 + v^2 + 1)^2} \delta_{uv}.$$

2.3 Basic definitions :

1. A smooth manifold M with given Riemannian metric is called Riemannian manifold.
2. Let M and M' be Riemannian manifolds. A diffeomorphism $g : M \to M'$ is called an isometry if :

$$\langle v, w \rangle_q = \langle dg_q(v), dg_q(w) \rangle_{g(q)}$$

for all $q \in M$ and v , w $\in T_q M$.

Local diffeomorphism and induced Riemannian metric: Let M and M' be Riemannian manifolds. Let $\pi : M' \to M$ be a diffeomorphism.Let $\langle , \rangle$ be a Riemannian metric on M then there exists a unique Riemannian metric $\langle , \rangle'$ on M' such that π is local isometry.

This follows from the fact that local diffeomorphisms are immersions.

2.4 Existence of Riemannian metric.

Proposition : Any smooth manifold M has a Riemannian metric.

Proof : We know that M has a smooth partition of unity sub-ordinate to any open cover. Let $\{V_i\}$ be an open cover of M , and let $\{\phi_i\}$ be a smooth partition of unity on M subordinate to $\{V_i\}$.Without loss of generality we may assume that V_i's are contained in coordinate neighborhoods $U_i's$,where $(U_i, \psi_i : V_i \to \mathbb{R}^n)$ is coordinate charts.Now, we know

1. $\phi_i \geqslant 0$, $\phi_i = 0$ on the $M - \overline{V_i}$

2. $\sum_i \phi_i(p) = 1$,$\forall p \in M$.

We can define a Riemannian metric on each V_i pulling back the metric from $\mathbb{R}^n$ since $\psi_i : V_i \to \mathbb{R}^n$ is diffeomorphism on an open set. Let $\langle , \rangle_i$ denote the inner product on each T_pM thus obtained.Then we can define a metric on M by setting

$$\langle v, w \rangle = \sum_i \phi_i(p) \langle v, w \rangle_i$$

$\forall p \in M$ and $u, v \in T_pM$. $\square$

Chapter 3

Connections on manifolds

3.1 Definitions

Affine connections: An Affine connection ∇ on a smooth manifold M is a mapping

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

$$(X, Y) \to \nabla_X Y$$

satisfying the following properties :

a) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$.

b) $\nabla_X (fY) = f\nabla_X Y + X(f)Y$.

c) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,

for all $f, g \in C^\infty(M)$ and $X, Y, Z \in \mathfrak{X}(M)$.

Lemma : If $X, Y \in \mathfrak{X}(M)$ and $X(p) = 0$ then $\nabla_X Y(p) = 0$.

Proof: Let U be a coordinate neighborhood of p and let $(x_1, .., x_n)$ denote the coordinates on U. Let f be a bump function at p where $\overline{V} \subseteq U$. This means $f = 1$ on a neighbourhood V of p and $f = 0$ on U^c. Now ,

$$\nabla_{f^2 X} Y = \nabla_{\sum f^2 X_i \frac{\partial}{\partial x_i}} Y$$

$$= \nabla_{(\sum fX_i)(f\frac{\partial}{\partial x_i})} Y$$

$$= \sum (fX_i) \nabla_{f\frac{\partial}{\partial x_i}} Y \quad \rightarrow (1).$$

Note : $X_i \in C^\infty(U)$, $fX_i \in C^\infty(M)$,$f\frac{\partial}{\partial x_i} \in \mathfrak{X}(M)$ where $\{\frac{\partial}{\partial x_1}, .., \frac{\partial}{\partial x_n}\}$ are coordinate vector fields.

Also

$$f^2 \nabla_X Y = \nabla_{f^2 X} Y \quad \rightarrow (2)$$

Now from equation (1) and (2) we have

$$f^2 \nabla_X Y = \sum (fX_i) \nabla_{f\frac{\partial}{\partial x_i}} Y$$

on M.

Now evaluate both sides at p :

LHS $= f(p)^2 \nabla_X Y|_p = \nabla_X Y|_p$

RHS $= \sum (fX_i)(p) \nabla_{f\frac{\partial}{\partial x_i}} Y|_p$

$= \sum f(p) X_i(p) \nabla_{f\frac{\partial}{\partial x_i}} Y|_p = 0$

since $f(p)X_i(p) = 0 \ \forall i$.

And similarly we can show that if $Y = 0$ on an open set $U \neq \emptyset$ then $\nabla_X Y = 0$ on U for all $X \in \mathfrak{X}(M)$.

Corollary : If $X, Y \in \mathfrak{X}(U)$ then $\nabla_X Y$ is a well defined element of $\mathfrak{X}(U)$. We also note that the properties (a),(b) and (c) of affine connection hold $\forall X, Y, Z \in \mathfrak{X}(U)$ and $f \in C^\infty(U)$.

Local Expression of $\nabla_X Y$: Let U be a coordinate neighborhood of p with coordinate $(x_1, .., x_n)$ and let $X_i = \frac{\partial}{\partial x_i}$. Let $X, Y \in \mathfrak{X}(U)$ then we have

$$X = \sum_i x_i X_i, Y = \sum_j y_j X_j.$$

Let $\nabla_{X_i} X_j = \sum_k {\Gamma_{ij}}^k X_k$ where ${\Gamma_{ij}}^k$ are smooth functions and then we have that

$$\nabla_X Y = \sum_k (\sum_{ij} x_i y_j {\Gamma_{ij}}^k + X(y_k)) X_k.$$

By the using properties (a),(b) and (c) of affine connections.

3.1.1 Covariant Derivative

Proposition : Let M be a smooth manifold with an affine connection ∇. There exist a unique vector field $\frac{DW}{dt}$ associated with each vector field W along a smooth curve $\gamma : I \to M$ such that the following hold:

If g is any smooth function on I and V is another vector field along γ then

a) $\frac{D}{dt}(gW) = \frac{dg}{dt}W + g\frac{DW}{dt}$

b) $\frac{D}{dt}(V+W) = \frac{DV}{dt} + \frac{DW}{dt}$

c) If W is given by $W(s) = X(\gamma(s))$ where $X \in \mathfrak{X}(M)$, then

$$\frac{DW}{dt} = \nabla_{\frac{d\gamma}{dt}} X.$$

Remark : $\frac{DW}{dt}$ is called the *covariant derivative* of W along γ.

Proof : First we prove the uniqueness of covariant derivative assuming the existence.Let U be a coordinate neighborhood with coordinates $(x_1, .., x_n)$.Suppose $\gamma(I) \cap U \neq \emptyset$. Let $\gamma(t) = (x_1(t), .., x_n(t))$. Then we can write the vector field W locally as $W = \sum_j w_j(t) X_j$, $j = 1, 2, .., n.$, where $X_j = X_j(\gamma(t))$.

By properties (a) and (b), we have

$$\frac{DW}{dt} = \sum_j \frac{dw^j}{dt} X_j + \sum_j w^j \frac{DX_j}{dt}.$$

By (c) and the properties of affine connection we have ,

$$\frac{DX_j}{dt} = \nabla_{\frac{d\gamma}{dt}} X_j$$

$$= \nabla_{(\sum_i \frac{dx_i}{dt} X_i)} X_j$$

$$= \sum_i \frac{dx_i}{dt} \nabla_{X_i} X_j$$

where $i, j = 1, 2, ..., n$.

Hence, we have

$$\frac{DW}{dt} = \sum_j \frac{dw^j}{dt} X_j + \sum_{i,j} \frac{dx_i}{dt} w^j \nabla_{X_i} X_j \quad \rightarrow (1).$$

The equation(1) shows that the operator $\frac{D}{dt}$ is unique.

Now we shall prove the existence of $\frac{DW}{dt}$.

To show existence define $\frac{DW}{dt}$ in U by equation (1). The properties (a), (b) and (c) can be checked without any difficulty. Let $U^{'}$ be another coordinate neighbourhood with $U \cap U^{'} \neq \emptyset$ then by the uniqueness of $\frac{DW}{dt}$ the definition agree in $U \cap U^{'}$. Hence we can extend the definition of $\frac{DW}{dt}$ to all of M. This completes the proof.□

3.1.2 Examples

Affine connection on $\mathbb{R}^n$:

Let $(x_1, .., x_n)$ be the usual coordinates on all of $\mathbb{R}^n$.Let $X, Y \in \mathfrak{X}(\mathbb{R}^n)$.Let $Y = \sum Y_j \frac{\partial}{\partial x_j}$. Then define

$$\nabla_X Y = \sum_j X(Y_j) \frac{\partial}{\partial x_j}.$$

Note that $X(Y_j)$ is the directional derivative of Y_j with respect to X.It is easy to verify the properties of affine connections.

3.2 Parallel transport

3.2.1 Definition :

Let M be a smooth manifold of dimension n with an affine connection ∇ and let $\gamma : I \rightarrow M$ be a smooth curve in M. Let W be a smooth vector field on M along this curve then we say that W is parallel along γ if $\frac{DW}{dt} = 0$,$\forall t \in I$.

Proposition: Let M be a smooth manifold of dimension n with an affine connection ∇. Let W_0 is a tangent vector in $T_{\gamma(t_0)}M$, $t_0 \in I$. where $\gamma : I \rightarrow M$ is a

smooth curve in M. Then there exist a unique parallel vector field W along γ such that $W(t_0) = W_0$.
we called this W(t)is the parallel transport of W_0 along the curve γ.

We omit the *Proof.*

Equation of parallel transport in coordinate system : Let M be a smooth manifold with an affine connection ∇. Let (U, ϕ) be a coordinate system in M. Suppose W is a parallel vector field along a smooth curve γ in U. Let $\gamma(t) = (x_1(t), .., x_n(t))$ and $X_j = \frac{\partial}{\partial x_j}$. Let $W = \sum_j w^j X_j$ then from equation (1) we have ,

$$\frac{DW}{dt} = \sum_j \frac{dw^j}{dt} X_j + \sum_{i,j} \frac{dx_i}{dt} w^j \nabla_{X_i} X_j = 0.$$

Now putting $\nabla_{X_i} X_j = \sum_k {\Gamma_{ij}}^k X_k$ and replacing j with k in the first sum we have a system of n differential equations in $w^k(t)$

$$0 = \frac{dw^k}{dt} + \sum_{i,j} \frac{dx_i}{dt} w^j {\Gamma_{ij}}^k, k = 1, 2, .., n.$$

Lemma : Suppose $(U, (x_1, .., x_n))$ is a coordinate system on a manifold M with an affine connection ∇.Suppose X is a vector field on M.We call it parallel on U if $\nabla_{X_i} X = 0 \ \forall i = 1, 2, .., n$. Given a function $f \in C^\infty(M)$ and $Y \in \mathfrak{X}(M)$ we have $\nabla_Y fX = Y(f)X$ on U if X is parallel in U.
Proof: Define Y in U as $Y = \sum_i Y_i \frac{\partial}{\partial x_i}$. Now $f : M \to \mathbb{R}$ is a smooth function so we have :

$$\begin{aligned} \nabla_Y(fX) &= \nabla_{\sum_i Y_i \frac{\partial}{\partial x_i}} fX \quad on\ U \\ = \sum Y_i f \nabla_{\frac{\partial}{\partial x_i}} X + \sum Y_i X \frac{\partial}{\partial x_i}(f) &\quad on\ U \\ &= \sum_i Y_i \frac{\partial}{\partial x_i}(f) X \quad on\ U \\ &= Y(f)X \quad on\ U. \end{aligned} \tag{3.1}$$

3.2.2 Parallel transport in $\mathbb{R}^n$:

Let $(x_1, .., x_n)$ be the usual coordinates on all of $\mathbb{R}^n$. Then $\frac{\partial}{\partial x_i}$ are parallel vector fields.This can be easily seen by the definition of affine connection on $\mathbb{R}^n$:

$$\nabla_X Y = \sum_j X(Y_j)\frac{\partial}{\partial x_j}.$$

Since Y_j are constant functions in this case. Hence

$$\nabla_X \frac{\partial}{\partial x_j} = 0$$

for all $X \in \mathfrak{X}(\mathbb{R}^n),\ 1 \leq i \leq n$.

3.3 Riemannian connection:

An affine connection ∇ on M is said to be *compatible* with the metric if for all $X, Y, Z \in \mathfrak{X}(M)$

$$X\langle Y, Z\rangle \ = \ \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

Remark : It is easy to see that this is equivalent to the following:
Given a curve $\alpha : I \to M$ and X ,Y two vector field on α then

$$\frac{d}{dt}\langle X, Z\rangle \ = \ \langle \frac{DX}{dt}, Y\rangle + \langle X, \frac{DW}{dt}\rangle.$$

3.3.1 Symmetric connection :

If ∇ is an affine connection on a smooth manifold M then it is called symmetric if $\nabla_X Y - \nabla_Y X = [X, Y]\ \ \forall X, Y \in \mathfrak{X}(M)$.

3.3.2 Levi-Civita connection or Riemannian connection:

Levi-Civita's theorem : On a Riemannian manifold M there exist a unique affine connection ∇ satisfying the following conditions :

(p). $\nabla_X Y - \nabla_Y X = [X, Y]$

(q) . $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

$\forall X, Y \in \mathfrak{X}(M)$.

Proof : To prove this theorem we will use the following lemma :

Let us denote $\mathfrak{X}^*(M)$ by the set of all dual form on M.

Lemma : Let M be a smooth manifold . For $X \in \mathfrak{X}(M)$ let X^* be the one form on M such that :

$$X^*(Y) = \langle X, Y\rangle \ , \forall\ Y \in M.$$

Then the function $X \mapsto X^*$ is $C^\infty(M)$ - linear isomorphism from $\mathfrak{X}(M)$ to $\mathfrak{X}^*(M)$.

Uniqueness : First we will prove the uniqueness of such connections by assuming the existence. Suppose D is another connection satisfying (p) and (q). Then:

$$X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle \quad (3.2)$$

$$Y\langle Z, X\rangle = \langle D_Y Z, X\rangle + \langle Z, D_Y X\rangle \quad (3.3)$$

$$Z\langle X, Y\rangle = \langle D_Z X, Y\rangle + \langle X, D_Z Y\rangle \quad (3.4)$$

Now by adding (3.2) and (3.3) and subtracting (3.4). We find the following expression:

$$2\langle D_X Y, Z\rangle = X\langle Y, Z\rangle + \ Y\langle Z, X\rangle - \ Z\langle X, Y\rangle$$

$$-\langle X, [Y, Z]\rangle + \ \langle Y, [Z, X]\rangle + \ \langle Z, [X, Y]\rangle.$$

which is called the *Koszul* formula. Hence we have

$$(\nabla_X Y, Z) = (D_X Y, Z)$$

Now from the lemma above we have $\nabla = D$.

Existence: Now we will prove the existence of a such ∇.

Denote the right hand side of equation(*) by $G(X, Y, Z)$.For fixed vector fields X ,Y in $\mathfrak{X}(M)$ the map $Z \to G(X, Y, Z)$ is clearly $C^\infty(M)$-linear.Hence by the above lemma there exist a unique vector field $\nabla_X Y$ such that $G(X,Y,Z) = \langle \nabla_X Y, Z\rangle, \forall Z \in \mathfrak{X}(M)$.Now we have to verify that $\nabla_X Y$ satisfies all all the necessary conditions.

All these properties are easy to prove .For instance , we prove the symmetry condition of ∇ as follows: Let $X, Y \in \mathfrak{X}(M)$.Then

$$2\ \langle \nabla_X Y - \nabla_Y X, Z\rangle \ = \ G(X,Y,Z) \ - \ G(Y,X,Z)$$

$$= \ \langle Z, [X,Y]\rangle - \ \langle Z, [Y,X]\rangle$$

$$= \ 2\ \langle [X,Y], Z\rangle.$$

$\forall Z \in \mathfrak{X}(M)$. This shows $\nabla_X Y - \nabla_Y X = \ [X,Y]$. □

REMARK : The connection ∇ given by $Livi - Civita's$ theorem called the $Livi - Civita$ connection.

3.4 Induced Connection:

Let M and N be smooth manifolds of dimensions m and n respectively. Let $F: M \to N$ be an immersion. If N has a Riemannian metric then F induces a Riemannian metric on M as mentioned before,

$$\langle v, w\rangle_p = \langle dF_p(v), dF_p(w)\rangle_{F(p)} \tag{3.5}$$

for all $v, w \in T_p(M)$.
Moreover F induces a connection on M as follows :

Let ∇' be the Riemannian connection on N and let (U,ϕ) be a coordinate system at p in M such that $\phi(U)$ is submanifold of N and is contained in a coordinate neighborhood V of N.Hence we pretend that $U \subseteq V$. Let $X, Y \in \mathfrak{X}(U)$. We can extend X , Y to the whole of V say to X' and Y' so that $X' \mid_p = X$ and $Y' \mid_p = Y$ $\forall p \in U$. Then we define

$$\nabla_X Y(p) = (\nabla'_{X'} Y')^T$$

where $(\nabla'_{X'}Y')^T$ denotes the tangential component of $(\nabla'_{X'}Y')$.

We can easily prove that ∇ is a well defined Riemannian connection on M with respect to the induced metric from N as follows:

First we prove that ∇ is compatible with the metric that is it satisfies :

$$X\langle Y, Z\rangle \;=\; \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

We have for all $p \in U$:

$$X\langle Y, Z\rangle(p) = X'\langle Y', Z'\rangle(p) = \langle \nabla_X{}'Y', Z'\rangle(p) + \langle Y', \nabla_X{}'Z'\rangle(p) \tag{3.6}$$

And we have $X'(p) = X$, $Y'(p) = Y$ and $Z'(p) = Z$ $\forall p \in U$ so equation(3.6) gives:

$$X\langle Y, Z\rangle(p) = \langle \nabla_X{}'Y', Z\rangle(p) + \langle Y, \nabla_X{}'Z'\rangle(p) \tag{3.7}$$

Now $\nabla_X Y(p) = (\nabla'_{X'}Y')^T$ is given and the inner product of vector field Z and Y with normal components give zero. Hence we have :

$$\begin{aligned} X\langle Y, Z\rangle(p) &= \langle (\nabla'_{X'}Y')^T, Z\rangle(p) + \langle Y, (\nabla'_{X'}Z')^T\rangle(p) \\ &= \langle \nabla_X Y, Z\rangle(p) + \langle Y, \nabla_X Z\rangle(p) \end{aligned} \tag{3.8}$$

This shows that ∇ satisfy the compatibility condition.

Now we will check the symmetry. For all $p \in M$ we have ,

$$(\nabla_X Y - \nabla_Y X)(p) = (\nabla'_{X'}Y' - \nabla'_{Y'}X')^T(p) = [X', Y']^T(p) = [X, Y](p) \tag{3.9}$$

The last equality of equation(3.9) can be proved in local coordinates: Let $(U, (x_1, .., x_n))$ be a system of coordinates.Then we can write $X = \sum_i X_i \frac{\partial}{\partial x_i}$ and $Y = \sum_j Y_j \frac{\partial}{\partial x_j}$ and similarly we can take $X' = \sum_i X'_i \frac{\partial}{\partial x_i}$ and $Y' = \sum_j Y'_j \frac{\partial}{\partial x_j}$.Now we have

$$\begin{aligned} [X', Y']^T &= (\sum_{i,j=1}^{n} \{X'_i \frac{\partial Y'_j}{\partial x_i} - Y'_i \frac{\partial X'_j}{\partial x_i}\} \frac{\partial}{\partial x_j})^T \\ &= (\sum_{i}^{m} \sum_{j}^{n} \{X_i \frac{\partial Y'_j}{\partial x_i} - Y_i \frac{\partial X'_j}{\partial x_i}\} \frac{\partial}{\partial x_j})^T \\ &= (\sum_{i,j=1}^{m} \{X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i}\} \frac{\partial}{\partial x_j}) \end{aligned} \tag{3.10}$$

since $\nabla_X Y(p)$ depends only on X(p) and Y along the integral curve $\gamma : I \to M$ of X through p. Then last equality of equation(3.10) gives :

$$[X', Y']^T = [X, Y]^T.$$

Thus ∇ satisfies the compatibility and symmetry so it is a Riemannian connection.

Lemma : Let $M^2 \subset \mathbb{R}^3$ be an embedded surface in $\mathbb{R}^3$ with induced Riemannian metric.Let $\gamma : I \to M$ be a smooth curve on M and let V be a vector field tangent to M along γ ; V can be thought of as a smooth function $V : I \to \mathbb{R}^3$, with $V(t) \in T_{\gamma(t)}(\mathbb{R}^3)$.V is parallel in M if and only if $\frac{dV}{dt}$ is perpendicular to $T_{\gamma(t)}(M) \subset T_{\gamma(t)}(\mathbb{R}^3)$ where $\frac{dV}{dt}$ is the usual derivative of V.
Proof : (a) We know that V is parallel if $\frac{DV}{dt} = 0$.Let $(U, (x_1, x_2, x_3))$ be a coordinate system on $\mathbb{R}^3$.Then we can write $V = \sum_i v_i X_i$ where $X_i = \frac{\partial}{\partial x_i}$, v_i's are smooth functions and $X_i = X_i(\gamma(t))$. Now from the equation of parallel transport in a coordinate system we have :

$$0 = \frac{DV}{dt} = \sum_i \frac{dv_i}{dt} X_i + \sum_{j,i} \frac{dx_j}{dt} v_i \nabla'_{X_j} X_i. \tag{3.11}$$

Rewrite as follows:

$$\begin{aligned} 0 = \frac{DV}{dt} = \ \nabla_{\frac{d\gamma}{dt}} V = \ (\nabla'_{\frac{d\gamma}{dt}} V)^T \\ = (\sum_i \frac{dv_i}{dt} X_i + \sum_{i,j} \frac{dx_j}{dt} v_i \nabla'_{X_j} X_i)^T \\ = (\sum_i \frac{dv_i}{dt} X_i)^T \\ = (\frac{dV}{dt})^T \end{aligned} \tag{3.12}$$

This shows that if V is parallel then $(\frac{dV}{dt})^T = 0$ this implies that

$$\frac{dV}{dt} \perp T_{\gamma(t)M}.$$

Let $S^2 = \{x \in \mathbb{R}^2 : |x| = 1\}$. Let $\gamma : \mathbb{R} \to S^2$ be the smooth curve given by,

$$t \mapsto (cost, sint, 0)$$

and

$$V = \frac{d\gamma}{dt} = (-sint, cost, 0).$$

Since

$$\frac{dV}{dt} = (-cost, -sint, 0)$$

is parallel to $(cost, sint, 0)$, $\frac{dV}{dt} = 0$ on S^2.

Christoffel symbols : Let M be a Riemannian manifold with Riemannian connection ∇. Let $(U, (x_1, ..., x_n))$ be a coordinate system and let $X_i = \frac{\partial}{\partial x_i}$, $1 \le i \le n$. Then $\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$ where Γ_{ij}^k's are smooth functions on U. Γ_{ij}^k's are called the *Christoffel symbol* and are given by:

$$\Gamma_{ij}^m = \frac{1}{2} \sum_k \{ \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \} g^{km} \tag{3.13}$$

where $g_{ij} = \langle X_i, X_j \rangle$ and (g^{km}) is the inverse of (g_{km}).

REMARK : $\Gamma_{ij}^k = \Gamma_{ji}^k$ by the symmetry property , $1 \le \forall i, j, k \le n$.

Example : We know that in $\mathbb{R}^n$ the $g_{ij} = \delta_{ij}$,$1 \le i, j \le n$ with respect to usual coordinate $(x_1, .., x_n)$. Hence $\Gamma_{ij}^k = 0$.

The next proposition shows that Riemannian connections are preserved under (local) isometries:

Proposition: Let $F : M \to N$ be an isometry then pull back of the Riemannian connection from N is the Riemannian connection of M that is $\nabla_X Y = \nabla_{dF(X)} dF(Y)$ for all $X, Y \in \mathfrak{X}(M)$.

Proof : Let X , Y be vector fields on M. Then $dF(X), dF(Y) \in \mathfrak{X}(N)$. Let $(U, (x_1, .., x_n))$ and $(V, (y_1, .., y_n))$ be coordinate systems at p and $F(p)$ respectively such that $x_i(q) = y_i(F(q))$ for all $q \in U$. We can do this since F is a diffeomorphism. It follows that we have $dF(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$. Let $X = \sum_i X_i \frac{\partial}{\partial x_i}$ and $Y = dF(X) = \sum_j Y_j \frac{\partial}{\partial x_i}$. Then

$$Y_i(F(q)) = Y_{F(q)}(y_i) = (dF(X_q))y_i = X_q(y_i \circ F) = X_q(x_i) = X_i(q). \tag{3.14}$$

It follows that the components of X and Y are also preserved by F.Since F is an isometry, metric is also preserved which means g_{ij}'s are preserved. So from equation(3.13) *Christoffel symbol* are also preserved.Then from the expression of $\nabla_X Y$ in a coordinate system we are *done*. □

Chapter 4

Geodesics

A parametrized curve $\gamma : I \to M$ is called a geodesic at a point $s \in I$ if $\frac{D}{dt}(\frac{d\gamma}{dt}) = 0$ at s. If $\frac{D}{dt}(\frac{d\gamma}{dt})$ is zero for all $s \in I$ then γ is called a geodesic.

REMARK : By the property (q) of $Levi - Civita$ connection we have that :

$$\frac{d}{dt}\langle\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle = 2\langle\frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle = 0. \tag{4.1}$$

This means γ has constant speed.

4.1 Equation of Geodesics in local coordinates

:

Let $(U, (x_1, ..x_n))$ be a coordinate system at $\gamma(s)$ in M. Let the local expression of $\gamma(t) = (x_1(t), .., x_n(t))$ .Then γ is a geodesic if and only if

$$\frac{D}{dt}(\frac{d\gamma}{dt}) = \sum_k(\frac{d^2x_k}{dt^2} + \sum_{ij}\Gamma_{ij}{}^k\frac{dx_i}{dt}\frac{dx_j}{dt})\frac{\partial}{\partial x^k} = 0 \tag{4.2}$$

by the equation of parallel transport in coordinate system.This implies

$$\frac{d^2x_k}{dt^2} + \sum_{ij}\Gamma_{ij}^k\frac{dx_i}{dt}\frac{dx_j}{dt} = 0, k = 1, 2, ..., n. \tag{4.3}$$

Hence a solution of the above second order differential equations give a geodesic.

4.2 Example

Geodesics in $\mathbb{R}^n$: We know that *Christoffel* symbols Γ^k_{ij} are zero for $\mathbb{R}^n$. So from equation(4.3) the equation of geodesics become $\frac{d^2x_k}{dt^2} = 0,\ 1 \leq k \leq n$. It follows that the geodesics of $\mathbb{R}^n$ are straight lines.

Homogeneity property of geodesic: Let us define $\gamma_v(t)$ as $\gamma_v(0) = p \in M$ and $\gamma'_v(0) = v$. If the geodesic $\gamma_v(t)$ is defined on the interval $I = [0, b]$ then the geodesic $\gamma_{cv}(t)$ for $c > 0$ is defined on the interval $[0, \frac{b}{c}]$ such that :

$$\gamma_{cv}(t) = \gamma_v(ct). \tag{4.4}$$

MAXIMAL GEODESIC : A geodesic $\gamma : I \to M$ with initial point p and initial velocity $v \in T_p(M)$ is called maximal in M if I is the largest possible domain, that is if $\gamma_1 : J \to M$ is another geodesic with initial point p and initial velocity v then we have $J \subset I$ such that $\gamma_1 = \gamma \mid_J$.

A Riemannian manifold M is called geodesically complete if every maximal geodesic is defined on $\mathbb{R}$.For example $\mathbb{R}^n$ is geodesically complete.

In the equation (4.3) of geodesics if we put $\frac{dx_k}{dt} = y_k$ then we have the following system of first order differential equation :

$$\begin{aligned} \frac{dx_k}{dt} &= y_k \\ \frac{dy_k}{dt} &= -\sum_{i,j} \Gamma^k_{ij} y_i y_j \end{aligned}. \tag{4.5}$$

So these second order differential equation in any coordinate neighbourhood U determines the first order differential equation in TU.

Lemma : There exists a unique vector field G on TM such that $\pi : TM \to M$ gives a one to one correspondence between the geodesics of M and the integral curves of G on TM.

Proof: We will use the following two results in this proof :

Result 1 :For any tangent vector $v \in T_pM$ there exist a neighborhood I of 0 in $\mathbb{R}$ and a neighbourhood N in TM and a C^∞ mapping $\gamma : I \times N \to M$ such that the curve $t \to \gamma_v(t)$ is the unique geodesic of M with $\gamma^{'}(0) = v$ and $\gamma(0) = p \ = \ \pi(0)$.

Result 2 : For any two geodesics $\gamma : I \to M$ and $\theta : J \to M$ if we have $\frac{d\gamma}{dt}(\alpha) = \frac{d\theta}{dt}(\alpha)$ for $\alpha \in I \cap J$. then we have $\gamma = \theta$ on $I \cap J$.

Suppose G_v is the initial velocity of the curve $t \to \gamma_v^{'}(t)$ for $v \in TM$ then by result(1) G is a smooth vector field on TM.

Claim (1): If $\gamma : I \to M$ is a geodesic in M , then we have γ' is an integral curve of G.

Proof: For all t , suppose $\alpha(t) = \gamma'(t)$ and for arbitrary fixed s suppose $\gamma^{'}(s) = w$ and $\beta(t) = \gamma_w^{'}(t)$. Then by result(2) we have $\gamma(s+t) = \ \gamma_w(t)$ this shows that we have $\alpha(s+t) = \ \beta(t)$. Then we have $\alpha^{'}(s+t) = \beta^{'}(t)$ by taking derivative.

So we have

$$\alpha^{'}(s) = \beta(0) = G_w = G_{\alpha(s)}. \tag{4.6}$$

This shows that $\gamma^{'}$ is an integral curve of G.

Claim (2): If $\gamma^{'}$ is an integral curve of G then $\pi \circ \gamma$ is a geodesic in M.

Proof: If $\alpha(0) = v$ then by the uniqueness of integral curve shows that we have $\pi \circ \alpha = \pi \circ \gamma^{'}{}_{v} = \ \gamma_v$ in a neighborhood of 0 since $t \to \gamma_v^{'}(t)$ is also integral curve of G by claim(1). For arbitrary s let η be another integral curve of G with initial velocity v. Then α and η have same velocity then we have $\alpha(s+t) = \eta(t)$ so we have $\pi \circ \alpha(s+t) = \pi \circ \eta(t) = \gamma_{\eta(0)}(t)$.
so the maps $\pi \circ \gamma^{'} = \ \gamma$ and $\gamma \to \gamma^{'}$ are inverses of eac other so we have if $\gamma^{'}$ is integral curve of G then $\pi \circ \gamma$ is geodesic in M.

4.3 Exponential map

Let $U_x \subset T_xM$ to be a set of vectors v in T_xM such that the geodesic with initial velocity v is defined on the interval $[0,1]$. Let $exp_x : U_x \to M$ be the map defined by

$$exp_x(v) = \gamma_v(1), \forall v \in T_xM. \tag{4.7}$$

This is called the exponential map of M on U_x.
If we define a geodesic as $t \to \gamma(at)$ where $a \in [0,1]$ then this geodesic has initial velocity av.Then by homogeneity of a geodesics we have

$$exp_x(av) = \gamma_{av}(1) = \gamma_v(a). \tag{4.8}$$

If a Riemannian manifold M is geodesically complete then $U_x = T_xM$.
Proposition: For each $x \in M$ exp_x is a diffeomorphism from an open neighborhood U' of 0 in T_xM to an open neighborhood U of x in M.

Proof : First we will prove that $exp_x : T_xM \to M$ is smooth on a neighborhood of $0 \in T_xM$.
We have $exp_x(v) = \gamma_v(1)$ for v in T_xM and $\gamma_v(1)$ is a geodesic which satisfy equation(4.3) which gives smooth solution and exponential map is just evaluation of γ at $t = 1$ which is clearly smooth.

Now we will prove the existence of U' :

We have $\pi : TM \to M$ which is the cannonical projection map which is smooth. Let $p \in M$.Let (U, ϕ) be a coordinate neighborhood of p.Then by the topology on TM we have $\pi^{-1}(U)$ is open in TM and denote this by V. So $U' = T_xM \cap \pi^{-1}(U)$ is an open neighbourhood in T_xM.

Now the differential of exponential map defined by $(dexp_x) : T_0(T_xM) \to T_xM$ is an isomorphism of $T_0(T_xM)$ to T_xM :

$$\begin{aligned} dexp_x(v) &= \frac{d}{dt}(exp_x(tv))|_{t=0} \\ &= \frac{d}{dt}(\gamma_{tv}(1))|_{t=0} \\ &= \frac{d}{dt}(\gamma_v(t))|_{t=0} \\ &= v. \end{aligned} \tag{4.9}$$

Now we will prove that there exist a neighbourhood W of x such that for all q in W the map $dexp_x|_q : T_0(T_xM) \to T_xM$ is an isomorphism.
This is an easy consequence of following lemma :

Lemma : Suppose M , N are smooth manifolds of dimension n and $f : M \to N$ is smooth. Let $p \in M$ be such that $df_p : T_pM \to T_{f(p)}N$ is an isomorphism.Then there exist a neighborhood $p \in U \subset M$ such that $df|_x : T_xM \to T_{f(x)}N$ is an isomorphism for all x in U.

So by this lemma the differential of exponential map is also an isomorphism in a neighbourhood of x. Now we apply the inverse function theorem to complete the proof.

Normal neighbourhood : In $\mathbb{R}^n$ a set U' containing 0 is called starshaped about 0 if for all $v \in U'$, $tv \in U'$ for all $t \in [0, 1]$. If $U' \subset T_xM$ is starshaped such that $\exp_x : U' \to U \subset M$ is diffeomorphism then U is called a normal neighbourhood of x.

Let $(e_1, .., e_n)$ be an orthonormal basis of T_xM. The normal coordinate system $(x_1, .., x_n)$ determined by $(e_1, .., e_n)$ assign to each point $p \in U$ the vector $v = {exp_x}^{-1}(p)$, since exp_x is diffeomorphism such a v exist can be written as

$$v = \ {exp_x}^{-1}(p) = \ \sum_i x_i(p)e_i.$$

Example : Exponential map for $\mathbb{R}^n$
The geodesics in $\mathbb{R}^n$ through p are given by $\gamma(t) = p + tv$ $\gamma'(0) = v$. So $exp_p : T_p(R^n) \to R^n$ is defined for any $v \in T_p(R^n)$.In fact we have

$$exp_p(v) = \ \gamma_v(1) \ = \ p + v.$$

Clearly this map is diffeomorphism from $T_p(\mathbb{R}^n) \to \mathbb{R}^n$.

4.4 Gauss Lemma.

Definition. A two parameter smooth mapping $f : W \subset \mathbb{R}^2 \to M$ is called a parametrized surface. If (u, v) are the usual coordinates on $\mathbb{R}^2$ then we call $u \mapsto y(u, v_0)$ the u -parameter curve for $v = v_o$.Similarly we define v-parameter curve for $u = u_o$.

Let $(x_1, .., x_n)$ be the coordinates on $f(W) \subset M$ then we can write,

$$\begin{aligned} f_u &= \frac{\partial f}{\partial u} = \sum_i \frac{\partial x_i}{\partial u} \frac{\partial}{\partial x_i} \\ f_v &= \frac{\partial f}{\partial v} = \sum_i \frac{\partial x_i}{\partial v} \frac{\partial}{\partial x_i}. \end{aligned} \tag{4.10}$$

Proposition . (*Symmetry*) Let M be a Riemannian manifold with connection ∇.Let $\frac{D}{\partial v}\frac{\partial f}{\partial u}$ denote the covariant derivative of $\frac{\partial f}{\partial u}$ along the v-parameter curves of the two-parameter parametrized surface $f : W \to M$.Similarly we define $\frac{D}{\partial u}\frac{\partial f}{\partial v}$. Then

$$\frac{D}{\partial v}\frac{\partial f}{\partial u} = \frac{D}{\partial u}\frac{\partial f}{\partial v}.$$

We omit the *Proof*.

Lemma. (*Gauss*) Let M be a Riemannian manifold and $p \in M$. Let z be a non zero tangent vector in T_pM and $v_z, w_z \in T_z(T_pM)$ then we have:

$$\langle dexp_p(v_z), dexp_p(w_z)\rangle = \langle v_z, w_z\rangle \tag{4.11}$$

where v_z is radial.

Proof . We can take $v = z$ since $v_z = tz$, for some $t > 0$. Let us define a parametrized surface in T_pM by

$$y(t, s) = t(v + sw).$$

Now we define a parametrized surface in M as follows:

$$\tilde{y}(t, s) = exp_p(t(v + sw)).$$

Now we have $y_t(1,0) = v_v$ and $y_s(1,0) = w_v$, hence $\tilde{y}_t(1,0) = dexp_p(v_v)$ and $\tilde{y}_s(1,0) = dexp_p(w_v)$. So we claim that

$$\langle \tilde{y}_t(1,0), \tilde{y}_s(1,0) \rangle = \langle v, w \rangle.$$

The mapping $t \to \tilde{y}(t,s)$ defines a geodesic.Hence its acceleration is zero that means $\tilde{y}_{tt} = 0$. So we have

$$\langle \tilde{y}_t, \tilde{y}_t \rangle = \langle v + sw, v + sw \rangle.$$

Now using previous *proposition* we have following :

$$\frac{\partial}{\partial t} \langle \tilde{y}_t, \tilde{y}_s \rangle = \langle \tilde{y}_t, \tilde{y}_{st} \rangle = \langle \tilde{y}_t, \tilde{y}_{ts} \rangle = \frac{1}{2} \frac{\partial}{\partial s} \langle \tilde{y}_t, \tilde{y}_t \rangle. \tag{4.12}$$

Now putting the value of $\langle \tilde{y}_t, \tilde{y}_t \rangle$ in above equation we have

$$(\frac{\partial}{\partial t} \langle \tilde{y}_t, \tilde{y}_s \rangle)(t,0) = \langle v, w \rangle \quad \forall t.$$

Now

$$\lim_{t \longrightarrow 0} \tilde{y}_s(t,0) = \lim_{t \longrightarrow 0} (dexp_p(tv_v))tw = 0.$$

Hence we have

$$\langle \tilde{y}_t, \tilde{y}_s \rangle (0,0) = 0.$$

Thus by some elementary calculus we have $\langle \tilde{y}_t, \tilde{y}_s \rangle (t,0) = t \langle v, w \rangle$. We put $t = 1$ to complete the proof.□

Chapter 5

Curvature

Let M be a Riemannian manifold with Riemannian connection ∇. The mapping $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$\begin{aligned} R(X,Y)Z &= [\nabla_Y, \nabla_X]Z + \nabla_{[X,Y]}Z \\ &= \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z. \end{aligned} \tag{5.1}$$

is called the *curvature* of M.

Proposition 1 :

(1.) If $f, g \in C^\infty(M)$ then ,

$$\begin{aligned} R(fX_1 + gX_2, Y_1) &= fR(X_1, Y_1) + gR(X_2, Y_1) \\ R(X_1, fY_1 + gY_2) &= fR(X_1, Y_1) + gR(X_1, Y_2). \end{aligned} \tag{5.2}$$

where $X_1, X_2, Y_1, Y_2 \mathfrak{X}(M)$.

(2.) If $f \in C^\infty(M)$ and $X, Y, Z, W \in \mathfrak{X}(M)$ then ,

$$\begin{aligned} R(X,Y)fZ &= fR(X,Y)Z \\ R(X,Y)(Z+W) &= R(X,Y)Z + R(X,Y)W. \end{aligned} \tag{5.3}$$

Proof : The proof is just simple calculation using the properties of Lie bracket and Riemannian connection ∇. Hence we omit it.

Bianchi Identity :

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0. \tag{5.4}$$

Proof : For proving this we will use *Jacobi* identity which is

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0. \tag{5.5}$$

Now, we have

$$R(X, Y)Z = [\nabla_Y, \nabla_X]Z + \nabla_{[X,Y]}Z. \tag{5.6}$$

$$R(Y, Z)X = [\nabla_Z, \nabla_Y]X + \nabla_{[Y,Z]}Z. \tag{5.7}$$

$$R(Z, X)Y = [\nabla_X, \nabla_Z]Y + \nabla_{[Z,X]}Z. \tag{5.8}$$

now adding (5.6) + (5.7) + (5.8) and using symmetry property ($\nabla_X Y - \nabla_Y X = [X, Y]$) we get the desired identity.□

Proposition 2. Let $\langle R(X, Y)Z, T\rangle = \ (X, Y, Z, T)$. Then we have :

(i). $(X, Y, Z, T) + (Y, Z, X, T) + (Z, X, Y, T) = 0.$

(ii). $(X, Y, Z, T) = -(Y, X, Z, T)$

(iii). $(X, Y, Z, T) = -(X, Y, T, Z)$

(iv) . $(X, Y, Z, T) = (Z, T, X, Y)$.

Proof : The proof is just simple calculation using properties of ∇ and the Lie bracket. Hence we omit the proof.

Expression of curvature in local coordinates :

Let (U, ϕ) be a coordinate system at $p \in M$. Let us denote $X_i = \frac{\partial}{\partial x_i}$. Then we have

$$R(X_i, X_j)X_k = \sum_l R^l_{ijk} X_l. \tag{5.9}$$

Using $\nabla_{X_i} X_j = \sum_k \Gamma^k_{ij} X_k$ and the expressions for $R(X_i, X_j)X_k$ and from the equation(5.1) we have,

$$R^s_{ijk} = \sum_l \Gamma^l_{ik}\Gamma^s_{jl} - \sum_l \Gamma^l_{jk}\Gamma^s_{il} + \frac{\partial}{\partial x_j}\Gamma^s_{ik} - \frac{\partial}{\partial x_i}\Gamma^s_{jk}. \tag{5.10}$$

We simply put

$$\langle R(X_i, X_j)X_k, X_s\rangle = \sum_l R^l_{ijk} g_{ls} = R_{ijks}. \tag{5.11}$$

Remark : The value of (X, Y, Z, T) at p depends only on the values of $X(p), Y(p), Z(p)$ and $T(p)$.

5.1 Sectional curvature

If $x, y \in T_pM$ then we define

$$|x \wedge y| = \sqrt{|x|^2|y|^2 - \langle x, y\rangle^2}.$$

This is the area of parallelogram in T_pM spanned by x,y.

Proposition 3. Let Ω be a 2 - dimensional subspace spanned by $v, w \in T_pM$. The real number

$$K(v,w) = \frac{(v,w,v,w)}{|v \wedge w|^2} \tag{5.12}$$

is independent of choice of the linearly independent vectors $v, w \in \Omega$. Hence it is reasonable to write $K(v,w) = K(\Omega)$.

REMARK : It is called the sectional curvature of Ω at p.

Proof: Let $\{v, w\}$ be a basis for Ω. Let $x, y \in \Omega$ be two linearly independent vectors. Then $x = \lambda_1 v + \lambda_2 w$ and $y = \mu_1 v + \mu_2 w$ for some $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ where $\lambda_1\mu_2 - \mu_1\lambda_2 \neq 0$. A simple calculation shows that

$$(x,y,x,y) = (\lambda_1\mu_2 - \mu_1\lambda_2)^2(v,w,v,w)$$

and

$$|x \wedge y| = (\lambda_1\mu_2 - \mu_1\lambda_2)^2|v \wedge w|.$$

Hence from (40) we have $K(x,y) = K(v,w)$.

Lemma ($K(\Omega)'s$ determine R :) Let $F : W \times W \times W \to W$ and $F' : W \times W \times W \to W$ be tri-linear mappings satisfying the properties of Proposition 2 , where W is a vector space of dimension ≥ 2. with an inner product.Let us denote $(x,y,w,z) = \langle F(x,y)w, z\rangle$ and $(x,y,w,z)' = \langle F'(x,y)w, z\rangle$. If $\{x, y\}$ are two linearly independent vectors then we define $K(x,y) = K(\Omega)$ and $K'(x,y) = K'(\Omega)$ by equation(5.12) , where Ω is two dimensional subspace of W. If $K(\Omega) = K'(\Omega)$ for all Ω then $F = F'$.

Lemma 4 : Let M be a Riemannian manifold.Define a tri-linear mapping $C : T_pM \times T_pM \times T_pM \to T_pM$ by

$$\langle C(X,Y,T), Z\rangle = \langle X,T\rangle\langle Y,Z\rangle - \langle Y,T\rangle\rangle\langle X,Z\rangle, \tag{5.13}$$

where $p \in M$ and $X, Y, T, Z \in T_pM$. Then M has constant sectional curvature equal to K_o if and only if $R = K_oC$ where R is the curvature of M.

Proof . The proof follows from an easy calculation by using Propostion 2 and the previous lemma.□

Corollary 5.: Let M be a Riemannian manifold with connection ∇. Let T_pM be a tangent plane at $p \in M$. We have $K(p, \Omega) = K_o$ for all 2-dimensional subspace $\Omega \subset T_pM$ if and only if

$$R_{ijkl} = K_o(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \tag{5.14}$$

where $R_{ijkl} = \langle R(e_i, e_j)e_k, e_l\rangle, i, j, k, l = 1, .., n$ where $(e_1, .., e_n)$ is an orthonormal basis of T_pM and $\delta'_{ij}s$ denote the *Kronecker delta.*
It follows that $K(p, \Omega) = K_o$ for all $\Omega \subset T_pM$ if and only if $R_{ijij} = K_o$ for all $i \neq j$ and $R_{ijkl} = 0$ for all other cases.

Theorem : Let M and N are Riemannian manifolds of dimension m and n respectively.Let $f : M \to N$ is an isometry.Then curvature is preserved by this isometry.
Proof : Proof follows from isometry preserves connection.

Examples :

1. Let $S^n := \{x := (x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$ with the induced metric from $\mathbb{R}^{n+1}$. Then S^n has constant sectional curvature equal to 1.
We demonstrate it for S^2.We use the same notation as in the example in section 2.2. Then

$$g_{11} = g_{22} = \frac{4}{(1 + u^2 + v^2)^2},$$

$$g_{12} = g_{21} = 0$$

and the *Christoffel symbols* are :

$$\begin{aligned}
-\Gamma_{11}^1 = \Gamma_{22}^1 &= \frac{2u}{1+u^2+v^2},\\
\Gamma_{11}^2 = -\Gamma_{22}^2 &= \frac{2v}{1+u^2+v^2},\\
\Gamma_{12}^1 = \Gamma_{21}^1 &= \frac{-2v}{1+u^2+v^2},\\
\Gamma_{12}^2 = \Gamma_{21}^2 &= \frac{-2u}{1+u^2+v^2}.
\end{aligned} \tag{5.15}$$

We have the following equations ,

$$\begin{aligned}
R_{ijks} &= \sum_l R_{ijk}^l g_{ls}\\
R_{ijk}^s &= \sum_l \Gamma_{ik}^l \Gamma_{jl}^s - \sum_l \Gamma_{jk}^l \Gamma_{il}^s + \frac{\partial}{\partial x_j}\Gamma_{ik}^s - \frac{\partial}{\partial x_i}\Gamma_{jk}^s.
\end{aligned} \tag{5.16}$$

we put $k = i = 1$ and $s = j = 2$ then using above equations we have

$$R_{1212} = R_{121}^1 g_{12} + R_{121}^2 g_{22} = R_{121}^2 g_{22} \tag{5.17}$$

since $g_{12} = 0$. And we have from the second part of equation(5.16) ,

$$R_{121}^2 = \Gamma_{11}^1\Gamma_{21}^2 + \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{21}^1\Gamma_{11}^2 - \Gamma_{21}^2\Gamma_{12}^2 + \frac{\partial}{\partial v}\Gamma_{11}^2 - \frac{\partial}{\partial u}\Gamma_{21}^2. \tag{5.18}$$

Now putting the values of all Γ_{ij}^k and after a simple computation we have ,

$$\begin{aligned}
R_{2121} = R_{1212} &= \frac{4}{(1+u^2+v^2)^2} \times \frac{4}{(1+u^2+v^2)^2}\\
&= K_o(g_{11}.g_{22}).
\end{aligned} \tag{5.19}$$

After putting the values of g_{11} and g_{22} we have $K_0 = 1$.Similarly we can show that R_{ijks} is zero for other cases. We note that $SO(3,\mathbb{R})$ acts transitively by isometry on S^2.Hence we are done by previous theorem.

2.

The upper half plane $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2; y > 0\}$ has constant sectional curvature equal to -1.

The metric on H^2 is given by

$$g_{11} = g_{22} = \frac{1}{y^2} \ , \ g_{12} = g_{21} = 0.$$

The *Christoffel symbols* are given by

$$\begin{aligned} \Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} &= 0, \\ \Gamma^2_{11} &= \frac{1}{y}, \\ \Gamma^1_{12} = \Gamma^2_{22} &= \frac{-1}{y}. \end{aligned} \tag{5.20}$$

We use similar calculation as we did in the previous example.
Now here we put $i = k = 2$ and $j = s = 1$. so we have

$$R_{1212} = R_{2121} = R^1_{212}\ g_{12} + R^2_{212}\ g_{22} = R^2_{212}\ g_{22}. \tag{5.21}$$

Since $g_{12} = 0$. Now we have ,

$$R^2_{212} = \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{21} - \Gamma^2_{12}\Gamma^1_{22} + \frac{\partial}{\partial x}\Gamma^1_{22} - \frac{\partial}{\partial y}\Gamma^1_{12}. \tag{5.22}$$

Now putting the values of Γ^k_{ij} and by simple computation we have

$$\begin{aligned} R_{2121} &= -\frac{1}{y^4} \\ &= K_o(g_{11}.g_{22}). \end{aligned} \tag{5.23}$$

Now putting the values of g_{11} and g_{22}.We have $K_o = -1$.Similarly we can show that R_{ijks} is zero for other cases.

Change of Curvature under scaling of metric :

Proposition 6. Suppose M is a Riemannian manifold with a Riemannian metric $\langle , \rangle$ and $k > 0$ is a constant. On M change the Riemannian metric $\langle , \rangle$ to $\langle , \rangle^{'}$ as follows :

$$\langle u, v\rangle^{'}_p = k\langle u, v\rangle_p \tag{5.24}$$

for all $p \in M$ and $u, v \in T_pM$. For any linearly independent $x, y \in T_pM$ if $K(x, y)$ is sectional curvature for the metric $\langle , \rangle$ and $K^{'}(x, y)$ is sectional curvature for the metric $\langle , \rangle^{'}$ then we have $K^{'}(x, y) = \frac{K(x,y)}{k}$.

Proof : Let $(U,(x_1,..,x_n))$ be a coordinate system on M. In U the *Christoffel symbols* with respect to $\langle,\rangle$ are given by ,

$$\Gamma_{ij}^m = \frac{1}{2}\sum_k\{\frac{\partial}{\partial x_i}g_{jk} + \frac{\partial}{\partial x_j}g_{ki} - \frac{\partial}{\partial x_k}g_{ij}\}g^{km} \tag{5.25}$$

where $g_{ij} = \langle\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_i}\rangle$ and (g^{km}) is the inverse of (g_{km}). Let us denote the metric tensor etc. by the same symbols with a prime for the new metric.Then $g'_{ij} = kg_{ij}$ this implies $(g'^{ij}) = (g'_{ij})^{-1} = \frac{1}{k}(g^{ij})^{-1}$. It is clear that ${\Gamma'}^k{}_{ij} = \Gamma_{ij}^k$ for all i,j,k. Hence $R'_{ijkl} = kR_{ijkl}$. Thus for any pair of linearly independent vectors $x,y \in T_pM$ we have $\langle R(x,y)x,y\rangle = \frac{1}{k}\langle R'(x,y)x,y\rangle$.

Therefore,

$$\begin{aligned} K(x,y) &= \frac{\langle R(x,y)x,y\rangle}{\langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2} \\ &= \frac{\langle R(x,y)x,y\rangle'/k}{\{\langle x,x\rangle'\langle y,y\rangle' - \langle x,y\rangle'^2\}/k^2} \\ &= k\frac{\langle R(x,y)x,y\rangle'}{\langle x,x\rangle'\langle y,y\rangle' - \langle x,y\rangle'^2}. \end{aligned} \tag{5.26}$$

Hence, $K'(x,y) = \frac{K(x,y)}{k}$. □

Example : We shall use Proposition (6) to calculate the sectional curvature of

$$S^2(r) = \{(x,y,z) \in \mathbb{R}^3\ ;\ x^2+y^2+z^2 \ = \ r^2\}.$$

Since the induced metric on $S^2(r)$ from $\mathbb{R}^3$ is equivalent to scaling the metric of $S^2(1)$ by a factor of r^2, $S^2(r)$ has constant curvature $\frac{1}{r^2}$.

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