Suslin Matrices and Spin Groups

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Certificate of Examination

This is to certify that the dissertation titled **"Suslin Matrices and Spin Groups"** submitted by **Dony Varghese** (Reg. No. MS12075) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 21, 2017

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Amit Kulshrestha at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

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Contents

Abstract

Clifford algebra of a quadratic space (V, q) is the quotient of the tensor algebra of V by the two-sided ideal $I(V, q)$, generated by $\{x \otimes x - q(x).1 \mid x \in V\}$.

In [Sus77], A.A. Suslin defined a sequence of matrices whose size doubles at each step. Using Suslin construction, for $v, w \in R^{n+1}$ we get a matrix of size $2^n \times 2^n$. Moreover, each Suslin matrix S has a conjugate Suslin matrix $\overline{S}$ such that $S\overline{S} = \overline{S}S = (v.w^T)I_{2^n}$. In [Chi15], V.R. Chintala showed that Suslin matrices can be used to construct Clifford algebra of $H(R^n)$ with the quadratic form determined by the bilinear form $b(v, w) = v.w^T$. Suslin identities are used to define standard involution on the Clifford algebra. As an application of Suslin matrices, we obtain a proof of the following exceptional isomorphism [Chi15],

$$Spin_4(R) \cong SL_2(R) \times SL_2(R) \ , \ Spin_6(R) \cong SL_4(R)$$

Suslin matrices are defined in an inductive way. We tried to generalize the idea of Suslin matrices to a more general set up of central simple algebras. For that, a new set was defined called Suslin set with certain properties that are satisfied by Suslin matrices. We looked at algebras that are isomorphic to $M_{2^n}(F)$. Let A be an algebra isomorphic to $M_{2^n}(F)$ by the map ϕ. Then, by taking inverse image of Suslin matrices under ϕ, we indeed obtain a Suslin set. We hope that Suslin sets could be useful to understand Suslin matrices.

Chapter 1

Quadratic Forms and Clifford Algebras

In this chapter we define some basic notions in the algebraic theory of quadratic forms. In first section, we recall quadratic forms and some of their basic properties. In next section, we discuss about Clifford algebras of quadratic spaces and involutions on it.

1.1 Quadratic Forms

Let F be a field with $\mathrm{char}(F) \neq 2$ and V be a vector space over F. An F valued function q on V is called a *quadratic form* on V if there exist a *symmetric bilinear form* b on V such that $q(x) = b(x, x)$, for all $x \in V$. A vector space equipped with a quadratic form is called a *quadratic space* (V, q). Each symmetric bilinear form on V defines a quadratic form on V.

Proposition 1.1.1. *Different symmetric bilinear forms defines different quadratic forms.*

Proof Let $q(x) = b(x, x)$, then we have

$$\begin{aligned} q(x+y) &= b(x+y, x+y) = b(x,x) + b(x,y) + b(y,x) + b(y,y) \\ &= q(x) + q(y) + 2b(x,y) \end{aligned}$$

from which we get $b(x, y) = \frac{1}{2}(q(x+y) - q(x) - q(y))$. Also, $b(x, y) = \frac{1}{2}(q(x+y) - q(x, y))$. These equations are called *Polarization Formulae* . Thus q determines b uniquely. ■

The symmetric bilinear form b is called the bilinear form associated with q. For each b, we can define a mapping, $l_b : V \to V^*$ as $l_b(v_1)(v_2) = b(v_1, v_2)$. Rank of b is considered as the rank of the linear transformation l_b. This same rank is taken as the rank of q. A quadratic form q is called *non-singular quadratic form* if the associated bilinear form is non-singular. Quadratic form is called *regular quadratic form* if it is non-singular. Then the quadratic space is called *regular* .

1.1.1 Orthogonality

Let (V, q) be a regular quadratic space, with associated bilinear form b. For $x, y \in V$, we say x is *orthogonal* to y and write $x \perp y$ if $b(x, y) = 0$. Since b is symmetric, $x \perp y$ if and only if $y \perp x$. Also, we have $x \perp y$ if and only if $q(x + y) = q(x) + q(y)$. If A is a subset of V, we define the orthogonal set $A^\perp$ by $A^\perp = \{x \mid x \perp a, \text{for all } a \text{ in } A\}$.

We can see that,

1. $A^\perp$ is a linear subspace of V.

2. If $A \subseteq B$, then $A^\perp \supseteq B^\perp$.

3. $A^{\perp\perp} \supseteq A$ and $A^{\perp\perp\perp} = A$.

4. $F = F^{\perp\perp}$ for F, a linear subspace of V.

5. $F \cap F^\perp = 0$ and $V = F \oplus F^\perp$ for F, a linear subspace of V.

1.1.2 Diagonalization

Theorem 1.1.2. *Every quadratic space (V, q) over F can be represented by orthogonal sum of one dimensional spaces.*

Proof We prove this by induction of V. If dim$V$=1, then there is nothing to prove. Suppose that dim$V = n$ and the theorem is true for $n-1$ dimensional spaces. Let us consider two cases.

Case I : If $q(v) = 0$ for all $v \in V$, then result is trivially true.

Case II : If $q(v) \neq 0$ for some $v \in V$, say v_1. Let $q(v_1) = a \neq 0$. Then, Let $W = \{v_1\}^{\perp}$. Then $V = span(v_1) \oplus W$, with $\dim(W) = n - 1$. Since restriction of q to W is a quadratic form on W, the result follows from inductive hypothesis. ■

So, there exists basis $(v_1, ...v_n)$ such that $q(v_i) = a_i$ where $a_i \in F$ and $b(v_i, v_j) = 0$ if $i \neq j$. Then for $x = \sum_{i=1}^{d} x_i v_i$,

$$q(x) = \sum_{i=1}^{n} a_i x_i^2.$$

We denote this as

$$q =< a_1, a_2, ..., a_n >$$

If (V_1, q_1) and (V_2, q_2) be two quadratic forms, then we can define quadratic form on $V_1 \oplus V_2$ by $q(x_1 + x_2) = q_1(x_1) + q_2(x_2)$.

Definition 1.1.3. A $2n$-dimensional quadratic form q is called *hyperbolic* if it can be represented as $n\times < 1, -1 >$.

1.1.3 Isotropy

A vector $v \in (V, q)$ is called *isotropic* if $q(x) = 0$ and *anisotropic* otherwise. We denote the set of all isotropic elements by $\mathrm{Iso}(V, q)$ and all anisotropic elements by $\mathrm{An}(V, q)$. If a linear subspace U of V is contained in $\mathrm{Iso}(V, q)$, then it is called *totally isotropic.*

1.1.4 Isometry

Let (V_1, q_1) and (V_2, q_2) be two quadratic spaces. A linear map $T : V_1 \to V_2$ is called *isometry* if

1. T is injective, and

2. $q_2(T(x)) = q_1(x)$ for all $x \in V_1$.

   By the polarization formula, (2) is equivalent to

3. $b_2(T(x), T(y)) = b_1(x, y)$ for all $x, y \in V_1$.

Isometry is an injective map. If the space (V, q) is regular, condition 2(or 3) implies the map is injective.

1.1.5 Orthogonal Group

Isometry of a space to itself is called a *orthogonal mapping* . T^{-1} is defined since the map is surjective. T^{-1} is also an isometry. Orthogonal mappings forms a group which is denoted as $O(V, q)$ and called *Orthogonal Group.*

1.2 Clifford Algebras

Let (V, q) be a quadratic space and A be a unital algebra. A *Clifford mapping* j is an injective linear mapping $j : V \to A$ such that

1. $1 \notin j(V)$, and

2. $(j(x))^2 = q(x).1 = q(x)$, for all $x \in V$.

If $j(V)$ generates A, then A together with the map j is called a *Clifford Algebra* for (V, q).

Let j be a Clifford mapping, then

$$
\begin{aligned}
j(x)j(y)+j(y)j(x) &= j(x+y)^2 - j(x)^2 - j(y)^2 \\
&= (q(x+y) - q(x) - q(y)).1 \\
&= 2b(x,y).1
\end{aligned}
$$

If $x \perp y$ then $xy = -yx$.

Example 1.2.1. Let us consider the simplest case when $q = 0$. Then take $A = \wedge^* V$ (Exterior Algebra), and $j(x) = x$. Since $x \wedge x = 0 = q(x)$ for all $x \in V$, j is a Clifford Mapping and since V generates $\wedge^* V$, Exterior Algebra is a Clifford Algebra of $q = 0$.

Example 1.2.2. Suppose (V, q) of one-dimension space with $q(v) = -1$. Let $A = \mathbb{C}$, where $\mathbb{C}$ denotes the algebra of Complex Numbers. Consider the map $j(\lambda v) = \lambda i$. Then, $j(\lambda v)^2 = -\lambda^2 = q(\lambda v)$. Then j is a Clifford mapping and $\mathbb{C}$ is a Clifford algebra of (V, q). We can also consider $\mathbb{C}$ as a subalgebra of $M_2(\mathbb{R})$ with $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so a typical element of A looks like $z = x + iy = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$.

Example 1.2.3. Let (V, q) be one-dimensional with some with $w \in V$ such that $q(w) = 1$. Here take $A = \mathbb{R}^2$ with multiplication defined as $(x_1, y_1)(x_2, y_2) = (x_1y_1, x_2y_2)$. Then, identity element will be $1 = (1, 1)$. Let $j(\lambda w) = (\lambda, -\lambda)$. Then $1 \notin j(V)$, and

$$
j(\lambda w)^2 = \lambda^2(1,1) = q(\lambda w).1
$$

so, $A = \mathbb{R} \oplus \mathbb{R}$ is a Clifford algebra with mapping j.

Here also, we can consider A as subalgebra of $M_2(\mathbb{R})$ with $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

A Clifford algebra $A(V, q)$ is called a *Universal Clifford Algebra* if whenever there is an isometry T from (V, q) to (W, r), it extends to an algebra homomorphism $\tilde{T}$ of Clifford algebras. That is, if $B(W, r)$ is a Clifford algebra of (W, r), then

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \downarrow \subset & & \downarrow \subset \\ A(V,q) & \xrightarrow{\tilde{T}} & B(W,r) \end{array}$$

Since V generates $A(V, q)$, $\tilde{T}$ is unique. If we take T as identity map on V, then we have universal Clifford algebra is unique.

Let V be of dimension d and $\{v_1, v_2, ..., v_d\}$ be a orthogonal basis. Then we have $v_i v_j = -v_j v_i$. It is easy see that $\{v_{i_1} v_{i_2} ... v_{i_k} \mid 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq d \text{ and } 1 \leq k \leq d\}$ is the basis for Universal Clifford algebra (now onwards Clifford algebra) and empty product as identity element. So, the dimension of Clifford algebra is $\sum_{k=0}^{d} \begin{pmatrix} d \\ k \end{pmatrix} = 2^n$.

We can consider the Clifford algebra $A(V, q)$ as a quotient of $\otimes^* V$ of the ideal I generated by all $x \otimes x - q(x)$ with $x \in V$.

Let i be the inclusion mapping $V \to \otimes^* V$, and let us denote the Clifford mapping from V to a universal Clifford algebra $A(V, q)$ by j_A. By universal property, there is a unique algebra homomorphism k_A from $\otimes^* V$ into $A(V, q)$ such that $k_A \circ i = j_A$, since V generates $A(V, q)$, k_A is surjective.

Let I_q be the ideal in V generated by the elements $x \otimes x - q(x).1$, let $C(V, q) = \otimes^* V / I_q$ be the quotient algebra, and let $\pi : \otimes^* V \to C(V, q)$ be the quotient mapping. Let us set $j_B = \pi \circ i$. Then I_q is in the null-space of k_A, and so there exists a unique algebra homomorphism

$$J : C(V, q) \to A(V, q)$$

such that $k_A = J \circ \pi$. Then $j_A = k_A \circ i = J \circ \pi \circ i = J \circ j_B$. Again, J is surjective. If $x \in V$, then $J(j_B(x)) = j_A(x)$, so that $1 \neq j_B(V)$. Further,

$$(j_B(x))^2 = \pi(x \otimes x) = \pi(x \otimes x + q(x).1) - \pi(q(x).1) = q(x).1$$

so that j_B is a Clifford mapping of V into $C(V,q)$. Since $i(V)$ generates $\otimes^* V$, $j_B(V) = \pi(i(V))$ generates $C(V,q)$, and so $C(V,q)$ is a Clifford algebra for (V,q). Since $A(V,q)$ is universal, there exists a unique unital algebra homomorphism $\rho : A(V,q) \to C(V,q)$ such that $\rho \circ j_A = j_B$. It follows that J is an algebra isomorphism of $C(V,q)$ onto $A(V,q)$, with inverse ρ. Thus $C(V,q)$ is a universal Clifford algebra for (V,q).

1.2.1 Involutions on Clifford algebra

Let (V,q) be a quadratic space, with Clifford algebra $Cl(V,q)$. Let $m(x) = -x$ for $x \in V$, so we have

$$\begin{array}{ccc} V & \xrightarrow{\quad m \quad} & V \\ \downarrow \subset & & \downarrow \subset \\ Cl(V,q) & \xrightarrow{\ \tilde{m}\ } & Cl(V,q) \end{array}$$

take $\tilde{m}(a) = a'$. The map $a \to a'$ is an automorphism, and $a'' = a$. So this is a involution, called *principal involution* .

Let us define

$$Cl^+ = \{a \mid a = a'\} \text{ and } Cl^+ = \{a \mid a = -a'\}$$

Then we have $Cl = Cl^+ \oplus Cl^-$, Cl^+ is a subalgebra of Cl called *even Clifford algebra* . We can see that

$$Cl^+.Cl^+ = Cl^-.Cl^- = Cl^+ \text{ and } Cl^+.Cl^- = Cl^-.Cl^+ = Cl^-$$

In tensor algebra of V, consider the anti-automorphism

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto v_n \otimes v_{n-1} \otimes \cdots \otimes v_1.$$

Since the ideal $I(V,q)$ is invariant under this reversal, this operation descends to an anti-automorphism of $Cl(V,q)$ called *transpose*, let us denote its image by x^t. We define $x^* = \overline{m}(x^t) = \overline{m}(x)^t$, this is an involution on $Cl(V,q)$ called *standard involution* .

Chapter 2

Suslin Matrices and Link to Clifford Algebras

In this chapter we discuss about Suslin matrices and their connection with Clifford algebra. In first section, we look at how the Suslin matrices are constructed and some of its properties. In next section, we discuss about how the Suslin matrices can be used to construct Clifford algebra of a particular quadratic space.

2.1 Suslin Construction

In [Sus77], A.A. Suslin defined a sequence of matrices called *Suslin matrices* with the property that, size of those doubles in each step. For each Suslin matrix S, he defined another matrix $\overline{S}$ such that $S\overline{S}$ and $S + \overline{S}$ are scalar matrices.

Let us check how to construct $S_n(v, w)$ of size $2^n \times 2^n$ from two vectors $v, w \in R^{n+1}$.

Let $v = (a_0, v_1)$, $w = (b_0, w_1)$ where $v_1, w_1$are vectors in R^n. Define

$$S_0(v,w) = a_0, \;\; S_1(v,w) = \begin{pmatrix} a_0 & v_1 \\ -w_1 & b_0 \end{pmatrix}$$

and

$$S_n(v,w) = \begin{pmatrix} a_0 I_{2^{n-1}} & S_{n-1}(v_1,w_1) \\ -S_{n-1}(w_1,v_1)^T & b_0 I_{2^{n-1}} \end{pmatrix}$$

and $\overline{S}$ is defined as

$$\overline{S}_n(v,w) = S_n(w,v)^T = \begin{pmatrix} b_0 I_{2^{n-1}} & -S_{n-1}(v_1,w_1) \\ S_{n-1}(w_1,v_1)^T & a_0 I_{2^{n-1}} \end{pmatrix}$$

Example : $v = (a_0, a_1)$ and $w = (b_0, b_1)$

$$S(v,w) = \begin{pmatrix} a_0 & a_1 \\ -b_1 & b_0 \end{pmatrix} \text{ and } \overline{S}(v,w) = \begin{pmatrix} b_0 & -a_1 \\ b_1 & a_0 \end{pmatrix}$$

Example : $v = (a_0, a_1, a_2)$ and $w = (b_0, b_1, b_2)$

$$S(v,w) = \begin{pmatrix} a_0 & 0 & a_1 & a_2 \\ 0 & a_0 & -b_2 & b_1 \\ -b_1 & a_2 & b_0 & 0 \\ -b_2 & -a_1 & 0 & b_0 \end{pmatrix} \text{ and } \overline{S}(v,w) = \begin{pmatrix} b_0 & 0 & -a_1 & -a_2 \\ 0 & b_0 & b_2 & -b_1 \\ b_1 & -a_2 & a_0 & 0 \\ b_2 & a_1 & 0 & a_0 \end{pmatrix}$$

Example : $v = (a_0, a_1, a_2, a_3)$ and $w = (b_0, b_1, b_2, b_3)$

$$S = \begin{pmatrix} a_0 & 0 & 0 & 0 & a_1 & 0 & a_2 & a_3 \\ 0 & a_0 & 0 & 0 & 0 & a_1 & -b_3 & b_2 \\ 0 & 0 & a_0 & 0 & -b_2 & a_3 & b_1 & 0 \\ 0 & 0 & 0 & a_0 & -b_3 & -a_2 & 0 & b_1 \\ b_1 & 0 & a_2 & a_3 & b_0 & 0 & 0 & 0 \\ 0 & b_1 & -b_3 & b_2 & 0 & b_0 & 0 & 0 \\ -b_2 & a_3 & a_1 & 0 & 0 & 0 & b_0 & 0 \\ -b_3 & -a_2 & 0 & a_1 & 0 & 0 & 0 & b_0 \end{pmatrix}$$

$$\overline{S} = \begin{pmatrix} b_0 & 0 & 0 & 0 & -a_1 & 0 & -a_2 & -a_3 \\ 0 & b_0 & 0 & 0 & 0 & -a_1 & b_3 & -b_2 \\ 0 & 0 & b_0 & 0 & b_2 & -a_3 & -b_1 & 0 \\ 0 & 0 & 0 & b_0 & b_3 & a_2 & 0 & -b_1 \\ -b_1 & 0 & -a_2 & -a_3 & a_0 & 0 & 0 & 0 \\ 0 & -b_1 & b_3 & -b_2 & 0 & a_0 & 0 & 0 \\ b_2 & -a_3 & -a_1 & 0 & 0 & 0 & a_0 & 0 \\ b_3 & a_2 & 0 & -a_1 & 0 & 0 & 0 & a_0 \end{pmatrix}$$

It is easy to see that $\overline{S}(v,w)$ is also a Suslin matrix. $\overline{S}(v,w) = S(v',w')$ where $v' = (b_0, -v_1)$ and $w' = (a_0, -w_1)$.

The following are properties of Suslin matrices ([Sus77], Lemma 5.1).

Theorem 2.1.1. *For $S_n = S_n(v,w)$, we have $S_n\overline{S}_n = (v.w^T)I_{2^n} = \overline{S}_nS_n$.*

Proof It is easy to see that by induction, $n = 0$ is clear. Let us assume for it is true till S_{n-1}, then

$$S_n(v,w) = \begin{pmatrix} a_0 I_{2^{n-1}} & S_{n-1}(v_1,w_1) \\ -\overline{S}_{n-1}(v_1,w_1) & b_0 I_{2^{n-1}} \end{pmatrix} \text{ and } \overline{S}_n(v,w) = \begin{pmatrix} b_0 I_{2^{n-1}} & -S_{n-1}(v_1,w_1) \\ \overline{S}_{n-1}(v_1,w_1) & a_0 I_{2^{n-1}} \end{pmatrix}$$

and

$$S_n\overline{S}_n = \overline{S}_n S_n = \begin{pmatrix} a_0b_0 + (v_1.w_1^T)I_{2^{n-1}} & 0 \\ 0 & a_0b_0 + (v_1.w_1^T)I_{2^{n-1}} \end{pmatrix}$$

■

Lemma 2.1.2. *Let* $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *where* $A, B, C, D \in M_k(R)$ *, are such that* $AB = BA$. *Then* $detS = detT$*, where* $T = DA - CB$.

Proof First of all, we have

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} .\det A = \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} .\det \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} = \det \begin{pmatrix} A & BA \\ C & DA \end{pmatrix}$$

then $\begin{pmatrix} A & BA \\ C & DA \end{pmatrix}$ can be transformed to $\begin{pmatrix} A & 0 \\ 0 & T \end{pmatrix}$ by elementary row transformations. Then we have

$$\det S.\det A = \det A.\det T$$

■

By using induction n and using above Lemma, we get

$$\det S_n = (v.w^T)^{2^{n-1}}$$

The set of Suslin Matrices of size $2^n \times 2^n$ is a R-module under matrix-addition and scalar multiplication given by $rS_n(v, w) = S_n(rv, rw)$.

2.1.1 Useful sequence of matrices J_n

In [Sus77], Suslin defines another sequence of matrices,

$$J_n = \begin{cases} 1 & \text{for } n=0 \\ \begin{pmatrix} J_{n-1} & 0 \\ 0 & -J_{n-1} \end{pmatrix} & \text{for } n \text{ even} \\ \begin{pmatrix} 0 & J_{n-1} \\ -J_{n-1} & 0 \end{pmatrix} & \text{for } n \text{ odd} \end{cases}$$

By simple calculations, we get

$$\det J = 1 \text{ and } J^T = J^{-1} = (-1)^{\frac{n(n+1)}{2}} J$$

so, J is symmetric for $n = 4k$ and $n = 4k+3$ and J is skew-symmetric for $n = 4k+1$ and $n = 4k+2$.

In ([Sus77], Lemma 5.3) it is noted that the following formulae are valid

$$\begin{aligned} \text{for } n{=}4k : (S_n(v,w)J_n)^T &= S_n(v,w)J_n \\ \text{for } n{=}4k+1 : S_n(v,w)J_nS_n(v,w)^T &= (v.w^T)J_n \\ \text{for } n{=}4k+2 : (S_n(v,w)J_n)^T &= -S_n(v,w)J_n \\ \text{for } n{=}4k+3 : S_n(v,w)J_nS_n(v,w)^T &= (v.w^T)J_n \end{aligned}$$

from which we can deduce,

$$J_nS_n^TJ_n^T = \begin{cases} S & \text{for } n \text{ even} \\ \overline{S} & \text{for } n \text{ odd} \end{cases}$$

this can be used to define involution on the Clifford algebra.

Remark 2.1.3. For simplicity, we drop the subscript and write J or S (or $S(v,w)$).

2.1.2 Fundamental Property of Suslin Matrices

Lemma 2.1.4. *Let R be a commutative ring and let $v, w, s, t \in R^{r+1}$. Let $v = (a_0, a_1, ..., a_r)$, $w = (b_0, b_1, ..., b_r)$. Then,*

$$\begin{aligned} S_r(v,w) + S(w,v)^T &= \{a_0 + b_0\} I_{2^r} \\ S_r(s,t)S_r(w,v)^T + S_r(v,w)S_r(t,s)^T &= \{< s,w > + < v,t >\} I_{2^r} \\ S_r(w,v)^T S_r(s,t) + S_r(t,s)^T S_r(v,w) &= \{< s,w > + < v,t >\} I_{2^r} \end{aligned}$$

Proof See [JR06], Lemma 3.1

Now, the *Fundamental property* of Suslin matrices.

Lemma 2.1.5. *Let* $v = (a_0, a_1, ..., a_r) = (a_0, v_1)$, $w = (b_0, b_1, ..., b_r) = (b_0, w_1)$, $s = (c_0, c_1, ..., c_r) = (c_0, s_1)$ *and* $t = (d_0, d_1, ..., d_r) = (d_0, t_1)$ *for some* $v_1, w_1, s_1, t_1 \in (R^r)$. *Then*

$$\begin{aligned} S_r(s,t)S_r(v,w)S_r(s,t) &= S_r(v',w') \\ S_r(t,s)S_r(w,v)S_r(t,s) &= S_r(w',v') \end{aligned}$$

for some $v', w' \in R^{r+1}$, *which depends linearly on* v, w *and quadratically on* s, t. *Consequently,* $v'.w'^T = (s.t^T)^2(v.w^T)$.

Proof See [JR10], Lemma 2.5

Definition 2.1.6. A Suslin matrix $S_r(v,w)$ is called *special* if $< v,w >= v.w^T = 1$.

Definition 2.1.7. The *Special Unimodular Vector Group* $SU_{m_r}(R)$ is the subgroup of $SL_{2^r}(R)$ generated by the *special Suslin matrices.*

An involution on $SU_{m_r}(R)$

Let us define an involution on $SU_{m_r}(R)$. Let $\alpha = \prod_{i=1}^{n} S_i$ where S_i are special Suslin matrices then define, $\hat{\alpha} = \prod_{i=n}^{1} S_i$.

When r is even, $\alpha \mapsto \hat{\alpha}$ is well defined involution, by Suslin identities

$$S_r(v,w) = J_r S_r(v,w)^T J_r^{-1}$$

Hence, $\hat{\alpha} = J_r \alpha^T J_r^{-1}$, only depends on α. So if it has some other representation, say $\alpha = \prod_{i=1}^n S_i = \prod_{i=1}^n S_i'$, then $\hat{\alpha} = \prod_{i=n}^1 S_i = \prod_{i=n}^1 S_i'$.

This applies only if r is even. When r is odd, this $\wedge$ can be only defined upto a unit u, with $u^2 = 1$.

Lemma 2.1.8. *Let$S_r(v,w)$, $r \geq 2$ be a Suslin matrix. If it has the following property that $S_r(x,y)S_r(v,w) = S_r(p,q)$ for any spacial Suslin matrix $S_r(x,y)$, then $S_r(v,w) = uI_{2^r}$.If $< v,w >= 1$, then $u^2 = 1$.*

Proof See [JR10], Lemma 3.1

Corollary 2.1.9. *Let $\alpha = \prod_{i=1}^n S_i$ where S_i are special Suslin matrices and $\hat{\alpha} = \prod_{i=n}^1 S_i$. If $\alpha = I_{2^r}$, r-odd, then $\hat{\alpha} = uI_{2^r}$ with $u^2 = 1$.(If r is even, $\hat{\alpha} = I_{2^r}$).*

Proof See [JR10], Corollary 3.2

2.2 Link to Clifford Algebras

As we can see Clifford algebra as quotient of tensor algebra, we have the following universal property.

For any associative algebra A over R with a linear map $j : V \to A$ such that $j(x)^2 = q(x)$ for all $x \in V$, then there is a unique algebra homomorphism

$$f : Cl(V,q) \to A \text{ such that } f \circ i = j.$$

Theorem 2.2.1. *Clifford algebra for $V = H(R^n)$, hyperbolic quadratic space equipped with the quadratic form $q(v,w) = v.w^T$ is $Cl(V,q) = M_{2^n}(R)$.*

Proof Consider $V = R^n$ with the quadratic form $q(v,w) = v.w^T$.

We have to find a linear map $\theta : V \to M_{2^n}(R)$ such that

1. $\theta(V)$ generates $M_{2^n}(R)$

2. $\theta(v)^2 = q(v)$ for all $v \in V$

Proceeding by induction, $n = 0$ case is trivial.

Let $V = H(R^{n-1})$ with $q(v,w) = v.w^T$ and there exists a θ as above, then take $\widetilde{V} = H(R^n) = V \oplus < x, y >$with $q(\lambda x + \mu y) = \lambda\mu$. Consider the map,

$$\begin{aligned}
\widetilde{\theta} : \widetilde{V} &\to M_{2^n}(R) \\
v &\mapsto \begin{pmatrix} \theta(v) & 0 \\ 0 & -\theta(v) \end{pmatrix} \text{ for } v \in V \\
x &\mapsto \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \\
y &\mapsto \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}
\end{aligned}$$

Arbitrary element of $M_{2^n}(R)$ looks like $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A, B, C, D \in M_{2^{n-1}}(R)$. A can be written in terms of $\theta(v)$, $v \in V$, then

$$\widetilde{\theta}(A) = \widetilde{A} = \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix}$$

using this,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \widetilde{A}\widetilde{\theta}(x)\widetilde{\theta}(y) + \widetilde{B}\widetilde{\theta}(x) + \widetilde{\theta}(y)\widetilde{C} + \widetilde{\theta}(y)\widetilde{D}\widetilde{\theta}(x)$$

So, $\{\widetilde{\theta}(v), v \in \widetilde{V}\}$ generates $H(R^n)$.

Now, $\widetilde{\theta}(v+\lambda x+\mu y) = \begin{pmatrix} \theta(v) & \lambda I \\ \mu I & -\theta(v) \end{pmatrix}$ and $\left(\widetilde{\theta}(v+\lambda x+\mu y)\right)^2 = \begin{pmatrix} q(v)+\lambda\mu & 0 \\ 0 & q(v)+\lambda\mu \end{pmatrix}$.

■

Let $\phi : H(R^n) \to M_{2^n}(R)$ be the linear map given by

$$\phi(v,w) = \begin{pmatrix} 0 & S_{n-1}(v,w) \\ \overline{S}_{n-1}(v,w) & 0 \end{pmatrix} \text{ where } \overline{S}_{n-1}(v,w) = S_{n-1}(w,v)^T$$

then $\phi(v,w)^2 = q(v,w)I_{2^n}$, then by universal property, ϕ extends to a homomorphism

$$\phi : Cl(H(R^n), q) \to M_{2^n}(R).$$

This map is given in [Chi15] and it is actually an isomorphism.

The quadratic space (V,q) is said to be *embedded* in A if $V \subseteq A$ and there is an isometry $\alpha : V \to V$ such that

$$v\alpha(v) = \alpha(v)v = q(v)$$

Our quadratic form in $H(R^n)$ is $q(v,w) = v.w^T$ and we can take our α as

$$\alpha : S_n(v,w) \to \overline{S}_n(v,w)$$

then $(v,w) \to S(v,w)$ is an embedding of $H(R^n)$ into $M_{2^{n-1}}(R)$.

With this embedding $\alpha : H(R^n) \to A$, consider the R-linear map

$$\begin{aligned} \psi : H(R^n) &\to M_2(A) \\ x &\mapsto \begin{pmatrix} 0 & x \\ \alpha(x) & 0 \end{pmatrix} \end{aligned}$$

Theorem 2.2.2. *The above defined ψ is injective, hence an isomorphism.*

Proof We know that the Clifford algebra of $H(R^n)$ with $q(v,w) = v.w^T$ is $M_{2^n}(R)$. So, we prove that the map $M_{2^n}(R) \to M_2(A)$ is injective. Every ideal of $M_{2^n}(R)$ is of the form $M_{2^n}(I)$ for some ideal I of R. Then $ker(\psi) = M_{2^n}(I)$, but the map ψ is R-linear and is identity on R which implies $I = 0$. By dimension arguments, it is clear that ψ is an isomorphism. ■

The following theorem is done in [JR10] is a consequence of properties of Clifford algebra, ([Chi15], Theorem 3.4)

Theorem 2.2.3. *Let X and Y be Suslin matrices, then XYX is also a Suslin matrix with $\overline{XYX} = \bar{X}\bar{Y}\bar{X}$.*

Proof For $z_1, z_2 \in H(R^n)$,

$$< z_1, z_2 >= z_1 z_2 + z_2 z_1 = (z_1 + z_2)^2 - z_1^2 - z_2^2$$

is an element in R. Multiplying by , we get

$$z_1 < z_1, z_2 >= z_1^2 z_2 + z_1 z_2 z_1$$

we have $z_1^2 = q(z_1)$ which implies $z_1 z_2 z_1 \in H(R^n)$. Take $z_1 = \begin{pmatrix} 0 & X \\ \overline{X} & 0 \end{pmatrix}$ and $z_2 = \begin{pmatrix} 0 & Y \\ \overline{Y} & 0 \end{pmatrix}$, then $z_1 z_2 z_1 = \begin{pmatrix} 0 & XYX \\ \bar{X}\bar{Y}\bar{X} & 0 \end{pmatrix}$. ■

2.2.1 Involution in Cl

We have $Cl(H(R^n), q)$ isomorphic to $M_{2^n}(R)$ via map ψ given by

$$\psi(v,w) = \begin{pmatrix} 0 & S_{n-1}(v,w) \\ \overline{S}_{n-1}(v,w) & 0 \end{pmatrix}$$

The map $(v,w) \to (-v,-w)$ induces standard involution in $Cl(H(R^n), q)$.

Theorem 2.2.4. *Let $M \in Cl \cong M_{2^n}(R)$. Then the standard involution* is given by*

$$M^* = J_n M^T J_n^T$$

Proof It is clearly an involution ($J_n^{-1} = J_n^T$). We know that vector space generates Clifford algebra, it is enough to check with (v, w). Therefore, it is enough to check with the matrices of the kind

$$\psi(v,w) = \begin{pmatrix} 0 & S_{n-1}(v,w) \\ \overline{S}_{n-1}(v,w) & 0 \end{pmatrix}$$

whether it is multiplication by -1.

Let us define a new Suslin matrix,

$$S_n^{'} = S_n(v,w) = \begin{pmatrix} 0 & S_{n-1}(v,w) \\ -\overline{S}_{n-1}(v,w) & 0 \end{pmatrix} \text{ where } v' = (0,v) \text{ and } w' = (0,w)$$

then,

$$\psi(v,w) = \lambda S_n^{'} = -S_n^{'}\lambda \text{ where } \lambda = \begin{pmatrix} I_{2^{n-1}} & 0 \\ 0 & -I_{2^{n-1}} \end{pmatrix}$$

observe that $\overline{S}_n^{'} = -S_n^{'}$.

$$(S_n^{'})^* = \begin{cases} S_n^{'} & \text{for } n \text{ even} \\ -S_n^{'} & \text{for } n \text{ odd} \end{cases}$$

when n is even,

$$\psi(v,w)^* = (\lambda S_n^{'})^* = (S_n^{'})^* \lambda^* = S_n^{'} \lambda = -\lambda S_n^{'}$$

when n is odd,

$$\psi(v,w)^* = (\lambda S_n^{'})^* = (S_n^{'})^* \lambda^* = (-S_n^{'})(-\lambda) = -\lambda S_n^{'}$$

so, for any n,

$$\psi(v,w)^* = -\lambda S_n^{'} = -\psi(v,w).$$

■

Let us find out what happens to an arbitrary element of Clifford algebra. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as 2×2 block matrix and using above theorem, we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{cases} \begin{pmatrix} D^* & -B* \\ -C^* & A^* \end{pmatrix} & \text{for } n \text{ odd} \\ \begin{pmatrix} A^* & -C^* \\ -B^* & D^* \end{pmatrix} & \text{for } n \text{ even} \end{cases}$$

Chapter 3

Action of Spin Group on Suslin Matrices

In this chapter we discuss about how Spin group acts Suslin matrices. In first section, we define Spin group of a Clifford algebra. In next section, we discuss about the action of Spin group. Here we consider two separate cases, when n is even and odd.

3.1 Spin Group

Let (V, q) be a quadratic space over k with orthogonal group $O(q)$ and Clifford algebra $Cl(q)$. Since $-id \in O(q)$, it induces an isomorphism in Clifford algebra level, $Cl(-id) : Cl(q) \to Cl(q)$. Let us denote it by γ. Consider the set :

$\Gamma = \{s \in C(q)^* : \gamma(s)Vs^{-1} = V\}$ where $C(q)^*$ is the group of invertible elements.

Γ is actually a subgroup of $C(q)$ called *Clifford group* of q.

So, each $s \in \Gamma(q)$ defines a automorphism α_s of V which can be defined as $x \mapsto$

$\Gamma(s)xs^{-1}$. This we have a group homomorphism $\alpha : \Gamma(q) \to \text{Aut}(V)$ which takes s to α_s. In fact α_s is an isometry and kernel of α mapping is k^*.

$$1 \to k^* \to \Gamma(q) \overset{\alpha}{\to} O(q) \to 1$$

We can define a canonical involution τ on $Cl(q)$ by $\tau(x_1x_2 \cdots x_k) = x_kx_{k-1} \cdots x_1$ where $x_i \in V$ and extend it to $Cl(q)$ linearly. From this we get *norm* N on $Cl(q)$ by $N(s) = \tau(s)s$. For $x \in V$, $N(x) = q(x)$. This norm can be also extended to $Cl(q)$. It is easy to check that if $s \in \Gamma(q)$, then $N(s) \in k^*$and $N(\gamma(s)) = N(s)$. We define $\text{Pin}(q) = ker(N : \Gamma(q) \to k^*)$. Then we have the following exact sequences,

$$1 \to k^* \to \Gamma(q) \overset{\alpha}{\to} O(q) \to 1$$

$$1 \to \text{Pin}(q) \to \Gamma(q) \overset{N}{\to} k^* \to 1$$

combining these results in

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pm 1 & \longrightarrow & Pin(q) & \longrightarrow & \alpha(Pin(q)) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & k^* & \longrightarrow & \Gamma(q) & \longrightarrow & O(q) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow N & & \downarrow sn & & \\
 & & (k^*)^2 & \longrightarrow & k^* & \longrightarrow & \frac{k^*}{(k^*)^2} & \longrightarrow & 1 \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & &
\end{array}$$

The homomorphism $sn : O(q) \to \frac{k^*}{k^{*2}}$ induced by the norm is called *spinor norm* .

The inverse image $\alpha^{-1}(SO(q))$ is called the *special Clifford group*and is denoted as $S\Gamma(q)$. In fact $S\Gamma(q) \subset Cl_0(q)$ and the intersection is $\text{Spin}(q) = S\Gamma(q) \cap \text{Pin}(q)$ is called the *spin group* of the quadratic space (V,q).

Now, we can define Spin group in our concerned Clifford algebra.

Recall that Clifford algebra is $\mathbb{Z}_2$ graded algebra $Cl = Cl_0 \oplus Cl_1$. Under the map ψ, elements of Cl_0 and Cl_1 corresponds to $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ and $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ respectively.

Consider the following groups,

$$U^0_{2n}(R) = \{x \in Cl_0 \mid xx^* = 1\} \text{ (Corresponding to Pin}(q))$$

and

$$\Gamma(R) = \{x \in Cl \mid x^* V x^{-1} = V\}$$

where V is $H(R^n)$. Taking intersection, we get spin group,

$$Spin_{2n}(R) = \{x \in U^0_{2n} \mid xVx^{-1} = V\}$$

Let $\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \in Spin_{2n}(R)$. Using last identities from last chapter,

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}^* = \begin{cases} \begin{pmatrix} g_1^* & 0 \\ 0 & g_2^* \end{pmatrix} & \text{for } n \text{ even} \\ \begin{pmatrix} g_2^* & 0 \\ 0 & g_1^* \end{pmatrix} & \text{for } n \text{ odd} \end{cases}$$

3.2 Action on Suslin Matrices

We can use the Suslin matrices to prove results of Spin groups. The following is from [Chi15].

We consider two cases when the n is odd and even.

3.2.1 When n is odd

Consider the group

$G_r(R) = \{g \in GL_{2^r}(R) \mid gS(v,w)g^* \text{ is a Suslin Matrix for all Suslin Matrices } S \in M_{2^r}(R)\}$

We also consider the subgroup $SG_r(R)$ consisting of $g \in G_r(R)$ which preserves the norm $v.w^T$ for all (v, w) through the action. It is actually isomorphic to corresponding Spin Group. Here $n = \dim V$ is odd and $r = n - 1$ is even.

For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2^n}(R)$, we have $M^* \begin{pmatrix} D^* & -B^* \\ -C^* & A^* \end{pmatrix}$. Therefore for $(g_1, g_2) = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \in U^0_{2n}(R)$, we have

$$(g_1, g_2)^* = (g_2^*, g_1^*)$$

since elements of $U^0_{2n}(R)$ has unit norm, that is $(g_1, g_2)(g_1, g_2)^* = 1$. Then,

$$g_2 = (g_1^*)^{-1}$$

Spin Group, $Spin_{2n}(R)$ is precisely the subgroup of $U^0_{2n}(R)$ which stabilizes $H(R^n)$ through conjugate action.

If $\big(g, (g^*)^{-1}\big) \in Spin_{2n}(R)$, then for any Suslin Matrix $S \in M_{2^{n-1}}(R)$ there exist a Suslin matrix $T \in M_{2^{n-1}}(R)$ such that,

$$\begin{pmatrix} g & 0 \\ 0 & (g_1^*)^{-1} \end{pmatrix} \begin{pmatrix} 0 & S \\ \overline{S} & 0 \end{pmatrix} \begin{pmatrix} g^{-1} & 0 \\ 0 & g^* \end{pmatrix} = \begin{pmatrix} 0 & T \\ \overline{T} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & gSg^* \\ (g_1^*)^{-1}\overline{S}g^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & T \\ \overline{T} & 0 \end{pmatrix}$$

Hence, for any Suslin Matrix $S \in M_{2^{n-1}}(R)$ and $\big(g, (g^*)^{-1}\big) \in Spin_{2n}(R)$, then gSg^* is a Suslin matrix.

Let us define,

$$g \bullet S = gSg^*$$

Consider,

$$G_{n-1}(R) = \{g \in GL_{2^{n-1}}(R) \mid g{\bullet}S \text{ is a Suslin Matrix for all Suslin Matrices } S \in M_{2^{n-1}}(R)\}$$

Then one can define a homomorphism,

$$\begin{aligned} \chi : Spin_{2n}(R) &\rightarrow G_{n-1}(R) \\ \left(g, (g^*)^{-1}\right) &\mapsto g \end{aligned}$$

But, the map is not surjective.

Now, we define a length function on the space of Suslin matrices as,

$$l(S) = S\overline{S} = v.w^T$$

then norm preserving subgroup of $G_{n-1}(R)$ will be

$$SG_{n-1}(R) = \{g \in G_{n-1}(R) \mid l(g \bullet S) = l(S) \text{ for all Suslin Matrices } S \in M_{2^{n-1}}(R)\}$$

Suppose $g \in SG_{n-1}(R)$, then $l(gg^*) = 1$. Then one can expect $\left(g, (g^*)^{-1}\right)$ to be an element of Spin group.

Theorem 3.2.1. *The homomorphism $\chi : Spin_{2n}(R) \rightarrow SG_{n-1}(R)$ is an isomorphism.*

Proof We first show that if $g \in SG_{n-1}(R)$, then $(g^*)^{-1} \in G_{n-1}(R)$.

Let $T = g \bullet S$, then $\overline{T} = T^{-1} = (g_1^*)^{-1} \bullet S$. Now, we can write a general Suslin matrix as a linear combination of unit-length Suslin matrices using the linearity of $\bullet$ it follows that $\overline{T} = (g_1^*)^{-1} \bullet S$ for a general Suslin matrix S. Consider the map $SG_{n-1}(R) \rightarrow Spin_{2n}(R)$ by $g \mapsto \left(g, (g^*)^{-1}\right)$, and it is actually inverse of χ.

3.2.2 When n is even

Let $(g_1, g_2) \in Spin_{2n}(R)$, then for any Suslin matrix S, there exist a Suslin matrix T,

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \begin{pmatrix} 0 & S \\ \overline{S} & 0 \end{pmatrix} \begin{pmatrix} g_1^{-1} & 0 \\ 0 & g_2^{-1} \end{pmatrix} = \begin{pmatrix} 0 & T \\ \overline{T} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & g_1 S g_2^{-1} \\ g_2 \overline{S} g_1^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & T \\ \overline{T} & 0 \end{pmatrix}$$

Here also we consider the homomorphism

$$\begin{aligned} \chi : Spin_{2n}(R) &\rightarrow G_{n-1}(R) \\ (g_1, g_2) &\mapsto g_1 \end{aligned}$$

This projection has non trivial kernel. Let $(1, g) \in ker(\chi)$. If we take $S = 1$, then g is a Suslin matrix. So, for any S, Suslin matrix g and Sg^{-1} are Suslin matrices. By Lemma 2.1.8, when $n > 2$, we have $g = uI$ where $u \in R$ and $u^2 = 1$.

If we take $\mu = \{u \in R \mid u^2 = 1\}$, we get a exact sequence when $n > 2$ and n even

$$1 \rightarrow \mu \rightarrow Spin_{2n}(R) \rightarrow GL_{2^{n-1}}(R)$$

When n is even, $(g_1, g_2)^* = (g_1^*, g_2^*)$.

3.2.3 $Spin_4(R)$

2×2 Suslin matrix for $v = (a_1, a_2)$ and $w = (b_1, b_2)$ is $S(v, w) = \begin{pmatrix} a_1 & a_2 \\ -b_2 & b_1 \end{pmatrix}$. So, for any pair (g_1, g_2), $g_1 S g_2^{-1}$is a Suslin matrix for $S \in M_2(R)$. Also, by definition norm for 2×2 matrices is determinant.

$$(g_1, g_2)(g_1^*, g_2^*) = (\det g_1, \det g_2)$$

then we have,

$$Spin_4(R) = SL_2(R) \times SL_2(R)$$

3.2.4 $Spin_6(R)$

Theorem 3.2.2. *$Spin_6(R) = SL_4(R)$*

Proof We show that $SG_2(R) = SL_4(R)$. First we show that, for any 4×4 matrix M, the product is MSM^* is a Suslin matrix is S is a Suslin matrix.

Take $S = \begin{pmatrix} a & S_1 \\ -\overline{S}_1 & b \end{pmatrix}$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. For 4×4 matrices,

$$M^* = \begin{pmatrix} A^* & -C^* \\ -B^* & D^* \end{pmatrix}$$

To show MSM^* is a Suslin matrix, it is enough to show with $M \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} M^*$ and $M \begin{pmatrix} 0 & S_1 \\ -\overline{S}_1 & 0 \end{pmatrix} M^*$ are Suslin matrices. We have

$$M \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} M^* = \begin{pmatrix} aAA^* - bBB^* & -aAC^* + bBD^* \\ aCA^* - bDB^* & -aCC^* + bDD^* \end{pmatrix}$$

and

$$M \begin{pmatrix} 0 & S_1 \\ -\overline{S}_1 & 0 \end{pmatrix} M^* = \begin{pmatrix} -B\overline{S}_1A^* - AS_1B^* & B\overline{S}_1C^* + AS_1D^* \\ -D\overline{S}_1A^* - CS_1B^* & D\overline{S}_1C^* + CS_1D^* \end{pmatrix}$$

We know, any 2×2 matrix $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ is a Suslin matrix and $X^* = \overline{X} = \begin{pmatrix} w & -y \\ -z & x \end{pmatrix}$. Also, $X + X^*$ and XX^* are scalar matrices.

Now, if $g \in SG_2(R)$, then $\det(g) = 1$. By definition, $l(S)^2 = \det(S)$ for any Suslin matrix $S \in M_4(R)$. From this, we get $\det(g) = l(gg*) = 1$.

For 4×4 matrices, it is easy to show that $l(MM^*) = \det(M)$. We have,

$$MM^* = \begin{pmatrix} AA^* - BB^* & -AC^* + BD^* \\ CA^* - DB^* & -CC^* + DD^* \end{pmatrix}$$

and

$$l(MM^*) = AA^*DD^* + BB^*CC^* + AC^*BD^* + DB^*CA^*$$

■

Chapter 4

Suslin Set

In this chapter we try to generalize Suslin matrices. First, we define a new set with some properties that are satisfied by Suslin matrices. In next section, we identify Suslin sets in split matrix algebras $M_{2^n}(F)$.

Let us consider a finite dimensional $F-$algebra, where F is a field. For any subset S of $F-$algebra A. we shall define

$$C_A(S) = \{a \in A \mid as = sa \text{ for all } s \in S\}$$

which is called the *centralizer* of S in A. It is always a subalgebra of A. As a special case of this, we shall define $Z(A) = C_A(A)$, called the *center* of the algebra A.

Definition 4.0.1. 1. A is called $F-$*central* (or central over F) if $Z(A) = F$ $(= F.1)$.

2. A is called *simple* if A has no two-sided ideal other than (0) and A.

3. A is called *central simple algebra (CSA)* over F if A satisfies both (1) and (2).

Basic examples are:

Example 4.0.2. For any $n-$dimensional $F-$vector space V, the endomorphism algebra $A = \text{End}(V) \cong M_n(F)$ is always a CSA over F.

Example 4.0.3. Let $a, b \in F^*$. We define the quaternion algebra $Q = (a, b)_F$ to be the $F-$algebra on two generators i, j with the defining relations,

$$i^2 = a,\ j^2 = b,\ ij = -ji.$$

Then, $Q = (a, b)_F$ is a $F-$algebra with $\dim Q = 4$.
Quaternion algebra $A = \left(\frac{a,b}{F}\right)$is also a CSA over F.

Clifford algebras are CSA over some fields [Lam05]. Suslin matrices are defined in an inductive way. We try to generalize the idea of Suslin matrices to a more general set up of central simple algebras.

Definition 4.0.4. Let (A, τ) denote a central simple algebra A with involution τ. A subspace S of A is called a *Suslin set* of (A, τ) if,

1. S is τ stable. That is for $x \in S$, $\tau(x) \in S$.
2. S generates A as algebra.
3. For all $X, Y \in S$, $XYX \in S$. (A special property seen in Suslin matrices Theorem 2.2.3)

Example 4.0.5. Let $S(n)$ be set of Suslin matrices of size $2^n \times 2^n$ and $S \in S(n)$. Suslin matrices form a vector subspace of $M_{2^n}(R)$. We know the matrices of the kind $\begin{pmatrix} 0 & S \\ \overline{S} & 0 \end{pmatrix}$ generates $M_{2^{n+1}}(R)$ which implies Suslin matrices generates $M_{2^n}(R)$. Consider the map $\tau : S \mapsto \overline{S}$, it is an involution. Since $M_{2^n}(R)$ is generated by Suslin matrices, we can consider the map τ in $M_{2^n}(R)$ as $\overline{X} = \overline{S_1 S_2 \cdots S_n} = \overline{S}_n \overline{S}_{n-1} \cdots \overline{S}_1$where $S_1, S_2, ..., S_n \in S(n)$. So, τ becomes an involution in $M_{2^n}(R)$. By Theorem 2.2.3, if $X, Y \in S(n)$, $XYX \in S(n)$. So, $S(n)$ is a Suslin set for $M_{2^n}(R)$ with involution τ.

Example 4.0.6. Let $Q = (a,b)_F$ is a quaternion algebra with standard involution ($i \mapsto -i$, $j \mapsto -j$, $k \mapsto -k$).

Then the following sets are the only Suslin sets,

- $Fi + Fj$
- $Fi + Fk$
- $Fj + Fk$
- $F + Fi + Fj$
- $F + Fi + Fk$
- $F + Fj + Fk$
- $Fi + Fj + Fk$

Remark 4.0.7. The above sets will be Suslin sets for orthogonal involution also.

4.1 Algebras isomorphic to $M_{2^n}(F)$

Let A be an algebra isomorphic to $M_{2^n}(F)$ through a map ϕ and $-$ denote the involution $S \mapsto \overline{S}$. Then there exists an involution τ on A which makes the following diagram commutative,

$$\begin{array}{ccc} A & \xrightarrow{\phi} & M_{2^n}(F) \\ \downarrow \tau & & \downarrow - \\ A & \xrightarrow{\phi} & M_{2^n}(F) \end{array}$$

By taking inverse image of Suslin matrices under ϕ, we get a Suslin set of A and an involution τ on A.

Proposition 4.1.1. *Consider $Q = (1,1)_F$. Then $\underbrace{Q \otimes Q \otimes \cdots \otimes Q}_{n \text{ times}} \cong M_{2^n}(F)$ and the involution τ as above corresponds to $\sigma \otimes \underbrace{id \otimes id \otimes \cdots \otimes id}_{n-1 \text{ times}}$, where σ is standard involution and id is identity map.*

Proof We have an isomorphism $Q \to M_2(F)$ by mapping
$i \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
By tensor product of matrices,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

we get the isomorphism $\underbrace{Q \otimes Q \otimes \cdots \otimes Q}_{n \text{ times}} \cong M_{2^n}(F)$.

- $M_2(F)$

Recall that 2×2 Suslin matrix for $v = (a_0, a_1)$ and $w = (b_0, b_1)$ is

$$S(v, w) = \begin{pmatrix} a_0 & a_1 \\ -b_1 & b_0 \end{pmatrix} \text{ and } \overline{S}(v, w) = \begin{pmatrix} b_0 & -a_1 \\ b_1 & a_0 \end{pmatrix}$$

Then

$$S = \frac{a_0}{2}(1 + i) + \frac{a_1}{2}(j + k) + \frac{b_1}{2}(k - j) + \frac{b_0}{2}(1 - i)$$

and

$$\overline{S} = \frac{a_0}{2}(1 - i) - \frac{a_1}{2}(j + k) - \frac{b_1}{2}(k - j) + \frac{b_0}{2}(1 + i)$$

then τ is $(i \mapsto -i \ , \ j \mapsto -j \ , \ k \mapsto -k)$, which is the standard involution in Q. But, any 2×2 matrices are Suslin matrices then the Suslin set is whole of Q.

- $M_4(F)$

Recall a 4×4 Suslin matrix for $v = (a_0, a_1, a_2)$ and $w = (b_0, b_1, b_2)$ is

$$S = \begin{pmatrix} a & S' \\ -\overline{S}' & b \end{pmatrix} \text{ and } \overline{S} = \begin{pmatrix} b & -S' \\ \overline{S}' & a \end{pmatrix}$$

where S' is a 2×2 Suslin matrix.

Then

$$\begin{aligned} S &= \frac{a}{2}(1 \otimes 1 + i \otimes 1) + \frac{b}{2}(1 \otimes 1 - i \otimes 1) + \frac{1}{2}(j \otimes S' + k \otimes S')\frac{1}{2}(k \otimes \overline{S}' - j \otimes \overline{S}') \\ \overline{S} &= \frac{a}{2}(1 \otimes 1 - i \otimes 1) + \frac{b}{2}(1 \otimes 1 + i \otimes 1) - \frac{1}{2}(j \otimes S' + k \otimes S') - \frac{1}{2}(k \otimes \overline{S}' - j \otimes \overline{S}') \end{aligned}$$

The involution τ is $\sigma \otimes id$, where σ is standard involution and id is identity map.

- $M_{2^n}(F)$

$2^n \times 2^n$ Suslin matrices look like,

$$S = \begin{pmatrix} a & S' \\ -\overline{S}' & b \end{pmatrix} \text{ and } \overline{S} = \begin{pmatrix} b & -S' \\ \overline{S}' & a \end{pmatrix}$$

where S' is a $2^{n-1} \times 2^{n-1}$ Suslin matrix and elements in $\underbrace{Q \otimes Q \otimes \cdots \otimes Q}_{n \text{ times}}$ will be

$$\begin{aligned} S &= \frac{a}{2}(\underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{n \text{ times}} + i \otimes \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{n-1 \text{ times}}) + \frac{b}{2}(\underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{n \text{ times}} - i \otimes \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{n-1 \text{ times}}) \\ &\quad + \frac{1}{2}(j \otimes S' + k \otimes S') + \frac{1}{2}(k \otimes \overline{S}' - j \otimes \overline{S}') \\ \overline{S} &= \frac{a}{2}(\underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{n \text{ times}} - i \otimes \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{n-1 \text{ times}}) + \frac{b}{2}(\underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{n \text{ times}} + i \otimes \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{n-1 \text{ times}}) \\ &\quad - \frac{1}{2}(j \otimes S' + k \otimes S') - \frac{1}{2}(k \otimes \overline{S}' - j \otimes \overline{S}') \end{aligned}$$

then here the involution τ is $\sigma \otimes \underbrace{id \otimes id \otimes \cdots \otimes id}_{n-1 \text{ times}}$, where σ is standard involution and id is identity map. ■

Remark 4.1.2. We can consider another isomorphism in 4×4 matrices. Here also, τ is $\sigma \otimes id$.

For $Q = (1,1)_F$, take the isomorphism,

$$\begin{aligned} \phi : Q \otimes Q &\rightarrow M_4(F) = End(Q) \\ x \otimes y &\mapsto \sigma_{x \otimes y} \end{aligned}$$

where $\sigma_{x\otimes y}(v) = x.v.y.$

4×4 Suslin matrix for $v = (a_0, a_1, a_2)$ and $w = (b_0, b_1, b_2)$ is

$$S(v, w) = \begin{pmatrix} a_0 & 0 & a_1 & a_2 \\ 0 & a_0 & -b_2 & b_1 \\ -b_1 & a_2 & b_0 & 0 \\ -b_2 & -a_1 & 0 & b_0 \end{pmatrix}$$

this corresponds to the following element of $Q \otimes Q$,

$$\frac{a_0}{2}(1\otimes1+i\otimes i)+\frac{b_0}{2}(1\otimes1-i\otimes i)-\frac{b_1}{2}(i\otimes k+j\otimes1)+\frac{a_1}{2}(j\otimes1-i\otimes k)-\frac{b_2}{2}(i\otimes j+k\otimes1)+\frac{a_2}{2}(i\otimes j-k\otimes1)$$

and

$$\overline{S}(v, w) = \begin{pmatrix} b_0 & 0 & -a_1 & -a_2 \\ 0 & b_0 & b_2 & -b_1 \\ b_1 & -a_2 & a_0 & 0 \\ b_2 & a_1 & 0 & a_0 \end{pmatrix}$$

this corresponds to the following element of $Q \otimes Q$,

$$\frac{a_0}{2}(1\otimes1-i\otimes i)+\frac{b_0}{2}(1\otimes1+i\otimes i)+\frac{b_1}{2}(i\otimes k+j\otimes1)-\frac{a_1}{2}(j\otimes1-i\otimes k)+\frac{b_2}{2}(i\otimes j+k\otimes1)-\frac{a_2}{2}(i\otimes j-k\otimes1)$$

The the involution in $Q \otimes Q$ is

$$\begin{aligned} 1 \otimes 1 &\mapsto 1 \otimes 1 \\ i \otimes i &\mapsto -i \otimes i \\ i \otimes j &\mapsto -i \otimes j \\ i \otimes k &\mapsto -i \otimes k \\ j \otimes 1 &\mapsto -j \otimes 1 \end{aligned}$$

this is $\sigma \otimes id$, where σ is standard involution and id is identity map.

We get the Suslin set in $Q \otimes Q$ as,

$$Span_F\{1 \otimes 1, i \otimes i, i \otimes j, i \otimes k, j \otimes 1, k \otimes 1\}.$$

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