Modeling Micro-gradient Magnetic Field Distribution with FEM

Parul Janagal

A dissertation submitted for the partial fulfillment of BS-MS dual degree in Science

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Certificate of Examination

This is to certify that the dissertation titled “Modeling Micro-gradient Magnetic Field Distribution with FEM” submitted by Ms. Parul Janagal (Reg. No. MS 12 042) for the partial fulfillment of BS-MS dual degree program of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Sanjeev Kumar   Dr. Abhishek Choudhary   Dr. S.K.Biswas

(Supervisor)

Dated: April 21, 2017

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Samir K. Biswas at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of the work done by me and all sources listed within have been detailed in the bibliography.

Parul Janagal

(Candidate)

Dated: April 21, 2017

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Samir K. Biswas

(Supervisor)

Acknowledgement

I would like to thank my guide Dr. Samir K. Biswas and IISER Mohali for providing me an opportunity to work and learn about the emerging field of Medical Physics. This wouldn't have been possible without the support of my friends, Adarsh and Harshita and my family. I would also like to thank IISER Mohali Library for its services and vast collection.

Parul Janagal

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Notation

I = Current

q_i = Charge

$\vec{F}$ = Force

$\vec{E}$ = Electric Field

$\vec{B}$ = Magnetic Field

$\rho(\vec{x}, t)$ = Charge density at position $\vec{x}$ at time t

$\vec{J}$ = Current Density

μ_0 = Permeability of free space

ϵ_0 = Permitivity of free space

∇ = Differential operator

$\delta(x - x_i)$ = Delta function at $\vec{x} = x_i$.

Contents

Abstract

One of the methods to image a sample is to send the radiation of a particular frequency towards the source. The atoms will resonate and send out the signal which can be used to locate the position of each atom in particular. A gradient field can be used to determine the position of atoms/molecules in the sample. Such a field can be created by a number of coil systems. Here we are going to find the field distribution produced by Helmholtz coils. For our purpose, we will use Finite Element Method as the technique to solve partial differential equations.

Chapter 1

Problem Statement

With the rising population of today's world, the number of diseases spreading across is also increasing. In this case, we regularly require new tools to diagnose such conditions. Although we already have many diagnostic tools available, the need for more accuracy and information is inevitable.

MRI is one such diagnostic tool which can be used to find known and unknown abnormalities such as, tumors, bleeding, injury, blood vessel diseases, or infection. It provides more information than other scans, like CT scan, X-ray, etc. It is currently the most sensitive non-invasive diagnostic tool.

MRI uses magnetic field and radio waves to generate the image of the body. Gradient magnetic field coils are used to produce the desired, unique over space, magnetic fields. Gradient coils provide controlled variations to the primary magnetic field to provide spatial localization of the signals.

A wide variety of coils is used to get the required gradient fields. One such simple system is of Helmholtz coils. In this design, current is in opposite directions. The coils are at a distance R from each other, where R is the radius of coil.

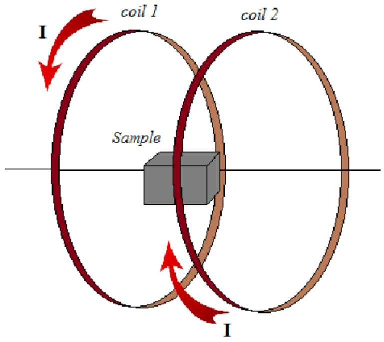


Figure 1.1: Helmholtz Coils

In this project, I will try to find the magnetic field distribution obtained by Helmholtz coils using the Finite Element Method.

Chapter 2

Theoretical Background

Electromagnetism

2.1 Electric and Magnetic Fields

To the best of our knowledge, there are four forces at play in the universe viz., gravitational force, electromagnetic force, weak force and strong force. The basic unit of electrodynamics is a charge. The charge experiences an electromagnetic force when placed in an electromagnetic field.

The **electric field** ($\vec{E}$) is a region around a charge (q_1) within which force would be exerted on other charged particles (q_2). This force is given by the Coulomb's Law -

$$\vec{F_{12}} = \frac{Kq_1q_2}{r^2} = q_2\vec{E_1} \tag{2.1}$$

Whereas if the charge is in motion, it creates a **magnetic field** ($\vec{B}$). However there are other materials as well which set up their magnetic fields, but our primary concern here is moving electric charges or currents. The most common source of a magnetic field is an electric current loop. The position $\vec{r}$(t) of a particle with charge q is dictated by the electric and magnetic fields through Lorentz force law -

$$\vec{F} = q(\vec{E} + \dot{\vec{r}} \times \vec{B}) \tag{2.2}$$

Roughly speaking, an electric field accelerates a particle in the direction of $\vec{E}$, while a magnetic field causes a particle to move in circles in the plane perpendicular to $\vec{B}$.

2.2 Charge and Current Density

Each particle carries with itself a number of properties which determine how it interacts with the four forces. For the force of electromagnetism, this property is a charge. It can be positive or negative, with a unit Coulomb (C).

To learn about the dynamics of continuous objects we now consider **Charge density** ($\rho(\vec{x}, t)$). Charge density is defined as charge per unit volume. The total charge Q in a region V is -

$$Q = \int_V \rho(\vec{x}, t) d^3x \tag{2.3}$$

The movement of such a charge density from one place to another gives rise to another quantity called as **Current density** ($\vec{J}(\vec{x}, t)$). For every surface S -

$$I = \int\limits_S \vec{J}.d\vec{S} \tag{2.4}$$

gives the charge per unit time passing through the surface S. I is called the current, i.e. charge per unit time. The quantity $\vec{J}$ is defined as the current per unit area.

$$\vec{J} = \frac{I}{S} = \rho\dot{\vec{x}} \tag{2.5}$$

2.3 Continuity Equation

The most important property of charge is that it is conserved *locally*, which implies that the charge cannot just vanish from one part of the universe and appear somewhere else. The property of local conservation means that charge density can only change over the course of time if there is a compensating current flowing in or out of the system. We can express this in the form of continuity equation -

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.\vec{J} = 0 \tag{2.6}$$

This equation arises whenever a quantity is locally conserved.

2.4 Maxwell's Equations

Until the 19^{th} century, Electricity and Magnetism were considered as different phenomenon, but are now known as components of a unified theory of **Electromagnetism**.

The complete unification of electric and magnetic phenomena in a mathematical theory was given by Maxwell, in a set of four equations. These four equations describe how electric and magnetic fields interact with the objects and how they propagate in space. Following is the set of four Maxwell's equations -

$$\nabla . E = \frac{\rho}{\epsilon_0} \tag{2.7}$$

$$\nabla . B = 0 \tag{2.8}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{2.9}$$

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \tag{2.10}$$

2.5 The potentials and Wave equation

Now, we want to solve the Maxwell's equations. We start with eqn.(2.8) -

$$\nabla . B = 0 \tag{2.11}$$

This implies that **B** is the curl of something

$$B = \nabla \times A \tag{2.12}$$

We can use this form of **B** in eqn.(2.9) -

$$\nabla \times E = -\frac{\partial}{\partial t} \nabla \times A \tag{2.13}$$

or

$$\nabla \times \left(E + \frac{\partial A}{\partial t} \right) = 0 \tag{2.14}$$

Since curl of the vector quantity $E + \frac{\partial A}{\partial t}$ is zero, we can define it equal to the gradient of something.

$$E + \frac{\partial A}{\partial t} = -\nabla \phi \tag{2.15}$$

or

$$E = -\nabla \phi - \frac{\partial A}{\partial t} \tag{2.16}$$

Here ϕ is called **Scalar Potential** and $\vec{A}$ is known as **Vector Potential**.

Now using both, eqn.(2.12) and eqn.(2.16) in eqn.(2.10) we get,

$$\nabla \times (\nabla \times A) = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\nabla \phi - \frac{\partial A}{\partial t} \right) \tag{2.17}$$

$$\nabla(\nabla.A) - \nabla^2 A = \mu_0 J - \mu_0\epsilon_0\frac{\partial}{\partial t}\nabla\phi - \mu_0\epsilon_0\frac{\partial^2 A}{\partial t^2} \tag{2.18}$$

$$\nabla\left(\nabla.A + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \mu_0 J \tag{2.19}$$

Using Lorentz gauge -

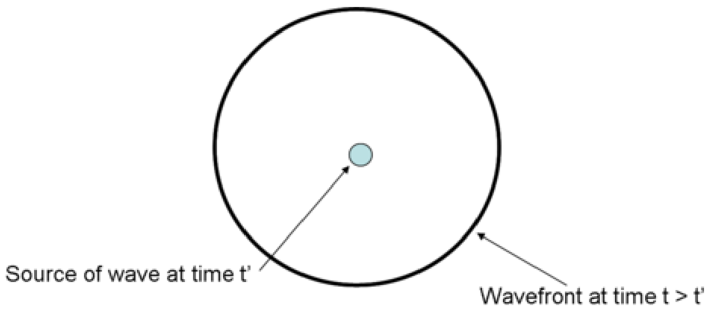
$$\nabla.A = -\frac{1}{c^2}\frac{\partial\phi}{\partial t} \tag{2.20}$$

we get the following equation, known as the 'Inhomogeneous Electromagnetic Wave Equation'-

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \mu_0 J \tag{2.21}$$

The solution of eqn.(2.21) is given as -

$$\vec{A}(\vec{r},t) = \int \frac{\delta(t' + \frac{|\vec{r}-\vec{r'}|}{c}) - t)}{|\vec{r}-\vec{r'}|}\frac{\vec{J}(\vec{r'},t)}{c}d^3r'dt' \tag{2.22}$$



The source of the wave occurs at time t' and propagates in a circular wavefront as time increases for $t > t'$.

Chapter 3

Solving The Wave Equation

In the last chapter, we arrived at the following differential equation -

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \mu_0 J \tag{3.1}$$

The solution of this equation provides us the information about how the wave propagates. We know the analytical solution of this as Eqn.(2.22), but now we wish to solve this numerically/computationally.

There are a variety of methods available to solve different kinds of PDEs. For eg., Finite Difference Method, Finite Volume Method, Gradient Discretisation Method, Finite Element Method, Monte Carlo Method, etc. Finite Difference Method can solve simple problems neatly, but as soon as complexity is added, it becomes unwieldy. FEM, however, is a sophisticated technique but can deal with complicated problems. Here we'll solve the above hyperbolic equation via Finite Element Method.

3.1 Finite Element Method

To analyze a system, a mathematical model is developed which describes the system. This mathematical model usually consists of differential equations and provided conditions. The solution to these equations is generally difficult to obtain. With the advancement of technology and high-performance computers, it is now possible to solve such differential equations. Several numerical techniques have been developed and applied to solve various mathematical problems to get their solutions.

The finite element method has been one of the major numerical solution methods. It can easily be used to analyze different kinds of problems. The FEM requires division of problem domain into many sub-domains. Each sub-domain is referred to as a *finite element*.

Since we don't know the solution of the equation we are solving, as a first step we approximate the solution. **Weight Residual** is one of such techniques.

3.1.1 Methods of Weighted Residual

The methods of weighted residual are useful to obtain approximate solutions to differential equations. To explain the method, we will use the following differential equation -

$$\frac{d^2u}{dx^2} - u = -x \tag{3.2}$$

where,

$$0 < x < 1$$

$$u(0) = 0 = u(1)$$

Step 1: We first need to assume a trial function, consisting of unknown coefficients to be determined later. The function is chosen such that it satisfies the boundary conditions. To improve the approximate solutions, we can add more terms to the trial function. For Eqn.(3.2) we choose the following function -

$$\tilde{u} = ax(1-x) \tag{3.3}$$

Generally, the accuracy of an approximated solution is dependent upon the proper selection of trial function.

Step 2: We know that the trial function will not solve the differential equation. There will be some residual which can be estimated by substituting $\tilde{u}$ in the main equation. The residual can be written as -

$$R = \frac{d^2\tilde{u}}{dx^2} - \tilde{u} + x = -2a - ax(1-x) + x \tag{3.4}$$

Step 3: To make the above residual zero we need to find the value of 'a,' which best approximates the trial solution to the exact solution. To this end, we select a weighting/test function and set the weighted residual over the problem domain to zero.

$$I = \int_0^1 wR dx = \int_0^1 w\left(\frac{d^2\tilde{u}}{dx^2} - \tilde{u} + x\right) dx \tag{3.5}$$

$$I = \int_0^1 w\left(-2a - ax(1-x) + x\right) dx = 0 \tag{3.6}$$

Step 4: Now we define the weighting/test function. There are a number of ways to define the test function.

1. Collocation Method : The Dirac Delta function is used as a .

$$w = \delta(x - x_i) \tag{3.7}$$

   The sampling point x_i must be within the domain.

2. Least Squares Method : The test function is defined from the Residual such that -

$$w = \frac{dR}{da} \tag{3.8}$$

3. Galerkin's Method : The test function is derived from the -

$$w = \frac{d\tilde{u}}{da} \tag{3.9}$$

Here we just have one unknown constant, so we need just one test function. In general, we need the same number of test function as that of unknown constants.

3.1.2 Strong and Weak Formulation

The formulation described in the last section is called the *strong formulation*. In Eqn.(3.5), it is required to solve the highest order of derivative term in the differential equation. This formulation puts conditions on the chosen trial function, such that it should be differentiable twice and the second derivative must not vanish.

In order to reduce the conditions on the trial function, integration by parts is applied to the strong formulation, i.e. eqn.(3.5) -

$$I = \int_0^1 w\left(\frac{d^2\tilde{u}}{dx^2} - \tilde{u} + x\right) dx \tag{3.10}$$

$$I = \int_0^1 \left(-\frac{dw}{dx}\frac{d\tilde{u}}{dx} - w\tilde{u} + xw\right) dx + \left[w\frac{d\tilde{u}}{dx}\right]_0^1 = 0 \tag{3.11}$$

In Eqn.(3.11), the trial function only needs to be differentiable once. As a result, the requirement for the trial function is reduced. This is called as *weak formulation*.

The weak formulation has an advantage for Galerkin's method. If the governing differential equation is a self-adjoint operator, Galerkin's method along with the weak formulation results in a symmetric matrix in terms of the unknown coefficients of the trial function. However, the same solution is obtained by both weak and strong formulations.

3.1.3 Trial Function

Regardless of the formulation used, the accuracy of solution heavily depends on the choice of trial function. Choosing the trial function can be difficult, especially if there is a significant variation over the problem domain.

To overcome such difficulties, a trial function can be described using piecewise continuous functions. For a one-dimensional domain, the piecewise linear functions can be defined as -

$$\phi_i(x) = \begin{cases} (x - x_{i-1})/h_i & x_{i-1} \leq x \leq x_i \\ (x_{i+1} - x)/h_{i+1} & x_i \leq x \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{3.12}$$

3.2 Solving Poisson's equation

Poisson's as well as Laplace's equations are common field-governing equations to describe various physical natures. These differential equations can represent wave equation, heat conduction etc.

$$\nabla^2 u = g \tag{3.13}$$

for Laplace's equation g = 0.

3.2.1 Poisson's Equation

In 2D, the Poisson's Equation can be written as -

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x, y) \quad \text{in} \quad \Omega \tag{3.14}$$

Here g(x,y) is the force term. u is the dependent variable which doesn't have different x and y components.

The boundary conditions are -

$$\frac{\partial u}{\partial n} = \tilde{q} \quad \text{on} \quad \Gamma_n \tag{3.15}$$

and

$$u = \tilde{u} \quad \text{on} \quad \Gamma_e \tag{3.16}$$

For the boundary -

$$\Gamma_e \cup \Gamma_n = \Gamma \tag{3.17}$$

and

$$\Gamma_e \cap \Gamma_n = \emptyset \tag{3.18}$$

3.2.2 Method of weighted residual for Poisson's equation

Applying the method of weighted residual and doing integration of the differential equation (eqn. 3.14) -

$$I = \int_\Omega w \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - g(x,y) \right) d\Omega - \int_{\Gamma_e} w \frac{\partial u}{\partial n} d\Gamma \tag{3.19}$$

We apply integration by parts to develop the weak form.

$$\begin{aligned} \int_\Omega w \left(\frac{\partial^2 u}{\partial x^2} \right) d\Omega &= \int_{y_1}^{y_2} \left(\int_{x_1}^{x_2} w \frac{\partial^2 u}{\partial x^2} dx \right) dy \\ &= -\int_\Omega \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} d\Omega + \oint_\Gamma w \frac{\partial u}{\partial x} n_x d\Gamma \end{aligned} \tag{3.20}$$

We get a similar expression for the y-component. Therefore, for the first two terms in eqn.(3.19) we get,

$$\int_\Omega w \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} \right) d\Omega = -\int_\Omega \left(\frac{\partial w}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial u}{\partial y} \right) d\Omega + \oint_\Gamma w \frac{\partial u}{\partial n} d\Gamma \tag{3.21}$$

This equation is also known as *Green's Theorem*.
Using eqn.(3.22) in eqn.(3.19) we get,

$$I = -\int_\Omega \left(\frac{\partial w}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial u}{\partial y} \right) d\Omega - \int_\Omega w g(x,y) d\Omega + \int_\Gamma w \frac{\partial u}{\partial n} d\Gamma \tag{3.22}$$

3.2.3 Linear Triangular Element

To discretize the 2D domain one can use rectangular or triangular elements. We want to use the *triangular linear element*. The triangular element has three nodes and the variable interpolation between the nodes is linear in *x* and *y*.

$$u = a_1 + a_2 x + a_3 y \tag{3.23}$$

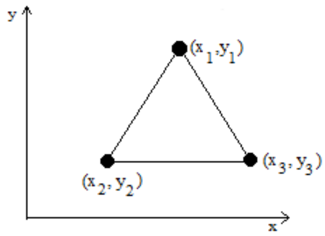


Figure 3.1: Linear Triangular Element

or

$$u = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{3.24}$$

a_i here is the constant, which is to be determined later. For the three nodes of a triangular element we can write -

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{3.25}$$

u_i here is the nodal variable with x_i and y_i being the coordinate values. We can find the value of constants a_i's by just taking the inverse of square matrix and multiplying it on both sides.

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{3.26}$$

where

$$A = \frac{1}{2} det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \tag{3.27}$$

Here A gives the area of the linear triangular element. However it's value is positive or negative depending upon the direction of nodes (anti-clockwise or clockwise respectively). Substituting the eqn.(3.26) in eqn.(3.24) we get -

$$u = H_1(x,y)u_1 + H_2(x,y)u_2 + H_3(x,y)u_3 \tag{3.28}$$

where $H_i(x, y)$ is called as shape/test/weighting function for linear triangular element, given as -

$$H_1 = \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y] \tag{3.29}$$
$$H_2 = \frac{1}{2A}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y] \tag{3.30}$$
$$H_3 = \frac{1}{2A}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y] \tag{3.31}$$

Shape functions also satisfy the following condition -

$$H_i(x_j, y_j) = \delta_{ij} \tag{3.32}$$

or

$$\sum_{i=1}^{3} H_i = 1 \tag{3.33}$$

3.2.4 System Matrix

The first integral in Eqn.(3.22) represents the *system matrix*. It contains the geometrical and material properties of the domain. It also represents the resistance put by elements to change when subjected to external force. System matrix, however, is the assembled form of *element matrices*. For a linear triangular element as shown in Fig.(3.1), the element matrix is computed as -

$$\begin{aligned}[K] &= \int_{\Omega_e} \left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} \right) d\Omega \\ &= \int_{\Omega_e} \left(\begin{Bmatrix} \frac{\partial H_1}{\partial x} \\ \frac{\partial H_2}{\partial x} \\ \frac{\partial H_3}{\partial x} \end{Bmatrix} \begin{Bmatrix} \frac{\partial H_1}{\partial x} & \frac{\partial H_2}{\partial x} & \frac{\partial H_3}{\partial x} \end{Bmatrix} + \begin{Bmatrix} \frac{\partial H_1}{\partial y} \\ \frac{\partial H_2}{\partial y} \\ \frac{\partial H_3}{\partial y} \end{Bmatrix} \begin{Bmatrix} \frac{\partial H_1}{\partial y} & \frac{\partial H_2}{\partial y} & \frac{\partial H_3}{\partial y} \end{Bmatrix} \right)\end{aligned} \tag{3.34}$$

Here $w = H_i$ is the shape function and Ω_e is the element domain. On performing the integration we get $[K]$ as a 3×3 matrix with elements consisting of x_i and y_i. By calculating the above element matrix for each triangular element in the domain, we can calculate the System Matrix (KK).

3.2.5 System Vector

The other domain integral which is to be calculated is -

$$[F] = \int_{\Omega} wg(x, y)d\Omega \tag{3.35}$$

$[F]$ here denotes the 'element vector.' This integral gives us the R.H.S. force term of the equation. The integral will be calculated for each triangular element, and everything is assembled at the end which provides us with the *System Vector* (FF).

3.2.6 Boundary Integral

For the boundary integral, we'd solve -

$$\int_{\Gamma_n} w\frac{\partial u}{\partial n} d\Gamma = \sum \int_{\Gamma_e} w\frac{\partial u}{\partial n} d\Gamma \tag{3.36}$$

Where Γ_n denotes the natural boundary of the domain and Γ_e denotes the element boundary. Over the element boundary, we first calculate the above integral (L.H.S.). So we find the value of above integral for all the boundary nodes and then add this value to its corresponding place in the KK matrix.

3.2.7 MATLAB : Data and Results

Following are the general steps followed to obtain a graphical solution on MATLAB.
Step 1 : Discretize the domain using the trial function elements. We use piecewise linear, linear triangular and tetrahedral elements quite often for 1D, 2D, and 3D respectively.
Step 2 : In a loop for all triangles, find Area Eqn.(3.27), shape function Eqn.(3.29-3.31), system matrix Eqn.(3.34) and system vector Eqn.(3.35).
Step 3 : Also in the same loop assemble K matrix and F vector over the full domain.
Step 4 : Apply boundary conditions. Add this matrix, B, to KK (assembled K) matrix.
Step 5 : To get the final answer, take inverse of KF (KK + B) matrix and multiply with FF (assembled F) vector.
Step 6 : Plot the solution on the discretized domain using appropriate command.

Following are the steps that I used to solve the Poisson equation, for constant source -

$$\nabla^2\phi = 3.03 \tag{3.37}$$

with the boundary condition -

$$\frac{\partial \phi}{\partial n} = -1.4\phi \tag{3.38}$$

1. Using the 'pdetool' in MATLAB, I first chose a domain and discretize it. There is a total of 709 nodes making 1328 triangles.

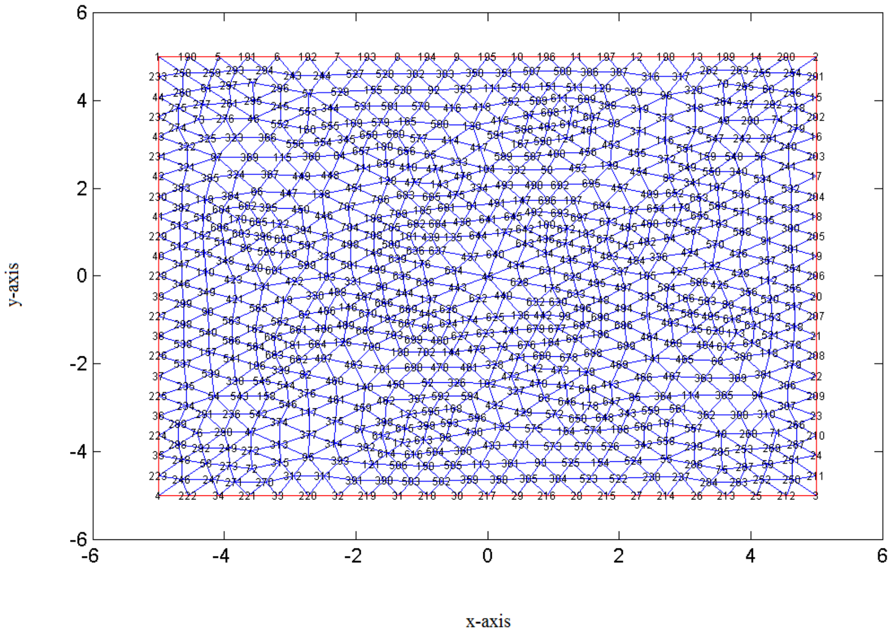


Figure 3.2: Mesh discretized with triangular element

2. At node 449, I put some value for the force vector. For rest of the nodes, it is zero.

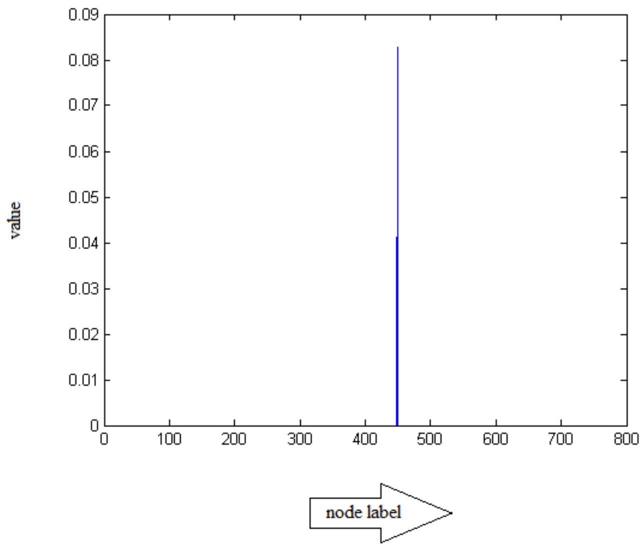


Figure 3.3: Force vector

3. The KF^{-1} matrix which is to be multiplied with the right-hand side is then obtained to be -

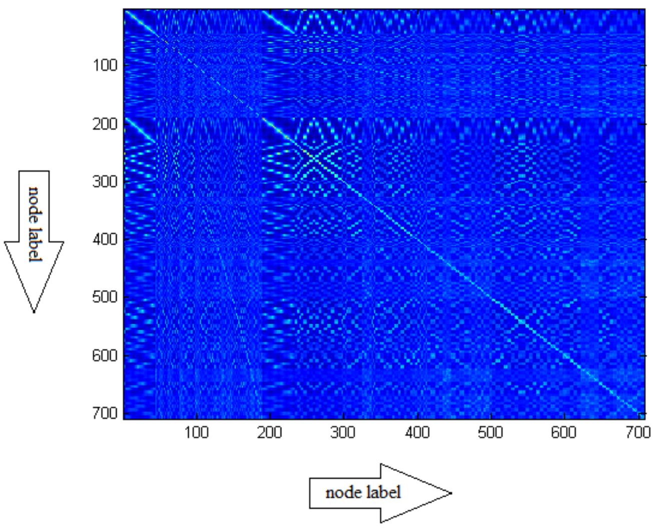


Figure 3.4: KK^{-1} matrix

4. The solution hence obtained $U = KF^{-1}FF$

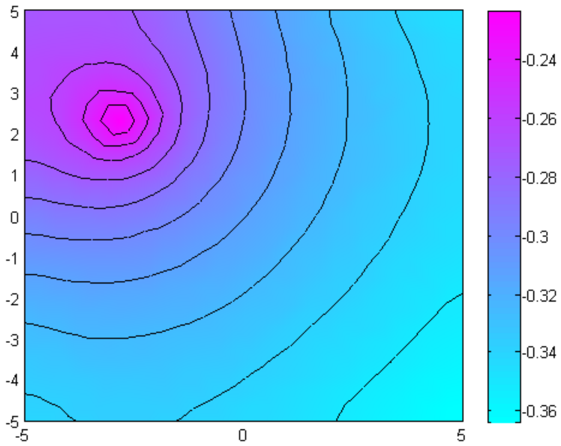


Figure 3.5: Solution of eqn.(3.37)

3.3 Solving the 'Inhomogeneous Electromagnetic Wave Equation'

The inhomogeneous EM wave equation as mentioned earlier is -

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \mu_0 J \tag{2.21}$$

To reduce the complexity of above equation I change it to -

$$\nabla^2 A + \frac{w^2}{c^2}A = -\mu_0 J \tag{3.39}$$

Let

$$A = u\hat{\imath} + v\hat{\jmath} \tag{3.40}$$

$$J = J_x\hat{\imath} + J_y\hat{\jmath} \tag{3.41}$$

Substituting A and J in eqn.(3.39) -

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{w^2}{c^2}(u+v) = -\mu_0 (J_x + J_y) \tag{3.42}$$

Using the method of weighted residual as in eqn.(3.5),

$$\begin{aligned} I = &\int_\Omega \left(\sigma_1 \frac{\partial^2 \vec{u}}{\partial x^2} + \sigma_2 \frac{\partial^2 \vec{v}}{\partial y^2}\right) d\Omega + \int_\Omega \left(\sigma_1 \frac{\omega^2}{c^2}\vec{u} + \sigma_2 \frac{\omega^2}{c^2}\vec{v}\right) d\Omega \\ &+ \int_\Omega \mu_0 \left(\sigma_1 \vec{J}_x + \sigma_2 \vec{J}_y\right) - \int_{\Gamma_e} (\sigma_1\phi_x + \sigma_2\phi_y)\, d\Gamma \end{aligned} \tag{3.43}$$

where σ_1 is the shape function for $\hat{x}$ component and σ_2 is for $\hat{y}$.

We used -

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} H_1 & 0 & H_2 & 0 & H_3 & 0 \\ 0 & H_1 & 0 & H_2 & 0 & H_3 \end{pmatrix} \tag{3.44}$$

and

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_1 \\ H_2 & 0 \\ 0 & H_2 \\ H_3 & 0 \\ 0 & H_3 \end{pmatrix} \tag{3.45}$$

Compute H_i from eqn.(3.29) - (3.31).

On formulating the weak form of eqn.(3.43), we get

$$\begin{aligned} I = &- \int_\Omega \left(\frac{\partial \sigma_1}{\partial x} \frac{\partial \vec{u}}{\partial x} + \frac{\partial \sigma_2}{\partial y} \frac{\partial \vec{v}}{\partial y} \right) d\Omega + \int_\Omega \left(\sigma_1 \frac{\omega^2}{c^2} \vec{u} + \sigma_2 \frac{\omega^2}{c^2} \vec{v} \right) \\ &+ \int_\Omega \mu_0 \left(\sigma_1 \vec{J}_x + \sigma_2 \vec{J}_y \right) d\Omega + \int_{\Gamma_e} \left(\sigma_1 \phi_x + \sigma_2 \phi_y \right) d\Gamma \end{aligned} \tag{3.46}$$

We can define -

$$K + C + B = -F \tag{3.47}$$

$$\begin{aligned} K &= - \int_\Omega \begin{bmatrix} \frac{\partial \sigma_1}{\partial x} \frac{\partial \vec{u}}{\partial x} \\ \frac{\partial \sigma_2}{\partial y} \frac{\partial \vec{v}}{\partial y} \end{bmatrix} d\Omega \\ &= - \int_\Omega \begin{pmatrix} \frac{\partial \sigma_1}{\partial x} & \frac{\partial \sigma_2}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial \vec{u}}{\partial x} \\ \frac{\partial \vec{v}}{\partial y} \end{pmatrix} d\Omega \end{aligned} \tag{3.48}$$

$$C = \frac{\omega^2}{4 * A * c^2} \begin{pmatrix} \sigma_1 & \sigma_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{3.49}$$

$$B = \int_\Gamma \begin{pmatrix} \sigma_1 & \sigma_2 \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial n} \\ \frac{\partial v}{\partial n} \end{pmatrix} d\Gamma \tag{3.50}$$

where $\frac{\partial u}{\partial n} = \phi_x$ and $\frac{\partial v}{\partial n} = \phi_y$

$$F = \frac{\mu_0}{2} \begin{pmatrix} \sigma_1 & \sigma_2 \end{pmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix} \tag{3.51}$$

To find the solution $\begin{pmatrix} u \\ v \end{pmatrix}$ we calculate $K + C + B$ and find the inverse of it. Then we multiply it with the RHS.

I followed the following steps in MATLAB to find the solution -

1. I used the following mesh with 312 triangles and 177 nodes -

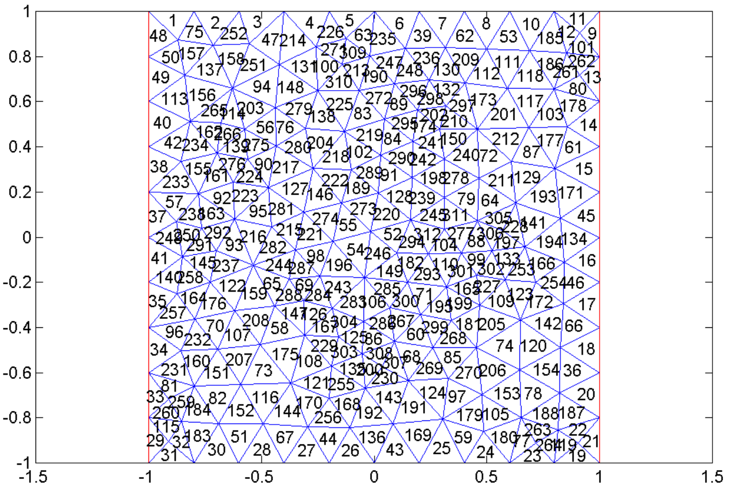


Figure 3.6: Mesh

2. I used the following parameters -

   (a) c = 3 $\times$ 10^8 m/s

   (b) μ_0 = 4π $\times$ 10^{-7} N/A^2

   (c) ω = 10 MHz

   (d) $\vec{J}$ = 1000 μA/μ m^2 = 10^9 A/m^2

3. Following is the solution obtained after computing K, C, B, F.

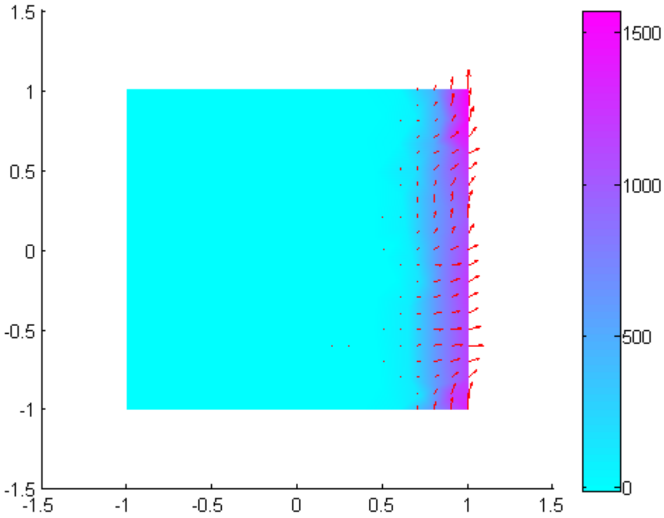


Figure 3.7: Solution

Shown in red is the vector flow. Vector direction depends on the location of point.

4. If another current density of same magnitude is placed on the other side, then we get the following results -

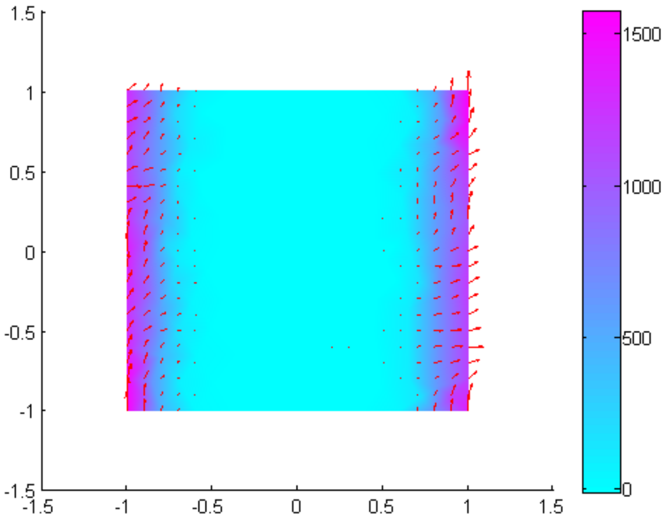


Figure 3.8: Solution for two current densities

The result shown in Fig.(3.8) is similar to what we would get if we take the projection of Helmholtz coils.

Chapter 4

COMSOL Multiphysics

COMSOL Multiphysics is a FEM based solver and simulation software. One can simulate the results by defining the type of physics they want to study and the domain of the geometry. Further one has to make a geometry and apply the physics and boundary conditions. One simple example is of a charge density in a 2D domain pointing out of the plane. For such a system the magnetic field distribution will be -

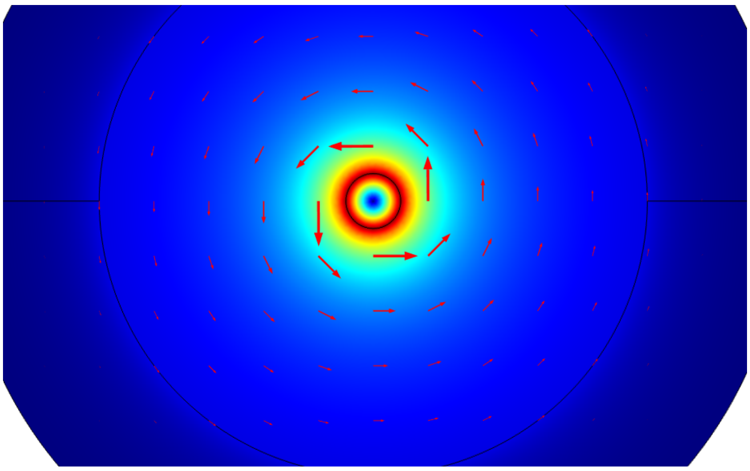


Figure 4.1: Magnetic field of a point source

For a pair of current densities pointing in opposite directions -

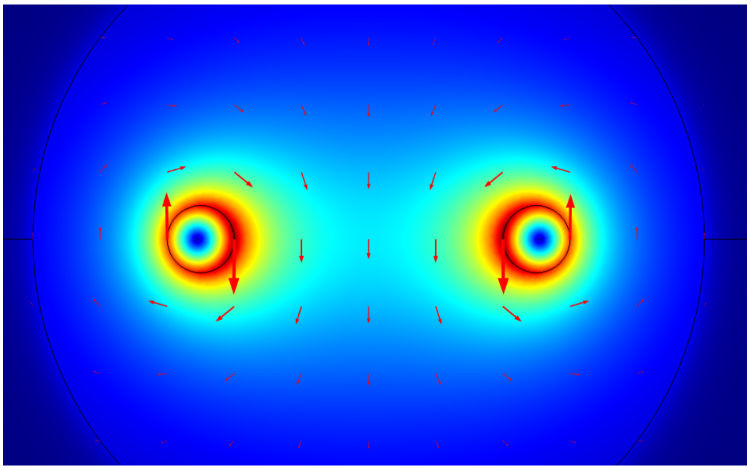


Figure 4.2: Magnetic field of two point sources

4.1 Helmholtz coil design

The main aim of the project was to find the magnetic field generated by a set of coils as shown in Fig.(1.1). For this purpose, I use COMSOL Multiphysics. Using the mf (magnetic fields) Physics in COMSOL I made the following 3D geometry including two coils and one sphere.

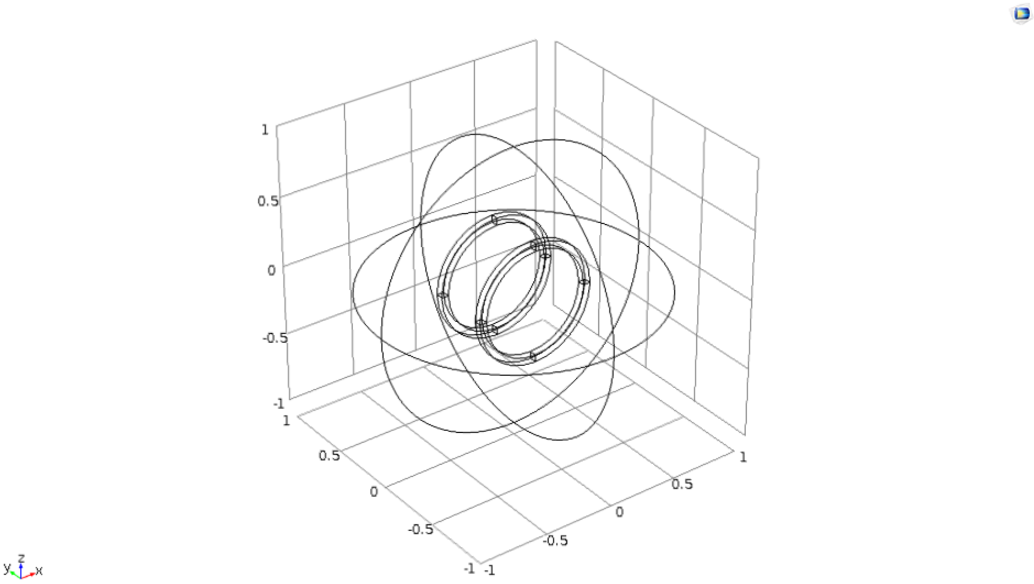


Figure 4.3: 3D coil geometry for Helmholtz coils

Whatever is inside the sphere makes the problem domain. Next, I discretized the domain using linear tetrahedral elements. I used air as the 'material' throughout the domains.

Coil Specifications -

1. Coil Radius = 0.5 [cm]
2. Distance between coils = 0.5 [cm]
3. Coil Current = 2.5 [A]
4. Number of turns in each coil = 5
5. Coil wire cross-section area = $1 \times 10^{-6}[m^2]$
6. Coil conductivity = $6 \times 10^7[S/m]$ (Copper)

For a system with current in same direction in both the coils we will get the following result -

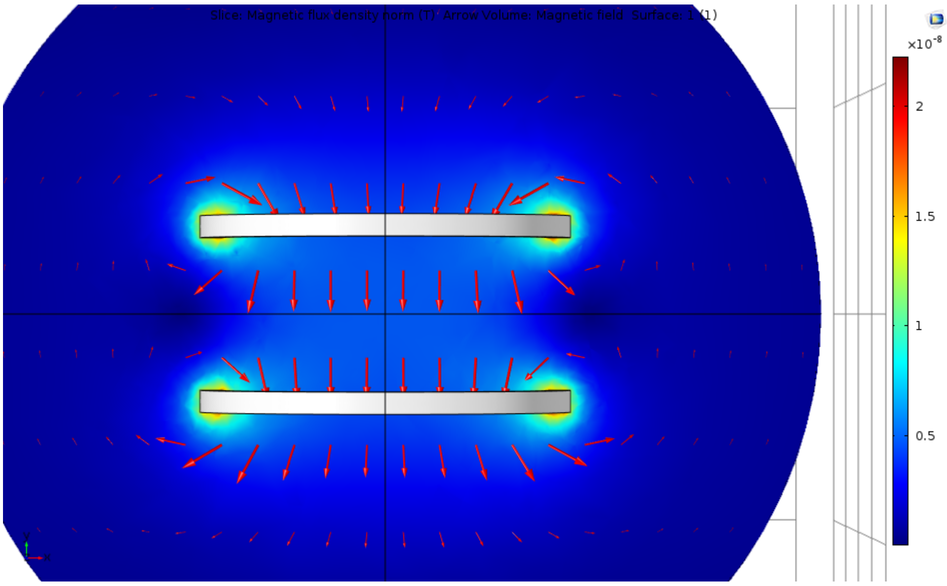


Figure 4.4: Field Distribution for Helmholtz coil with currents in same direction

For a pair with currents in opposite direction, with following coil specifications -

1. Coil Radius = 1 [cm]
2. Distance between coils = 1 [cm]
3. Coil Current = 1.0 [A]

4. Number of turns in each coil = 1

5. Coil wire cross-section area = $1 \times 10^{-6} [m^2]$

6. Coil conductivity = $6 \times 10^{7} [S/m]$ (Copper)

I obtained -

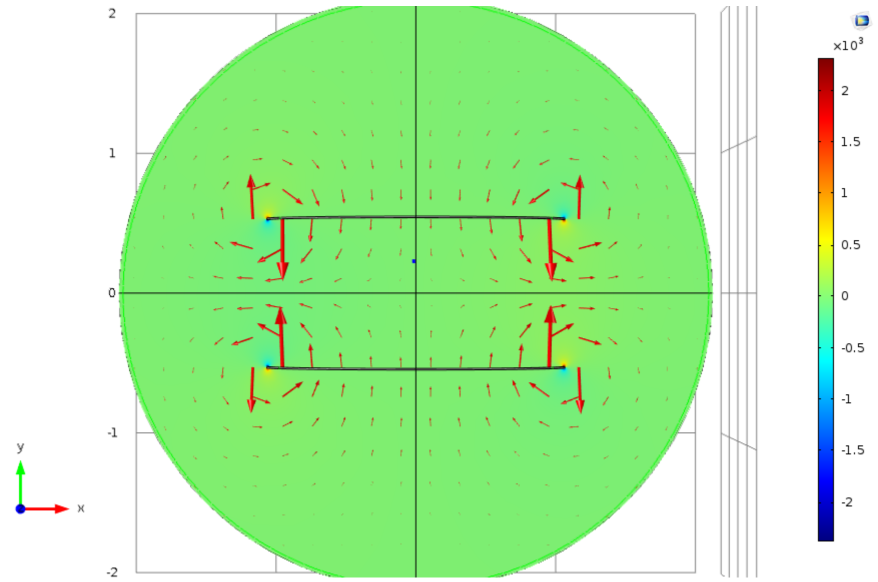


Figure 4.5: Field Distribution for Helmholtz coil with currents in opposite direction

Following is the line graph of magnetic field distribution z-axis -

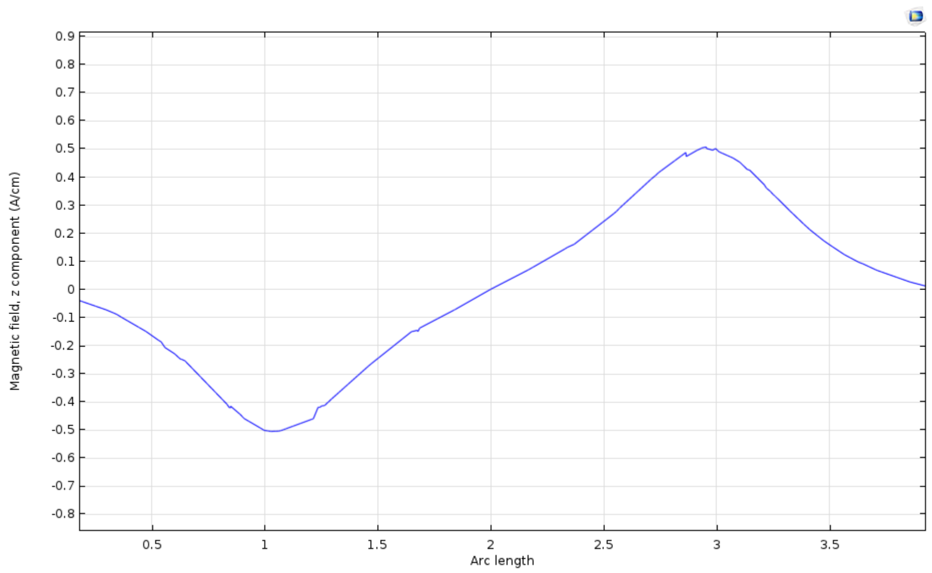


Figure 4.6: Field Distribution along z-axis

The gradient thus obtained here is 5.33 [mT/m] or 5.33 [μT/mm].

For similar coil design with following specifications -

1. Coil Radius = 1 [cm]
2. Distance between coils = 2 [cm]
3. Coil Current = 1.0 [A]
4. Number of turns in each coil = 2
5. Coil wire cross-section area = $1 \times 10^{-6}[m^2]$
6. Coil conductivity = $6 \times 10^7[S/m]$ (Copper)

I obtained -

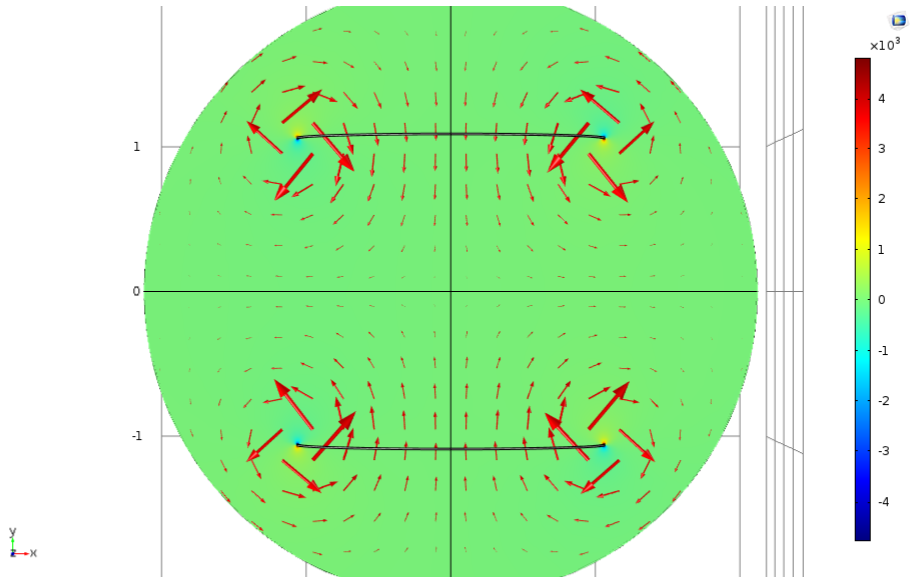


Figure 4.7: Field Distribution for a coil with currents in opposite direction

Following is the line graph of magnetic field distribution z-axis -

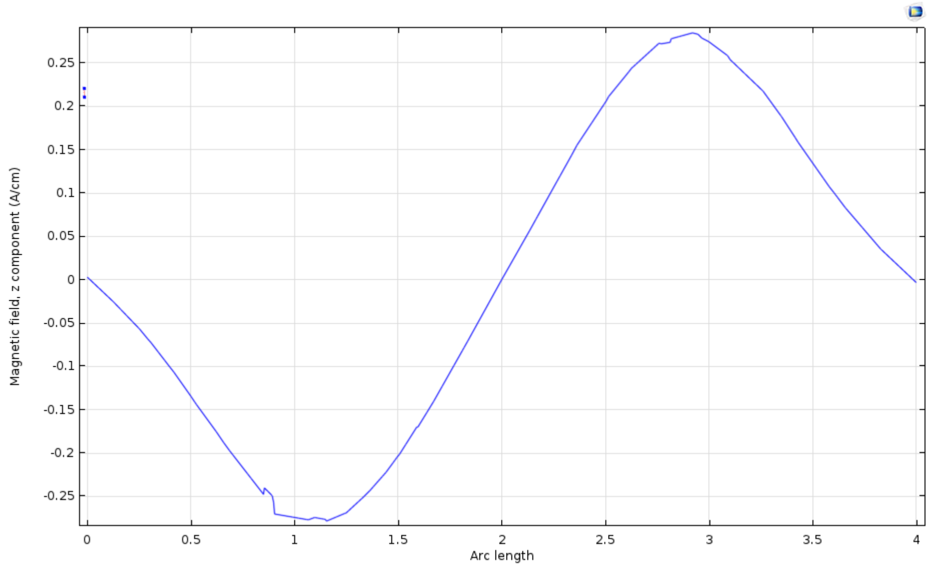


Figure 4.8: Field Distribution along z-axis

The gradient thus obtained here is 1.254 [mT/m] or 1.254 [μT/mm].

4.2 Other Coil Designs

With the above mentioned design of coil, we got a gradient in only one direction, i.e. y-direction. We can add two more pairs of coils to this design and can obtain a gradient in all the three directions.

As shown below I added 4 other coils, with each pair having currents in opposite direction -

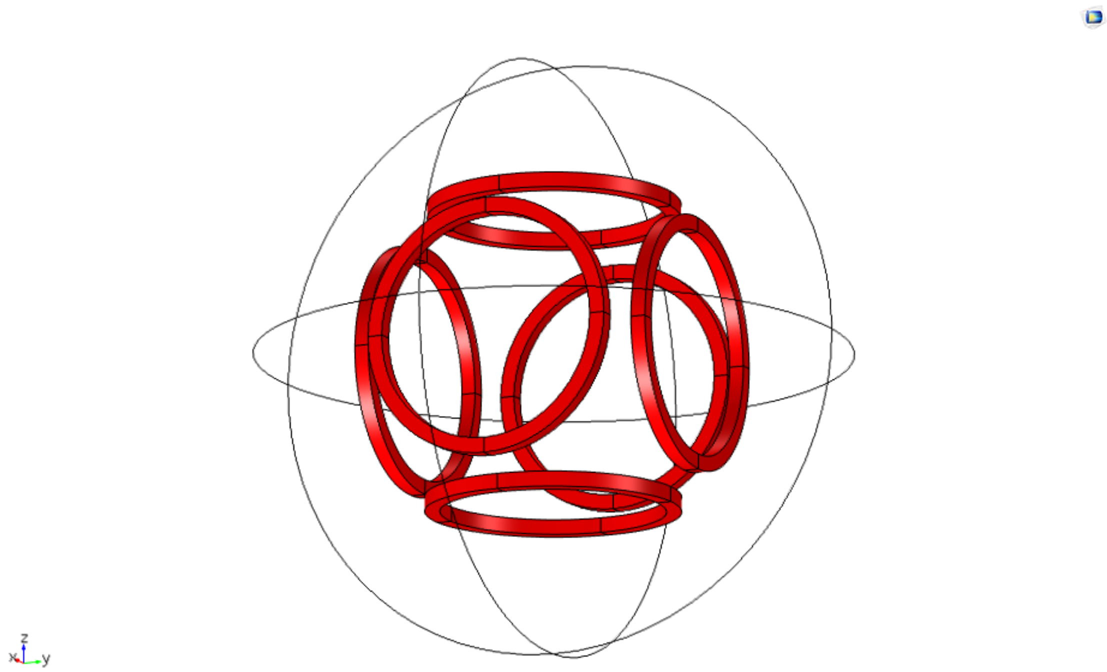


Figure 4.9: 6 coils design

Coil Specifications -

1. Coil Radius = 0.4 [cm]
2. Distance between coils = 1.0 [cm]
3. Coil current = 1 [A]
4. Number of turns in each coil = 2
5. Coil wire cross-section area = $1 \times 10^{-6} [m^2]$
6. Coil conductivity = $6 \times 10^7 [S/m]$ (Copper)

Following is the field distribution obtained in xy plane -

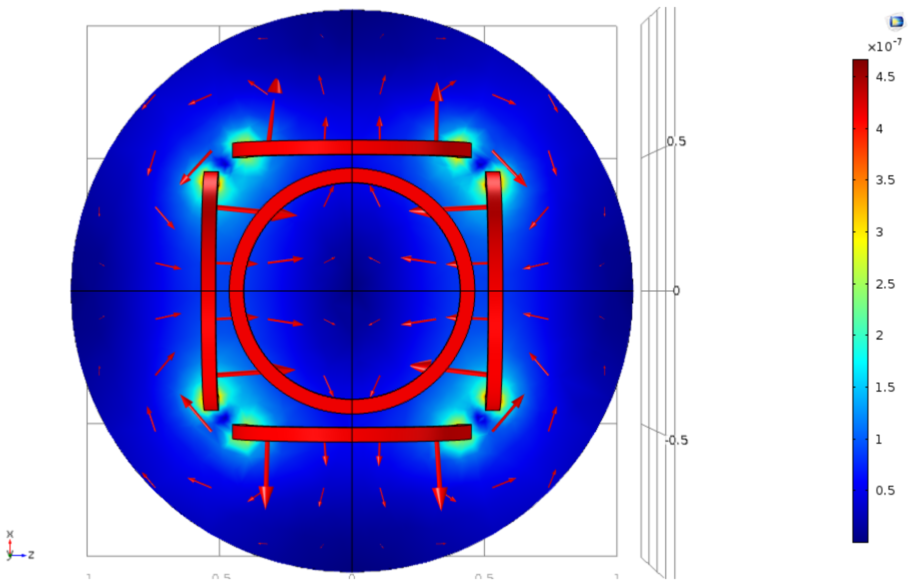


Figure 4.10: Field Distribution in xy-plane

be adjusted accordingly.Following is the field distribution along the axes -

1. Field Distribution along x-axis

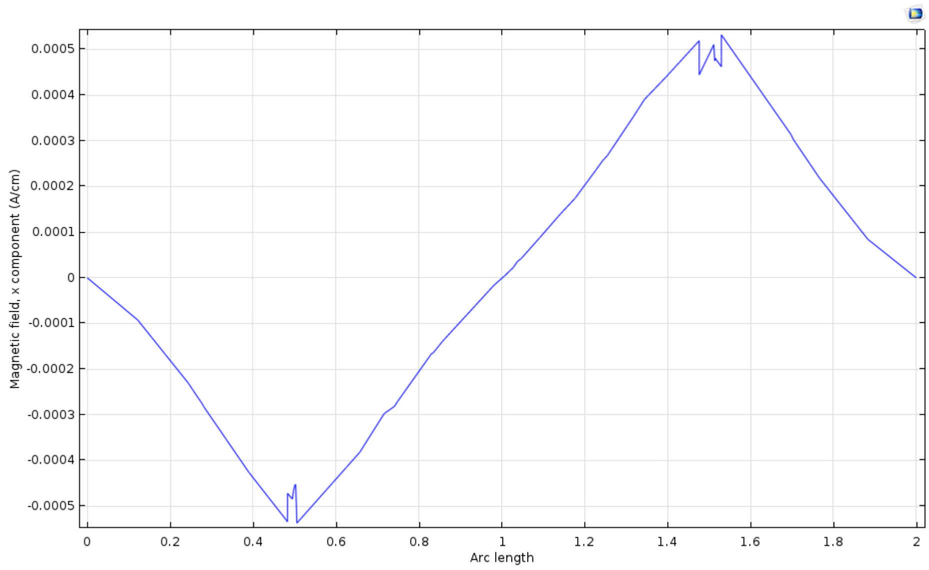


Figure 4.11: Field Distribution along x-axis

The gradient thus obtained along x-axis is 12 [μT/m] or 12 [μT/nm].

2. Field Distribution along y-axis

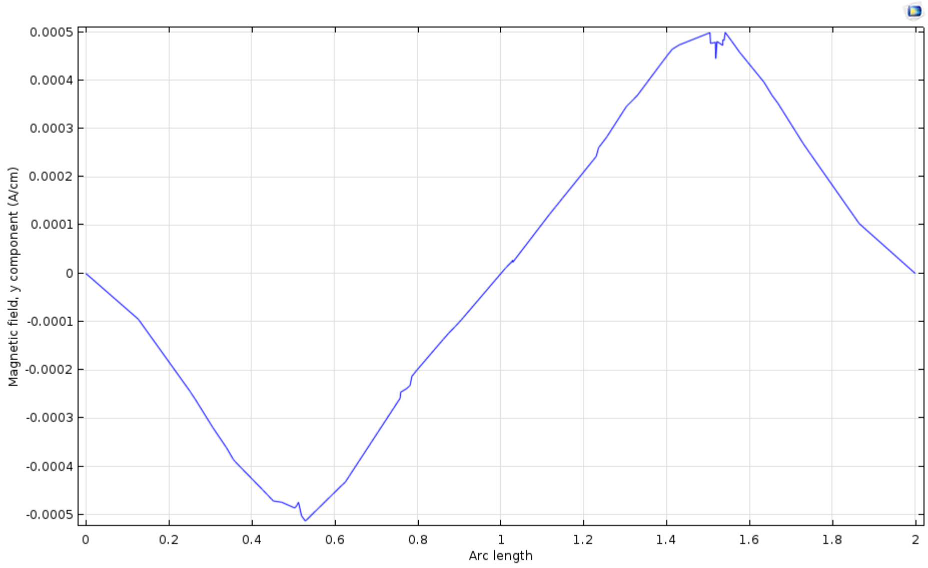


Figure 4.12: Field Distribution along y-axis

The gradient thus obtained along y-axis is 12 [μT/m] or 12 [μT/nm].

3. Field Distribution along z-axis

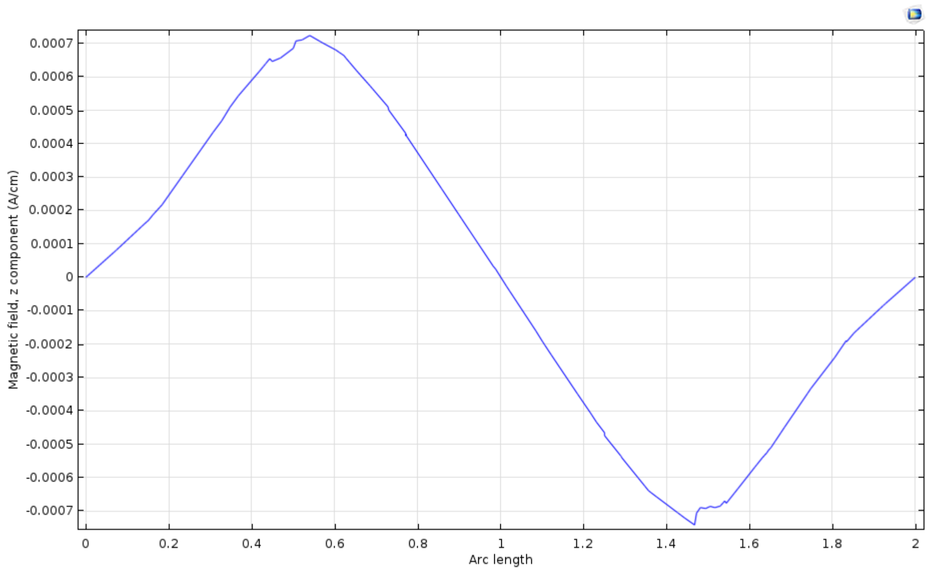


Figure 4.13: Field Distribution along z-axis

The gradient thus obtained along z-axis is 50 [μT/m] or 50 [μT/nm].

4.3 Conclusion

For a gradient close to μT/m the coil should be large in size and the distance between coils is needed to be large. The coil current and number of turns can be adjusted accordingly.

Index

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