Monodromy Groups of Fuchsian Differential Equations

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Certificate of Examination

This is to certify that the dissertation titled **Monodromy Groups of Fuchsian Differential Equations** submitted by **Rishabh Dhiman** (Reg. No. MS12082) for the fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 20, 2017

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Varadharaj R. Srinivasan at the Indian Institute of Science Education and Research, Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. This work is based on the research article "Two Generator Subgroups of $\mathrm{SL}_2(\mathbb{C})$ and the Hypergeometric, Riemann, and Lamé equations" authored by R.C. Churchill appeared in *J. Symb. Comp*,(1999) **28**, 521-545. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Varadharaj R. Srinivasan
(Supervisor)

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Abstract

In this thesis, we study the monodromy groups of Fuchsian Differential Equations and its properties. We find circuit matrices at all singularities of a Fuchsian differential equation. These circuit matrices forms a group called monodromy group. In a Fuchsian differential equation, if there are three singularities then we can predict the properties of its monodromy group by finding the trace of circuit matrices at all singularities.

Chapter 1 deals with basic definitions and terminologies. In Chapter 2, we provide a formula to calculate the traces of the circuit matrices at singular points which depends on analytic coefficients of our Fuchsian differential equation. We state our main theorem in Chapter 3 and discuss few examples. In Chapter 4 we prove several interesting group theoretic lemmas that are needed for the main theorem and outline the proof of our main theorem. All our proofs and examples can be found in [Chu99].

Contents

Chapter 1

Important Statements and results

This chapter contains definitions and few results which will be used later.

1.1 Important Definitions

Regular Singular Point: Consider the differential equation

$$D^n y + P_1(z)D^{n-1}y + ... + P_{n-1}(z)Dy + P_n(z)y = 0 \tag{1.1}$$

where

$$D = \frac{d}{dz}, \;\; P_i(z) \in \mathbb{C}(z)$$

then a point $z_0 \in \mathbb{C}$ is said to be **regular singular** iff $\lim\limits_{z \to z_0} (z - z_0)^i P_i(z)$ exists $\forall i = 1, 2, ..., n$. ∞ is said to be regular iff $\lim\limits_{z \to \infty} z^i P_i(z)$ exists $\forall\ i = 1, 2, 3..., n$.

Fuchsian Differential Equation: It is a linear homogeneous ordinary differential equation with analytic coefficients in the complex domain whose all singular points are regular singular points.

Monodromy Group/ Circuit Matrix: Upon analytic continuation of solutions of fuchsian differential equation, suppose that the solutions $y = f_i(z)$ are taken to the solutions $y = g_i(z)$. Since, solutions are linearly independent, therefore, there exists a matrix $[m_{ij}]_{i,j=1}^n$ such that

$$g_i = \sum_{j=1}^{n} m_{ij} f_j \quad \forall i$$

The matrix $[m_{ij}]_{i,j=1}^n$ is called the monodromy or circuit matrix.

Diagonalizable Group: Suppose $G \subset SL(2, \mathbb{C})$, then G is said to be diagonalizable if it has matrix representation in D (where D forms the group of all unimodular diagonal matrices)

Reducible Group: Suppose $G \subset SL(2, \mathbb{C})$, then G is said to be reducible if it has the matrix representation in group

$$\left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \text{such that } \lambda \in \mathbb{C} - \{0\} \right\}$$

Imprimitive/DP Group: Suppose $G \subset SL(2, \mathbb{C})$, then G is said to be Imprimitive/DP if it has matrix representation in $D \cup P$ (where P is the permutation group which contains unimodular matrices of the form

$$\left\{ \begin{bmatrix} 0 & \eta \\ -\eta^{-1} & 0 \end{bmatrix} \text{such that } \eta \in \ \mathbb{C} - \{0\} \right\}$$

Quaternionic Group: Suppose $G \subset SL(2, \mathbb{C})$, then G is said to be quaternionic if G is isomorphic to quarternion group.

Zariski closed sets: Let $S \subset k[x_1, x_2, ..., x_n]$ be a set of polynomials and let $V(S) \subset A_n$ (where A_n is an affine n-space) denotes common zeroes of elements of S. i.e.

$$V(S) = \{a \in A_n \mid \ f(a) = 0 \ \ \forall f \in S\}$$

Then, a subset X of A_n of the form $V(S)$ is said to be Zariski closed in A_n.

Zariski closure: In a topological space X, the Zariski closure $\overline{F}$ of $F \subset X$ is the smallest closed set in X such that $F \subset \overline{F}$.

Now, suppose that $\overline{H}$ denote the normal subgroups of $GL(2, \mathbb{C})$. The elements of $\overline{H}$ are of the form $(\mathbb{C} - \{0\})I$. Let $G \subset GL(2, \mathbb{C})$,and, suppose $H_G = G \cap \overline{H}$. So, **projective group** for G is given as G/H_G. Now, let $G \in GL(2, \mathbb{C})$, then

- G is **Projectively Dihedral** if G/H_G is isomorphic to the dihedral group D_n where $n > 2$
- G is **Projectively Tetrahedral** if G/H_G is isomorphic to A_4.
- G is **Projectively Octadral** if G/H_G is isomorphic to S_4.
- G is **Projectively icosahedral** if G/H_G is isomorphic to A_5.

1.2 Analytic Continuation

It is a technique of extending the domain of analytic function over which it is defined. (from reference [Ahl78])

1.2.1 Continuation Principle

The function obtained by analytic continuation of any solution of an analytic differential equation, along any path in the complex plane, is a solution of the differential equation along the same path.

1.2.2 Branch Point

A Branch Point of an analytic function is a point in the complex plane whose complex argument can be mapped from a single point in the domain to the multiple points in range. for example, consider the function

$$f(z) = z^a \quad a \in \mathbb{C} \;\; and \;\; a \notin \mathbb{Z}$$

Write $z = e^{i\theta}$ where $\theta \in [0, 2\pi)$, then at $\theta = 0$, $f(e^{0i}) = 1$ and at $\theta = 2\pi$, $f(e^{2\pi i}) = e^{2\pi i a}$ So, the values of $f(z)$ at $arg(z) = 0$ and $arg(z) = 2\pi$ are different, despite the fact that they correspond to the same point in the domain. In this way we have also extended the domain in which function is defined.

1.2.3 Analytic continuation of $f(z) = z^p$

Firstly, we will look at the analytic continuation of the function $g(z) = ln(z)$ in complex plane. Let $\theta \in [0, 2\pi)$ be the argument of z. write z in polar form $z = Re^{i\theta}$, then $lnz = lnR + i\theta$. For, the same point $z = Re^{i\theta + 2n\pi i}$, $lnz = lnR + i\theta + 2n\pi i$. Thus, for the same point, function $g(z) = ln(z)$ has different values.

Now consider the function $f(z) = z^p$ (where $z, p \in \mathbb{C}$) in complex plane, then

$$\begin{aligned} f(z) = z^p = z^{\alpha + i\beta} &= e^{\alpha lnz} e^{i\beta lnz} \\ &= e^{\alpha(ln|z| + iArgz)} e^{i\beta(ln|z| + iArgz)} \\ &= e^{\alpha(ln|z|)} e^{i\beta(ln|z|)} e^{\alpha(iArg(z))} e^{i\beta(iArg(z))} \end{aligned}$$

Since, $Arg(z)$ and $Arg(z + 2n\pi)$ corresponds to the same point in domain. Therefore, for the same point in domain

$$\begin{aligned} f(z) &= e^{(\alpha + i\beta)ln|z|} e^{i\alpha(Arg(z) + 2n\pi)} e^{-\beta(Arg(z) + 2n\pi)} \\ &= e^{(\alpha + i\beta)ln|z|} e^{(i\alpha - \beta)(Argz + 2n\pi)} \\ &= e^{p.ln|z|} e^{ip(Argz + 2n\pi)} \\ &= e^{p[ln|z| + iArgz] + 2n\pi ip} = e^{p.logz} e^{2n\pi ip} \\ &= z^p e^{2\pi nip} = f(ze^{2n\pi i}) \end{aligned} \tag{1.2}$$

So, for the same point in domain, $f(z) = z^p$ has different values. Hence, Analytic continuation of $f(z)$ *is* $f(ze^{2n\pi i})$.

Remark 1. *Analytic continuation of* $f(z) = z^p$ *is* $f(z)$ *if* $p \in \mathbb{Z}$.

1.3 Normal Form of second order Fuchsian Equation

Suppose

$$\frac{d^2y}{dx^2} + A_1(x)\frac{dy}{dx} + A_0(x)y = 0 \tag{1.3}$$

is defined on the Riemann sphere P^1 having rational function coefficients. Let $y = ye^{-\int A_1(x)/2}$, then, by substituting the value of y in (1.3), we can find the normal form which is given as (from reference [Kov86])

$$\frac{d^2y}{dx^2} + ry = 0 \tag{1.4}$$

where $r = \frac{A_1(x)^2}{4} + \frac{A_1(x)'}{2} - A_0(x)$.

Chapter 2

Calculation of Circuit Matrix

In this chapter, we will define the general form of Fuchsian Differential Equation and will calculate the circuit matrix and its trace at regular singular points of a second order Fuchsian Differential Equation.

The most general second order Fuchsian differential equation on Riemann sphere P^1 is of the form

$$y'' + \left(\sum_{j=1}^{m} \frac{A_j}{x - a_j}\right) y' + \left(\sum_{j=1}^{m} \frac{B_j}{(x - a_j)^2} + \sum_{j=1}^{m} \frac{C_j}{x - a_j}\right) y = 0 \tag{2.1}$$

such that $a_1, a_2 ...., a_m$ are distinct complex numbers and A_j, B_j, C_j are complex constants such that $\sum_{j=1}^{m} C_j = 0$(from reference [BR89]). The normal form of (2.1) is

$$y'' + \left(\sum_{j=1}^{m} \frac{\overline{B_j}}{(x - a_j)^2} + \sum_{j=1}^{m} \frac{\overline{C_j}}{x - a_j}\right) y = 0 \tag{2.2}$$

where

$$\overline{B_j} = \frac{1}{4}(1 + 4B_j - (1 - A_j)^2))$$

$$\overline{C_j} = C_j - \frac{1}{2}\left(\sum_{i \neq j} \frac{A_j}{a_j - a_i}\right).$$

2.1 Circuit matrix for m=2

Fuchsian Equation (2.1) takes the form

$$y'' + \left(\sum_{j=1}^{2} \frac{A_j}{x-a_j}\right) y' + \left(\sum_{j=1}^{2} \frac{B_j}{(x-a_j)^2} + \sum_{j=1}^{2} \frac{C_j}{x-a_j}\right) y = 0 \tag{2.3}$$

For point a_1, let us assume a series solution

$$y = \sum_{n=0}^{\infty} c_n (x-a_1)^{n+r} \tag{2.4}$$

$$y' = \sum_{n=0}^{\infty} c_n (n+r)(x-a_1)^{n+r-1} \tag{2.5}$$

$$y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1)(x-a_1)^{n+r-2} \tag{2.6}$$

Putting (2.4),(2.5) and (2.6) in (2.3), we get the indicial equation as follows:

$$r^2 + r(A_1 - 1) + B_1 = 0$$

Therefore,

$$r = \frac{1}{2}\left(1 - A_1 \pm \sqrt{(A_1-1)^2 - 4B_1}\right) \tag{2.7}$$

These are called characteristic exponents. Generator exponents are given as:

$$w_1(x) = x^{\frac{1}{2}(1-A_1+\sqrt{(1-A_1)^2-4B_1})}$$

$$w_2(x) = x^{\frac{1}{2}(1-A_1-\sqrt{(1-A_1)^2-4B_1})}$$

So, the solutions for (2.3) will be of the form:

$$y = (x^{\frac{1}{2}(1-A_1\pm\sqrt{(1-A_1)^2-4B_1})})(c_0 + c_1(x-a_1) + c_2(x-a_1)^2 + ...)$$

By Remark(1) in chapter 2, Analytic continuation of function $f(z) = z^p$ is $f(z)$ if $p \in Z$. So, we don't need to do analytic continuation of whole solution instead we only need to do it for generator exponents. Analytic Continuation of generator exponents:

$$w_1(xe^{2\pi i}) = x^{\frac{1}{2}(1-A_1+\sqrt{(1-A_1)^2-4B_j})}e^{i\pi(1-A_1+\sqrt{(1-A_1)^2-4B_j})}$$

$$w_2(xe^{2\pi i}) = x^{\frac{1}{2}(1-A_1-\sqrt{(1-A_1)^2-4B_j})}e^{i\pi(1-A_1-\sqrt{(1-A_1)^2-4B_j})}$$

So, by Continuation principle

$$w_1(xe^{2\pi i}) = w_1(x)e^{i\pi(1-A_1+\sqrt{(1-A_1)^2-4B_1})} + w_2(x) \times 0$$

Similarly,

$$w_2(xe^{2\pi i}) = w_1(x) \times 0 + w_2(x)e^{i\pi(1-A_1-\sqrt{(1-A_1)^2-4B_1})}$$

Hence, circuit matrix (t_1) is:

$$\begin{bmatrix} e^{i\pi(1-A_1+\sqrt{(1-A_1)^2-4B_1})} & 0 \\ 0 & e^{i\pi(1-A_1-\sqrt{(1-A_1)^2-4B_1})} \end{bmatrix}$$

Therefore,

$$trace(t_1) = -2e^{-i\pi A_1} Cos\pi\sqrt{(A_1-1)^2-4B_1} \tag{2.8}$$

Similarly, we can find circuit matrix and its trace for point a_2. In the same way, for equation (2.1), we can find circuit matrix and trace for any $x = a_j$(singularity) where $j = 1, 2, ..., m+1$. So, trace at any singularity is given as

$$trace(t_j) = -2e^{-i\pi A_j} Cos\pi\sqrt{(A_j-1)^2-4B_j}. \tag{2.9}$$

Denote $\infty \in P^1$ by a_{m+1} and set

$$A_{m+1} = A_\infty = 2 - \sum_{j=1}^{m}(A_j), \;\; B_{m+1} = B_\infty = \sum_{j=1}^{m}(B_j + C_j a_j). \tag{2.10}$$

For equation (2.3), $a_3 = a_\infty$ *and* $b_3 = b_\infty$.

For normal form (2.2) of general Fuchsian Equation trace(t_j) for any a_j where $j = 1, 2.., m+1$ is given as

$$t_j = -2Cos\pi\sqrt{(A_j - 1)^2 - 4B_j}. \tag{2.11}$$

2.2 ∞ as regular singular point

Let $u = \frac{1}{x}$. Then,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -u^2\frac{dy}{du} \tag{2.12}$$

$$\frac{d^2y}{dx^2} = 2u^3\frac{dy}{du} + u^4\frac{d^2y}{du^2} \tag{2.13}$$

Put (2.12) and (2.13) in (2.1), and then calculate

$\lim_{u\to 0} u\times$ coefficient of $\frac{dy}{du}$ which gives, $A_\infty = 2 - \sum_{j=1}^{m} A_j$ which is finite. Similarly calculate,

$\lim_{u\to 0} u^2\times$ coefficient of y which gives, $B_\infty = \sum_{j=1}^{m} B_j + C_j a_j$ which is also finite.

Hence, ∞ is a regular singular point of general Fuchsian differential equation.

Chapter 3

The Main Theorem

In the previous chapter, we found the formula for calculating the trace of circuit matrix at regular singular points. In this chapter, we shall provide a group theoretic formulation of the main theorem (whose proof will be given in chapter 4). We shall also determine the properties of monodromy group of certain Fuchsian Differential Equations.

3.1 The Main Theorem

Theorem. *Assume that the Fuchsian Equation (2.1) has three regular singularities, a_1, a_2 and $a_3 = a_\infty$, and $G \subset SL(2, \mathbb{C})$ is the monodromy group of its normal form which is given by Equation (2.2). For j=1, 2, 3 set*

$$t_j = -2Cos\pi\sqrt{(A_j - 1)^2 - 4B_j}$$

such that $A_3 = A_\infty$ and $B_3 = B_\infty$ are as above in (2.10), and t_j denotes the trace of monodromy matrix at singularity a_j. Then, we can say that:

(a) G is ***reducible*** *if and only if one of the two equivalent conditions is satisfied :*

1. $t_1{}^2 + t_2{}^2 + t_3{}^2 - t_1 t_2 t_3 = 4.$
2. *at least one of the four possible cases of the expression* $\sqrt{(A_1 - 1)^2 - 4B_1} + \sqrt{(A_2 - 1)^2 - 4B_2} + \sqrt{(A_3 - 1)^2 - 4B_3}$ *is odd.*

(b) If G is irreducible. Then, G is ***DP*** *if and only if one of the two equivalent conditions is satisfied :*

1. *At least two of* t_1, t_2 *and* t_3 *are zero.*
2. *At least two of* $2\sqrt{((A_1-1)^2-4B_1)}$, $2\sqrt{((A_2-1)^2-4B_2)}, 2\sqrt{((A_3-1)^2-4B_3)}$ *are odd.*

Further, in the irreducible DP case G is ***finite*** *if and only if :*

1. *all three* t_j *becomes zero for* $j = 1, 2, 3$. *or, equivalently all three of the expressions above in (2) gives us odd integers, and, G becomes quaternionic in this case; or*
2. *the non-zero trace is resonant, G becomes* ***projectively dihedral*** *in this case.*

(c) G is finite but irreducible and not DP if and only if G is ***projectively tetrahedral***, ***projectively octahedral***, ***projectively icosahedral***. *Further,*

1. *The group G is* ***projectively tetrahedral*** *if and only if* ${t_1}^2+{t_2}^2+{t_3}^2-t_1t_2t_3 = 2$ *such that* $t_1, t_2, t_3 \in \{0, \pm 1\}$.
2. *The group G is* ***projectively octahedral*** *if and only if* ${t_1}^2+{t_2}^2+{t_3}^2-t_1t_2t_3 = 3$ *such that* $t_1, t_2, t_3 \in \{0, \pm 1, \pm\sqrt{2}\}$.
3. *The group G is* ***projectively icosahedral*** *if and only if* ${t_1}^2+{t_2}^2+{t_3}^2-t_1t_2t_3 \in \{2-\mu_2, 3, 2+\mu_1\} = \{1+{\mu_2}^2, 3, 1+{\mu_1}^2\}$ *such that* $t_1, t_2, t_3 \in \{0, \pm\mu_2, \pm 1, \pm\mu_1\}$, *where* $\mu_1 = \frac{1}{2}(1+\sqrt{5})$ *and* $\mu_2 = -\frac{1}{2}(1-\sqrt{5})$.

(d) If none of (a) to (c) hold, then, the Zariski closure of G is $SL(2, \mathbb{C})$.

Proof. (b) Assume that for some $t_j = -2Cos\pi\sqrt{(A_j-1)^2-4B_j} = 0$ for $j = 1, 2, 3$ which is possible if $\sqrt{(A_j-1)^2-4B_j}$ = odd integer/2. Hence, $2\sqrt{(A_j-1)^2-4B_j}$ must be an odd integer.

Now assume that $2\sqrt{(A_j-1)^2-4B_j}$ is an odd integer for some $j = 1, 2, 3$. It implies that $\sqrt{(A_j-1)^2-4B_j}$ = odd integer/2. Hence, $t_j = -2Cos\pi\sqrt{(A_j-1)^2-4B_j} = 0$. □

The rest of the proof will be given by analogous theorm proved in chapter 4.

Now, we will give some examples of The Main Theorem.

3.2 Examples of Main Theorem

Example 3.1 (The Hypergeometric Equation)**.** *The Hypergeometric equation is a Fuchsian equation of the form*

$$y'' + \frac{\gamma - (\alpha+\beta+1)x}{x(1-x)}y' - \frac{\alpha\beta}{x(1-x)}y = 0 \tag{E}$$

defined on the Riemann Sphere P^1*, where* $\alpha, \beta and \gamma$ *are arbitrary complex constants. Hypergeometric equation can also be written as*

$$y'' + \left(\frac{\gamma}{x} - \frac{\gamma}{x-1} + \frac{\alpha+\beta+1}{x-1}\right)y' + \left(\frac{-\alpha\beta}{x} + \frac{\alpha\beta}{x-1}\right)y = 0.$$

The normal form of (E) is

$$y'' + \frac{1}{4}\left(\frac{1-\lambda^2}{x^2} + \frac{1-\nu^2}{(x-1)^2} - \frac{\lambda^2-\mu^2+\nu^2-1)}{x} + \frac{\lambda^2-\mu^2+\nu^2-1}{x-1}\right)y = 0$$

with parameters λ, μ, ν *defined by*

$$\begin{aligned} \lambda &= 1-\gamma \\ \nu &= \gamma - (\alpha+\beta) \\ \mu &= \pm(\alpha-\beta) \end{aligned} \tag{3.1}$$

Characteristic exponents at singularity a_1*=0 are 0 and* $1-\gamma$*, at singularity* a_2*=1 are 0 and* $\gamma - (\alpha+\beta)$ *and at singularity* a_3 *=*a_∞ $= \infty$ *are* α *and* β*. The traces at singularities* a_1, a_2, a_∞ *are as follows:*

$$\begin{aligned} t_1 &= -2Cos\pi(\gamma-1) = -2Cos\pi\lambda \\ t_2 &= -2Cos\pi(\gamma-(\alpha+\beta)) = -2Cos\pi\nu \\ t_\infty &= -2Cos\pi(\alpha-\beta) = -2Cos\pi\mu \end{aligned} \tag{3.2}$$

Proposition 3.1. *The monodromy group G of hypergeometric equation (E) is:*

(a) reducible if and only if at least one of $\alpha, \beta, \gamma-\alpha$ *and* $\gamma-\beta$ *is an integer or, parellely, if and only if at least one determination of* $\pm\lambda\pm\nu\pm\mu$ *is an odd integer; and*

(b) DP but irreducible if and only if at least two of $\lambda - \frac{1}{2}, \nu - \frac{1}{2}$ *and* $\mu - \frac{1}{2}$ *are integers*

and third has not the form $n - \frac{1}{2}$ for some integer n. Further, G is quaternionic if and only if all three of $\lambda - \frac{1}{2}, \nu - \frac{1}{2}$ and $\mu - \frac{1}{2}$ are integers.

Proof. (a) From main theorm, we know that monodromy group is reducible iff one of the possible determinations of expression

$\sqrt{(A_1-1)^2-4B_1} + \sqrt{(A_2-1)^2-4B_2} + \sqrt{(A_3-1)^2-4B_3}$
is an odd integer . In view of proposition let us assume that $\lambda + \mu + \nu$ is an odd integer which implies that $1 - 2\beta$ is an odd integer,which is only possible if β is an integer. Similarly, we can prove for other cases. (from reference [BC90])

(b) From main theorm we know that monodromy group is DP but not reducible iff at least two of three traces vanishes. Suppose that $\lambda - \frac{1}{2}$ is an integer, then $\lambda =$ odd integer/2, which implies $t_1 = -2Cos\pi\lambda = 0$. Similarly, we can show for other cases also. □

Example 3.2. ***The Legendre Equation:***

The Legendre equation is of the form

$$y'' - \frac{2x}{1-x^2}y' + \frac{\lambda}{1-x^2}y = 0 \tag{F}$$

where λ is real. It can also be written as

$$y'' + \left(\frac{1}{x-1} + \frac{1}{x+1}\right)y' + \left(\frac{-\lambda/2}{x-1} + \frac{\lambda/2}{x+1}\right)y = 0 \tag{G}$$

Hence, it is Fuchsian. The normal form is given as

$$y'' + \frac{1}{4}\left(\frac{1}{(x-1)^2} + \frac{1}{(x+1)^2} - \frac{2\lambda+1}{x-1} + \frac{2\lambda+1}{x+1}\right)y = 0 \tag{H}$$

For $a_1 = 1$, characteristic exponents are $r^{\pm} = \dfrac{1}{2}(1 - 1 \pm \sqrt{(1-1)^2 - 0})$ Hence, trace of circuit matrix is $t_1 = -2$. Similarly, for $a_2 = -1$, $t_2 = -2$. Now, $A_\infty = 0, B_\infty = -\frac{\lambda}{2} - \frac{\lambda}{2} = -\lambda$. So, $r^{\pm} = \dfrac{1}{2}(1 \pm \sqrt{1+4\lambda})$. Therefore, trace at $a_3 = a_\infty$ is $t_3 = -2Cos\pi\sqrt{1+4\lambda}$. Hence,

$$t_1{}^2 + t_2{}^2 + t_3{}^2 - t_1t_2t_3 = 4(Cos(\pi\sqrt{1+4\lambda}) + 1)^2 + 4 \tag{3.3}$$

By main theorem, we can show that

- *the corresponding monodromy group is reducible iff* $Cos(\pi\sqrt{1+4\lambda}) = -1$*, which is only possible iff* $\lambda = k(k+1)$.
- *Clearly, G is not DP/Imprimitive as any two traces does not vanish.*
- *Clearly, equation (5.1)* $\neq 2 or 3$. *Hence, Monodromy group can't be projectively tetrahedral or octahedral.*
- *Monodromy group can't be projectively icosahedral because* ${t_1}^2+{t_2}^2+{t_3}^2-t_1t_2t_3 \notin (2-\mu_2, 3, 2+\mu_2)$ *where* $\mu_1 = -\frac{1}{2}(1-\sqrt{5}), , \mu_2 = \frac{1}{2}(1+\sqrt{5})$*, and* $2-\mu_2 = \frac{5-\sqrt{5}}{2} < 4, 2+\mu_2 = \frac{5+\sqrt{5}}{2} < 4$ *and* $3 < 4$.
- *Hence,monodromy group is either reducible or Zariski closure of G is* $SL(2,\mathbb{C})$.

Example 3.3. ***Riemann's Equation.***

It is the Fuschsian equation of the form $y'' + c_1(x)y' + c_2(x)y = 0$*, where*

$$c_1(x) = \frac{1-\eta_1-\mu_1}{x} + \frac{1-\eta_2-\mu_2}{x-1} \tag{3.4}$$

$$c_2(x) = \frac{\eta_1\mu_1}{x_2} + \frac{\eta_2\mu_2}{(x-1)^2} + \frac{\eta_3\mu_3-\eta_1\mu_1-\eta_2\mu_2}{x(x-1)}. \tag{3.5}$$

Complex constants η_j, μ_j *are subject to the constraint*

$$\sum(\eta_j + \mu_j) = 1.$$

Its normal form is given as

$$y'' + \frac{1}{4}\left(\frac{1-(\eta_1-\mu_1)^2}{x^2} + \frac{1-(\eta_2-\mu_2)^2}{(x-1)^2} + \frac{\nu}{x} - \frac{\nu}{x-1}\right) y = 0 \tag{I}$$

where $\nu = 1-(\eta_1-\mu_1)^2-(\eta_2-\mu_2)^2+(\eta_3-\mu_3)^2$.

The traces at singularities $a_1 = 0, a_2 = 1$ *and* $a_3 = \infty$ *are* $t_j = -2Cos\pi(\eta_j - \mu_j)$ *where* $j = 1, 2, \infty$ *and* $\eta_\infty = \eta_3$, $\mu_\infty = \mu_3$.

Proposition 3.2. *The monodromy group G of Riemann equation (I) is :*

(a) reducible if and only if one of $\eta_1+\eta_2+\eta_3, \eta_1+\eta_2+\mu_3$, $\eta_1+\mu_2+\eta_3, \mu_1+\eta_2+\eta_3$ *is an integer; and*

(b) DP but irreducible if and only if at least two of the three expressions $2(\eta_j - \mu_j)$ are odd integers and third is an even integer. Furthermore, G is quaternionic if and only if all three expressions are odd integers.

Proof. (a) We know that monodromy group G is reducible if and only if one of the four expressions

$$\sqrt{(A_1-1)^2-4B_1}+\sqrt{(A_2-1)^2-4B_2}+\sqrt{(A_3-1)^2-4B_3}$$

is an odd integer.

For equation (I)

$$\sqrt{(A_1-1)^2-4B_1}+\sqrt{(A_2-1)^2-4B_2}+\sqrt{(A_3-1)^2-4B_3}$$

$$= \eta_1+\eta_2+\eta_3-(\mu_1+\mu_2+\mu_3) \tag{3.6}$$

We know that $\sum(\eta_j+\mu_j)=1$. Let $\eta_1+\eta_2+\eta_3= n$(integer). Therefore $\mu_1+\mu_2+\mu_3$=1 - n. So, equation (3.6) becomes 2n-1 which is an odd integer.

Similarly, we can show for other possible expressions. (from reference [BC90])

(b) By main theorem, we know that G is DP but not reducible if and only if at least two of

$$2\sqrt{(A_j-1)^2-4B_j}$$

where $j=1,2,\infty$ are odd integers.

For equation (I)

$$2\sqrt{(A_j-1)^2-4B_j}=2(\eta_j-\mu_j)$$

where $j=1,2,\infty$. Hence by main theorem, it is clear that monodromy group is DP but not reducible iff at least two of the above integers are odd. If all three are odd, then monodromy group is quaternionic. □

Example of Riemann Equation.

Consider the Riemann Fuchsian Equation:

$$y'' - (\frac{2/3}{x} + \frac{4/3}{x-1})y' + (\frac{2/3}{x^2} + \frac{4/3}{(x-1)^2} - \frac{5/16}{x(x-1)})y = 0.$$

Observe that $1 - \eta_1 - \mu_1 = -\frac{2}{3}$ and $\eta_1\mu_1 = \frac{2}{3}$. So, $(\mu_1, \eta_1) = (1, \frac{2}{3})$ or $(\mu_1, \eta_1) = (\frac{2}{3}, 1)$.

Also, $1 - \eta_2 - \mu_2 = -\frac{4}{3}$ and $\eta_2\mu_2 = \frac{4}{3}$. So, $(\mu_2, \eta_2) = (1, \frac{4}{3})$ or $(\mu_2, \eta_2) = (\frac{4}{3}, 1)$.

Now, $\eta_3\mu_3 - \eta_1\mu_1 - \eta_2\mu_2 = -\frac{5}{16}$ and $\eta_1 + \mu_1 + \eta_2 + \mu_2 + \eta_3 + \mu_3 = 1$. So, $(\mu_3, \eta_3) = (-\frac{9}{4}, -\frac{3}{4})$ and $(\mu_3, \eta_3) = (-\frac{9}{4}, -\frac{3}{4})$.

For singularities $a_1 = 0, a_2 = 1$ and $a_3 = \infty$ the traces are as following:

$$trace(t_1) = -2Cos\pi(\eta_1 - \mu_1) = -1 \tag{3.7}$$

$$trace(t_2) = -2Cos\pi(\eta_2 - \mu_2) = -1 \tag{3.8}$$

$$trace(t_3) = -2Cos\pi(\eta_3 - \mu_3) = 0 \tag{3.9}$$

Hence, $t_1^2 + t_2^2 + t_3^2 - t_1t_2t_3 = 2$. Therefore, Monodromy Group of normal form is projectively tetrahedral.

Chapter 4

Reduction to Group Theory

The Main Theorem in previous chapter can be reduced to a purely group- theoretic result which we will discuss in this chapter.

Theorem. *Assume that* $S, T \in SL(2, \mathbb{C})$ *such that* $G = \langle S, T \rangle$ *Then, we can say that:*

(a) G is ***reducible*** *if and only if* ${t_S}^2 + {t_T}^2 + {t_{ST}}^2 - t_S t_T t_{ST} = 4$.

(b) If G is irreducible. Then, G is DP if and only if two of the three traces t_S, t_T *and* t_{ST} *becomes zero. Further, in the irreducible DP case G is finite if and only if:*

   1. All of the three traces become zero, and, G becomes quaternionic in this case.

   2. non - zero trace is resonant, G becomes projectively dihedral in this case.

(c) Suppose G is finite but irreducible and not DP, then, G is ***projectively tetrahedral****,* ***projectively octahedral****,* ***projectively icosahedral****. Further,*

   1. G is ***projectively tetrahedral*** *if and only if* ${t_S}^2 + {t_T}^2 + {t_{ST}}^2 - t_S t_T t_{ST} = 2$ *such that* $t_S, t_T, t_{ST} \in \{-1, 0, +1\}$.

   2. G is ***projectively octahedral*** *if and only if* ${t_S}^2 + {t_T}^2 + {t_{ST}}^2 - t_S t_T t_{ST} = 3$ *where* $t_S, t_T, t_{ST} \in \{-\sqrt{2}, -1, 0, 1, \sqrt{2}\}$.

3. *G is **projectively icosahedral** if and only if* ${t_S}^2 + {t_T}^2 + {t_{ST}}^2 - t_S t_T t_{ST} \in \{2-\mu_2, 3, 2+\mu_1\} = \{1+{\mu_2}^2, 3, 1+{\mu_1}^2\}$ *and* $t_S, t_T, t_{ST} \in \{-\mu_1, -1, -\mu_2, 0, \mu_2, 1, \mu_1\}$, *where* $\mu_1 = \frac{1}{2}(1+\sqrt{5})$ *and* $\mu_2 = (\mu_1)^{-1} = -\frac{1}{2}(1-\sqrt{5})$.

(d) If no case from (a) to (c) holds, then, Zariski closure of G *is* $SL(2, \mathbb{C})$.

Now, we will give proof for part(b) of the theorem.

Proof. Suppose G is DP but not reducible. So, it means G is matrix subgroup of $D \cup P$. Let $S, T \in SL(2, \mathbb{C})$ and suppose that both are in D.

Let $S = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}$ and $T = \begin{bmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{bmatrix}$ where $\alpha, \beta \in \mathbb{C}$. Then $ST = \begin{bmatrix} \alpha\beta & 0 \\ 0 & \alpha^{-1}\beta^{-1} \end{bmatrix}$

So, ${t_S}^2 + {t_T}^2 + t_{ST}^2 - t_S t_T t_{ST} = 4$. Therefore (by theorem2(a)), G is reducible which is a contradiction. Hence, at least one of S and T must be in Permutation Group(P).

Without loss of generality, suppose that S$\in P$, T$\in D$.

Let S$=\begin{bmatrix} 0 & \eta \\ -\eta^{-1} & 0 \end{bmatrix}$, T $= \begin{bmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{bmatrix}$ where $\eta, \gamma \in \mathbb{C}$. So, ST$= \begin{bmatrix} 0 & \eta\gamma^{-1} \\ -\eta^{-1}\gamma & 0 \end{bmatrix}$
Now, $t_S = t_{ST} = 0$. Hence, two of the traces vanish.

Now, we will prove the reverse assertion. Without loss of generality assume that two of the traces $t_T = t_{ST} = 0$. Observe that $t_S = t_S^{-1}$ (because, $S \in (SL(2, \mathbb{C}))$. Also, since G is not reducible, it implies that $t_S \neq \pm 2$.

We claim that if $t_S = \pm 2$, then S is not diagonalizable.

Proposition 4.1. *If* $t_M = \pm 2$*, then* M *is not diagonilasable.*

Proof.

$$\text{Let } M = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ such that (a,b,c,d)} \in \mathbb{C} - \{0\} \textit{ and } ad - bc = 1 \right\}$$

.

Firstly, we will find its eigen values. Let $\lambda \in \mathbb{C}$, so

$$M - \lambda I = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} =$$

Characteristic Equation is given as follows

$\lambda^2 - \lambda(a+d) + 1 = 0$. So, $\lambda = \dfrac{a+d \pm \sqrt{(a+d)^2-4}}{2}$.

If $a + d = trace(M) = 2$, then $\lambda = 1$, For $\lambda = 1$, we will find eigen vectors

$$\begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, it implies $(a-1)x + by = 0$ and $cx + (d-1)y = 0$.

We can show that both the above equations are identical. So, eigen vector(v) is given as $r\begin{pmatrix} \frac{b/a-1}{1} \end{pmatrix}$, where r is a constant. Since, all the other vectors are not independent of v, so, we can't find any other eigen vector. Hence,we can't diagonalize M. Similarly, we can prove that M is not diagonalizable for eigen value $\lambda = -1$

Also, since we can find two different eigen vectors corresponding to two different eigen values if $trace(M) \neq \pm 2$. Hence, M is diagonalizable if $trace(M) \neq \pm 2$. □

Now, we know that $t_S \neq \pm 2$, so, S can be diagonalized and also $det(S) = 1$.

So, we can take $S = \left\{ \begin{bmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{bmatrix} \ where \ \mu \in \ \mathbb{C} - \{0\} \right\}$

Let $T = \left\{ \begin{bmatrix} x & y \\ z & -x \end{bmatrix} \ where \ x, y \in \ \mathbb{C} \right\}$. So, $ST = \begin{bmatrix} \mu x & \mu y \\ \mu^{-1}z & -\mu^{-1}x \end{bmatrix}$

Now, $t_{ST} = \mu x - \mu^{-1}x = \mu^{-1}(\mu^2 - 1)x = 0$. If $\mu = \pm 1$, then $t_S = \pm 2$. So, x must be 0. Hence, $T = \begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix}$. Since, $det(T) = 1$. It implies T$\in P$.

Hence, G is a DP group.

Now, we will prove part(1) of this assertion

Assume that all the three traces vanish. Let S be a matrix such that $t_S = 0$ and diagonal elemements of S$\neq 0$ i.e.

$$S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ where } a, b, c, d \in \mathbb{C},\ \ a, d \neq 0,\ \ a + d = 0 \text{ and } ad - bc = 1 \right\}$$

Now, we will diagonalise the matrix S. Firstly, we will find the eigen values for S. Let λ be a complex constant. Then

$$S - \lambda I = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} =$$

The characteristic equation is given as $\lambda^2 + 1 = 0$, which implies $\lambda = \pm i$. Now let us find the eigen vectors corresponding to eigen values $\pm i$

For $\lambda = i$

$$\begin{bmatrix} a - i & b \\ c & d - i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which implies, $(a - i)x + by = 0$ and $cx + (d - i)y = 0$

We can show that both the above equations are identical. Hence, eigen vector at $\lambda = i$ is $s\left(\frac{1}{-(a-1)/b}\right)$, where s is a complex constant.

For $\lambda = -i$

$$\begin{bmatrix} a + i & b \\ c & d + i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, $(a + i)x + by = 0$ and $cx + (d + i)y = 0$

Both the above equations are identical. Hence, eigen vector at $\lambda = -i$ is $t\left(\frac{1}{-(a+1)/b}\right)$, where t is a complex constant. Now, Digonalisation matrix is given as

$$M = \begin{bmatrix} 1 & 1 \\ -\frac{(a-i)}{b} & -\frac{(a+i)}{b} \end{bmatrix}$$

Now, $M^{-1} = \frac{-b}{2i} \begin{bmatrix} -\frac{(a+i)}{b} & -1 \\ \frac{(a-i)}{b} & 1 \end{bmatrix}$ and $M^{-1}SM = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

Therefore, digonalisation of S is

$$\begin{bmatrix} \pm i & 0 \\ 0 & \mp i \end{bmatrix}$$

Assume that $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ then, $TST^{-1} \in \begin{bmatrix} \pm i & 0 \\ 0 & \mp i \end{bmatrix}$.

Hence, we can rescale the second basis element without altering the form of S. G is thus Quaternionic.

To see the converse firstly we will show that for any matrix $M \in (SL(2, \mathbb{C}))$

$$t_{M^4} = {t_M}^4 - 4{t_M}^2 + 2$$

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Firstly, observe that $t_{M^2} = {t_M}^2 - 2$ Now,

$$M^4 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & d^2 + bc \end{bmatrix} \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & d^2 + bc \end{bmatrix}$$

$$\begin{aligned} t_{M^4} &= (a^2 + bc)^2 + (ab + bd)(ac + cd) + (ac + cd)(ab + bd) + (d^2 + bc)^2 \\ &= a^4 + d^4 + 6a^2d^2 + 4a^3d + 4ad^3 - 4a^2 - 4d^2 - 8ad + 2 \\ {t_M}^4 &= (a^2 + d^2 + 2ad)^2 \\ &= a^4 + d^4 + 6a^2d^2 + 4a^3d + 4ad^3 \end{aligned}$$

Hence, $t_{M^4} = {t_M}^4 - 4{t_M}^2 + 2$. Now, if $M \in Quaternion\ Group$, then, it implies that either $t_M = 0$ or $t_M = \pm 2$.

If $t_M = \pm 2$, then M must be $\pm I$, otherwise Jordan form would be

$$\begin{bmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{bmatrix}$$

forcing M to have an infinite order which is a contradiction.

Now suppose $\langle S, T \rangle >\subseteq SL(2, \mathbb{C})$ is quaternionic. We have already seen that if $t_S \neq 0 \neq t_T$, then $\langle S, T \rangle \subseteq \{I, -I\}$. So, without loss of generality, we may assume that $t_S = 0$. So, S can be identified with the matrix

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

S has order 4, containing $-I$. If $t_T = \pm 2$, then $T = \pm I$, which has infinite order. Hence, $\langle S, T \rangle$ cannot have order 8 unless $t_T = 0$. Therefore, T can be identified with the matrix

$$\left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \text{ where a,b,c} \in \mathbb{C} \right\}$$

Now, ST $= \begin{bmatrix} ia & ib \\ -ic & ia \end{bmatrix}$. If $t_{ST} \neq 0$, it implies $ST = \pm I$. Hence, t_{ST} must be 0 to make it of finite order. Therefore, a=0.

Hence, $t_S = t_T = t_{ST} = 0$. □

Proof of Theorem $2(c)$ The proof of part (c) rests on some group theoretic results.

Notation: A_n is alternating group and S_n is symmetric groups; $n \geq 1$

Proposition 4.2. *(a) Assume that $g, h \in A_4$ such that $g \notin < h >$ and $h \notin < g >$. Then, $A_4 =< g, h >$ if and only if g is a 3-cycle or h is a 3-cycle. Further, if both are 3-cycles, then, $\{hg, hg^{-1}\}$, consists of a 3-cycle and a product of disjoint transpositions.*

(b) Assume that $g, h \in S_4$ such that $g \notin < h >$ and $h \notin < g >$. Then, $S_4 =< g, h >$ if and only if at least one of g, h and hg is a 4-cycle and at most one is of order 2.

To prove this proposition, we can assume that $|g| \geq |h| \geq |gh|$, which can be achieved by switching generators, or, replacing generators by inverses.

Proof. (a) We know that $A_4 \subset S_4$ has identity, all 3- cycles and all products of disjoint transpositions. Suppose none of g, h and gh is a 3-cycle. If, $|g| = |h| = |gh| = 2$, then $|G| = | < g, h > | = 4 \neq 12 = |A_4|$.

If g and h are 3-cycles, then, without loss of generality let $h = (123), g = (234)$, so, $\{g, g^{-1}\} = \{(234), (243)\}$ and $\{hg, hg^{-1}\} = \{(12)(34), (124)\}$. Hence, the final assertion follows. If $|g| = |h| = |gh| = 3$, It is clear that $G =< g, h >= A_4$. Now, suppose that $|g| = |h| = 3$ and $|gh| = 2$. It is clear that $|G| = 6$ or 12. We claim that $|G| \neq 2, 3, 4$(Since, $g \notin < h >$ and $h \notin < g >$ and $|g| = |h| = 3 = |gh| = 3$, hence G has more than 4 elements.) Also, by Langrange's theorem $|G| \neq (5, 7, 8, 9, 10, 11)$.

Claim: A_4 doesn't contain any subgroup of order 6.

Suppose to the contrary that A_4 contains a subgroup G of order 6. Then, G is isomorphic to $\mathbb{Z}_6$ or S_3. But, A_4 doesn't contain any element of order 6. Therefore $G \cong S_3$. Now, S_3 contains 3 elements of order 2. Therefore, $(12)(34), (13)(24), (14)(23)$ which have order 2 must lie in G. Observe that these three elements and identity forms a subgroup of G of order 4. But, $4 \nmid 6$ which is a contradiction. Therefore, A_4 does not contain any subgroup of order 6.

Hence, $|G| = 12 = |A_4|$.

(b)Let $g, h \in S_4$ such that $g \notin < h >, h \notin < g >$ and w.l.o.g assume that $|g| \geq |h| \geq |gh|$. Firstly, we will prove that one of g, h and gh must be a 4-cycle and at most one can be a 2-cycle.

If $|g| = |h| = 3$, then $g, h \in A_4$, and, it is clear from part (a) that $G =< g, h >= A_4 \neq S_4$. If $|g| = |h| = |gh| = 2$, then $|G| = 4 \neq |S_4|$. If $|g| = 3, |h| = |gh| = 2$. Suppose $g = (i_1 i_2 i_3)$, then, for this case, h must be $(i_j i_k)$ where $j, k \in \{1, 2, 3\}$, otherwise $|gh| \neq 2$. Hence, $|G| \leq 12 \neq 24 = |A_4|$. If $|g| = 4, |h| = |gh| = 2$. In this case, without loss of generality let $g = (1234), h = (13)$, then $G =< g, h >= \{(1), (1234), (13), (14)(23), (24), (12)(34), (13)(24), (1432)\}$, hence, $|G| = 8 < 24 = |S_4|$

Now, we will assume that g is 4-cycle and will consider different possible cases assuming that h is at least a 3-cycle

If $|g| = 4, |h| = 3$, then, $|G| = | < g, h > |$ can be 12 or 24.

Now, A_4 is the only subgroup of order 12 in S_4. Also, A_4 doesn't contain elements of order 4. Therefore,$|G| = 24 = |S_4|$. Hence, $G =< g, h >= S_4$.

If $|g| = 4, |h| = 4$, then, without loss of generality assume that $g = (1234)$, so $h \in \{(1324), (1342), (1243), (1423), (1432)\}$. But, it implies that gh is a 3-cycle in all the cases. Hence, G has elements of order 3 and 4 both. Hence, $G = S_4$. □

Lemma 1. *Assume that $g, h \in A_5$ are 5 cycles such that $g \notin < h >, h \notin < g >$. Then, at least one of hg and hg^{-1} is not a 5-cycle.*

Proof. Assume that both hg and hg^{-1} are 5-cycles, then, without loss of generality assume that $h = (12345)$. Since, hg is a 5-cycle, so it has no fixed point. Hence, there does not exist any map $g : l \rightarrowtail l - 1 = l + 4$ for any $l \in \{1, 2, 3, 4, 5\}$; Observe that hg^{-1} is such that it has no fixed point, therefore, for any $l \in \{1, 2, 3, 4, 5\}$ there does not exist any map $g : l \rightarrowtail l + 1$; also, g cannot has any fixed point, so, for no $l \in \{1, 2, 3, 4, 5\}$, there exist any map$g : l \rightarrowtail l$. Hence, $\forall l \in \{1, 2, 3, 4, 5\}$, we must have map as $g : l \rightarrowtail \{l + 2, l + 3\}$. Since, $g \notin < h >$, g cannot move cyclically i.e. k cannot advance all the points by same ammount. So, now assume by cyclically permuting labels that $g : 1 \rightarrowtail 3$ and $g : 2 \rightarrowtail 5$. But, the possibilities for 3 are $g : 3 \rightarrowtail \{5, 1\}$, hence $g : 3 \rightarrowtail 1$, which is not possible, because, g is a 5-cycle. □

Lemma 2. *Suppose $g, h \in A_5$ are 5-cycles such that $g \notin < h >$ and $h \notin < g >$, then:*

(a) When hg is a 3-cycle, then, for some $1 \leq j \leq 4$, $h^j g$ is a product of disjoint transpositions.

(b) When hg is the product of disjoint transpositions, then, the element $h^{-1}g$ is a 3-cycle.

Proof. (a) Without loss of generality suppose $h = (12345)$ and assume that for hg fix 1 and 2 both. Hence, we have $g : 1 \rightarrowtail 5$ and $g : 2 \rightarrowtail 1$. So, $g = (15342)$ or $g = (15432) = h^{-1} \in < h >$. Hence, $g = (15342)$. Now, $h^4 g = h^{-1} g = (14)(25)$ is a product of disjoint transpositions. In, the same way we can prove for the cases if hg fixes 3,4,5.

(b) Without loss of generality suppose $h = (12345)$ and assume that hg fixes 1. So, $hg \in \{(23)(45), (24)(35), (25)(34)\}$. When $hg = (23)(45)$, then $g = h^{-1}.hg = (15432)(23)(45) = (153)$, which is a contradiction, since, g is a 5-cycle. If $hg = (24)(35)$, then $g = h^{-1}.hg = (15432)(24)(35) = (15234)$, which is not a contradiction. Now, $h^{-1}g = h^4 g = h^3.hg = (14253)(24)(35) = (145)$, as required. If $hg = (25)(34)$, then $g = h^{-1}.hg = (15432)(25)(34) = (15)(24)$,which is again a contradiction, since, g is a 5-cycle. □

Proposition 4.3. *Assume that $g, h \in A_5$ such that $g \notin < h >$ and $h \notin < g >$. Then, $A_5 =< g, h >$ if and only if at least one of g, h and hg is a 5-cycle and at most one is of order 2.*

To prove this proposition, again, we can assume that $|g| \geq |h| \geq |gh|$, which can be achieved by switching generators, or, replacing generators by inverses.

Proof. A_5 consists of identity, 5-cycles, 3-cycles and a product of disjoint transpositions. Therefore, for non identity element, $|g|, |h|, |gh| \in \{5, 3, 2\}$. Without loss of generality assume that $|g| \geq |h| \geq |gh|$. Firstly, we will show that $|g| = 5$.

If $|g| = |h| = |gh| = 2$, then $|G| =< g, h >= 4 \neq 60 = |A_5|$. If $|g| = 3, |h| = |gh| = 2$, then, $G = \{1, g, g^2, h, gh, g^2h\}$. Hence, $|G| \leq 6 \neq |A_5|$. Let $|g| = |h| = 3, |gh| = 2$. In this case, if $g = (i_1i_2i_3)$, then h must be $i_ji_ki_l$ such that one of $j, k, l \notin \{1, 2, 3\}$ and other two $\in \{1, 2, 3\}$ and they should be consecutive i.e they can be 12,23 or 31. So, without loss of generality let $g = (234), h = (345)$, then $G = \{(1), (234), (243), (345), (354), (235), (253), (245), (254),$
$(23)(45), (24)(35), (25)(34)\}$. Hence, $|G| \leq 12 \neq |A_5|$. Assume that $|h| = |g| = |gh| = 3$. We know that $g \notin < h >$, hence, it implies that there must be a point which is transposed by g but not by h, but, since both g and h are of order 3, it also implies that there must be a point transposed by both. So, without loss of generality suppose that $g = (123)$, then h is contained in the set $\{(234), (243), (235), (253), (245), (254)\}$. Hence, hg for each case is as follows: $(234)(123) = (13)(24); (243)(123) = (143);$
$(235)(123) = (13)(25), (253)(123) = (153); (245)(123) = (14523); (254)(123) = (15423).$
Since, we assumed that $|hg| = 3$, so we will deal with second and fourth cases in the list. Also, we can see that $h^{-1}g$ is a product of disjoint transpositions in these two cases, hence these are of order 2.

Hence, $|g|$ must be 5.

In this case assume that $|h| = |gh| = 2$, then $G =< g, h >= \{1, g, g^2, g^3, g^4, h, gh,$
$g^2h, g^3h, g^4h\}$. Hence, $|g| \leq 10 \neq |A_5|$. Hence, $|h| \geq 3$.

Now, let us fix $|g| = 5$ i.e. g is a 5-cycle. Since, we know that A_5 has no proper subgroups of order > 12, then, if $|h| = 3$, then by Langrange's theorem it is clear that $G =< g, h >= A_5$.

In the same case assume that $|h| = 5$, then Lemmas 1 and 2 gives elements of order 3 and 2 as well. Hence, by the same arguments as above we can easily show that $G = A_5$. □

Remark 2. *Assume that $S \in SL(2, \mathbb{C})$, and let I be the identity matrix, then S satisfies following properties:*
(a) The jordan form of S is either diagonal or one of $\begin{bmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{bmatrix}$. Particularly, Jordan form is diagonal if and only if $t_S \neq \pm 2$ or $S = \pm I$; any finite order element is diagonalisable. (b) $S^2 = I$ iff $t_S = \pm 2$. (c) $S^2 = -I$ iff $t_S = 0$. (d) $S^3 = I$ iff $t_S = -1$. (e) $S^3 = -I$ iff $t_S = 1$. (f) $S^4 = I$ iff $S^2 = -I$, in which case (c) applies. (g) $S^4 = -I$ iff $t_S = \pm\sqrt{2}$. (h) $S^5 = I$ iff $t_S = -\frac{1}{2}(1 - \sqrt{5})$ or $t_S = \frac{1}{2}(1 + \sqrt{5})$.

Fricke-Klein formulae

Assume that $S, T \in SL(2, \mathbb{C})$ and let I be the identity matrix, then,

$$\begin{aligned} &(a)\ t_I = 2 \\ &(b)\ t_{ST} = t_{TS} \\ &(c)\ t_{(S,T)} = t_S^2 + t_T^2 + t_{ST}^2 - t_S t_T t_{ST} - 2 \end{aligned} \tag{4.1}$$

Consequences of Fricke-Klein formulae:

$$\begin{aligned} &(a)\ t_{TST^{-1}} = t_S \\ &(b)\ t_{(S^{-1}T, T^{-1})} = t_{(S,T)} \\ &(c)\ t_{(T,TS)} = t_{(S,T)} \\ &(d)\ t_S = t_S^{-1} \\ &(e)\ t_S^2 = {t_S}^2 - 2 \\ &(f)\ (t_{S^{-1}T} - t_{ST})(t_{S^{-1}T} + t_{ST} - t_S t_T) = 0 \\ &(g)\ (t_{S^2T} - t_T)(t_{S^2T} + t_T - t_S t_{ST}) \end{aligned} \tag{4.2}$$

Lemma 3. *Assume that $S, T \in SL(2, \mathbb{C})$, $t_{ST} = 0$ and $G = \langle S, T \rangle$. Then, G is a DP group if and only if two of the three traces t_S, t_T and t_{ST} are zero and absolute value* *of third is* $\sqrt{2}$*. Particularly, when* $G = \langle S, T \rangle$ *and* $t_{ST} = 0$*, it should be the case that* $\{t_S, t_T, t_{ST}\} \notin \{-1, 0, 1\}$*.*

Proof. By theorem 2(a), we can easily see that G is not reducible and by 2(b), we can see that G is a DP group and since $t_{S,T} = 0$, so by 5.1(c), third trace has absolute value $\sqrt{2}$. Hence, $\{t_S, t_T, t_{ST}\} \notin \{-1, 0, 1\}$. □

Lemma 4. *If* $S, T \in SL(2, \mathbb{C})$*,then, the condition* $t_{ST} \neq t_{S^{-1}T}$*, implies that* $t_{S^{-1}T} = t_S t_T - t_{ST}$*.*

On the other hand, if $t_{(S,T)} = 0$ *and* $t_S = \pm 1$ *the opposite condition* $t_{ST} = t_{S^{-1}T}$ *implies* $t_T \in \{\pm\frac{2}{3}\sqrt{3}\}$*.*

Proof. Since, $t_{ST} \neq t_{S^{-1}T}$, so by 5.2(f), we may see that $t_{S^{-1}T} + t_{ST} - t_S t_T$ must be 0. Hence, $t_{S^{-1}T} = t_S t_T - t_{ST}$.
To prove other assertion of lemma, since $t_S \neq \pm 2$, so without loss of generality assume that $S = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ and $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now, since, $t_{(S,T)} = 0$, so, by, theorem 2(a), G is irreducible. Also, G is not DP by theorm 2(b) and by two trace assumption.
Claim: $bc \neq 0$
Assume that $bc = 0$, we know that $det(T) = 1$, so, $ad - bc = 1$, if $bc = 0$, either $b = 0$ or $c = 0$
w.l.o.g assume that $b = 0$, then,
$T = \begin{bmatrix} a & 0 \\ c & a^{-1} \end{bmatrix}$ has infinite order, which is a contradiction, because G is neither reducible nor DP. By, rescaling the basis, we may then achieve $T = \begin{bmatrix} a & ad-1 \\ 1 & d \end{bmatrix}$.
Now ,the condition $t_{S^{-1}T} = t_{ST}$, implies that $(\lambda^2 - 1)(a - d) = 0$, so, if $\lambda = \pm 1$, then $t_S = \pm 2$ but $t_S = \pm 1$. Hence, $a = d$. Therefore, $T = \begin{bmatrix} a & a^2-1 \\ 1 & a \end{bmatrix}$
Now, $t_S^2 + t_T^2 + t_{ST}^2 - t_S t_T t_{ST} - 2 = 3a^2 - 1 = 3\left(\frac{t_T}{2}\right)^2 - 1$. Hence, $t_T \in \{\pm\frac{2}{3}\sqrt{3}\}$ □

We need the following proposition to give proof of theorem 2(c)

Proposition 4.4. *Assume that* $G \subset SL(2, \mathbb{C})$*, then* G *is either: (a) reducible in which diagonalizable and abelian cases are included; or (b) DP but irreducible in which quaternionic and projectively dihedral cases are included; or (c) if none of above cases* *occurs,then projectively tetrahedral, projectively octahedral or projectively icosahedral; or (d) When none of above cases occurs ,then, Zariski closure of G is $SL(2,\mathbb{C})$.*

Next proposition gives the proof for theorem 2c(1)

Proposition 4.5. *Suppose $S, T \in SL(2,\mathbb{C})$ and $G = \langle S, T\rangle$. Then, G is projectively tetrahedral iff $t_S^2 + t_T^2 + t_{ST}^2 - t_S t_T t_{ST} = 2$ and $t_S, t_T, t_{ST} \in \{-1, 0, 1\}$.*

Proof. Assume that G is projectively tetrahedral group, so let $H \subset G$, such that $G/H \simeq A_4$. Let $[P]$ denote the equivalence class in G/H of $P \in G$. So, if, $P \notin H$, then, the posssibilities are $[P]^3 = 1$ or $[P]^2 = 1$. It implies that $P^3 = \pm I, P^2 = -I$ and $P^2 = I$ is not possible (since, $P \in SL(2,\mathbb{C})$ and $P \notin H$). So, by Remark2 [(c) - (e)], we conclude that $t_P \in \{-1, 0, 1\}$. Hence $t_S, t_T, t_{ST} \in \{-1, 0, 1\}$.

We claim that $\sigma = t_S^2 + t_T^2 + t_{ST}^2 \in \{1, 2, 3\}$. Since, $G =< S, T >$, therefore, $G/H =< [S], [T] >$. Also, $G/H \simeq A_4$. Hence, by proposition 4.2(a), at least one of S and T must be a 3-cycle. Assume that S is 3-cycle. then, $S^3 = \pm I$, which implies that $t_S = \pm 1$ (by remark 2[(d),(e)]). Assume that S and ST are 2-cycles, then, $T^2 = (ST)^2 = -I$,which implies $t_T = t_{ST} = 0$ (by remark 2(c)), therefore, $\sigma = t_S^2 + t_T^2 + t_{ST}^2$ is at least 1. Other conditions then can be verified.

Case1: If $\sigma = 1$, then elements $[S], [T]$ and $[ST]$, after relabelling satisfies the relation $s^2 = t^2 = (ts)^3 =$, which implies $|G/H| \leq 6$, which implies $|G| \leq 12$, which is a contradiction. Since, $|G| = 24$. Hence, $\sigma = 1$ is not possible.

Case2: If $\sigma = 2$, then $t_S^2 + t_T^2 + t_{ST}^2 = 2$, which is only possible iff two of t_S, t_T, t_{ST} are 3-cycles and one is 2-cycle. So, by remark (2), $t_S t_T t_{ST} = 0$. Hence , G is projectively tetrahedral.

Case(3): If $\sigma = 3$, then $[S], [T], [ST]$ are 3-cycles, hence, by proposition 4.2(a) $[TS^{-1}]^2 = e$, which implies $(TS^{-1})^2 = -I$, hence, by Remark2(c) and 4.1(b), $t_{TS^{-1}} = t_{S^{-1}T} = 0$. Since, $[ST]$ is a 3-cycle, therefore, $t_{ST} \neq 0$. Hence, by lemma 4, $t_{ST} = t_S t_T$, which implies $t_S t_T t_{ST} = 1$ (by 5.2(d))Hence, $t_S^2 + t_T^2 + t_{ST}^2 - t_S t_T t_{ST} = \sigma - 1 = 2$. Now by 5.1(c), $t_{(S,T)} = 0$, so, by Remark 2(c), $(S, T)^2 = -I \in G$, hence $H \subset G$. Clearly, by theorem2(b), G is not reducible and by lemma(3), G is not DP. Hence, by proposition 4.4(c), $|G| \geq 24$ (which include the possibility $|G| = \infty$). Assume that $P \in G$ and $t_P = -1$, then by substituting P by $-P$, we can assume that $t_P = 1$. Observe that replacements with $P = S, T$ do not differ the value of $t_S t_T t_{ST}$. Then, by relabelling, we get reduce to the possibility $t_S = t_T = 1$ and $t_{ST} = 1$ or 0. Now, suppose $t_{ST} = 1$, then by lemma(4), $t_{S^{-1}T} = 0$. Replace S with S^{-1},we need to only consider the second.

So, by Remark2[(c) to (e)], we note that $[S]^3 = [T]^3 = [TS]^2 = e$. So, it implies that $|G/H| \leq 12$. But we have already shown that $|G| \geq 24$, so, $|G/H| \geq 12 = |A_4|$, therefore, the map $\phi : A_4 \to G/H$ is isomorphism. Hence, G is projectively tetrahedral. □

Next proposition gives the proof for theorem 2c(2)

Proposition 4.6. *Assume that $S, T \in SL(2, \mathbb{C})$ such that $G = \langle S, T \rangle$. Then, G is projectively octahedral iff $t_S^2 + t_T^2 + t_{ST}^2 - t_S t_T t_{ST} = 3$ and $t_S, t_T, t_{ST} \in \{-\sqrt{2}, -1, 0, 1, \sqrt{2}\}$.*

Proof. Assume that G is projectively octahedral, so, let $H \subset G$, such that $G/H \simeq S_4$. For, some, $F \in G$, suppose $[F]$ denotes its equivalence class. If $F \notin H$, then the possibilities are $[F]^4 = e, [F]^3 = e$ and $[F]^2 = e$ which is only case iff $F^4 = -I, F^3 = \pm I$ or$F^2 = -I$. Observe that $F^4 = F^2 = I$ is not possible (since $F \in SL(2, \mathbb{C})$ and $F \notin H$). Hence, by Remark2 [(c),(d),(e) and (g)], $t_S \in \{0, \pm 1, \pm\sqrt{2}\}$.

relabelling, we may assume that $|t_S| \geq |t_T| \geq |t_{ST}|$.

Since, $G = \langle S, T \rangle$, therefore, $G/H = \langle [S], [T] \rangle$. Also, $G/H \simeq S_4$. So, by proposition 4.2(b), at least one of S, T and ST is a 4-cycle and atmost one is of order 2. Hence, by Remark2(g) $|t_S| = \sqrt{2}$ and by Remark2[(d),(e) and (g)], $|t_T| = 1$ or $\sqrt{2}$.

Hence, $\sigma = t_S^2 + t_T^2 + t_{ST}^2 \in \{3, 4, 5, 6\}$.

Case1: $\sigma = 3$. The only possibility is $t_S = \sqrt{2}, t_T = 1$ and $t_{ST} = 0$. It implies $t_S t_T t_{ST} = 0$. Hence, G is projectively octahedral.

Case2: $\sigma = 4$. There are two possibilities in this case.

(a) $|t_S| = |t_T| = \sqrt{2}, t_{ST} = 0$ and (b) $|t_S| = \sqrt{2}, |t_T| = 1, |t_{ST}| = 1$ For (a), $t_{(S,T)} = 2$, G is therefore reducible and by lemma 7(c) $G/H \simeq S_4$ is impossible, which is a contradiction to the assumption. For (b), $t_{(S,T)} = 2 \pm \sqrt{2} \notin \{0, \pm 1, \pm\sqrt{2}\}$, which is also impossible.

Case3: $\sigma = 5$. The only possibility is $|t_S| = |t_t| = \sqrt{2}, |t_{ST}| = 1$. So, $t_S t_T t_{ST} = \pm 2$ If $t_S t_T t_{ST} = -2$. it implies $t_{(S,T)} = 5 \notin \{0, \pm 1, \pm\sqrt{2}\}$, which is impossible. If $t_S t_T t_{ST} = 2$. it implies $t_{(S,T)} = 1 \in \{0, \pm 1, \pm\sqrt{2}\}$, which gives us desired result.

Case(4): $\sigma = 6$. The only possibility is $|t_S| = |t_t| = \sqrt{2}$, $|t_{ST}| = \sqrt{2}$. So, $t_S t_T t_{ST} = \pm 2\sqrt{2}$ So, $t_{ST} = 4 \pm 2\sqrt{2} \notin \{0, \pm 1, \pm\sqrt{2}\}$. Hence this case is impossible.

To, prove the reverse assertion firstly it is clear that G is not reducible(by theorem2(b)). Suppose that G is DP, so, assume that $S \in D$ and $T \in P$. Then, $ST \in P$, which implies $t_T = t_{ST} = 0$, since $t_S, t_T, t_{ST} \in \{0, \pm 1, \pm\sqrt{2}\}$, therefore, $t_S^2 + t_T^2 + t_{ST}^2 < 3$. Hence,$G$ is not DP. Clearly, by proposition 4.4, G is not tetrahedral. Since G is not DP,reducible and projectively tetrahedral. Hence, by propostion 4.4(c), if G is finite, then G has either 48 or 120 elements.

Observe that $-I \in G$. If $0 \in \{t_S, t_T, t_{ST}\}$, then it is immediate from Remark(2c), so assume otherwise that $t_S^2 + t_T^2 + t_{ST}^2 > 3$ implying $|t_S| = \sqrt{2}$, and 4.3(d) gives $t_{S^2} = t_S^2 - 2 = 0$, and by Remark(2c) again $S^4 = -I$.

We know that, when $t_P < 0$, for $P \in SL(2, \mathbb{C})$, then, by subtituting P by -P, we can get $t_P > 0$ and such substitutions with $P = S, T$ don't differ the value for $t_S t_T t_{ST}$. Now, by relabelling, we can assume that $t_S \geq t_T \geq t_{ST}$ which implies that atleast $t_S \geq t_T \geq 0$. So, by the identity $t_S^2 + t_T^2 - t_S t_T t_{ST} = 3$, we reduce to two cases $(t_S, t_T, t_{ST}) = (\sqrt{2}, \sqrt{2}, 1)$ and $(t_S, t_T, t_{ST}) = (\sqrt{2}, 1, 0)$. If $(t_S, t_T, t_{ST}) = (\sqrt{2}, \sqrt{2}, 1)$, since $t_S \neq \pm 2$, we can choose a basis so to identify S with $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$, where $\lambda = \frac{\sqrt{2}}{2}(1 \pm i)$. Now, G is irreducible and not DP, so by lemma(4), we can write $T = \begin{bmatrix} a & ad - 1 \\ 1 & d \end{bmatrix}$. So, $d = \sqrt{2} - a$, and, after solving $t_{ST} = 1$, $a = \frac{\sqrt{2}}{2}$. Since, $t_{S^2T} = 0$, and, since $\langle S, T \rangle = \langle S, ST \rangle$, so, by replacing T by ST this case is boiled down to case 2. Now, suppose $(t_S, t_T, t_{ST}) = (\sqrt{2}, 1, 0)$, then, by Remark(2), we note that $[S]^4 = [T]^3 = [ST]^2 = e$, so, it implies $|G/H| \leq 24$, since, it is the standard representation for S_4. We also know that $|G| \geq 48$. So, the map, $\phi : S_4 \to G/H$ is isomorphism. Hence, G is projectively octahedral. □

We need the following lemma to prove proposition 2c(3)

Lemma 5. *Assume that $S, T \in SL(2, \mathbb{C})$ such that $t_S \notin \{-2, 2\}$. The following conditions hold:(a) $t_T = t_{ST} = t_{S^{-1}T}$ implies $t_T = 0$; (b) $t_{ST} = 1, t_{S^2T} = t_S$ implies*

$t_T = 0$; (c) $t_S \neq 0, t_{S^2T} = 1$ *and* $t_{ST} = (t_S)^{-1}$ *implies* $t_T = 0$; (d) $t_S = (t_T)^{-1} \neq 0$ *implies* $t_{s^{-1}T} = 1$.

Proof. Since, $t_S \notin \{2, -2\}$, so, let us assume $S = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ and $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) Since, $t_{ST} = t_{S^{-1}T}$, it implies, $(\lambda^2 - 1)(a - d) = 0$, since, $t_S \notin \{-2, 2\}$, therefore $\lambda = \pm 1$ is not possible, so, $a = d$. Now, $t_{ST} = t_T$ and $a = d$, therefore, it implies, $a(\lambda^2 - 1) = 0$. Hence, $a = 0$.

(b) Since, $t_{ST} = 1$ and $t_{S^2T} = t_S$, therefore,

$$\lambda^2 a + d = \lambda, \lambda^4 a + d = \lambda^3 + \lambda$$

which gives us the unique solution $a = \frac{\lambda}{\lambda^2 - 1} = -d$, which implies $t_T = 0$.

(c) Since, $t_{S^2T} = 1$, it implies $\lambda^2 a + \lambda^{-2} d = 1$. We also know that $t_{ST} = (t_S)^{-1}$, therefore, it implies $\lambda^2 a + \lambda^{-2} d + a + d = 1$, hence, $a + d = t_T = 0$.

(d) Since, $t_S = (t_T)^{-1}$, it implies $\lambda + \lambda^{-1} = (a + d)^{-1}$ which implies $\lambda a + \lambda^{-1} a + \lambda^{-1} d + \lambda d = t_{ST} + t_{S^{-1}T} = 1$. Hence, $t_{S^{-1}T} = 1$. □

Next proposition gives the proof for theorem 2c(3)

Proposition 4.7. *Assume that* $S, T \in SL(2, \mathbb{C})$ *and* $G = \langle S, T \rangle$. *Let* $\mu_1 = \frac{1}{2}(1 + \sqrt{5})$, $\mu_2 = \mu_1^{-1} = -\frac{1}{2}(1 - \sqrt{5})$. *G is projectively icosahedral if and only if* $t_S^2 + t_T^2 + t_{ST}^2 - t_S t_T t_{ST} \in \{2 - \mu_2, 3, 2 + \mu_1\}$ *such that* $t_S, t_T, t_{ST} \in \{-\mu_1, -1, -\mu_2, 0, \mu_2, 1, \mu_1\}$.

Proof. Suppose that G is projectively icosahedral i.e. if $H \subset G$, then $G/H \cong A_5$. Now, let $[F]$ denote the equivalence class of F in G/H. So, for $F \in G/H$, possibilities are $[F]^5 = e, [F]^3 = e, [F]^2 = e$, which is the case if and only if $F^5 = \pm I, F^3 = \pm I, F^2 = I$, again observe that $F^2 = I$ is not possible, since, $F \in SL(2, \mathbb{C})$ and $F \notin H$. Hence, from remark(2), we can conclude that $t_F \in \{-\mu_1, -1, -\mu_2, 0, \mu_2, 1, \mu_1\}$.

Now, by proposition 4.3 and remark2[(h),(i)], $t_S \in \{\mu_1, 1, \mu_2\}$ and if $t_S = 1$ then $\{t_T, t_{ST}\} \cup \{\pm\mu_1, \pm\mu_2\} \neq \phi$. Again by proposition(5) at most one of t_T and t_{ST} $\in \{-2, 0, 2\}$. Now, if we list all the possibilities along with the condition $t_{(S,T)} = t_S^2 + t_T^2 + t_{ST}^2 - t_S t_T t_{ST} \in \{2 - \mu_2, 3, 2 + \mu_1\} \in \{0, \pm 1, \pm\mu_1, \pm\mu_2\}$, then we reduce to the following ten possibilities.

Case	t_S, t_T, t_{ST}	$t_{(S,T)}$
1	(μ_1, μ_1, μ_1)	μ_1
2	$(\mu_1, \mu_1, 1)$	μ_1
3	$(\mu_1, 1, 1)$	1
4	$(\mu_1, 1, \mu_2)$	1
5	$(\mu_1, 1, 0)$	μ_1
6	$(\mu_1, \mu_2, 0)$	1
7	$(1, 1, -\mu_2)$	1
8	$(1, \mu_2, \mu_2)$	-μ_2
9	$(1, \mu_2, 0)$	-μ_2
10	$(\mu_2, \mu_2, -\mu_2)$	-μ_2

For all the possibilities above it is clear that $t_S^2+t_T^2+t_{ST}^2-t_St_Tt_{ST} \in \{2-\mu_2, 3, 2+\mu_1\}$.

Now, we will prove the opposite side of proposition. Since $t_{ST} \neq 2$, so, it is clear that G is irreducible. We claim that G is not DP. Assume that G is DP and let $t_T = t_{ST} = 0$ and $t_S \neq 0$, then $t_S^2 \in \{2-\mu_2, 3, 2+\mu_1\}$, which is not possible if $t_S \in \{-\mu_1, -1, -\mu_2, 0, \mu_2, 1, \mu_1\}$.Hence, G is not DP. Thus, we are reduced to irreducible non-DP group G. So, we need to examine the ten cases above.

Case 1: Substitute S^{-1} in place of S. Since, $t_T \neq 0$, so, by lemma(5a) $t_{ST} \neq t_{S^{-1}T}$. By (4.2f), we have $t_{S^{-1}T} = t_St_T - t_{ST} = {\mu_1}^2 - \mu_1 = 1$. So, we are reduced to case 2.

Case 2: In this case let us substitute S^{-1} in place of S and ST in place of T. Since, $t_T \neq 0$,by lemma(5b), we can see that $t_{S^2T} \neq t_S$ and by (4.2g) $t_{S2T} = t_St_{ST} - t_T = \mu_1 - \mu_1 = 0$. So, we are boiled down to case 5.

Case 3: Substitute S^{-1} in place of S. Since, $t_T \neq 0$, so, by lemma(5a) $t_{ST} \neq t_{S^{-1}T}$. By (4.2f), we have $t_{S^{-1}T} = t_St_T - t_{ST} = \mu_1 - 1 = \mu_2$. Thus, we are boiled to case 4.

Case 4: Substitute ST in place of S. Since, $t_T \neq 0$, by lemma(5c), $t_S^2T \neq t_T$, so, by (4.2g), we have $t_S^2T = t_St_{ST} - t_T = \mu_1\mu_2 - 1 = 0$. This case is reduced to case 9.

Case 6: Substitute $S^{-1}T$ in place of T. Observe that $t_S = (t_T^{-1}) \neq 0$ and $t_{ST} = 0$, by lemma(5d), it implies that $t_{S_{-1}}T = 1$. So, again we are reduced to case 4.

Case 7: Since $t_S \neq \pm 2$, so, assume that $S = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ and $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now, $S^2 \neq I$, since G is irreducible. So, $t_{S^{-1}T} = t_{ST}$ iff $a = d$. So, $t_{ST} = a(\lambda + \lambda^{-1})$, since, $t_S = t_T = 1$, it implies that $t_{ST} = a = \frac{1}{2} \neq -\mu_2$. Hence, $t_{S^{-1}T} \neq t_{ST}$. So, $t_{S-1T} = t_S t_t - t_{ST} = 1 + \mu_2 = \mu_1$. Replacing S by S^{-1}, thus, reduces us to case 3.

Case 8: We have already shown that $t_{S^{-1}T} = t_{ST}$ iff $a = d$ in case 7. In this case $t_T = t_{ST}$, so, $2a = t_T = a(\lambda + \lambda^{-1}) = at_S = a$, which is a contradiction. Hence, $t_{S^{-1}T} = t_S t_t - t_{ST} = 0$. Hence, if we replace S by S^{-1}, we are reduced to case 9.

Case 10: From case 7, we have seen that $t_{S^{-1}T} = t_{ST}$ iff $a = d$. So, $\mu_2 = t_S = \lambda + \lambda^{-1} = -t_{ST} = -a(\lambda + \lambda^{-1})$, which implies that $a = -1$, thus, $t_T = -2 \neq \mu_2$. Hence, as we have seen earlier that $t_{S^{-1}T} = t_S t_t - t_{ST} = \mu^2 + \mu_2 = \mu_2\mu_1 = 1$. So, if we substitute S^{-1} in place of S, we are reduced to case 8.

Hence, for all the above cases, we are reduced to cases 5 and 9. So, we have boiled down all the possibilities to $(\mu_i, 1, 0)$, such that $i = 1$ or 2. Observe that $-I \in G$, by Remark (2c). Also, $[S]^5 = [T]^3 = [TS]^2 = e$ in G/H (by remark2), so, it implies $|G/H| \leq 60$, and, since, A_5 has elements of orders $2, 3$ and 5, therefore $|G/H|$ must be divisible by 30. Hence, by proposition(3c), the order must be 60, which implies that the map, $\phi : A_5 \to G/H$, is an isomorphism.

□

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