# Branch Groups

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# Certificate of Examination

This is to certify that the dissertation titled **"Branch Groups"** submitted by **Ms. Karthika Rajeev** (Reg. No. MS12095) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Prof. Kapil Hari Paranjape Dr. Pranab Sardar Prof. I. B. S. Passi

(Supervisor)

Dated: April 21, 2017

# Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. I. B. S. Passi at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Karthika Rajeev
(Candidate)
Dated: April 21, 2017

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. I. B. S. Passi
(Supervisor)

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I place my acknowledgement to the Department of Mathematics IISER, Mohali for the support. I also like to remember and thank the Central Library, IISER Mohali for their kind services and the facilities provided.

Finally, I would like to acknowledge that the material presented in this thesis is based on other people's work. If I have made then it is the selection, presentation and some basic calculations of the materials from different sources which are listed in the bibliography.

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# Contents

# Abstract

This dissertation is an exposition of the theory of branch groups. We organized this study in two parts. In first part, we discuss the construction and distinctive properties of a finitely generated branch group, the Grigorchuk group. The second part is the study of the characterization theory of branch group. We also see the classification of the just infinite group in which branch groups form a partition.

# Introduction

The concept of branch groups is introduced by R. I. Grigorchuk at St. Andrews Group Theory Conference in Bath 1997. He defines branch group as a group whose lattice of subnormal subgroups is similar to the structure of a spherically homogeneous rooted tree. In [Gri00], he gave a second, equivalent definition, based on a geometrical point of view: branch groups are groups acting spherically transitively on a spherically homogeneous rooted tree $T$ and having subnormal subgroups similar to the corresponding structure in the full automorphism group $Aut(T)$ of the tree T.

The class of branch group is an important area of research. This class contains groups which are answers to a number of long-standing problems in group theory. They provide easiest counterexamples to general Burnside problem, solutions to John Milnor's question on the existence of a group with intermediate word growth, examples of groups which are amenable but not elementary amenable, examples of groups with finite width etc.

Branch groups are defined in both abstract and profinite cases. The interest of this dissertation is limited to the study abstract branch groups. Our primary objective is to discuss the characterization theory of branch group developed by P. D. Hardy in [Har02].

This thesis is organized into three chapters. The first chapter is an introduction to spherically homogeneous rooted trees. Here, we see some definitions of subgroups of a group acting on a rooted tree and discuss their properties.

The second chapter is a detailed survey of the Grigorchuk group $G$, which is treated as one of the most famous examples of a finitely generated branch group. The Grigorchuk group is an automorphism group of the regular binary rooted tree. It first appeared in [Gri80] by Grigorchuk at 1980, as a counterexample to general Burnside problem. We see that the Grigorchuk group is a finitely generated infinite group in which every element has finite order. We study some of the normal subgroups of $G$ and show that $G$ has congruence property, the word problem of $G$ is solvable and $G$ is not finitely presented. This chapter also deals with the description of the terms of the derived series of $G$ [Gri00] and compute their indices in $G$.

The Grigorchuk group has other remarkable properties like; it is the first constructed group with intermediate growth [GP08],[Gri85], it has bounded width property [BG$^+$00], it is an example of amenable group which is not elementary amenable [Gri98], etc. Prominent examples of branch groups are studied in [BGŠ03].

The theory of branch group is studied in the last chapter of this dissertation. This chapter is divided into two parts, titled as just infinite groups and structure theory of branch groups.

Branch group naturally arises as a subclass of just infinite group, these are the infinite groups in which every proper quotient is finite. The class of just infinite group is of great importance because every finitely generated infinite group can be mapped onto a just infinite group (see Proposition (3): [Gri00]). In the first part we discuss the fundamental work of Wilson [Wil71], [Wil00] on the classification of just infinite groups. We also study the following trichotomy established by Grigorchuk in [Gri00]: every just infinite group G is either a branch group, or G contains a finite index normal subgroup that is isomorphic to the direct product of a finite number of copies of a group L, where L is either simple or hereditarily just infinite (residually finite groups with all finite index normal subgroups are just infinite).

The second part of the third chapter is a generalization of Wilson's work on just infinite groups to a larger class of groups. We end this study with the proof of the characterization of branch group.

The class of branch group is connected to other fields such as analysis, geometry, combinatorics, probability and computer science. Many branch groups show the property of self-similarity; hence it is the starting point of the study of the fractal theory [Nek05].

# Chapter 1

# Spherically homogeneous rooted trees

In this chapter, we define some of the subgroups of a group acting on a rooted tree and study their properties. We follow [Nek05] for definitions and notations. For further details one may refer to [BGŠ03] and [Gri00].

## 1.1 The trees

Let $\bar{m} = \{m_n\}_{n=1}^{\infty}$ be a sequence of integers with $m_n \geq 2$ and let $\bar{X} = \{X_n\}_{n=1}^{\infty}$ be a sequence of alphabets with $|X_n| = m_n$.

We define a *word* $w$ of length $n$ over $\bar{X}$ as a sequence of letters of the form $w = x_1 x_2 \ldots x_n$ where $x_i \in X_i$ for all $i$. The length of the word $w$ is denoted by $|w|$. Let $\bar{X}^*$ denotes the set of all finite words over $\bar{X}$, including the unique zero length word, the *empty word*, $\emptyset$.

We introduce a partial order on $\bar{X}^*$ by the *prefix relation* $\leq$, i.e., $u \leq v$ if $u = u_1 \ldots u_n$, $v = v_1 \ldots v_k$, $n \leq k$, then $u_i = v_i$ for all $i \leq n$, where $u, v \in \bar{X}^*$. The partially ordered set of words $\bar{X}^*$ is called a *spherically homogeneous rooted tree,* $T_{\bar{m}}$. The sequence $\bar{m}$ is called the *branching index* of $T_{\bar{m}}$. The set of all words of length $n$ forms the *nth level* of $T_{\bar{m}}$ and is denoted by $L_n$. The product $N_n = m_1 \cdots m_n$ gives the number of vertices of the level $L_n$.

If all $m_i$ are equal, to say $m$, and all $X_i$ are equal, to say $X$, then the resulting tree is said to be a *regular rooted tree* and it is denoted by $T_m$.

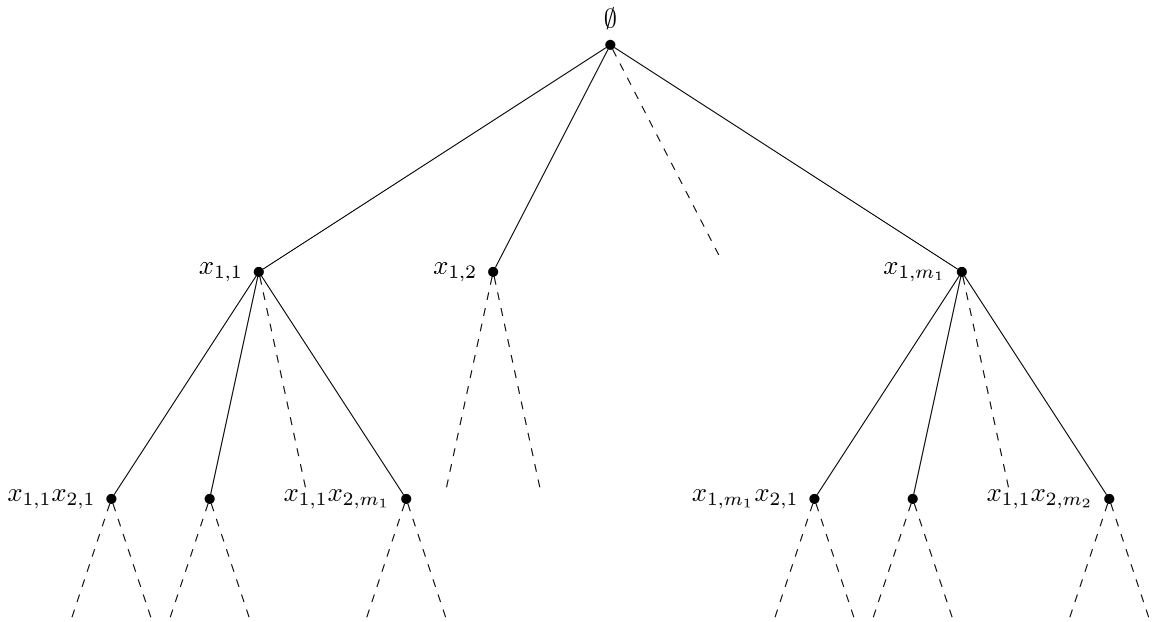


Figure 1.1: The tree $T_{\bar{m}}$, where $X_i = \{x_{i,1}, \ldots, x_{i,m_i}\}$.

For any $u \in \bar{X}^*$, $T_u$ denotes the subtree hanging below the vertex $u$ with $u$ as the root. If $|u| = n$, then for any $v \in L_n$ we have $T_u \cong T_v$. We use the notation $T_{\langle n \rangle}$ to denote the tree isomorphic to all subtrees with root vertex $u$, where $u \in L_n$.

A map $f : \bar{X}^* \longrightarrow \bar{X}^*$ is an *endomorphism* of the tree $T_{\bar{m}}$ if $f$ preserves the prefix relation and the root. A bijective endomorphism is called an *automorphism* of the tree $T_{\bar{m}}$. It can be easily observed that $f$ permutes the vertices of $L_n$ of the tree $T_{\bar{m}}$ for every $n \geq 1$. The set of all automorphisms of the tree $T_{\bar{m}}$ forms a group and is denoted by $\mathrm{Aut}T_{\bar{m}}$. The automorphism group of the subtree $T_u$ ($T_{\langle n \rangle}$) is denoted by $\mathrm{Aut}(T_u)$ ($\mathrm{Aut}T_{\langle n \rangle}$ respectively) for all vertex $u \in L_n$.

## 1.2 Group acting on a rooted tree

Let $G$ be a group acts on a rooted tree $T = T_{\bar{m}}$ by the automorphism of the tree $T$. The action of $G$ on $T$ is said to be *faithful* if the kernel of the action is trivial. Then $G$ can be viewed as a subgroup of $\mathrm{Aut}(T)$.

**Definition 1.2.1** Let $G \leq \mathrm{Aut}(T)$ be an automorphism group of the rooted tree $T = T_{\bar{m}}$.

1. An action of a group $G$ by automorphisms of the tree $T$ is said to be *level-transitive* if it is transitive on every level $L_n$ of the tree $T$.

2. The *vertex stabilizer* is the subgroup $St_G(v) = \{g \in G : g(v) = v\}$, where $v$ is a vertex of $T$.

3. The *nth level stabilizer* is the subgroup $St_G(n) = \bigcap_{v \in L_n} St_G(v)$.

4. The *rigid vertex stabilizer* of a vertex $v$ of $T$ is the subgroup $RiSt_G(v) = \{g \in G : g(u) = u \text{ for all } u \notin T_v\}$.

5. The *nth level rigid stabilizer* $RiSt_G(n)$ is the subgroup $\langle RiSt_G(v) : v \in L_n \rangle$ generated by the union of rigid stabilizers of the vertices of the $n$th level.

**Proposition 1.2.2** *Let $G$ be a level transitive automorphism group of the rooted tree $T = T_{\bar{m}}$. Then*

1. *A vertex stabilizer $St_G(v)$ for a vertex $v$ of $T$ is a subgroup of index $|L_n|$ in $G$.*

2. *For every vertex $v$ of $T$ and $g \in G$ , $g \cdot St_G(v) \cdot g^{-1} = St_G(g(v))$ and $g \cdot RiSt_G(v) \cdot g^{-1} = RiSt(g(v))$.*

3. *The level stabilizers $St_G(n)$ are normal finite index subgroups of $G$ and $\bigcap_{n=1}^{\infty} St_G(n) = 1$.*

4. *If $u, v$ be two vertices of $T_{\bar{m}}$ and $u \leq v$ then $St(v) \leq St(u)$ and $RiSt(v) \leq RiSt(u)$.*

5. *If $u, v$ be two vertices of $T_{\bar{m}}$ which are not comparable then $RiSt(u) \bigcap RiSt(v) = [RiSt(u), RiSt(v)] = 1$.*

6. *The level rigid stabilizer $RiSt_G(n)$ is a normal subgroup, which is equal to the direct product $\prod_{v \in L_n} RiSt(v)$ of its subgroups.*

*Proof:*

(1) It can be easily proved that $St_G(v)$ for a vertex $v$ of $T$ is a subgroup of $G$. Consider the action of $G$ on $L_n$. Clearly, for all $v \in L_n$, stabilizer of $v$ under the action of $G$ on $L_n$ is the subgroup $St_G(v)$. By orbit-stabilizer theorem we have

$$\frac{|G|}{|\text{stab}_G(v)|} = |\text{orbit}(v)|$$

But orbit$(v) = L_n$ since the action of $G$ on $L_n$ is transitive. Hence we get,

$$\frac{|G|}{\text{Stab}_G(v)} = |L_n|.$$

(2) Let $g \in G$ and $h \in St_G(v)$. Consider the action of $ghg^{-1}$ on $g(v)$.

$$ghg^{-1}(g(v)) = gh(g^{-1}g(v)) = gh(v) = g(v).$$

Since $g$ and $h$ were arbitrary, we have $g \cdot St_G(v) \cdot g^{-1} \leq St_G(g(v))$.
Now, consider $g' \in St_G(g(v))$. Then

$$\begin{aligned} g'(g(v)) &= g(v) \\ g^{-1}(g'(g(v))) &= g^{-1}(g(v)) \\ g^{-1}g'g(v) &= v. \end{aligned}$$

This gives $g^{-1}g'g \in St_G(v)$ and $g' \in g \cdot St_G(v) \cdot g^{-1}$. Hence $St_G(g(v)) \leq g \cdot St_G(v) \cdot g^{-1}$ and we get the equality.
Similarly, we can show that $g \cdot RiSt_G(v) \cdot g^{-1} = RiSt(g(v))$ for $v \in T_{\bar{m}}$.

(3) Let $g \in G$ and $g' \in St_G(n)$. Consider the action of $gg'g^{-1}$ on a vertex $v \in L_n$.

$$gg'g^{-1}(v) = gg'(g^{-1}(v)) = g(g^{-1}(v)) = v,$$

since $g^{-1}(v) \in L_n$ and $g'$ acts trivially on it. The set of all conjugates of the elements of $St_G(n)$ by the elements of $G$ is in $St_G(n)$, hence $St_G(n) \trianglelefteq G$.
We define a homomorphism $\phi$ from $G$ to symmetric group of $|L_n|$ elements. The function is well defined since the action of $G$ on $L_n$ is transitive. Clearly, the kernel of this homomorphism is the subgroup $St_G(n)$. By first isomorphism theorem,

$$\frac{G}{\ker(\phi)} = \text{image}(\phi),$$

where image$(\phi)$ is a subgroup of Sym$(|L_n|)$. Hence the subgroup $St_G(n)$ has finite index in $G$.
Let $1 \neq g \in St_G(n)$. Then there exists some vertex $x$ in $T$ with $|x| = m$ such that $g(x) \neq x$. Hence $g \notin St_G(m)$, so that $g \notin \bigcap_{n=1}^{\infty} St_G(n)$. Therefore $\bigcap_{n=1}^{\infty} St_G(n)$

(4) Let $u, v \in T$ and $u \leq v$. If $g \in G$ fixes $v$, then it has to fix $u$. Thus $St_G(v) \leq St_G(u)$. Similarly, $RiSt_G(v) \leq RiSt_G(u)$.

(5) The result is an easy consequence of the definition of the subgroup $RiSt_G(v)$ of a vertex $v$ of the tree $T$.

(6) The normality of the subgroup $RiSt_G(n)$ can be proved as similar to that of the subgroup $St_G(n)$. It follows from the result above that

$$RiSt_G(n) = \langle RiSt_G(v) : v \in L_n \rangle = \prod_{v \in L_n} RiSt(v).$$

$\square$

# Chapter 2

# The Grigorchuk first group

Here we discuss the construction and some of the distinctive properties of the Grigorchuk group. All the discussions of this chapter are based on chapter VIII of [dLH00]. The chapter does not include the study of the growth property of the Grigorchuk group. One may refer to [Gri85] and [dLH00] for the details.

## 2.1 Preliminaries

Let $G$ be a group and $S$ be a non-empty subset of $G$.

The *normal closure* of the set $S$ in $G$ is the smallest normal subgroup of $G$ which contains $S$. It is denoted by $\langle S\rangle^G$. If $S$ is subgroup of $G$, then we use the expression $S^G$ to denote the normal closure of $S$ in $G$. We write $N_G(S)$ for the normalizer of $S$ in $G$.

The set $S$ is called a *generating set* of $G$ if every element of $G$ can be expressed as a product of finitely many members of $S$ and their inverses. We write $G = \langle S\rangle$ and

$$G = \{g \in G : g = s_1^{\epsilon_1} \cdots s_i^{\epsilon_i}, s_i \in S, \epsilon_i \in \{1, -1\}\}.$$

The group $G$ is said to be finitely generated if the cardinality of $S$ is finite.

Let $F$ be a free group on a set $X$ of generators. A *letter* is an element of the set $X \cup X^{-1}$. A *word* $w$ is a finite string of letter of the form $w = x_1x_2\cdots x_n$, where

$x_i \in X \cup X^{-1}$. We denote the identity element of $F$ by 1. Each element of $F$ other than the identity element can be represented by a unique element called the *reduced word.* A reduced word $w$ is of the form $w = x_1x_2 \cdots x_m$ in which no two successive letters $x_ix_{i+1}$ form an inverse pair $x_ix_i^{-1}$ or $x_i^{-1}x_i$. We denote the length of a reduced word $w$ by $|w|$.

**Presentation of groups:** Suppose there is a one to one correspondence between the set of generators, $X$, of $F$ and the set of generators, $S$, of $G$. The identification of the set generators of $F$ with $G$ extends to an epimorphism $\phi : F \longrightarrow G$. By first isomorphism theorem $G$ is isomorphic to the quotient of $F$ by the kernel of $\phi$. If there exists a subset $R$ of $F$ such that $\langle R\rangle^F$ is equal to the kernel of $\phi$, then the group $G$ can be determined up to isomorphism by the sets $X$ and $R$. The expression $\langle X|R\rangle$ is called a *presentation* of $G$, where $X$ is identified with a set of generators of $G$ and $R$ is identified with a set of relators of $G$. If the cardinality of $X$ and $R$ are finite then $G$ is said to be finitely presentable.

**Free products:** Let $G = \langle X_G|R_G\rangle$ and $H = \langle X_H|R_H\rangle$ be two groups with $X_G \cap X_H = \phi$. The free product of $G$ and $H$ is the group $G * H$ which is defined to be

$$G * H = \langle X_G \cup X_H | R_G \cup R_G\rangle.$$

Let $1 \neq g \in G * H$. The element $g$ has a unique expression of the form

$$g = g_1g_2 \dots g_n,$$

where $g_i \in (G \cup H) - \{1\}$ and $g_i, g_{i+1}$ do not belong to the same group $G$ or $H$.
For further clarification in the definitions of group presentation and free product, one may refer to [Bog08].

Let $H$ be a subgroup of $G$. We use the notation $H \leq_f G$ to denote $H$ is a subgroup of finite index in $G$ and we write $H \trianglelefteq_f G$ for $H \trianglelefteq G$ and $H \leq_f G$.

**Definition 2.1.1** A group $G$ is said to be *residually finite* if for any $1_G \neq g \in G$, there exists a homomorphism $\psi : G \longrightarrow A$ such that $\psi(g) \neq 1_A$, where $A$ is a finite group.

**Proposition 2.1.2** *$G$ is residually finite if $\bigcap_{H \trianglelefteq_f G} H = 1$.*

*Proof:* For any $1 \neq g \in G$ there exists a normal subgroup $H$ of finite index in $G$ such that $g \notin H$. Then the epimorphism from $G$ to $G/H$ maps $g$ to a non-trivial element in $G/H$. $\square$

**Definition 2.1.3** A group $G$ is said to be *Hopfian* if every epimorphism from $G$ to $G$ is an isomorphism.

**Proposition 2.1.4** *Every finitely generated residually finite group is Hopfian.*

*Proof:* Suppose that $G$ be a finitely generated residually finite group and $\phi : G \longrightarrow G$ be an epimorphism which is not injective. Let $g_0$ be a non-trivial element in $G$ such that $g_0 \in ker(\phi)$. Since $G$ is residually finite there exists a homomorphism, say $\pi$, from $G$ to a finite group $A$ such that image of $g_0$ in $A$ is non-trivial. For each $n \geq 1$, we may choose $g_n \in G$ such that $\phi^n(g_n) = g_0$, where $\phi^n$ is the composition of $\phi$ $n$ times. Define a homomorphism $\pi_n : G \longrightarrow A$ as $\pi_n(g) = \pi(\phi^n(g))$ for any $g \in G$. Then we have

$$\pi_n(g_n) = \pi(\phi^n(g_n)) = \pi(g_0) \neq 1.$$

Consider $m > n$. Then $\pi_m(g_m) \neq 1$.

$$\pi_m(g_n) = \pi(\phi^m(g_n)) = \pi(\phi^{m-n-1}(\phi(\phi^n(g_n)))) = \pi(\phi^{m-n-1}(\phi(g_0))) = 1.$$

Thus all $\pi_i$ are distinct. This gives infinite number of homomorphisms from $G$ to $A$. Since $G$ is finitely generated and $A$ finite, only finite number of homomorphisms are possible from $G$ to $A$. Hence we get a contradiction, and the result follows. $\square$

## 2.2 The Grigorchuk group

The Grigorchuk group $G$ acts faithfully on a regular binary rooted tree $T$ and can thus be regarded as a subgroup of the full automorphism group $\mathrm{Aut}(T)$. Let $X = \{0, 1\}$ be a 2-element alphabet. The set $X^*$ of words over $X$ (i.e., finite sequence of symbols in $X$, including the empty word) can be identified naturally with the vertices of $T$. The action of $G$, generated by $a, b, c, d \in \mathrm{Aut}(T)$, is given recursively as follows:

$$\begin{array}{llll} a(0w) = 1w, & a(1w) = 0w; & b(0w) = 0a(w), & b(1w) = 1c(w); \\ c(0w) = 0a(w), & c(1w) = 1d(w); & d(0w) = 0w, & d(1w) = 1b(w), \end{array}$$

for any $w \in X^*$.

The element $a$ permutes the vertices in the first level. The element $b$ fixes the first level and acts as $(a, c)$; i.e., on the first tree hanging from the root $b$ acts as $a$, and on the second tree it acts as $c$. Similarly, the elements $c$ and $d$ fix the first level and act as $(a, d)$ and $(1, b)$ respectively.

Using induction the following can be verified:

$$a^2 = b^2 = c^2 = d^2 = 1,$$

$$bc = cb = d, bd = db = c, cd = dc = b.$$

This implies the subset $\{1, b, c, d\}$ forms a subgroup of $G$ which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the Klein four-group $(V)$. Every element of $G$ can be expressed as a reduced word of the form

$$s_0 a s_1 a s_2 a \cdots s_{m-1} a s_m,$$

where $s_i \in \{b, c, d\}$ for $i = 1, \dots, m-1$ and $s_0, s_1 \in \{1, b, c, d\}$. The set of all reduced words in letters of $\{a, b, c, d\}$ can be seen as a free product of $\{1, a\}$ and $\{1, b, c, d\}$. Hence $G$ is isomorphic to a quotient of the free product $(\mathbb{Z}/2\mathbb{Z}) * V$.

Denote $St_G(1)$ to be the subgroup, stabilizer of first level, of $G$ which is defined as

$$St_G(1) = \{g \in G : g(x) = x \text{ for all } x \in X\}.$$

Clearly, $a \notin St_G(1)$ and it consists of those elements of $G$ which have even number of occurrence of the element $a$. Hence the subgroup $St_G(1)$ of $G$ is generated by the elements $b, c, d, aba, aca$ and $ada$. We have,

$$\begin{aligned} b &= (a, c) & \qquad aba &= (c, a) \\ c &= (a, d) & \qquad aca &= (d, a) \\ d &= (1, b) & \qquad ada &= (b, 1). \end{aligned}$$

This induces a homomorphism from $St_G(1)$ into $G \times G$ as follows:

$$\psi_1 = (\phi_0, \phi_1) : St_G(1) \longrightarrow G \times G,$$

$$\begin{array}{llll} \phi_0(b) = a & \phi_1(b) = c & \phi_0(aba) = c & \phi_1(aba) = a \\ \phi_0(c) = a & \phi_1(c) = d & \phi_0(aca) = d & \phi_1(aca) = a \\ \phi_0(d) = 1 & \phi_1(d) = b & \phi_0(ada) = b & \phi_1(ada) = 1. \end{array}$$

The homomorphism $\psi_1$ defined as above is injective and $\phi_i : St_G(1) \longrightarrow G$ is onto for $i = \{0, 1\}$. Injectivity is clear from the definition of $\psi_1$. Surjectivity is followed from the fact that the image of $St_G(1)$ under $\psi_i$ contains the elements $a, b, c, d$ which generate the whole group $G$. Then group $G$ has a proper subgroup which is mapped onto $G$ so that $G$ can not be finite. Hence the Grigorchuk group $G$ is infinite.

We can generalize the notion of the level stabilizer $St_G(1)$ and the homomorphism $\phi_1$ associated to it. We define $St_G(n)$, the $n$th level stabilizer, as the subgroup of $G$ consists of all those elements that stabilize the vertices of the $n$th level. The homomorphism $\psi_n$ is defined as

$$\psi_n = (\phi_{0,\dots,0}, \dots, \phi_{1,\dots,1}) : \begin{cases} St_G(n) & \longrightarrow G \times \cdots \times G \\ g & \longmapsto (g_{0,\dots,0}, \dots, g_{1,\dots,1})_n \end{cases},$$

where the subscript $n$ indicates that there are $2^n$ copies of $G$, where $2^n$-uplets are indexed by the $\{0, 1\}$ sequence of length $n$, and where $\phi_{i_1,\dots,i_n}$ is in fact the composition of $\phi_{i_n} \circ \cdots \circ \phi_{i_1}$. As in the case of $n = 1$ above, $\psi_n$ is injective and $\phi_{i_1,\dots,i_n} : St_G(n) \longrightarrow G$ is onto.

**Proposition 2.2.1** $\bigcap_{n=1}^{\infty} St_G(n) = 1$.

*Proof:* By Proposition 1.2.2 (3). □

**Corollary 2.2.2** *$G$ is residually finite.*

*Proof:* For each $n \geq 1$, the normal subgroups $St_G(n)$ are of finite index in $G$. Hence the result follows by Proposition 2.1.2. □

**Corollary 2.2.3** *$G$ is Hopfian.*

*Proof:* By Proposition 2.1.4. □

**Theorem 2.2.4** *Every $g \in G$ there exists $n \in \mathbb{N}$ such that $g^{2^n} = 1$.*

*Proof:* For any $g \in G$ we have the following representation:

$$g = s_0 a s_1 a s_2 a \cdots s_{m-1} a s_m,$$

where $s_i \in \{b, c, d\}$ for $i = 1, \dots, m-1$ and $s_0, s_m \in \{1, b, c, d\}$. We define *length of $g$* as the number of non-unit generators in the shortest representation of g. We will proceed by inducting on the length $k$ of $g$.

$1, a, b, c, d$ are the elements of length one in $G$. Clearly, $g^2 = 1$ for $k = 1$. For $k = 2$ consider the elements $ab, ac$ and $ad$.

$$(ad)^4 = ((b,1)(1,b))^2 = (b,b)^2 = 1$$
$$(ac)^8 = ((d,a)(a,d))^4 = (da,ad)^4 = 1$$
$$(ab)^{16} = ((c,a)(a,c))^8 = (ca,ac)^8 = 1.$$

The other elements of length 2 are the conjugates of above elements hence have the same order.

For $k \geq 3$, assume that for any $g \in G$ of length less than equal to $k$ there exists $n \in \mathbb{N}$ such that $g^{2^n} = 1$. Let $g = s_0 a s_1 a s_2 a \cdots s_{m-1} a s_m$ be the shortest representation for $g$. There are three possibilities:

1. $g$ starts and ends with $a$: take the conjugate of $g$ by $a$ which is of length $k-2$.

2. first and last letter of $g$ belong to $\{b, c, d\}$: consider the conjugate of $g$ by first letter of $g$ which represent a word of length $k-1$ or $k-2$.

   In both cases, induction hypothesis apply.

3. $g$ starts with $a$ and ends with an element of $\{b, c, d\}$ (after conjugating, if necessary $g$ by $b, c$ or $d$).

Then,

$$g = as_1as_2a \cdots as_{k/2}, \text{ for } s_i \in \{b, c, d\}.$$

Now, there are two more cases.

**Case 1:** $k/2$ is even.

We can write $g$ as $g = (as_1a) \cdot s_2 \cdot (as_3a) \cdots s_{k/2} = (g_0, g_1)$, where $as_ia \in \{b, c, d\} \times \{a, 1\}$ and $s_i \in \{a, 1\} \times \{b, c, d\}$. The length of $g_i \leq k/2$ for $i = \{0, 1\}$. Then by induction hypothesis, there exists $n \in \mathbb{N}$ such that $g_0^{2^n} = g_1^{2^n} = 1$

$$g^{2^n} = (g_0^{2^n}, g_1^{2^n}) = 1$$

**Case 2:** $k/2$ is odd.

We have $g^2 = (as_1a) \cdot s_2 \cdot (as_3a) \cdots (as_{k/2}a) \cdot s_1 \cdot (as_2a) \cdots s_{k/2} = (h_0, h_1)$, and the length of $h_i \leq 2 \cdot k/2 = k$ for $i = \{0, 1\}$.

We will consider three more cases.

**(i)** Some of $s_j$ is equal to $d$.

We will have once $d = (1, b)$ and once $ada = (b, 1)$. Then the length of $h_i \leq k - 1$. By induction hypothesis there exists $n \in N$ such that $h_0^{2^n} = h_1^{2^n} = 1$. This implies $g^{2^{n+1}} = 1$.

**(ii)** Some of $s_j$ is equal to $c$.

Then $s_j = (a, d)$ and $as_ja = (d, a)$. Each of $h_i$ has length less than $k$ or is equal to a word of length $k$ involving $d$. In the first case we apply induction hypothesis, for second case use **(i)**.

**(iii)** If neither **(i)** not **(ii)** holds, then $g = abab \cdots ab$, which is of order at most 16.

Every element of $G$ is of order a power of 2. Hence $G$ is an infinite 2-group. $\square$

**Remark 2.2.5** It can be observed that for $1 \neq g \in St_G(1)$ and $\psi_1(g) = (g_0, g_1)_1$, we have $l(g_i) \leq \frac{l(g)+1}{2}$ for $i \in \{0, 1\}$ and $g_0 \neq g$ and $g_1 \neq g$.

## 2.3 Some normal subgroups of $G$

We write $g^h$ to denote the conjugation of $g$ by $h$ for $g, h \in G$ and $\langle g \rangle^G$ to denote the normal closure of $g$ in $G$.

Let $x = (ab)^2, y = (bd^a)^2, z = (b^a d)^2$ be the elements of $G$. We define two normal subgroups of $G$ as $B = \langle b \rangle^G, K = \langle (ab)^2 \rangle^G$ and $D = \langle a, d \rangle$. $D$ is a copy of the dihedral group of order 8.

**Proposition 2.3.1** *The group $B$ is generated by the elements $b, b^a, y, z$ and has index 8 in $G$.*

*Proof:* Cleraly, $b, b^a \in B$. We can write $y = [b, d^a]$ and $z = [b, d^a]^a$. Hence $y, z \in B$ because B contains the subgroup $[B, G]$.

Let $B_1 = \langle b, b^a, y, z \rangle$. Then $B_1 \leq B$. We will show that $B_1$ is normal in $G$. Then by minimality of $B$ we get $B_1 = B$. To prove the normality of $B_1$, it suffices to show that the set of all conjugates of the generators of $B_1$ by the generators of $G$ is in $B_1$. Since $c = bd$ we only need to check for the conjugation by $a, b$ and $d$.

Clearly, all the conjugates of $b$ are in $B_1$. An easy verification shows that the conjugation of the generators of $B_1$ by $a$ and $b$ are also in $B_1$.

We have,

$$\psi_1(x) = (ca, ac)$$
$$\psi_1(y) = (x, 1)$$
$$\psi_1(z) = (1, x).$$

Now,

$$db^a d = b^a(b^a d)^2 \in B_1$$
$$dyd = (1, b)(x, 1)(1, b) = (x, 1) = y \in B_1$$
$$dzd = (1, b)(1, x)(1, b) = (1, (ba)^2) = (1, (ab)^{14}) = z^7 = z^{-1} \in B_1.$$

The last step follows from the fact that $ab$ is of order 16. This completes the first part of the proof.
Now we will show that $B$ has index 8 in $G$.

Let $\overline{G}$ denotes the quotient of $G$ by the kernel of the canonical homomorphism $\pi$ from $G$ to the symmetric group of 8 vertices of the third level of the binary rooted tree. Let $1, 2, \ldots, 8$ denote the vertices $(0,0,0), (0,0,1), \ldots (1,1,1)$ of the third level. Then,

$$\pi(a) = (1,5)(2,6)(3,7)(4,8)$$
$$\pi(b) = (1,3)(2,4)(5,6)$$
$$\pi(c) = (1,3)(2,4)$$
$$\pi(d) = (5,6).$$

This generate the group $\overline{G} = G/St_G(3)$ and is isomorphic to $\big((\mathbb{Z}/2\mathbb{Z})\wr(\mathbb{Z}/2\mathbb{Z})\big)\wr(\mathbb{Z}/2\mathbb{Z})$. Let $\overline{B}$ denotes the image of $B$ under $\pi$. The group $\overline{B}$ is generated by the elements

$$\pi(b) = (1,3)(2,4)(5,6)$$
$$\pi(b^a) = (1,2)(5,7)(6,8)$$
$$\pi(y) = (1,2)(3,4)$$
$$\pi(z) = (5,6)(7,8).$$

The elements $\pi(y)$ and $\pi(z)$ are contained in the centre of $\overline{B}$, and they generate a group of order 4. Furthermore, $(\pi(b)\pi(b^a))^2 = \pi(y)\pi(z)$. Thus $\overline{B}$ is a group of order

16, so that $|\overline{G} : \overline{B}| = 2^7/2^4 = 2^3$. Hence

$$|G : B| \geq 2^3 \tag{2.1}$$

Now consider a map from $G$ to $G/B$. Clearly, $b$ maps to the identity element. Since $c = bd$, the image of $c$ in $G/B$ equals to the image of $d$ in $G/B$. Thus the group $G/B$ is generated from the image of $a$ and $d$. Therefore,

$$|G : B| \leq 2^3, \tag{2.2}$$

since $a, d$ is the dihedral group of order 8. From the equations (2.1) and (2.2) we get

$$|G : B| = 2^3.$$

This completes the proof. $\square$

**Proposition 2.3.2** *The group $K$ is generated by the elements $x, y, z$ and has index 16 in $G$.*

*Proof:* The elements $y, z$ are in $K$ since $y = ab[(ab)^2, d]ba$ and $z = y^a$. Let $K_1$ denotes the group generated by the elements $x, y, z$. Then $K_1 \leq K$ and it suffices to show that $K_1 \lhd G$. The following calculations show that the set of all conjugates of the generators of $K_1$ by the generators of $G$ are in $K_1$.

$$\begin{array}{ccc} a(ab)^2a = (ba)^2, & a(bd^a)^2a = (b^ad)^2, & a(b^ad)^2a = (bd^a)^2, \\ b(ab)^2b = (ba^2), & b(bd^a)^2b = (bd^a)^{-2} & b(b^ad)^2b = (ab)^{-2}(b^ad)^{-2}(ab)^2, \\ d(bd^a)^2d = (bd^a)^2, & d(bd^a)^2d = (bd^a)^2, & d(b^ad)^2d = (b^ad)^{-2}. \end{array}$$

Then $K_1$ is normal in $G$, so is in $B$. Now consider the group $B/K$. The group is generated by the image of $b$ which is of order two in $B/K$. Thus $|B : K| = 2$ and

$$|G : K| = |G : B||B : K| = 2^3 \cdot 2 = 2^4.$$

$\square$

We use the notation $K_m$ to denote the direct product of $2^m$ copies of $K$.

**Proposition 2.3.3**

1. $St_G(3) \leq K \leq St_G(1)$.

2. *$K$ is self-replicating, that is $K$ geometrically contains the direct product $K \times K$ (i.e., $K \succ K \times K$).*

3. *For the element $(ac)^4 \in K$ we have the following relations:*

$$(ac)^2 \equiv (ad)^2 \mod K,$$

$$(ab)^4 \equiv (ac)^4 \mod K_1.$$

4. *The group $K/K_1$ is isomorphic to the cyclic group of order four and is generated by the image of $(ab)^2$.*

5. *The equality $K' = K_2$ holds and $St_G(5) \leq K' \leq St_G(3)$.*

6. *For each $n \geq 1$, $St_G(n)$ contains the subgroup $K_n$ which is the product of $2^n$ copies of $K$ acting on the subtrees of the binary tree which begin at the $n$th level.*

*Proof:* (1) The generators of $K$ are contained in the subgroup $St_G(1)$, hence $K \leq St_G(1)$. Let $\overline{K}$ denote the image of $K$ under $\pi$, where $\pi$ is the homomorphism as defined earlier. We have $|\pi(G)| = 2^7$ and $|\pi(K)| = 2^3$.

$$|\pi(G) : \pi(K)| = |G/St_G(3) : (K \cdot St_G(3))/St_G(3)| = |G : K \cdot St_G(3)| = 2^4.$$

But, $K$ is of index $2^4$ in $G$ and $K \leq K \cdot St_G(3)$. This implies $K = K \cdot St_G(3)$ hence $St_G(3) \leq K$.

(2) We have $y = ((ab)^2, 1) \in K$. Since $\phi_i$ is an epimorphism from $St_G(1)$ to $G$, for any $g \in G$ there exists an element $\gamma$ in $St_G(1)$ (hence in $K$) such that $\psi_1(\gamma) = (g, *)$, where $*$ be an element of $G$. Then,

$$\psi_1(\gamma y \gamma^{-1}) = (g(ab)^2 g^{-1}, 1).$$

This implies $(\langle (ab)^2 \rangle^G, 1)$ is contained in $\psi_1(K)$, i.e., $K \times 1 \leq \psi_1(K)$. Similarly, for $z = (1, (ab)^2)$ we get $1 \times K \leq \psi_1(K)$. Hence $K \times K \leq \psi_1(K)$. We say that $K$ contains $K \times K$ geometrically and is denoted by $K \succ K \times K$.

(3) Note that $(ad)^2 \notin K$. If $(ad)^2 \in K$ then $ad \equiv da (\mathrm{mod} K)$. The elements $a$ and $d$ commute in $G/K$. This is a contradiction to the fact that index of $K$ in $G$ is 16. Consider the conjugate of $(ab)^2$ by $ca$.

$$ac(abab)ca = acabada = acacdada = (ac)^2(ad)^{-2}.$$

Then $(ac)^2(ad)^{-2}$ is a conjugate of $(ab)^2$ in $K$, hence the relation $(ac)^2 \equiv (ad)^2 \mod K$. Thus $(ac)^2 \notin K$. Since $(ab)^4 = ((ca)^2, (ac)^2)$, we have $(ab)^4 \notin K$. Also since $(ac)^4 = ((da)^2, (ad)^2)$, it follows that $(ab)^4 \equiv (ac)^4 \mod K_1$. Therefore $(ac)^4 \in K$, and $(ab)^8 = ((ca)^4, (ac)^4) \in K_1$.

(4) Consider the quotient of $K$ by $K_1$. Clearly, the elements $y$ and $z$ are identity in $K/K_1$ and the group is generated by image of $(ab)^2$. We see that $(ab)^4 \notin K_1$ but $(ab)^8 \in K_1$. Hence $(ab)^2$ is of order 4 in $K/K_1$. The quotient $K/K_1$ is isomorphic to the cyclic group of order 4.

(5) The derived subgroup $K'$ of $K$ is generated from the elements $[x, y], [x, z]$ and $[y, z]$. They have the following decomposition;

$$\begin{aligned}
[x, y] &= ((ba)^2, 1, 1, 1)_2; \\
[x, z] &= (1, 1, 1, ((b^a d)^2)^{-1}(ab)^2)_2; \\
[y, z] &= (1, 1, 1, 1)_2.
\end{aligned}$$

Clearly, the above elements belong to $K \times K \times K \times K = K_2$, hence $K' \leq K_2$. Using the argument as in the proof of Proposition 2.3.3 (2), we can show that $(\langle (ba)^2 \rangle^G, 1, 1, 1)$ is contained $K'$ since $((ba)^2, 1, 1, 1) \in K'$. Thus $K \times \{1\} \times \{1\} \times \{1\} \leq K'$. Furthermore, we have

$$b[x, y]b = (1, (ba)^2, 1, 1)_2; \tag{2.3}$$

$$a[x, y]a = (1, 1, (ba)^2, 1)_2; \tag{2.4}$$

$$ab[x, y]ba = (1, 1, 1, (ba)^2)_2. \tag{2.5}$$

The equations (2.3) to (2.5) together imply that $K_2 \leq K'$. Hence we get the equality $K' = K_2$.

It is easy to see that

$$\psi_n(St_G(n + k)) = St_G(k) \times \cdots \times St_G(k),$$

($2^n$ factors respectively for each vertex of level $n$).

From Proposition 2.3.3 (1) we get,

$$\psi_2(St_G(5)) = St_G(3) \times St_G(3) \times St_G(3) \times St_G(3) < K \times K \times K \times K = K'.$$

From the decomposition of the generating elements of $K'$ we can see that they belong to $St_G(3)$ . Consequently, $St_G(5) \leq K' \leq st_G(3)$.

(6) The proof proceeds by induction on $n$ of $St_G(n)$. For $n = 1$ we have seen that $K < St_G(1)$ and $K \times K \leq \psi_1(K)$. Thus $St_G(1)$ contains $K \times K$. Assume that $K_n$ is contained in $St_G(n)$.

Let

$$x_n = ((bd^a)^2, 1, \ldots, 1)_n \in K_n \leq St_G(n).$$

$x_n \in St_G(n+1)$ since $(bd^a)^2 \in St_G(1)$. So we can apply $\psi_{n+1}$ to $x_n$ and get

$$\psi_{n+1}(x_n) = ((ab)^2, 1, \ldots, 1)_{n+1}.$$

This gives $K \times \{1\} \cdots \times \{1\} \leq St_G(n+1)$. Thus the direct product of $2^{n+1}$ copies of $K$ is contained in $St_G(n+1)$ follows from the fact that $G$ acts transitively on $T$. $\square$

**Theorem 2.3.4** *For any $n \in \mathbb{N}$, there exists $g \in G$ such that $g^{2^n} \neq 1$.*

*Proof:* We will first prove the following result.

For an integer $n \geq 1$ and an element $g_n \in K$ such that $g_n^{2^n} \neq 1$, then there exists an element $h_n \in St_G(5) < K$ such that

$$\psi_5(h_n) = (g_n, 1, \ldots, 1)_5 \in K_5$$

and the element $g_{n+1} = (ab)^8 h_n \in K$ satisfies $g_{n+1}^{2^{n+1}} \neq 1$.

Suppose $g_n \in K$ such that $g_n^{2^n} \neq 1$. From Proposition 2.3.3 (6) we have $K_5 \leq St_G(5)$. In particular,

$$K \times \{1\} \times \cdots \{1\} \leq St_G(5),$$

with $2^5 - 1$ copies of $\{1\}$. Thus we can choose $h_n \in St_G(5)$ such that

$$\psi_5(h_n) = (g_n, 1, \ldots, 1)_5 \in K_5.$$

For convenience we may write

$$\psi_4(h_n) = ((g_n, 1), (1, 1), \ldots, (1, 1))_4 \in K_4.$$

Set $g_{n+1} = (ab)^8 h_n \in K$. We have,

$$\begin{aligned}
\psi_4((ab)^8) &= (a, c, a, c, a, c, a, c, a, c, a, c, a, c, a, c)_4, \\
\psi_4(g_{n+1}^2) &= (a(g_n, 1)a(g_n, 1), c(1, 1)c(1, 1), \ldots, a(1, 1)a(1, 1), c(1, 1)c(1, 1))_4 \\
&= ((1, g_n)(g_n, 1), (1, 1), \ldots, (1, 1))_4 \\
&= (g_n, g_n, 1, \ldots, 1)_5.
\end{aligned}$$

Consequently,

$$\psi_5(g_{n+1}^{2^{n+1}}) = (g_n^{2^n}, g_n^{2^n}, 1, \ldots, 1)_5 \neq 1 \in G^{32}.$$

Hence $g_{n+1}^{2n+1} \neq 1$ in $G$. Now, the proof of the theorem follows by induction on $n$. $\square$

## 2.4 Congruence subgroups of $G$

We define a *congruence subgroup* of $G$ as a subgroup which contains $St_G(n)$ for large $n$.

**Theorem 2.4.1** *Any normal subgroup of $G$ distinct from $\{1\}$ is a congruence subgroup.*

*Proof:* Let $1 \neq g \in G$ and $N$ be the normal closure of $g$ in $G$. By Proposition 2.2.1 we can find an integer $n \geq 1$ such that $g \in St_G(n)$ but $g \notin St_G(n+1)$. Let

$$\psi_n(g) = (g_{0,\ldots,0}, \ldots, g_{1,\ldots 1})_k.$$

Since $g \notin St_G(n+1)$, at least one component of $g$ which is not contained in $St_G(1)$. We may assume that $g_{0,\ldots,0} \notin St_G(1)$. Thus we can write

$$g_{0,\ldots,0} = ha,$$

where $h \in St_G(1)$ and $\psi_1(h) = (h_0, h_1)$. By Proposition 2.3.3 (6) we can choose $k \in K$ and $u \in St_G(n+1)$ such that

$$\psi_{n+1}(u) = (k, 1, \ldots, 1)_{n+1} = ((k, 1), (1, 1), \ldots, (1, 1))_n.$$

Then

$$\psi_n([g, u]) = (g_{0,\ldots,0}^{-1}(k, 1)^{-1} g_{0,\ldots,0}(k, 1), (1, 1) \ldots, (1, 1))_n,$$

where

$$\begin{aligned} g_{0,\dots,0}^{-1}(k,1)^{-1}g_{0,\dots,0}(k,1) &= (ha)^{-1}(k,1)^{-1}(ha)(k,1) \\ &= a(h_0,h_1)^{-1}(k,1)^{-1}(h_0,h_1)a(k,1) \\ &= a(h_0^{-1}k^{-1}h_0,1)a(k,1) \\ &= (1,h_0^{-1}k^{-1}h_0)(k,1) \\ &= (k,h_0^{-1}k^{-1}h_0). \end{aligned}$$

Hence

$$\psi_{n+1}([g,u]) = (k, h_0^{-1}k^{-1}h_0, 1, \dots, 1)_{n+1}.$$

Now, we choose $l \in K$ and $v \in St_G(n+1)$ such that

$$\psi_{n+1} = (l, 1, \dots, 1)_{n+1}.$$

Then

$$\psi_{n+1}([[g,u],v]) = ([k,l], 1 \dots, 1)_{n+1}.$$

Since $N$ is normal in $G$, $[N,G] \leq N$. Hence we have $[[g,u],v] \in N$ and for any element $m \in K'$ we can find $t \in N$ such that

$$\psi_{n+1}(t) = (m, 1, \dots, 1)_{n+1},$$

(as $k$ and $l$ are arbitrary in $K$).

$$\text{Hence,} \quad K' \times \{1\} \times \cdots \times \{1\} \leq N,$$

with $2^{n+1} - 1$ copies of $\{1\}$. It follows from the fact that $G$ acts transitively on the levels of $T$,

$$K' \times \cdots \times K' \leq N$$

with $2^{n+1}$ copies of $K'$. By Proposition 2.3.3 (5),

$$St_G(5) \times \cdots \times St_G(5) \leq N$$

with $2^{n+1}$ copies of $St_G(5)$. But we have

$$\psi_{n+1}(St_G(n+6)) = St_G(5) \times \cdots \times St_G(5),$$

thus

$$St_G(n+6) \leq N.$$

□

## 2.5 The derived series of $G$

The derived series $G^{(n)}$, $n \geq 1$ of the group $G$ is defined as follows:

$$G^{(1)} = G' = [G, G],$$

$$G^{(n+1)} = [G^{(n)}, G^{(n)}].$$

**Theorem 2.5.1** *The following relations hold:*

1. $G^{(1)} = \langle [a, d], K\rangle$ *and* $G^{(1)}/K \cong C_2$.

2. $G^{(2)} = \langle y^2, yz, K^{(1)}\rangle$ *and* $G^{(2)}/K^{(1)} \cong C_2 \times C_4$.

3. $K_1 = \langle y, G^{(2)}\rangle$ *and* $K_1/G^{(2)} \cong C_2$.

4. $G^{(n)} = K_{2n-3}$ *if* $n \geq 3$.

*The indices satisfies*

$$[G : G^{(n)}] = \begin{cases} 2^3 \ \textit{if}\ n = 1 \\ 2^7 \ \textit{if}\ n = 2 \\ 2^{2^{2n-2}+2} \ \textit{if}\ n \geq 3. \end{cases}$$

*Proof:* Through out in this proof we assume that $G$ is a 3 generated group and $G$ is generated from the elements $a, b, d$.

(1) Let $P := \langle [a, d], K\rangle$, where $K$ is generated as the normal closure of the element $(ab)^2$ in $G$. Since $(ab)^2 = [a, b] \in G^{(1)}$, it is clear that $P \leq G^{(1)}$. We have $G^{(1)} = \langle [a, d], [a, b]\rangle^G$. We will prove that $P$ is normal in $G$, then $G^{(1)} \leq P$, since $G^{(1)}$ is the minimal normal subgroup of $G$ containing $[a, d]$ and $[a, b]$. To prove the normality of $P$, it suffices to show that the conjugates of $[a, d]$ and $[a, b]$ by the generates of $G$ are in $P$:

$$[a, d]^a = [d, a]; \quad [a, d]^d = [d, a]; \quad [a, d]^b = (b^a d)^2 [a, d] \in P.$$

As $[a, b] = (ab)^2 \in K$ and $K \trianglelefteq G$, the set of all conjugates of $[a, b]$ are in $P$. Hence the result follows and $G^{(1)} = P$.

Now, we have $G^{(1)} = \langle [a, d], K\rangle$. Consider the canonical epimorphism from $G^{(1)}$ to $G^{(1)}/K$. Then, $G^{(1)}/K$ is is isomorphic to the image of $[a, d]$ in $G^{(1)}/K$, since $[a, d] \notin K$ (from Proposition 2.3.3 (3)), i.e., $G^{(1)}/K \cong C_2$.

(2) Let $Q := \langle y^2, yz, K^{(1)}\rangle$. It follows from the fact that $G^{(1)} = \langle [a,d], K\rangle$ and $K = \langle x, y, z\rangle$;

$$\begin{aligned} G^{(2)} &= [G^{(1)}, G^{(1)}] = [\langle [a,d], K\rangle, \langle [a,d], K\rangle] \\ &= \langle [[a,d], K], K^{(1)}\rangle \\ &= \langle \langle [[a,d], x], [[a,d], y], [[a,d], z]\rangle^G, K^{(1)}\rangle. \end{aligned}$$

But $[[a,d], z] = (1, (ab)^4) = [[a,d], y]^a$, so

$$G^{(2)} = \langle \langle [[a,d], x], [[a,d], y]\rangle^G, K^{(1)}\rangle.$$

Since $\langle \langle [[a,d], x], [[a,d], y]\rangle^G$ and $K^{(1)}$ are normal in $G$ so are in $G^{(2)}$. Thus we have,

$$G^{(2)} = \langle [[a,d], x], [[a,d], y]\rangle^G \cdot K^{(1)}.$$

An easy computation shows that $([[a,d], x])^{bd} = yz$ and $[[a,d], y] = y^2$. This gives the one way inclusion, $Q \leq G^{(2)}$.

The reverse inclusion follows from the normality of $Q$ in $G$. If we can show that $Q$ is normal in $G$ then $\langle \langle [[a,d], x], [[a,d], y]\rangle^G \leq Q$, and since $K' \leq Q$ we have $G^{(2)} \leq Q$. We prove the normality of $Q$ by showing that $[Q, G] \leq Q$. Since $K^{(1)}$ is normal in $G$ it suffices to show that the set of all conjugates of $y^2$ and $yz$ by the generators of $G$ are in $Q$. We do the computations as similar to the case of $P$ and get the equality, $G^{(2)} = Q$.

We have $G^{(2)} = \langle y^2, yz, K^{(1)}\rangle$. Consider a canonical epimorphism from $G^{(2)}$ to $G^{(2)}/K^{(1)}$. Then $G^{(2)}/K^{(1)}$ is generated from the image of $yz$ and $y^2$ in $G^{(2)}/K^{(1)}$. But,

$$\begin{aligned} y^4 &= (x^4, 1) = ((ca)^4, (ac)^4, 1, 1) \equiv 1 \bmod K^{(1)}, \\ (yz)^4 &= (x^4, x^4) \equiv (1,1) = 1 \bmod K^{(1)} (\text{from Proposition 2.3.3(3)}). \end{aligned}$$

And since $[yz, y^2] \equiv 1 \bmod K^{(1)}$, the image of $yz$ and $y^2$ commute in $G^{(2)}/K^{(1)}$. Thus $G^{(2)}/K^{(1)} = \langle \text{image}(y^2)\rangle \times \langle \text{image}(yz)\rangle = C_2 \times C_4$, as image$(y^2)$ has order 2 in $G^{(2)}/K^{(1)}$ and image$(yz)$ has order 4 in $G^{(2)}/K^{(1)}$.

(3) Let $R := \langle y, G^{(2)}\rangle$, with $G^{(2)} = \langle y^2, yz, K^{(1)}\rangle$.
It follows from Proposition 2.3.3 (2) that $K^{(1)} = K_2 \leq K_1$. Furthermore, $y = (x, 1)_1 \in K_1$ and $z = (1, x)_1 \in K_1$. So the group $R$ is contained in the group $K_1$.

Since $K_1$ contains the elements $y$ and $x$, $K_1$ together with $x$ generates $K$. Then we have,

$$\begin{aligned}
K &= \langle x, K_1 \rangle \\
K_1 &= K \times K = \langle x, K_1 \rangle \times \langle x, K_1 \rangle \\
K_1 &= \langle (x,1)_1, (1,x)_1, K_2 \rangle \\
&= \langle y, z, K^{(1)} \rangle.
\end{aligned}$$

Clearly, $K_1 \leq R$. Thus the equality follows. Now consider the canonical epimorphism from $K_1$ to $K_1/G^{(2)}$. Then the group $K_1/G^{(2)}$ is generated from the image of $y$ in $K_1/G^{(2)}$. From Theorem 2.5.1 (2) above it is clear that $y \notin G^{(2)}$. Hence $K_1/G^{(2)} \cong C_2$.

(4) The proof proceeds by induction on $n$, of $G^{(n)}$.

Let $n = 3$. We will prove that $G^{(3)} = K_3$. By definition $G^{(3)} = [G^{(2)}, G^{(2)}]$. Using the expression for $G^{(2)}$ and the fact that $K^{(1)} = K_2 = \langle (x,1,1,1)_2, (y,1,1,1)_2 \rangle^G$, we get

$$\begin{aligned}
G^{(3)} &= [\langle y^2, yz, K^{(1)} \rangle, \langle y^2, yz, K^{(1)} \rangle] \\
&= \langle [y^2, yz], [y^2, K^{(1)}, ][yz, K^{(1)}], K^{(2)} \rangle \\
&= \langle [y^2, K^{(1)}, ][yz, K^{(1)}], K^{(2)} \rangle \quad (\because [y^2, yz] = 1) \\
&= \langle \langle [y^2, \eta], [y^2, \zeta], [yz, \eta], [yz, \zeta] \rangle^G, K^{(2)} \rangle \\
&= \langle [y^2, \eta], [y^2, \zeta], [yz, \eta], [yz, \zeta] \rangle^G \cdot K^{(2)} \quad (\because K^{(2)} \trianglelefteq G),
\end{aligned}$$

where $\eta = (x,1,1,1)_2$ and $\zeta = (y,1,1,1)_2 = (x,1,1,1,1,1,1,1)_3$.

We have the following decomposition:

$$\begin{aligned}
[y^2, \eta] &= [(x^2,1)_1, (x,1,1,1)_2] = [(ca)^2, (ab)^2], 1, 1, 1] \\
&= (x^{-1}, x, 1, 1, 1, 1, 1, 1)_3 \in K_3 \\
[yz, \eta] &= [(x,x)_1, (x,1,1,1)_2] = ([(ca)^2, (ab)^2], 1, 1, 1)_2 \\
&= (y^{-1}, 1, 1, 1)_2 = (x^{-1}, 1, 1, 1, 1, 1, 1, 1)_3 \in K_3,
\end{aligned}$$

and $[y^2, \eta] \cdot [yz, \eta] = (1, x, 1, 1, 1, 1, 1, 1)_3$. As $G$ acts transitively on the levels of $T$, $K_3 \leq G^{(3)}$.

Moreover,

$$\begin{aligned}[y^2, \zeta] &= [(x^2, 1)_1, (x, 1, 1, 1, 1, 1, 1, 1)_3] = [[(ca)^2, (bd^a)^2], 1, 1, 1] \\ &= (z^{-1}x^2, 1, 1, 1, 1, 1, 1, 1)_3 \in K_3 \\ [yz, \zeta] &= [(x, x)_1, (x, 1, 1, 1, 1, 1, 1, 1)_3] = [[ca, (bd^a)^2], 1, 1, 1] \\ &= (x, x, 1, 1, 1, 1, 1, 1)_3 \in K_3. \end{aligned}$$

For the reverse inclusion it remains to show that $K^{(2)} \leq K_3$. Indeed we will show that $K^{(2)} = K_4 \leq K_3$.

$$\begin{aligned} K^{(2)} &= [K^{(1)}, K^{(1)}] = [K_2, K_2] \\ &= [\langle (x, 1, 1, 1)_2, (y, 1, 1, 1)_2 \rangle^G, \langle (x, 1, 1, 1)_2, (y, 1, 1, 1)_2 \rangle^G] \\ &= \langle [(x, 1, 1, 1)_2, (y, 1, 1, 1)_2] \rangle^G, \end{aligned}$$

where

$$\begin{aligned} [(x, 1, 1, 1)_2, (y, 1, 1, 1)_2] &= ([x, y], 1, 1, 1)_2 \\ &= (y, 1, 1, 1, 1, 1, 1, 1)_3 \\ &= (x, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)_4. \end{aligned}$$

Consequently, $K^{(2)} = K_4$, which implies $G^{(3)} \leq K_3$ and the equality $G^{(3)} = K_3$. Now, assume for $n \geq 3$, $G^{(n)} = K_{2n-3}$. We will prove for $n+1$, $G^{(n+1)} = K_{2(n+1)+3}$.

$$\begin{aligned} G^{(n+1)} &= [G^{(n)}, G^{(n)}] = [K_{2n-3}, K_{2n-3}] \\ &= [\langle (x, 1, \ldots, 1)_{2n-3}, (y, 1, \ldots, 1)_{2n-3} \rangle^G, \langle (x, 1, \ldots, 1)_{2n-3}, (y, 1, \ldots, 1)_{2n-3} \rangle^G] \\ &= \langle [(x, 1, \ldots, 1)_{2n-3}, (y, 1, \ldots, 1)_{2n-3}] \rangle^G, \end{aligned}$$

where

$$\begin{aligned} [(x, 1, \ldots, 1)_{2n-3}, (y, 1, \ldots, 1)_{2n-3}] &= ([x, y], 1, \ldots, 1)_{2n-3} \\ &= (y, 1, \ldots, 1)_{2n-2} \\ &= (x, 1, \ldots, 1)_{2n-1} \in K_{2n-1}. \end{aligned}$$

Thus $G^{(n+1)} = K_{2n-1} = K_{2(n+1)-3}$. Hence the proof is completed by induction.

We have $K \trianglelefteq G^{(1)} \trianglelefteq G$. This induces a canonical epimorphism from $G/K$ to $G/G^{(1)}$ with the kernel $G^{(1)}/K$. Then the first isomorphism theorem implies that,

$$\begin{aligned}
\frac{G/K}{G^{(1)}/K} &\cong G/G^{(1)} \qquad \text{and,} \\
[G:G^{(1)}] = |G/G^{(1)}| &= \frac{|G/K|}{|G^{(1)}/K|} \\
&= 2^4/2 = 2^3 \qquad\qquad \text{(from } \textit{Theorem } 2.5.1(1) \text{ and } (2)).
\end{aligned}$$

Now, we get a chain of subgroups with finite indices

$$G^{(2)} \trianglelefteq_2 K_1 \trianglelefteq_{2^2} K \trianglelefteq_2 G^{(1)} \trianglelefteq_{2^3} G.$$

Thus,

$$[G:G^{(2)}] = 2 \cdot 2^2 \cdot 2 \cdot 2^3 = 2^7.$$

For $n \geq 3$, $G^{(n)} = K_{2n-3}$ and observe that $|K_n : K_{n+1}| = 4^{2^n}$. Then one may get the following chain;

$$K_{2n-3} \trianglelefteq_{4^{2^{2n-4}}} K_{(2n-4)} \trianglelefteq_{4^{2^{2n-5}}} \ldots K_3 \trianglelefteq_{4^{2^2}} K_2 \trianglelefteq_{4^2} K_1 \trianglelefteq_4 K \trianglelefteq_{2^4} G.$$

$$\begin{aligned}
[G:G^{(n)}] &= [G:K_{2n-3}] \\
&= 2^4 \cdot 4 \cdot 4^2 \cdot 4^{2^2} \cdots 4^{2^{2n-5}} \cdot 4^{2^{2n-4}} \\
&= 2^4 \cdot 4^{(2^0+2^1+2^2+\cdots 2^{2n-5}+2^{2n-4})} \\
&= 2^4 \cdot 4^{(\frac{(1-2^{2n-3})}{1-2})} \\
&= 2^4 \cdot 4^{(2^{2n-3}-1)} \\
&= 2^4 \cdot 2^{2^{2n-2}-2} \\
&= 2^{2^{2n-2}+2}.
\end{aligned}$$

The theorem is thus proved. $\square$

## 2.6 Word problem for $G$

Let $S = \{a, b, c, d\}$ be the set of generators of $G$. Let $S^*$ denotes the set of all finite words in $S$ including the empty word $\emptyset$, and $S^*_{\text{red}}$ denotes the set of all reduced words of $S^*$. The set $S^*_{\text{red}}$ is in one to one correspondence with the free product $(\mathbb{Z}/2\mathbb{Z}) * V$. For a word $w \in S^*$, $\underline{w}$ denotes the element in $G$ represented by $w$.

A word problem for a group G with respect to a set of relations of its elements is the algorithmic problem of deciding whether two given words $w, w' \in S^*$ represent the same element $\underline{w} = \underline{w'}$ in $G$ (or equivalently the word $w \in S^*$ represents $\underline{w} = 1 \in G$).

Let $|w|_i$ denotes the number of occurrence of the letter $i$ in a word $w \in S^*$, where $i \in \{a, b, c, d\}$. We define a length function, $l$, for any $w \in S^*$ and $g \in G$ as follows,

$$l : S^* \longrightarrow \mathbb{N}$$
$$l(w) = |w|_a + |w|_b + |w|_c + |w|_d,$$

and

$$l : G \longrightarrow \mathbb{N}$$
$$l(g) = \inf\{l(w)|w \in S^*, \underline{w} = g\}.$$

Let $r : S^* \longrightarrow S^*_{\text{red}}$ denotes the reduction map and $S^{*,a\equiv 0}_{\text{red}} := \{w \in S^*_{\text{red}} : |w|_a \text{is even}\}$. The set $S^{*,a\equiv 0}_{\text{red}}$ can be identified with the set of words in $\{b, c, d, aba, aca, ada\}$. Let $\tilde{\phi}_0, \tilde{\phi}_1 : S^{*,a\equiv 0}_{\text{red}} \longrightarrow S^*$ be two maps defined recursively by

$$\begin{aligned} \tilde{\phi}_0(b) &= a & \qquad \tilde{\phi}_1(b) &= c \\ \tilde{\phi}_0(c) &= a & \tilde{\phi}_1(c) &= d \\ \tilde{\phi}_0(d) &= \emptyset & \tilde{\phi}_1(d) &= a \\ \tilde{\phi}_0(aba) &= d & \tilde{\phi}_1(aba) &= a \\ \tilde{\phi}_0(aca) &= d & \tilde{\phi}_1(aca) &= d \\ \tilde{\phi}_0(ada) &= b & \tilde{\phi}_1(ada) &= \emptyset \end{aligned}$$

and $\tilde{\phi}_j(uv) = \tilde{\phi}_j(u)\tilde{\phi}_j(v)$ for $u \in \{b, c, d, aba, aca, ada\}$, and $v$ is a word in $\{b, c, d, aba, aca, ada\}$, where $|v| < |uv|$.

The following observations can be easily verified.

**Observations:**

1. $\underline{w} \in St_G(1) \Longleftrightarrow w \in S^{*,a\equiv 0}_{\text{red}}$.

2. For $w \in S^{*,a\equiv 0}_{\text{red}}$, we have
   $\underline{\tilde{\phi}_0(w)} = \phi_0(\underline{w}) \quad$ and $\quad \underline{\tilde{\phi}_1(w)} = \phi_1(\underline{w})$.

3. For $w \in S^*_{\text{red}}$, we have
   $|w|_a$ is even and $\underline{\tilde{\phi}_j(w)} = 1 \in G$ for $j \in \{0, 1\} \Longleftrightarrow \underline{w} = 1 \in G$.

4. For $\emptyset \neq w \in S^{*,a\equiv 0}_{\text{red}}$, set $w_j = r(\tilde{\phi_j}(w)), g = \underline{w}$ and $g_j = \phi_j(g)(= \underline{w_j})$ for $j = 0, 1$.

   (a) We have $l(w_j) \leq \frac{l(w)+1}{2}$.

   (b) If $w$ starts with $a$ and ends with some $u \in \{b, c, d\}$, then $l(w_j) \leq \frac{l(w)}{2}$.

   (c) $l(g_j) \leq \frac{l(g)+1}{2}$.

   (d) If $g$ is not a conjugate to an element of shorter length in $G$, then $l(g_j) \leq \frac{l(g)}{2}$.

For any $w \in S^*$ we develop an algorithm to decide whether $\underline{w}$ represents 1 of $G$ or not as follows:

1. Compute $|w|_a$,

   If $|w|_a$ is odd then $\underline{w} \neq G$.

   If $|w|_a$ is even, compute $r(w)$,

   if $r(w)$ is empty then $\underline{w} = 1 \in G$,

   if $l(r(w)) \geq 1$ go to next step.

2. Compute $w_j = r(\phi_i(w))$ for $j = \{0, 1\}$ and return to step (1) and repeat the algorithm for $w_j$.

It is clear from the observations above that the algorithm terminates. Thus the word problem for $G$ is solvable.

## 2.7 Presentation of $G$

Let $\pi : (\mathbb{Z}/2\mathbb{Z}) * V \longrightarrow G$ defines an epimorphism by mapping $w \in (\mathbb{Z}/2\mathbb{Z}) * V$ to $\underline{w} \in G$, as the set $S^*_{\text{red}}$ is in one to one correspondence with $(\mathbb{Z}/2\mathbb{Z}) * V$.

For each $w \in S^*_{\text{red}}$, $w_{\underline{j}}$ with $\underline{j} = j_1, \ldots, j_k$ where $j_i \in \{0, 1\}$ defines a set of reduced words as follows

$$w_\emptyset = w$$

$$w_{j_1} = \begin{cases} r(\tilde{\phi}_{j_1}(w)) \text{ if } w \in S^{*,a\equiv 0}_{\text{red}} \\ \text{is not defined otherwise} \end{cases}$$

$$\ldots$$

$$w_{j_1,\ldots,j_{k-1},j_k} = \begin{cases} r(\tilde{\phi}_{j_k}(w_{j_1,\ldots,j_{k-1}})) \text{ if } w_{j_1,\ldots,j_{k-1}} \text{ is defined and } w_{j_1,\ldots,j_{k-1}} \in S^{*,a\equiv 0}_{\text{red}} \\ \text{is not defined otherwise .} \end{cases}$$

Let $K_n := \{w \in S^*_{\text{red}} : w_{j_1,\ldots,j_n} = \emptyset, \quad \forall \quad j_1,\ldots,j_n \in \{0,1\}\}$, for each $n \geq 1$. Clearly, $K_n \leq ker(\pi)$ for each $n \geq 1$.

**Lemma 2.7.1** *For each $n \geq 0$, the set $K_n$ is a normal subgroup of $(\mathbb{Z}/2\mathbb{Z}) * V$ and*

$$K_0 := \{1\} \leq K_1 \leq \cdots \leq K_n \leq K_{n+1} \leq \cdots \leq \bigcup_{n=0}^{\infty} K_n = ker(\pi).$$

*Proof:* Set $F = (\mathbb{Z}/2\mathbb{Z}) * V$.

It is clear from the definition that $K_n \leq K_{n+1}$ for $n \geq 0$. We will prove the lemma in the following three steps.

1. For each $n \geq 0, K_n$ is a subgroup of $F$.

   The proof proceeds by induction. For $n = 0, K_0 = \{1\}$ is a subgroup of $F$. Assume that $K_{n-1}$ is a subgroup of $F$. We identified $K_n$ with a subset of $F$ as,

   $$K_n = \{w \in F : |w|_a \text{ is even and } w_0, w_1 \in K_{n-1}\},$$

   where $w_j = r(\tilde{\phi}_j(w))$ for $j = \{0,1\}$.

   Let $w \in K_n$ with $w_0, w_1 \in K_{n-1}$. Then $w_0^{-1}, w_1^{-1} \in K_{n-1}$ as $K_{n-1}$ is a subgroup of $G$. Observe that $w_0^{-1} = (w^{-1})_0$ and $w_1^{-1} = (w^{-1})_1 \Rightarrow w^{-1} \in K_n$.

   For any $w, w' \in K_n$, we have $w_0, w_1, w'_0, w'_1 \in K_{n-1}$. Hence $ww' \in K_n$ since $w_0w'_0, w_1w'_1 \in K_{n-1}$. Thus by induction, we get $K_n$ as a subgroup of $F$ for each $n \geq 0$.

2. For each $n \geq 0, K_n$ is a normal subgroup of $F$.

   The proof proceeds by induction. For $n = 0, K_0 = \{1\} \lhd F$. Assume that $K_{n-1}$ is a normal subgroup of $F$ and we have

   $$K_n = \{w \in F : |w|_a \text{ is even and } w_0, w_1 \in K_{n-1}\}.$$

For any $w \in K_n$ we will show that conjugates of $w$ by the set of generators $\{a, b, c, d\}$ of $F$ is in $K_n$. This gives that the conjugate of $w \in K_n$ by any word in $\{a, b, c, d\}$ is in $K_n$. Thus $K_n \trianglelefteq F$.

For $w \in K_n$ , we will compute $|awa|_a$, $|bwb|_b$, $|cwc|_c$ and $|dwd|_a$.

Note that for any $w \in K_n$, $|w|_a$ is even.

We get

$$|awa|_a \equiv |w|_a (\text{mod } 2),$$

$$|bwb|_b = |cwc|_c = |dwd|_a = |w|_a$$

and

$$\begin{aligned} (awa)_0 &= w_1 & (awa)_1 &= w_0 \\ (bwb)_0 &= aw_0a & (bwb)_1 &= cw_1c \\ (cwc)_0 &= aw_0a & (cwc)_1 &= dw_1d \\ (dwd)_0 &= w_0 & (dwd)_1 &= bw_1b. \end{aligned}$$

It follows from the induction hypothesis that $awa, bwb, cwc, dwd \in K_n$. Hence the result follows.

3. $\bigcup_{n=0}^{\infty} K_n = ker(\pi)$.

   Clearly, $\bigcup_{n=0}^{\infty} K_n \leq ker(\pi)$. Let $w \in ker(\pi)$. Then $\underline{w} = 1 \in G$. By the algorithm developed for the word problem of $G$ above there exists $n \geq 1$ such that $w_{j_1,\ldots,j_n} = \emptyset$. Thus $w \in K_n$ and $ker(\pi) \leq \bigcup_{n=0}^{\infty} K_n$. Hence the equality follows.

Thus the proof is completed. $\square$

**Lemma 2.7.2** *For each $n \geq 0$, $K_n$ is strictly contained in $K_{n+1}$.*

*Proof:* Let $\tilde{\sigma}$ be a function defined on the set of generators $\{a, b, c, d\}$ of $S^*_{\text{red}}$ as,

$$\tilde{\sigma}(a) = aca, \quad \tilde{\sigma}(b) = d, \quad \tilde{\sigma}(c) = b, \quad \tilde{\sigma}(d) = c.$$

This will extends as a transformation $\tilde{\sigma} : S^*_{\text{red}} \longrightarrow S^*_{\text{red}}$ of the set of all reduced words. Observe that,

1. The image of $\tilde{\phi}_0 \circ \tilde{\sigma}$ in $S^*_{\text{red}}$ is in bijection with $\mathbb{Z}_2 * \mathbb{Z}_2$ as

$$\tilde{\phi}_0 \circ \tilde{\sigma}(a) = d, \quad \tilde{\phi}_0 \circ \tilde{\sigma}(b) = 1, \quad \tilde{\phi}_0 \circ \tilde{\sigma}(c) = a, \quad \tilde{\phi}_0 \circ \tilde{\sigma}(d) = a.$$

2. The image of $\tilde{\phi_1} \circ \tilde{\sigma}$ is the identity map as,

$$\tilde{\phi_1} \circ \tilde{\sigma}(a) = a, \quad \tilde{\phi_1} \circ \tilde{\sigma}(b) = b, \quad \tilde{\phi_1} \circ \tilde{\sigma}(c) = c, \quad \tilde{\phi_1} \circ \tilde{\sigma}(d) = d.$$

Let $\tilde{\sigma}^n$ denotes the $n$th iterate of $\tilde{\sigma}$. For all $n \geq 0$, we will prove that $\tilde{\sigma}^n((ad)^4)$ is in $K_{n+1}$ and not in $K_n$ (for each $n \geq 0, K_n$ is viewed as a subset of $S^*_{\text{red}}$).
For $n = 0$, consider $(ad)^4 = (b^2, b^2)_1 = (1,1)_1$, which is contained in $K_1$ and not in $K_0$.
For $n = 1$, consider $\tilde{\sigma}((ad)^4) = (acac)^4 = (da, ad)_1^4$, which is contained in $K_2$ and not in $K_1$.
For $n = 2$, consider $\tilde{\sigma}^2((ad)^4)$,

$$\tilde{\sigma}^2((ad)^4) = \tilde{\sigma}((acac)^4) = (acab)^8 = (da, ac)_1^8 = (1, 1, (da)^4, (ad)^4)_2 = (1, 1, 1, 1, 1, 1, 1, 1)_3.$$

Thus $\tilde{\sigma}^2((ad)^4) \in K_3$ and not in $K_2$.
Assume that $n \geq 3$ and the hypothesis is valid upto $n - 1$.
Set $w = \tilde{\sigma}^n((ad)^4)$. We use the algorithm developed for solving word problem of $G$ to compute $w$. By the definition of $\tilde{\sigma}$, $|w|_a$ is a multiple of 8. Since $|w|_a$ is even, go to the second step of the algorithm and compute $w_j = r(\tilde{\phi_j}(w))$ for $j = 0, 1$. It can be seen from the observation (1) above that $w_0$ is a reduced word in $a$ and $d$ of exponent 4. Hence $\underline{w_0} \in \langle a, d \rangle$, which of exponent 4 in $G$. We get $\underline{w_0} = 1$. Observation (2) provides $w_1 = r(\tilde{\sigma}^{n-1}(ad)^4)$. By induction hypothesis, $w_1$ is contained in $K_n$ and not in $K_{n-1}$. Hence we get $w \in K_{n+1}$ and $w \notin K_n$. Since we can identify the subset $K_n$ with the subgroup $K_n$ of $F$, the proof is completed by induction. □

**Remark 2.7.3** The containments in the lemma Lemma 2.7.1 are strict.

**Theorem 2.7.4** *The group $G$ is not finitely presentable.*

*Proof:* Suppose $G$ has a finite presentation. That is $G$ can be presented by the finite set of generators $\{a, b, c, d\}$ and a finite set of relators $\{r_1, r_2, \ldots, r_k\}$,

$$G = \Big\langle a, b, c, d, |a^2 = b^2 = c^2 = d^2 = bcd = r_1 = r_2 = \cdots = r_k = 1 \Big\rangle.$$

But this would be a contradiction to Lemma 2.7.1 and Lemma 2.7.2. Thus $G$ does not have a finite presentation. □

The group $G$ has a finite recursive presentation of the form

$$G = \left\langle a, b, c, d, | a^2 = b^2 = c^2 = d^2 = bcd = 1, w_n^4 = (w_n w_{n+1})^4 = 1 (n \geq 0) \right\rangle$$

where $(w_n)_{n \geq 0}$ is a sequence of words defines as $w_0 = ad$ and $w_{n+1} = \tilde{\sigma}(w_n)$ (refer [Lys85]).

# Chapter 3

# Branch groups

In this chapter, we state the two equivalent definitions of branch group. Here, we explicitly discuss the classification of just infinite group by John Wilson and [Wil00] R. I. Grigorchuk [Gri00]. We will see how this work of Wilson is generalized by P. D. Hardy to derive the structure theory of branch group [Har02].

**Definition 3.0.1 (R. I. Grigorchuk, [Gri00])** The group $G$ is a *branch group* if it has trivial centre and contains descending chains of subgroups $\{H_n\}_{n=1}^{\infty}, \{L_n\}_{n=1}^{\infty}$ where each $L_n$ is a subgroup of $H_n$ and such that the following hold:

1. $H_n$ is normal in $G$, the index $|G : H_n|$ is finite and $\bigcap_{n=1}^{\infty} H_n = \{1\}$.
2. There is a sequence $\{N_n\}_{n=1}^{\infty}$ of natural numbers such that $N_n$ divides $N_{n+1}$, and there are subgroups $L_n^{(1)}, \dots, L_n^{(N_n)}$ of $G$, each isomorphic to $L_n = L_n^{(1)}$, such that $H_n$ can be represented as the direct product
$$H = L_n^{(1)} \times \cdots \times L_n^{(N_n)} \tag{3.1}$$
in such a way that the product decomposition (3.1) of $H_{n+l}$ refines that of $H_n$; that is, each factor $L_n^{(i)}$ from (3.1) contains a product of $m_{n+1} = N_{n+1}/N_n$ factors $L_{n+1}^{(j)}, (i-1)m_{n+1} + 1 \leq j \leq im_{n+1}$ from the corresponding decomposition of $H_{n+l}$.
3. When $G$ acts on itself by conjugation, for each $n = 1, 2, \dots$ the factors in (3.1) are permuted transitively among themselves.

**Definition 3.0.2 (R. I. Grigorchuk, [Gri00])** A faithful action of a group $G$ on a rooted tree $T = T_{\bar{m}}$ is said to be a *branch action* if $G$ satisfies the following conditions:

1. The action of $G$ on $T$ is level-transitive,
2. $|G : RiSt_G(n)| < \infty$, for all $n \geq 1$.

A group $G$ is said to be a *branch group* if there is a branch action of $G$ on some rooted tree $T$.

**Remark 3.0.3** Definition 3.0.1 and Definition 3.0.2 are equivalent.

**Example 3.0.4** The first Grigorchuk group is a branch group. It follows from Proposition 2.3.3 that each subgroup $St_G(n)$ of $G$ contains $2^n$ copies of the subgroup $K = \langle (ab)^2 \rangle^G$, for all $n \geq 1$. We have seen that $St_G(n)$ are normal subgroups of finite index in $G$, and $\bigcap_{n=1}^{\infty} St_G(n) = 1$. Therefore by Definition 3.0.1, Grigorchuk group is a branch group.

One may refer to [BGŠ03] for more examples of branch groups.

## 3.1 Just infinite branch groups

A group $G$ is said to be *just infinite* if $G$ is infinite and all of its proper quotients are finite. A residually finite group $G$ is *hereditarily just infinite* if every normal subgroup of finite index is just infinite [see [Gri00]].
The main theorem of this section is the ternary classification of the just infinite groups by John Wilson and Grigorchuk. The structure theory of just infinite group was developed by J. S. Wilson [see [Wil71], [Wil00]]. According to his work the just infinite group splits into two classes. The trichotomy of just infinite groups were later proved by Grigorchuk in the paper [Gri00]. The notations, definitions and results we follow here are based on the paper [Wil00].

### 3.1.1 Subnormal subgroups

Let $G$ be a group and $T$ be a subset of $G$. For a subgroup $H$ of $G$ we denote $H^T$ for the subgroup $\langle H^t | t \in T \rangle$. We use the expression $core_K(H)$ to denote the subgroup $\bigcap(H^t | t \in T)$.

A subgroup $H$ of $G$ is said to be *subnormal* ($H$ sn $G$) if there exists a finite chain of subgroups of the form

$$H = G_n \lhd G_{n-1} \lhd \cdots \lhd G_1 \lhd G_0 = G. \tag{3.2}$$

We write $H \lhd^n G$ and least such $n$ is called the *defect* of $H$ in $G$. We recursively define the terms of a *normal closure series* for the subgroup $H$ in $G$ as $H_0 = G$ and $H_{r+1} = H^{H_r}$ for $r \geq 1$. Clearly, $H_{r+1} \lhd H_r$. It can be easily observed that if the chain (3.2) exists then $H_r \leq G_r$ for each $r$ and $H$ is subnormal in $G$ if and only if $H_n = H$.

The next result is by Wielandt [see chapter 1 of [Sto87]].

**Lemma 3.1.1** *Let $H, K$ sn $G$ and suppose that $H \lhd \langle H, K\rangle$. Then $\langle H, K\rangle$ sn $G$.*

*Proof:* Let $H \lhd^n G$ and

$$H = H_n \lhd H_{n-1} \lhd \cdots \lhd H_1 \lhd H_0 = G \tag{3.3}$$

be the normal closure series of $H$ in $G$. We claim that $K$ normalizes each term of the normal closure series (3.3). Clearly, $K$ normalizes $G$ and observe that $K$ normalizes $H$ since $H \lhd \langle H, K\rangle$. Thus for each $r \geq 1$, we get

$$H_r^K = (H^{H_{r-1}})^K = (H^K)^{H_{r-1}} = (H^{H_{r-1}}) = H_r.$$

Hence for each $r \geq 1$, $\langle H_r, K\rangle = H_r K$ and $H_{r+1} \lhd H_r K$. Since $K$ sn $H_r K$ we get $H_{r+1}K$ sn $H_r K$. Consequently,

$$\langle H, K\rangle = H_n K \text{ sn } \cdots H_1 K \text{ sn } H_0 K = G.$$

$\square$

**Lemma 3.1.2** *Suppose that each subnormal subgroup of $G$ has only finitely many conjugates in $G$. If $H, K$ sn $G$ then $\langle H, K\rangle$ sn $G$.*

*Proof:* Let $J = \langle H, K\rangle$ and

$$H = H_n \lhd \cdots \lhd H_1 \lhd J$$

be the normal closure series of $H$ in $J$. The proof proceeds by induction on the defect $n$ of $H$ in $J$. If $n = 0$ then $H = J$ and $J$ sn $G$. If $n = 1$ then $H \lhd J$ and the result follows from Lemma 3.1.1. Assume that $n \geq 2$. Since $H$ has only finitely many conjugates in $G$, $H$ has only finitely many conjugates in $H_{n-2}$. As $H \lhd H_{n-1}$ each of these conjugates is normal in $H_{n-1}$. Thus $H_{n-1}$ is the product of finitely many conjugates of $H$ in $H_{n-2}$. Then it follows from finitely many application of Lemma 3.1.1 that $H_{n-1}$ sn $G$. Hence proof is completed by induction. $\square$

One may refer to the paper [Sto87] for more details of subnormal subgroups.

### 3.1.2 $B$-closed and $N_0$-closed classes

Let $\mathcal{X}$ be a class of groups. As follows from [Wil00] the class $\mathcal{X}$ of groups is said to be $B$-closed if whenever $H, K$ are normal subgroups of their join and both $H \cap K$ and $HK$ are in $\mathcal{X}$, then $H \in \mathcal{X}$. We recursively define a set of classes of groups by setting $\mathcal{X}^0 = \mathcal{X}$ and $\mathcal{X}^{(n+1)} = (\mathcal{X}^{(n)})'$ for $n \geq 0$, where $\mathcal{X}'$ is defined to be the class of groups all of whose normal subgroups are in $\mathcal{X}$.
All the results we state here are from [Wil00] except Theorem 3.1.8 which is proved in [Wil70].

**Lemma 3.1.3** *If $\mathcal{X}$ is $B$-closed, then so is $\mathcal{X}^{(n)}$ for each $n \geq 0$.*

*Proof:* As $\mathcal{X}^{(n)}$ are defined recursively it suffices to show that $\mathcal{X}'$ is $B$-closed. Assume that $H, K \lhd \langle H, K\rangle$ and both $H \cap K$ and $HK$ are in $\mathcal{X}'$. Let $L \lhd H$. We will show that $L \in \mathcal{X}$. Set $L_1 = L^K$. Since $HK = KH$, we get

$$L_1^H = (L^K)^H = (L^H)^K = L^K = L_1.$$

Therefore $L_1 \lhd HK$. Consequently, $L_1 \in \mathcal{X}$ as $HK \in \mathcal{X}'$. Similarly, $L \cap K \in \mathcal{X}$ as $L \cap K \lhd H \cap K \in \mathcal{X}'$. Now, consider the subgroups $L(L_1 \cap K)$ and $L \cap (L_1 \cap K)$. As $L \leq L_1 \leq LK$, we have $L(L_1 \cap K) = L_1 \cap LK = L_1$ and $L \cap (L_1 \cap K) = (L \cap L_1) \cap K = L \cap K$. Observe that $L_1$ and $L \cap K$ are in $\mathcal{X}$, and, $L$ and $L_1 \cap K$ are normal in $L_1 = \langle L, L_1 \cap K\rangle$. Since $\mathcal{X}$ is $B$-closed $L \in \mathcal{X}$. Hence we can show that all normal subgroups of $H$ are in $\mathcal{X}$. That is $H \in \mathcal{X}'$ so that $\mathcal{X}'$ is $B$-closed. $\square$

**Lemma 3.1.4** *The class of just infinite group is $B$-closed.*

*Proof:* Let $\mathcal{X}$ be the class of just infinite groups. Assume that $H, K \lhd \langle H, K\rangle$ and $HK$ and $H \cap K$ are just infinite, so that $H \cap K \lhd_f HK$. Let $1 \neq L \lhd H$ and suppose

$L \cap K = 1$. We have $|LK : K| \leq |HK : K| < \infty$. By second isomorphism theorem,

$$\frac{LK}{K} \cong \frac{L}{L \cap K} \cong L.$$

Since $L \lhd H$ and $|HK : H| < \infty$ $L$ has only finitely many conjugates in $HK$ and all of them are normal in $H$. Set $L_1$ as the join of all conjugates of $L$ in $HK$. Then $L_1$ is a finite normal subgroup of $HK$. This is a contradiction to the just infiniteness of $HK$. Thus $L \cap K \neq 1$. As $L \lhd H$ and $H \cap K$ is just infinite, we get $|H \cap K : L \cap K| < \infty$. This gives $|H : L| < \infty$ and hence $H \in \mathcal{X}$. So, $\mathcal{X}$ is $B$-closed. □

Let $H \lhd^2 G$ and let $T$ be a subgroup of $G$. Then $H^t \lhd H^G$ for each $t \in T$, and the group $H^T$ is the product of these subgroups. We use the following result of [Wil70] to prove Proposition 3.1.6

**Lemma 3.1.5** *Let $H \lhd^2 G$ and let $T'$ be any set of elements of $G$. Write $T = T' \cup \{1\}$ and $H^* = H \cap (H^{T'})$. Then $(H^{*T'}H)(H^{T'}) = H^T$, and $(H^{*T'}H) \cap (H^{T'}) = H^{*T}$ .*

**Proposition 3.1.6** *Suppose that $\mathcal{X}$ is $B$-closed. Let $G \in \mathcal{X}'$ and $G_0 \lhd_f G$. Then $G_0 \in \mathcal{X}'$.*

*Proof:* Let $1 \neq H \lhd G_0$ and $S$ be a transversal to $G_0$ in $G$. Then $H^S \lhd G$ and thus $H^S \in \mathcal{X}$. Choose a non-empty subset $T$ of smallest cardinality such that $H^T \in \mathcal{X}$ for all $H \lhd G_0$. We replace $T$ by $Tt_0^{-1}$ for some $t_0^{-1} \in T$ and assume that $1 \in T$. Set $T' = T \setminus \{1\}$ and $H^* = H \cap H^{T'}$. Then $H^T \in \mathcal{X}$ and $H^{*T} \in \mathcal{X}$. It follows from Lemma 3.1.5 and the fact that $\mathcal{X}$ is $B$-closed, $H^{T'} \in \mathcal{X}$ for all $H \lhd G_0$. This contradicts the minimality of $T$. Hence $T' = T \setminus \{1\}$ is empty, so that $T = \{1\}$. Therefore, for all $H \lhd G_0$, $H \lhd G$. Thus $G_0 \in \mathcal{X}'$. □

**Corollary 3.1.7** *Let $\mathcal{X}$ be a $B$-closed class. If $G$ is just infinite and every normal subgroup of $G$ is in $\mathcal{X}$, then every subnormal subgroup of $G$ is in $\mathcal{X}$.*

*Proof:* Given that $G$ is just infinite and $G \in \mathcal{X}'$. Assume that $G \in \mathcal{X}^{(n)}$ for some $n \geq 1$. It is clear from Lemma 3.1.3 that $\mathcal{X}^{(n-1)}$ is $B$-closed. Since $\mathcal{X}^{(n)} = (\mathcal{X}^{(n-1)})'$ and $G$ is just infinite, by Proposition 3.1.6 we get each normal subgroup of $G$ is in $\mathcal{X}^{(n)}$. Therefore $G \in \mathcal{X}^{(n+1)}$, which means all subnormal subgroups of $G$ of defect $n+1$ are in $\mathcal{X}$. Now, the result follows by induction. □

Let $G$ be a group. We write $G$ satisfies max-$n$, if $G$ satisfies the maximal condition for normal subgroups.

**Theorem 3.1.8 (J. S. Wilson, [Wil70])** *Let $G$ be a group. If $G$ satisfies max-$n$, then so is each normal subgroup $G_0$ of finite index.*

*Proof:* Let $(H_i)$ be an ascending chain of normal subgroups of $G_0$. Suppose that $G_0$ does not satisfy the maximal condition for normal subgroups. Since $G_0 \lhd_f G$, we can choose a finite set $T$ of elements of $G$ such that $H_i^T \lhd G$ for each normal subgroup $H_i$ of $G_0$. As $G$ satisfies max-$n$, $(H_i^T)$ stabilizes after finite length. Pick a subset $T$ with minimal cardinality such that for any ascending chain $(H_i)$ of normal subgroups of $G_0$, the chain $(H_i^T)$ stabilizes after finite length. We may assume that $1 \in T$ and by the ongoing assumption on $G_0$, the subset $T' = T \setminus \{1\}$ is non-empty. Set $H_i^* = H_i \cap H_i^{T'}$. Then $(H_i^*)$ is an ascending chain of normal subgroups of $G_0$ and by the definition of $T$ the chain $(H_i^{*T})$ stabilizes after finite length. By the application of Lemma 3.1.5 shows that $(H_i^{T'})$ stabilizes after finite length. This contradicts the minimality of $T$ and thus $G_0$ satisfies max-$n$. $\square$

**Theorem 3.1.9** *Let $G$ be a just infinite group. Then every subnormal subgroup of $G$ satisfies max-$n$, and for each $n$ the group $G$ satisfies the maximal condition on subgroups $K$ satisfying $K \lhd^n G$.*

*Proof:* Let $\mathcal{X}$ be the class of groups satisfy max-$n$. It can be seen that $\mathcal{X}$ is $B$-closed. Assume that $G$ is just infinite. Clearly, $G \in \mathcal{X}$ and it follows from Theorem 3.1.8 that all normal subgroups of $G$ are in $\mathcal{X}$. Then by Corollary 3.1.7 all subnormal subgroups of $G$ are in $\mathcal{X}$. Therefore every subnormal subgroup of $G$ satisfies max-$n$. Now, the second assertion of the theorem is easily followed from this. $\square$

As defined in [Wil00] we say a class of groups $\mathcal{X}$ is $N_0$ closed if $HK \in \mathcal{X}$, whenever $H, K$ are normal subgroups of a group and $H, K \in \mathcal{X}$.

**Lemma 3.1.10** *Let $G$ be a just infinite group, let $H$ sn $G$ and let $\mathcal{X}$ be an $N_0$-closed class containing $H$. Then the normal subgroup of $G$ generated by $H$ is in $\mathcal{X}$.*

*Proof:* Assume that the defect of $H$ in $G$ is $n$ and

$$H = H_n \lhd \cdots \lhd H_1 \lhd H_0 = G \tag{3.4}$$

be the normal closure series of $H$ in $G$. We will prove the result by induction on $n$. For $n = 0$ and $n = 1$ the results trivially holds. Let $n \geq 2$. Assume that for all subgroups $K$ of $G$ of defect less than or equal to $n-1$ the hypothesis holds. Let $g \in H_{n-2}$ of the chain (3.4). Then $H^g \lhd H_{n-1}$, so that $H \lhd \langle H, H^g \rangle$. We get an ascending chain of normal subgroups of $H_{n-1}$ formed by the product of conjugates of $H$ by elements of $H_{n-2}$. Being a subnormal subgroup of $G$, $H_{n-1}$ satisfies max-$n$, so that the chain of normal subgroups of $H_{n-1}$ get stabilized after finite length. Hence set $U$ as the normal subgroup of $H_{n-1}$ which is maximal subject to being a product of finitely many conjugates of $H$ in $H_{n-2}$. As $\mathcal{X}$ is $N_0$-closed, $U \in \mathcal{X}$. But $H_{n-1} = U$ since $H_{n-1} = H^{H_{n-2}}$ and $U \lhd H_{n-2}$. Thus by induction hypothesis $H_{n-1}^G = H_1 \in \mathcal{X}$, and hence completes the proof. $\square$

**Corollary 3.1.11** *Let $G$ be a just infinite group. Then*

1. *$G$ has no non-trivial finite subnormal subgroups.*

2. *If $G$ has a non-trivial abelian subnormal subgroup, then $G$ has a finitely generated abelian normal subgroup of finite index.*

*Proof:*

(1) Let $\mathcal{X}$ be the class of finite groups. It is easy to see that $\mathcal{X}$ is $N_0$ closed. Assume that $K$ be a non-trivial finite subnormal subgroup of $G$ and $K \in \mathcal{X}$. Then by Lemma 3.1.10, $K^G \in \mathcal{X}$. But $K^G \lhd_f G$, since $G$ is just infinite. This is a contradiction to finiteness of $K^G$. Thus $G$ has no non-trivial finite subnormal subgroups.

(2) Let $\mathcal{X}$ be the class of nilpotent groups. It follows from Fitting's theorem that the class $\mathcal{X}$ is $N_0$ closed. Suppose $H$ is a non-trivial abelian subnormal subgroup of $G$. Clearly, $H \in \mathcal{X}$. By Lemma 3.1.10, $H^G$ is a non-trivial nilpotent subgroup of $G$. Since non-trivial nilpotent group has a non-trivial centre, the centre of $H^G$ is a normal abelian subgroup of $G$ which is finitely generated as $G$ satisfies the maximal condition on normal subgroups. $\square$

Here onwards we assume that $G$ is a just infinite group having no non-trivial abelian subnormal subgroups. For the details of just infinite groups with non-trivial abelian normal subgroups, one can refer to the paper [McC68].

### 3.1.3 Near complements for subnormal subgroups

We are following the results of [Wil00]. Let $G$ be a just infinite group having no non-trivial abelian subnormal subgroup.

**Definition 3.1.12** Let $H$ be a subnormal subgroup of $G$. A *near complement* to $H$ in $G$ is a subnormal subgroup $D$ of $G$ such that

$$\langle H, D\rangle = H \times D \text{ and } |G : H \times D| < \infty.$$

**Lemma 3.1.13** *Let $1 \neq H \lhd^2 G$. Then $H$ has a near complement $D$ in $G$ satisfying $D \lhd H^G$.*

*Proof:* Let $|G : H| < \infty$. Then we may choose $D = 1$ as a near complement to $H$ in $G$. Assume that $|G : H|$ is not finite. As $G$ satisfies the maximal condition for subgroups $M$ such that $M \lhd^2 G$, we may suppose that each subgroup satisfying $H < M$ has a near complement $N$ with $N \lhd M^G$. Since $G$ is just infinite we have $|G : H^G| < \infty$. Hence $H$ has only finitely many conjugates in $G$ and they intersect trivially. Let $K$ be an intersection of conjugates of $H$ such that $K \neq 1$ and $H \cap K = 1$. Thus $\langle H, K\rangle = HK = H \times K$. Since $H < HK \lhd^2 G$, $HK$ has a near complement in $G$. Let $N$ be the near complement of $HK$ in $G$ such that $N \lhd \langle HK\rangle^G = H^G$. That is we have,

$$\langle HK, N\rangle = HK \times N \quad \text{and} \quad |G : HK \times N| < \infty.$$

But $HK = H \times K$, so that

$$\langle H \times K, N\rangle = H \times K \times N = \langle H, KN\rangle \quad \text{and} \quad |G : H \times K \times N| < \infty.$$

Thus $KN$ is a near complement to $H$ in $G$ and $KN \lhd H^G$. $\square$

**Proposition 3.1.14** *Let $H \lhd^n G$ with $n \geq 3$. Then there is a subgroup $K \lhd_f H$ with $K \lhd^2 G$.*

*Proof:* Suppose that defect of $H$ in $G$ is $n$ and let

$$H = H_n \lhd H_{n-1} \lhd \cdots \lhd H_1 \lhd H_0 = G \tag{3.5}$$

be the normal closure series of $H$ in $G$. We proceed by induction on $n$. As $G$ satisfies the maximal condition for subgroups $M$ such that $M \lhd^n G$, we may further assume that each subgroup $M$ such that $H < M$ satisfies the conclusion. We write $H_1 = H^G$ and $H_2 = H^{H_1}$.

Case 1: Let $H_2 \leq_f G$. Set $N = \bigcap_{g \in G} H_2^g$. Then $1 \neq N \lhd_f G$, so that $N \cap H \leq_f H$. Take the intersection of each term in (3.5) with $N$, we get

$$H \cap N = H_n \cap N \lhd H_{n-1} \cap N \lhd \cdots \lhd H_0 \cap N = G \cap N = N. \tag{3.6}$$

Therefore $H \cap N \lhd^{n-2} N \lhd G$, as $N \leq H_2$. By induction hypothesis, there exists a subgroup $L \lhd_f H \cap N$ with $L \lhd^2 G$. Thus we have $L <_f H \cap N \leq_f H$. Take $K = \bigcap_{h \in H} L^h$ and observe that $K \lhd_f H$ and $K \lhd^2 G$.

Case 2: Let $|G : H_2|$ is infinite. It follows from Lemma 3.1.13 that $G$ has a subnormal subgroup $1 \neq D$ such that $D$ is a near complement to $H_2$ and $D \lhd H^G = H_1$. The normal closure series (3.5) gives

$$H \times D \lhd^{n-2} H_2 \times D \lhd H_1 \lhd G. \tag{3.7}$$

Since all the subnormal subgroups of $G$ larger than $H$ satisfies the result we can find a subnormal subgroup $L$ of $G$ such that $L \lhd_f H \times D$ and $L \lhd^2 G$. Now, set $C_L(D)$ as the centralizer of $D$ in $L$. The $C_L(D)$ has the requires properties, hence completes the proof. $\square$

Let $A, B$ be two subgroups of $G$. The commutator of the subgroup $A$ and $B$ is denoted by $[A, B]$, which is defined as

$$[A, B] = \Big\langle [a, b] : a \in A, b \in B \Big\rangle.$$

For $a, a' \in A$ and $b \in B$, we have the commutator identity

$$[aa', b] = [a, b]^{a'} [a', b].$$

Thus we get,

$$[a, b]^{a'} = [aa', b][a', b]^{-1} \in [A, B].$$

Therefore $[A, B]$ is normalized by $A$. Similarly, observe that $[A, B]$ is normalized by $B$.

**Lemma 3.1.15** *Let $K$ sn $G$ and let $|H : K|$ be finite. Then any subnormal subgroup $D$ of $G$ satisfying $[D, K] = 1$ also satisfies $[D, H] = 1$.*

*Proof:* Let $|H : K| = n$. We will prove for $n = 1$, then the result follows from induction. Assume $K \lhd H$.

Let $C = C_G(K)$. Consider $C \cap K$, which is equal to $C_K(K)$. The subgroup $C_K(K)$ is normal in $K$ and $K$ is subnormal in $G$, so that $C \cap K$ is subnormal in $G$. Then by ongoing assumption on $G$ the subgroup $C \cap K$ must be trivial. As $K \lhd_f G$, we get $1 = C \cap K \leq_f C \cap H$ and so that $C \cap H$ is finite. Then by Corollary 3.1.11 $C \cap H$ is also trivial.

Suppose $D$ be a subnormal subgroup of $G$. Since $H$ sn $G$, $H$ is subnormal in $\langle H, D\rangle$ and let

$$H = H_n \lhd H_{n-1} \lhd \cdots \lhd H_1 \lhd H_0 = \langle H, D\rangle \tag{3.8}$$

be the normal closure series of $H$ in $\langle H, D\rangle$. We recursively define a set of commutator subgroups as $\lambda_0 = D$ and $\lambda_r = [\lambda_{r-1}, H]$ for $r \geq 1$. The assumption $[D, K] = 1$ implies that $D \leq C$. Observe that $C$ is normalized by $H$, as $C = C_G(K) \lhd N_G(K)$ and $H \leq N_G(H)$. Hence we get $\lambda_1 = [D, H] \leq [C, H] \leq C$. Another observation shows that $\lambda_1 = [D, H] \lhd \langle H, D\rangle$. Consequently,

$$\lambda_2 = [\lambda_1, H] = [[D, H], H] \leq [D, H] = \lambda_1.$$

Now, it follows from an easy induction that $\lambda_{r+1} \leq \lambda_r$ for each $r \geq 1$.
We have, $H_1 = H^{H_0} = H^{\langle H,D\rangle} = H[D, H] = H\lambda_1$. Assume that $H_r = H\lambda_r$ for some $r \geq 1$, then

$$H_{r+1} = H^{H_r} = H^{H\lambda_r} = H^{\lambda_r} = H[\lambda_r, H] = H\lambda_{r+1}.$$

Then by induction, $H_r = H\lambda_r$ for each $r \geq 1$. In particular, $H_n = H\lambda_n$. But $H = H_n$, so that $\lambda_n \leq H$. Thus $\lambda_n \leq C \cap H = 1$.

Suppose that $n \geq 2$ and set $F = \lambda_{n-1}$. Then $[F, H] = [\lambda_{n-1}, H] = \lambda_n = 1$. Observe that

$$[[F, H], \lambda_{n-2}] = [\lambda_n, \lambda_{n-2}] = 1,$$
$$[[F, \lambda_{n-2}], H] = [[\lambda_{n-1}, \lambda_{n-2}], H] \leq [\lambda_{n-1}, H] = 1.$$

It follows from three subgroup lemma that $[[\lambda_{n-2}, H], F] = [F, F] = 1$, i.e., $F$ is abelian.

Since defect of $H$ in $\langle H, D\rangle$ is $n$, we can find an element $u \in H_{n-2}$ such that $H^u \nleq H$. As $H$ is subnormal in $G$, $H^u$ is subnormal in $G$. Furthermore $H, H^u \lhd \langle H, H^u\rangle$, so that $\langle H, H^u \rangle = HH^u$ sn $G$ by Lemma 3.1.1. We have $F = \lambda_{n-1} = [\lambda_{n-2}, H] \lhd \langle H, \lambda_{n-2} \rangle = H\lambda_{n-2} = H_{n-2}$ and $HH^u \lhd H_{n-2}$. Hence we get $HH^u \cap F \lhd HH^u$. As $HH^u \leq H_{n-1} = H\lambda_{n-1} = HF$, we can write $HH^u = HH^u \cap HF = H(HH^u \cap F)$. Since $H$ is strictly contained in $HH^u$, $HH^u \cap H$ is non-trivial. Thus $HH^u \cap F$ is a non-trivial abelian subnormal subgroup which is a contradiction to the assumption on $G$. This implies $n = 1$ and $H \lhd \langle H, D \rangle$, so that $[D, H] \leq C \cap H = 1$. The proof is thus completed. □

**Corollary 3.1.16** *If $H$ sn $G$ then $H$ has a near complement in $G$.*

*Proof:* Let $H \lhd^n G$. If $n = 1$, then $D = 1$ is a near complement to $H$, as $H \lhd_f G$. For $n = 2$, the result follows from Lemma 3.1.13. Assume that $n \geq 3$. By Proposition 3.1.14 $H$ has a normal subgroup $K$ of finite index such that $K \lhd^2 G$. Then Lemma 3.1.13 implies that $K$ has a near complement $D$ in $G$ with $D \lhd K^G$, i.e.,

$$\langle K, D \rangle = K \times D \quad \text{and} \quad |G : K \times D| < \infty.$$

Thus $[K, D] = 1$ and it follows from Lemma 3.1.15 that $[H, D] = 1$. Observe that

$$\langle H, D \rangle = H \times D \quad \text{and} \quad |G : H \times D| < \infty.$$

Hence $D$ is a near complement to $H$. □

**Corollary 3.1.17**

1. *Each subnormal subgroup $H$ of $G$ has only finitely many conjugates in $G$.*

2. *The join of two subnormal subgroups of $G$ is again subnormal.*

*Proof:*

(1) Let $H$ be subnormal subgroup of $G$ and $D$ be a near complement to $H$ in $G$. We have $|G : H \times D| < \infty$. Thus $H \lhd H \times D \leq_f G$ and so $H$ has only finitely many conjugates in $G$.

(2) Let $H$ and $K$ be two subnormal subgroups of $G$. The first part of this corollary and Lemma 3.1.2 together imply that $\langle H, D \rangle$ sn $G$. □

**Corollary 3.1.18** *Let $H, K$ be two subnormal subgroups of $G$. Then $H \cap K = 1$ if and only if $[H, K] = 1$.*

*Proof:* Suppose that $[H, K] = 1$. Then $H \cap K$ is a non-trivial abelian subnormal subgroup of $G$, which is a contradiction to the ongoing hypothesis on $G$. Thus we get $H \cap K = 1$.

Now, let $H \cap K = 1$. There exists finite index subgroups $H_0$ and $K_0$ in $H$ and $K$ respectively such that $H_0 \lhd^2 G$ and $K_0 \lhd^2 G$. Set $N = H_0^G \cap K_0^G$. If $H_0^G \cap K_0^G$ is trivial set $N$ as the subgroup $\langle H_0^G, K_0^G \rangle^G$. Clearly, $N$ is a normal subgroup of finite index in $G$ and $H_0$ and $K_0$ are normalized by $N$. Now, consider the subgroups $H_0 \cap N$ and $K_0 \cap N$. Replace $H_0$ by $H_0 \cap N$ and $K_0$ by $K_0 \cap N$. Then $H_0, K_0 \lhd N$. Since $H \cap K = 1$, $H_0 \cap K_0 = 1$ and we get $[H_0, K_0] = 1$. As $H_0 \lhd_f H$ sn $G$ and $[H_0, K_0] = 1$, we get $[H, K_0] = 1$ (by Lemma 3.1.15). Again by the application of Lemma 3.1.15 to the subnormal subgroup $K$ we get $[H, K] = 1$. □

**Corollary 3.1.19**

1. *If $H_1, H_2, K$ are subnormal subgroups and $H_1 \cap K = H_2 \cap K = 1$ then $\langle H_1, H_2 \rangle \cap K = 1$.*

2. *If $H_1, \ldots, H_n$ are subnormal subgroups such that $H_i \cap H_j = 1$ for $i \neq j$ then*

$$\langle H_1, \ldots, H_n \rangle = H_1 \times \cdots \times H_n.$$

*Proof:*

(1) By Corollary 3.1.18 $H_1 \cap K =$ if and only if $[H_1, K] = 1$ and $H_2 \cap K = 1$ if and only if $[H_2, K] = 1$. Thus $[\langle H_1, H_2 \rangle, K] = 1$ as $K$ commutes with both $H_1$ and $H_2$. Then by Corollary 3.1.17 and Corollary 3.1.18 we get $\langle H_1, H_2 \rangle \cap K = 1$.

(2) Let $H_1, H_2, H_3$ be subnormal in $G$. Since $H_1 \cap H_2 = 1$ we have $[H_1, H_2] = 1$, and $\langle H_1, H_2 \rangle = H_1 \times H_2$. We also have $H_1 \cap H_3 = 1$ and $H_2 \cap H_3 = 1$. Thus $\langle H_1, H_2 \rangle \cap H_3 = 1$ so that $[\langle H_1, H_2 \rangle, H_3] = 1$. Hence $\langle H_1, H_2, H_3 \rangle = H_1 \times H_2 \times H_3$. Similarly, the result follows for all finite values of $n$. □

### 3.1.4 The classification of just infinite groups

We are now in a position to prove the main theorem of this section: the trichotomy of just infinite groups.

**Theorem 3.1.20 (R. I. Grigorchuk, [Gri00])** *Let $G$ be a just infinite group. Then either $G$ is a branch group, or $G$ contains a normal subgroup of finite index which is* *isomorphic to the direct product of a finite number of copies of a group $L$, where $L$ is either simple or hereditarily just infinite.*

*Proof:* We continue to assume that $G$ is a just infinite group having no non-trivial abelian subnormal subgroups. We recall the following results:

1. Each subnormal subgroup $H$ of $G$ has only finitely many conjugates in $G$ (Corollary 3.1.17)

2. If $H_1, \ldots, H_n$ are subnormal subgroups such that $H_i \cap H_j = 1$ for $i \neq j$ then $\langle H_1, \ldots, H_n \rangle = H_1 \times \cdots \times H_n$ (Corollary 3.1.19).

Assume that $G$ is not hereditarily just infinite and contains no simple normal subgroup of finite index. Then we can find two subgroups $K$ and $Q$ of $G$ such that $K \lhd_\infty Q \lhd_f G$. We may further assume that $G$ acts on its subgroups by conjugation. For the subgroup $K$, $O(K)$ denotes the orbit of $K$ under the action of $G$. By Corollary 3.1.17 $O(K)$ is finite. We choose a subset $R \subseteq O(K)$ such that $R$ is maximal subject to $\bigcap\{K^g : K^g \in R\} \neq 1$. Now, set

$$L_1 = \bigcap_{K^g \in R} K^g.$$

Let $L_1^g, L_1^h$ be two disjoint conjugates of $L_1$ in $G$. Then $L_1^g \cap L_1^h = 1$, otherwise, it is a contradiction to the maximality of $R$. Therefore the intersection of any pairs of disjoint conjugates of $L_1$ is trivial. By Corollary 3.1.19

$$\langle L_1^g : L_1^g \in O(L_1) \rangle = Dr_{L_1^g \in O(L_1)} L_1^g,$$

where $Dr$ stands for the abbreviation of the direct product. Set

$$H_1 = Dr_{L_1^g \in O(L_1)} L_1^g.$$

Clearly, $H_1$ contains more than one term. Otherwise, $L_1$ becomes normal in $G$ and hence $L_1 \lhd_f G$, so that $K \leq_f G$. Observe that $H_1 \lhd_f G$.
We will proceed by induction on $n$ of $H_n$. For some $n \geq 1$ assume that

$$G \rhd_f H_n = L_n^{(1)} \times \cdots \times L_n^{(N_n)},$$

where $L_n^{(i)}$ are the isomorphic copies of a subgroup $L_n$ and $L_n^{(1)}, \ldots, L_n^{(N_n)}$ are permuted transitively under the action of $G$. Then there are three different cases for $L_n$:

(1) $L_n$ has a simple normal subgroup of finite index. In this case, we get one of the conclusions of the theorem.

(2) $L_n$ has a non-trivial normal subgroup $K_n$ of infinite index. In this case, we treat $K_n$ as we did for $K$. Since $K_n$ is subnormal, $O(K_n)$ is finite. We find a subset $R' \subseteq O(K_n)$ such that $R'$ is maximal subject to $\bigcap\{K_n^g : K_n^g \in O(K_n)\} \neq 1$. Set $L_{n+1} = \bigcap_{K_n^g \in R'} K_n^g$, and we get

$$H_{n+1} := \langle L_{n+1}^f : L_{n+1}^f \in O(K_n) = Dr_{L_{n+1}^f \in O(K_n)} L_{n+1}^f.$$

Observe that $H_{n+1} \lhd_f G$. It is also easy to observe that the decomposition of $H_{n+1}$ refines the decomposition of $H_n$. Thus we obtain a strictly decreasing chain $\{H_n\}_{n=1}^{\infty}$ of normal subgroups of finite index in $G$. The subgroup $H_n$ has the required decomposition as in the Definition 3.0.1 and the factors of $H_n$ are permuted transitively under the action of $G$. Furthermore $\bigcap_{n=1}^{\infty} H_n$ is trivial as $G$ is just infinite. Therefore, in this case, $G$ is a branch group.

(3) Suppose that neither case (1) nor case (2) holds. Then $L_n$ contains proper subgroups of finite index, and every such subgroup is just infinite. We now consider the intersection of all finite index subgroups of $G$ and denotes it by $H_*$. Assume that $H_*$ is non-trivial. Clearly, $H_*$ is a normal subgroup of $G$, hence $H_*$ has finite index in $G$. We have $H_n \leq H_*$. Set $M = H_* \cap L_n$. Observe that $1 \neq M \lhd_f L_n$. Thus $M$ is just infinite and there exists a proper subgroup $P$ of $M$ such that $P \lhd_f M$. Thus the subgroup $H_n$ contains the direct product $D := Dr_{1 \leq i \leq N_n} P^{(i)}$, where $P^{(i)}$'s are the distinct conjugates of $P$ under the action of $G$ and $P^{(i)} < L_n^{(i)}$ for each $1 \leq i \leq N_n$. But as $D$ is normal in $G$, $D$ has finite index in $G$, so that $H_* \leq D$. This implies $P = M$, which is a contradiction to the choice of $P$. Hence $H_* = 1$ and $G$ is residually finite. So in this case $G$ is hereditarily just infinite.

Thus the proof is completed. □

### 3.1.5 A criterion for a branch group to be just infinite

We have seen that just infinite groups split into branch groups. But this does not mean that every branch group is just infinite. Here we will prove the theorem by Grigorchuk given in the paper [Gri00], this gives a criterion under which a branch group becomes just infinite. The second part of this theorem is an important result on branch group.

**Theorem 3.1.21 (Grigorchuk, [Gri00])** *Let $G$ be a branch group acting on a rooted tree $T = T_{\bar{m}}$, with branch structure $(\{L_n\}_{n=1}^{\infty}, \{H_n\}_{n=1}^{\infty})$. The group $G$ is just infinite if and only if for each $n \geq 1$, the index of the commutator subgroup $L_n'$ in $L_n$ is finite. Moreover, every non-trivial normal subgroup $P$ of $G$ contains a subgroup $H_n'$ for some $n$.*

*Proof:* Suppose that $G$ is just infinite. Further, assume that commutator subgroup $L_n'$ has infinite index in $L_n$ for some $n$. Then the subgroup $H_n' = L_n' \times \cdots \times L_n'$ is a normal subgroup of infinite index in $G$. This is a contradiction to the just infiniteness of $G$. Thus one direction of the proof is clear.
To prove the reverse case, suppose that $L_n'$ has finite index in $L_n$ for all $n \geq 1$. Let $1 \neq g \in G$ and $P$ denotes the normal closure of the element $g$ in $G$. By Proposition 1.2.2 we can find $n$ such that $g \in St_G(n)$ but $g \notin St_G(n+1)$. Let

$$g = (g_1, \ldots, g_l, \ldots, g_{N_n})_n$$

be the decomposition of $g$. Since $g \notin St_G(n+1)$, at least one factor of $G$ does not belong to the subgroup $St_{\mathrm{Aut}T_{\langle n\rangle}}(1)$. Assume that $g_l$ be the factor of $g$ which is not contained in $St_{\mathrm{Aut}T_{\langle n\rangle}}(1)$. Select the vertex $u$ of length $n$ of the tree $T$ such that $g$ acts as $g_l$ on the subtree $T_u$. Thus put $g_l = g_u$. Then we can write $g_u = ha$, where $h \in St_{\mathrm{Aut}T_{\langle n\rangle}}(1)$ and $a$ is a non-trivial element of the symmetric group of $m_{n+1}$ elements. Since $a$ is non-trivial we can find two distinct letters $x, y$ in the alphabet $X_{n+1}$ such that $a(x) = y$. As defined for the subgroups of Grigorchuk group, we define the homomorphism $\psi_n : St_G(n) \longrightarrow G^{N_n}$ for each $n \geq 1$, where $G^{N_n}$ denotes the $N_n$ copies of the group $G$. Observe that the homomorphism $\psi_n$ is injective and it is surjective on each component of $G$. Therefore for any arbitrary element $\zeta \in L_{n+1}$ we can find an element $f \in St_G(n+1)$ such that

$$f = (1, \ldots, 1, f_u, 1, \ldots, 1)_n, \tag{3.9}$$

$$f_u = (1, \ldots, 1, \zeta, 1, \ldots, 1)_1, \tag{3.10}$$

where $f_u$ is in position $u$ in (3.9) and $\zeta$ is in position $x$ in (3.10). Consider the element $[g, f]$;

$$\begin{aligned}[g, f] &= (1, \ldots, 1, [g_u, f_u], 1, \ldots, 1)_n \\ &= (1, \ldots, 1, a^{-1}h^{-1}f_u^{-1}haf_u, 1, \ldots, 1)_n \\ &= (1, \ldots, 1, \zeta^{-h_{\{x\}}}, 1, \ldots, \zeta, 1, \ldots, 1)_{n+1},\end{aligned}$$

where the element $h_{\{x\}}$ denotes the factor of $h$ in position of $x$ and the elements $\zeta^{-h_{\{x\}}}, \zeta$ are in position of the vertices $uy$ and $ux$ respectively. Furthermore the element $[g, f] \in P$ as $P$ is normal in $G$.
Now consider another elements $\eta \in L_{n+1}$ and $h \in St_G(n+1)$ such that

$$h = (1, \ldots, 1, \eta, 1, \ldots, 1)_{n+1}$$

where $\eta$ is in position of the vertex $ux$. Then,

$$[[g, f], h] = (1, \ldots, 1, [\zeta, \eta], 1, \ldots, 1)_n.$$

Since $\zeta$ and $\eta$ were arbitrary elements of $P$, we get

$$P \geq 1 \times \cdots \times 1 \times L'_{n+1} \times 1 \times \cdots \times 1.$$

As the action of $G$ on each level is transitive we conclude that $P$ contains $N_{n+1}$ factors of the subgroup $L'_{n+1}$, i.e.,

$$P \geq L'_{n+1} \times \cdots \times L'_{n+1} \times \cdots \times L'_{n+1} = H'_{n+1}.$$

It follows from our assumption on the index of $L'_{n+1}$ that $H'_{n+1}$ has finite index in $G$. Thus the normal subgroup $P$ is of finite index in $G$. Hence the result follows. $\square$

## 3.2 Structure theory of branch group

In the paper [Wil00] John Wilson developed a structure lattice (a quotient of lattice of subnormal subgroups) for a just infinite group and characterized just infinite groups with finite and infinite structure lattice. Later, P. D. Hardy in his Ph.D. thesis [Har02] generalized the results of the paper [Wil00] together with results of the paper [GW03] to a more large class of groups.

Let $G$ be a group and $\mathcal{P}$ be a property. The group $G$ is said to be *just non-($\mathcal{P}$)* if all proper quotients of $G$ have the property $\mathcal{P}$, but $G$ does not have the property $\mathcal{P}$. In the paper [GW03] Grigorchuk and Wilson showed that branch groups belong to the class of just non-(virtually abelian) groups having no non-trivial virtually abelian normal subgroups. Hardy used the methods of [Wil00] to this class of groups and derived a structure theory of branch group which is purely based on the internal group-theoretical structure of the group. The aim of this section is to prove the structure theory of branch groups developed in [Har02].

We will first prove the result: every branch group $G$ is a just non-(virtually abelian) group having no non-trivial abelian normal subgroups.

**Theorem 3.2.1 (Grigorchuk, Theorem 4: [Gri00])** *Let $G$ be a branch group acting on a rooted tree $T = T_{\bar{m}}$. Every non-trivial normal subgroup of $G$ contains the commutator subgroup $(RiSt_G(n))'$ for some $n$. Consequently, every proper quotient of $G$ is virtually abelian.*

*Proof:* Let $(\{L_n\}_{n=1}^{\infty}, \{RiSt_G(n)\}_{n=1}^{\infty})$ be a branch structure of $G$, where $L_n$ is the rigid vertex stabilizer of a vertex in the $n$th level of the tree $T_{\bar{m}}$. It follows from Theorem 3.1.21 that each non-trivial normal subgroup of $G$ contains the commutator subgroup $(RiSt_G(n))'$ for some $n$.
Let $P$ be a non-trivial normal subgroup of $G$. Suppose that $(RiSt_G(n))' \leq P$ for some $n$. Observe that

$$\frac{RiSt_G(n) \cdot P}{P} \leq_f \frac{G}{P}.$$

Since

$$\frac{RiSt_G(n) \cdot P}{P} \cong \frac{RiSt_G(n)}{RiSt_G(n) \cap P}$$

and $(RiSt_G(n))' \leq RiSt_G(n) \cap P$, we get $RiSt_G(n) \cdot P/P$ is abelian. Therefore $G/P$ contains an abelian subgroup of finite index. Hence $G/P$ is virtually abelian. $\square$

**Lemma 3.2.2 (Grigorchuk, Wilson, Lemma 2: [GW03])** *Let $G$ be a branch group acting on a rooted tree $T = T_{\bar{m}}$ and let $K$ be a subgroup of $G$. If $K \lhd K^G$ and $K$ is abelian then $K = 1$.*

*Proof:* Suppose that $K \neq 1$. Set $Q = K^G$. Let $1 \neq k \in K$. Then we can find one vertex $u$ of $T$ with length $n$, such that $k(u) \neq u$. Therefore $k \notin RiSt_G(u)$. Choose an element $1 \neq f \in RiSt_G(u) \cap N_G(K)$. Now, consider the elements $k^{-1}, K^f \in K$. For any vertex $v$ of the subtree $T_u$ we have

$$k^f k^{-1}(v) = f^{-1}kfk^{-1}(v) = f^{-1}kk^{-1}(v) = f^{-1}(v),$$

(since $k^{-1}$ does not fix the vertex $v$, $f$ acts trivially on it and thus the third equality follows)

$$k^{-1}k^f(v) = k^{-1}f^{-1}kf(v) = k^{-1}kf(v) = f(v),$$

(since $kf(v) \notin T_u$, $f^{-1}$ acts trivially on it, thus the third equality follows).
Since $K$ is abelian the elements $k^{-1}$ and $k^f$ commute, so that

$$f^{-1}(v) = f(v).$$

As $v$ was arbitrary we get $f^{-1} = f$ for all $f \in RiSt_G(u) \cap N_G(K)$. Therefore the order of any non-trivial element of the group $RiSt_G(u) \cap N_G(K)$ is 2, hence $RiSt_G(u) \cap N_G(K)$ is an elementary abelian 2-group. Observe that the group $RiSt_G(u) \cap Q$ is also elementary abelian 2-group as $Q \leq N_G(K)$. Similarly, we can show that $RiSt_G(g(u)) \cap Q$ is elementary abelian 2-group for all elements $g(u)$ in the orbit of $u$ under the action of $G$. Therefore the group

$$A := \prod_{g(u) \in \mathrm{Orbit(u)}} (RiSt_G(g(u)) \cap Q)$$

is abelian. Then,

$$\begin{aligned} A &= (\prod_{g(u) \in \mathrm{Orbit(u)}} RiSt_G(g(u))) \cap Q \\ &= L \cap Q, \end{aligned}$$

where $L = \prod_{g(u) \in \mathrm{Orbit(u)}} RiSt_G(g(u)) = RiSt_G(n)$.
Suppose that $A$ is non-trivial. That is $G$ has a non-trivial abelian normal subgroup. Let $1 \neq a \in A$. Then we can find a vertex $u'$ of length $m$ in the tree $T$ such that $a(u') \neq u'$. As similar to the above argument for $A$, we get a subgroup

$$A' := (\prod_{g(u') \in \mathrm{Orbit(u')}} RiSt_G(g(u'))) \cap N_G(A),$$

is abelian. But $N_G(A) = G$ and $\prod_{g(u')\in \text{Orbit(u')}} RiSt_G(g(u')) = RiSt_G(m)$, hence we get that the subgroup $RiSt_G(m)$ is abelian. Choose a vertex $v'$ in the subtree $T_{u'}$. Then the subgroup $RiSt_G(v')$ is normal in $RiSt_G(m)$ as $RiSt_G(m)$ is abelian. We now have the condition

$$RiSt_G(v') \lhd RiSt_G(m) \lhd_f G.$$

The subgroup $RiSt_G(v')$ has at most $f$ conjugates in $G$. But $RiSt_G(v')^g = RiSt_G(g(v'))$ for any $g \in G$. This implies the orbit of $v'$ under the action of the group $G$ on the tree $T$ contains at most $f$ elements, which is a contradiction since $G$ acts transitively on each level of the tree $T$. Thus the subgroup $A$ must be trivial.
Assume that $A = \{1\}$. Thus we get,

$$\begin{aligned} \frac{L \cdot Q}{Q} &\cong \frac{L}{L \cap Q} \\ &= L. \end{aligned}$$

By Theorem 3.2.1, $L \cdot Q/Q$ is virtually abelian in $G$, so that $L$ is virtually abelian in $G$. Since $L$ is a normal subgroup of finite index in $G$, we get $G$ itself is virtually abelian. That is not the case because, if $G$ has a non-trivial abelian subgroup then by the same argument of Corollary 3.1.11, $G$ contains a non-trivial normal abelian subgroup. Hence we conclude that $K = 1$. □

**Remark 3.2.3** Let $G$ e a branch group acting on a rooted tree $T = T_{\bar{m}}$. Then the group $G$ is a just non-(virtually abelian) group having no non-trivial virtually abelian normal subgroups (follows from Theorem 3.2.1 and Lemma 3.2.2).

Here onwards we assume that $G$ is a just non-(virtually abelian) group having no non-trivial virtually abelian subgroups, unless otherwise stated. Observe that a just infinite group with no non-trivial virtually abelian normal subgroup belongs to the class of just non-(virtually abelian) groups having no non-trivial virtually abelian subgroups. Thus the results of the paper [Har02] is generalization of the paper [Wil00] and the results of the paper [Har02] apply to just infinite group with no non-trivial virtually abelian normal subgroups.

### 3.2.1 The lattice $\mathbb{L}$

Let $G$ be a just non-(virtually abelian) group having no non-trivial virtually abelian normal subgroups. We will define a lattice $\mathbb{L}$ for the subnormal subgroups of the group $G$. We restate the results of the sections (3.1.1), (3.1.2) and (3.1.3) for the members of the lattice $\mathbb{L}$, without detailed proofs. For further description, one can refer to the paper [Har02].

Let $\mathbb{L} = \mathbb{L}(G)$ denotes the collection of subnormal subgroups of $G$ which have only finitely many conjugates.

**Lemma 3.2.4** *If $H, K \in \mathbb{L}$ then $\langle H, K\rangle \in \mathbb{L}$.*

*Proof:* The result is a direct consequence of Lemma 3.1.2. □

For the subgroups $H, K \in \mathbb{L}$ we define $H \vee K = \langle H, K\rangle$ as the join of the subgroups and $H \wedge K = H \cap K$ as the meet of the subgroups. Then the collection $\mathbb{L}$ becomes a lattice of subnormal subgroups of $G$ with respect to subgroup inclusion.

Let $\mathcal{X}$ be a class of groups. We recall the definition of $N_0$-closed class: if whenever $H, K$ are normal subgroups of a group and $H, K \in \mathcal{X}$ then $HK \in \mathcal{X}$. Then we have the following result.

**Lemma 3.2.5** *If $H \in \mathbb{L}$ and $\mathcal{X}$ is an $N_0$-closed class containing $H$ then $H^G \in \mathcal{X}$.*

*Proof:* Refer Lemma 3.1.10 □

**Proposition 3.2.6** *If $\mathbb{L}$ contains a non-trivial virtually abelian element $H$ then $G$ has a non-trivial virtually abelian normal subgroup.*

*Proof:* The result directly follows from Corollary 3.1.11 as $H$ contains a non-trivial abelian subgroup of finite index. □

Let $H, K \in \mathbb{L}$ and let $K \leq H$. We use the expression $K \leq_{\mathrm{va}} H$ if $K$ contains a commutator group $H_1'$, where $H_1 \leq_f H$. That is $H/K$ is virtually abelian. We write $K \trianglelefteq_{\mathrm{va}} H$ if $K \leq_{\mathrm{va}} H$ and $K \trianglelefteq H$. For a subgroup $K \leq G$, we use the notation $C_G(K)$ for the centralizer of $K$ in $G$.

**Theorem 3.2.7** *Let $H, K \in \mathbb{L}$ with $K \trianglelefteq_{va} H$. Then $C_G(K) = C_G(H)$.*

*Proof:* Write $C = C_G(K)$, and the result follows by replacing the subgroup $D$ by $C$ in the proof of Lemma 3.1.15. $\square$

We now have the following two corollaries which are same as Corollary 3.1.18 and Corollary 3.1.19, the only difference is the assumption about $G$.

**Corollary 3.2.8** *Let $H, K \in \mathbb{L}$. Then $H \cap K = 1$ if and only if $[H, K] = 1$.*

**Corollary 3.2.9** *Let $H_1, \ldots, H_n \in \mathbb{L}$.*

1. *If $K \in \mathbb{L}$ and $H_i \cap K = 1$ for each $i$, then $\langle H_1, \ldots, H_n, K\rangle = H_1 \times \cdots \times H_n \times K$.*
2. *If $H_i \cap H_j = 1$ for all $i \neq j$, then $\langle H_1, \ldots, H_n\rangle = H_1 \times \cdots \times H_n$.*

### 3.2.2 Pseudo-complements

Let $G$ be a just non-(virtually abelian) group having no non-trivial abelian normal subgroups. As defined in [Har02] a *pseudo-complement* for an element $H \in \mathbb{L}$ is a subgroup $D \in \mathbb{L}$ such that $\langle H, D\rangle = H \times D \leq_{\text{va}} G$.

**Proposition 3.2.10** *Each $H \in \mathbb{L}$ has a pseudo-complement.*

*Proof:* Assume that $H = 1$. Then $G$ itself is a pseudo-complement for $H$ in $\mathbb{L}$. Suppose that $H \neq 1$ and $core_G(H) \neq 1$. Since $G$ is just non-(virtually abelian), $core_G(H) \trianglelefteq_{\text{va}} G$. Therefore $H \leq_{\text{va}} G$ as $core_G(H) \leq H$. Then we may choose $1 \in \mathbb{L}$ as a pseudo-complement for $H$ in $G$. Now, consider the case $H \neq 1$ and $core_G(H) = 1$. Thus we can find a subgroup $H_0 \in \mathbb{L}$ such that $H_0$ is the intersection of distinct conjugates of $H$ in $G$ and it is maximal subject to the condition $1 < H_0 \leq H$. Set $M$ as a conjugate of $H_0$ which is not contained in $H$. Clearly, $H_0 \cap M$ is trivial otherwise, we get a contradiction to the maximality of $H_0$. Let $D$ denotes the join of all conjugates of $M$. Observe that $D \in \mathbb{L}$ and $\langle H, D\rangle = H \times D$ (by Corollary 3.2.9). As $D$ is the join of all conjugates of $M$ that are not contained in $H$, $H \times D$ contains all conjugates of $M$ in $G$. That is we have $H \times D \geq M^G \trianglelefteq_{\text{va}} G$. Hence $H \times D \leq_{\text{va}} G$ and $D$ is a pseudo-complement for $H$ in $\mathbb{L}$. $\square$

**Proposition 3.2.11** *Let $H, K \in \mathbb{L}$, with $K \leq H$. The following are equivalent:*

1. *$K \leq_{va} H$,*

2. *Each series $K = K_n \trianglelefteq K_{n-1} \trianglelefteq \cdots \trianglelefteq K_0 = H$ of members of $\mathbb{L}$ has virtually abelian factors,*

3. *There exists $K_i \in \mathbb{L}$ such that $K = K_n \trianglelefteq_{va} K_{n-1} \trianglelefteq_{va} \cdots \trianglelefteq_{va} K_0 = H$.*

*If $H = G$ then these conditions are equivalent to the condition $core_H(K) = 1$.*

*Proof:* For any element $K \in \mathbb{L}$ with $K \leq H$ has only finitely many conjugates in $H$, thus there exists a finite series of the form

$$K = K_n \trianglelefteq K_{n-1} \trianglelefteq \cdots \trianglelefteq K_0 = H. \tag{3.11}$$

Assume that (1) holds, i.e., $K \leq_{\text{va}} H$. That is there exists a subgroup $M$ of finite index in $H$ with $M' \leq K$. Observe that for all $K \in \mathbb{L}$, $N_G(K)$ has finite index in $G$, as $K$ has only finitely many conjugates in $G$. Thus $N_G(K_i) \trianglelefteq_f G$ for each term $K_i$ of the series (3.11). We may choose the subgroup $M$ as the intersection of H with the intersection of all $N_G(K_i)$. Then $M$ has the properties as described above and $M$ normalizes each term of the series (3.11). Thus $K_i \trianglelefteq MK_i$ for each $K_i$. We get $MK_i/K_i \cong M/(M \cap K_i)$ and the quotient $M/(M \cap K_i)$ is abelian as $M' \leq (M \cap K_i)$. Since $K_i \trianglelefteq K_{i-1} \cap MK_i \leq_f K_{i-1}$ and $(K_{i-1} \cap MK_i)/K_i \leq MK_i/Ki$ is abelian, and so $K_{i-1}/K_i$ is virtually abelian. Thus (1) implies (2).
The implication of (2) to (3) is clear from the existence of the series (3.11).
Now assume that (3) holds. There exists a subgroup $D \in \mathbb{L}$, such that $D$ is a pseudo-complement for $K$. Therefore $\langle K, D \rangle = K \times D \leq_{\text{va}} G$ (by Proposition 3.2.10). By repeated application of Theorem 3.2.23 to the series (3.11), we get $C_G(K) = C_G(H)$. This implies $[H, D] = 1$, so that $H \cap D = 1$ by Corollary 3.2.8. Since $\langle K, D \rangle = K \times D \leq_{\text{va}} G$, we can find a subgroup $B$ of finite index in $G$ and $B'$ is contained in $\langle K, D \rangle$. Then

$$(B \cap H)' \leq B' \cap H \leq (K \times D) \cap H = K(D \cap H) = K,$$

(second last equality follows from the modular law: Let $a, b, c \in \mathbb{L}$ then $(a \vee b) \wedge c = a \vee (b \wedge c)$). As $(B \cap H) \leq_f H$, we have $K \leq_{\text{va}} H$. Thus (3) implies (1).
Now consider the case $H = G$. Let $K \leq_{\text{va}} H = G$. Then $K$ contains the commutator subgroup of a subgroup $H_0$ of finite index in $G$. Observe that $1 \neq H_0' \leq core_G(K)$, as $G$ is not virtually abelian and $H_0'$ is normal in $G$. Now suppose that $core_G(K) \neq 1$.

By the ongoing assumption on $G$, we get $G/core_G(K)$ is virtually abelian. That is there exists a abelian subgroup $P/core_G(K)$ of finite index in $G/core_G(K)$. Therefore $P \leq_f G$ and $P' \leq core_G(K) \leq K$. Hence $K \leq_{\mathrm{va}} G$ and the proof is completed. □

**Corollary 3.2.12** *Let $H, K, L \in \mathbb{L}$ and suppose that $K \leq H$.*

1. *$K \leq_{va} H$ if and only if $C_G(H) = C_G(K)$.*

2. *$L \leq_{va} K$ and $K \leq_{va} H$ then $L \leq_{va} H$.*

*Proof:*

(1) Let $K \leq_{\mathrm{va}} H$. Then Proposition 3.2.11 implies the existence of the series

$$K = K_n \trianglelefteq_{\mathrm{va}} K_{n-1} \trianglelefteq_{\mathrm{va}} \cdots \trianglelefteq_{\mathrm{va}} K_0 = H.$$

By the repeated application of Theorem 3.2.23, we get $C_G(H) = C_G(K)$.
Now assume that $C_G(H) = C_G(K)$. Let $D \in \mathbb{L}$ be a pseudo-complement for $K$, such that $\langle K, D\rangle = K \times D \leq_{\mathrm{va}} G$. Then $[H, D] = 1$ which implies $H \cap D = 1$ by Corollary 3.2.8. Choose a subgroup $B$ of finite index in $G$, such that $B'$ is contained in $\langle K, D\rangle$. Then

$$(B \cap H)' \leq B' \cap H \leq (K \times D) \cap H = K(D \cap H) = K,$$

by modular law. As $(B \cap H) \leq_f H$, we have $K \leq_{\mathrm{va}} H$.

(2) The result is a clear consequence of Proposition 3.2.11. □

### 3.2.3 The structure lattice of $G$

**Proposition 3.2.13** *The following are equivalent for $H, K \in \mathbb{L}$:*

1. *$H \cap K \leq_{va} H, K$,*

2. *$H, K \leq_{va} \langle H, K\rangle$,*

3. *$H$ and $K$ have a common pseudo-complement,*

4. *The set of pseudo-complements for $H$ and $K$ coincide.*

*Proof:* Assume that $H \cap K \leq_{\mathrm{va}} H, K$. Choose a pseudo-complement $D \in \mathbb{L}$ for $H$. By definition of $D$, we have $\langle H, D\rangle = H \times D$ and $H \times D \leq_{\mathrm{va}} G$, so that

$\langle H\cap K, D\rangle = (H\cap K)\times D$. Since $(H\cap K)\leq_{\text{va}} H$, $(H\cap K)\times D\leq_{\text{va}} H\times D\leq_{\text{va}} G$. Therefore $D$ is a pseudo-complement for $H\cap K$. But $H\cap K\leq_{\text{va}} K$, so that by Corollary 3.2.12 $C_G(H\cap K) = C_G(K)$. Thus $[K, D] = 1$, which implies $K\cap D = 1$ by Corollary 3.2.8. Consequently $\langle K, D\rangle = K\times D$. As $K\times D\geq (H\cap K)\times D\leq_{\text{va}} G$, $K\times D\leq_{\text{va}} G$. Hence $D$ is a pseudo-complement for $K$. Thus (1) implies (3) and which implies (4).

Now assume that (3) holds. That is $H$ and $K$ have a common pseudo-complement, say $D$. Then we have $\langle H, D\rangle = H\times D\leq_{\text{va}} G$ and $\langle K, D\rangle = K\times D\leq_{\text{va}} G$. We can find a subgroup $B$ of finite index in $G$ and $B'$ is contained in $\langle K, D\rangle$. Then

$$(B\cap\langle H, K\rangle)'\leq B'\cap\langle H, K\rangle\leq (K\times D)\cap\langle H, K\rangle = K(D\cap\langle H, K\rangle) = K,$$

and $B\cap\langle H, K\rangle\leq_f\langle H, K\rangle$. Thus $K\leq_{\text{va}}\langle H, K\rangle$. Now replace $K$ by $H$ we get $H\leq_{\text{va}}\langle H, K\rangle$. Hence (3) implies (2).

Now assume that (2) holds. By Proposition 3.2.11 we have

$$H = H_n\trianglelefteq_{\text{va}} H_{n-1}\trianglelefteq_{\text{va}}\cdots\trianglelefteq_{\text{va}} H_0 = \langle H, K\rangle. \tag{3.12}$$

Take the intersection of each term of the series (3.12) with $K$, we get $H\cap K\leq_{\text{va}} K$. By the same argument, we can show that $H\cap K\leq_{\text{va}} H$. Thus (2) implies (1), and the proof is completed. □

Let $H, K\in\mathbb{L}$. If $H, K$ satisfy any of the equivalent statements of Proposition 3.2.22, then we say $H$ is equivalent to $K$ and is denoted by $H\sim K$.

**Lemma 3.2.14**

1. *$\sim$ is an equivalence relation.*
2. *Let $H_1, H_2, K_1, K_2\in\mathbb{L}$ and suppose that $H_1\sim H_2$,$K_1\sim K_2$.*
   (a) $H_1\cap K_1\sim H_2\cap K_2$.
   (b) $\langle H_1, K_1\rangle\sim\langle H_2, K_2\rangle$.

*Proof:*

(1) Clearly, $H\sim K$ is an equivalence relation because $H$ and $K$ have the same set of pseudo-complements.

(2) (a) As $H_1 \sim H_2$, we have $H_1 \cap H_2 \leq_{\text{va}} H_1$. Then Proposition 3.2.11 there exists a sequence of the form

$$H_1 \cap H_2 = N_n \trianglelefteq_{\text{va}} N_{n-1} \trianglelefteq_{\text{va}} \cdots \trianglelefteq_{\text{va}} N_0 = H_1. \tag{3.13}$$

By intersecting each term of the series (3.13) with $K_1$ and by Corollary 3.2.12 we get $H_1 \cap H_2 \cap K_1 \leq_{\text{va}} H_1 \cap K_1$. Similarly, consider the case $K_1 \cap K_2 \leq_{\text{va}} K_1$, intersect the corresponding series of $K_1 \cap K_2$ in $K_1$ with $H_1$ and get $H_1 \cap K_1 \cap K_2 \leq_{\text{va}} H_1 \cap K_1$. But

$$(H_1 \cap H_2) \cap (K_1 \cap K_2) = (H_1 \cap H_2 \cap K_1) \cap (H_1 \cap K_1 \cap K_2) \leq_{\text{va}} H_1 \cap K_1,$$

so that $(H_1 \cap H_2) \cap (K_1 \cap K_2) \leq_{\text{va}} H_1 \cap K_1$. Similarly, we can show that $(H_1 \cap H_2) \cap (K_1 \cap K_2) \leq_{\text{va}} H_2 \cap K_2$. This implies that $H_1 \cap K_1 \sim H_2 \cap K_2$.

(b) Set $M = \langle H_1 \cap H_2, K_1 \cap K_2 \rangle$. Let $D$ be a pseudo-complement for $M$ in $\mathbb{L}$. That is $\langle M, D \rangle = \big\langle \langle H_1 \cap H_2, K_1 \cap K_2 \rangle, D \big\rangle = M \times D \leq_{\text{va}} G$. Thus $(H_1 \cap H_2) \cap D = 1$,which gives $[H_1 \cap H_2, D] = 1$ by Corollary 3.2.8. But $H_1 \cap H_2 \leq_{\text{va}} H_1$, so that $[H_1, D] = 1$ (by Corollary 3.2.12). Again by Corollary 3.2.8, $H_1 \cap D = 1$. In the same way we get $K_1 \cap D = 1$. Now Corollary 3.2.9 implies that $\langle H_1, K_1, D \rangle = \langle H_1, K_1 \rangle \times D$, and $\langle H_1, K_1 \rangle \times D \leq_{\text{va}} G$ since $\langle H_1, K_1 \rangle \times D \geq M \times D \leq_{\text{va}} G$. Hence $D$ is a pseudo-complement for $\langle H_1, K_1 \rangle$. Similarly, we get $D$ as a pseudo-complement for $\langle H_2, K_2 \rangle$. As $\langle H_1, K_1 \rangle$ and $\langle H_2, K_2 \rangle$ have a common pseudo-complement, $\langle H_1, K_1 \rangle \sim \langle H_2, K_2 \rangle$.

$\square$

Observe that $\sim$ is a congruence relation on the members of $\mathbb{L}$. We will now define another collection of subnormal subgroups of $G$ as the quotient set $\mathcal{L} = \mathbb{L}/\sim$. For an element $H \in \mathbb{L}$, the equivalence class $[H]$ of $H$ represents the corresponding element in $\mathcal{L}$. For $[H], [K] \in \mathcal{L}$, we define

$$[H] \vee [K] = [H \vee K]$$

as their join and

$$[H] \wedge [K] = [H \wedge K]$$

as their meet. Clearly, the join and meet of two elements in $\mathcal{L}$ is well defined by Lemma 3.2.14. Then $\mathcal{L}$ becomes a lattice and is called the *structure lattice* of $G$

(refer [Wil00], [Har02]). The partial ordering on the lattice $\mathcal{L}$ is given by $[K] \leq [H]$ if and only if $[H] \wedge [K] = [K]$. This is equivalent to say that $[K] \leq [H]$ if and only if $H \cap K \sim K$. Observe that $[G]$ and $[1]$ are the unique greatest and least elements of $\mathcal{L}$.

Now consider the equivalence class $[1]$. Let $1 \neq K \in \mathbb{L} \in [1]$. That is $1 \sim K$. This implies $1 \leq_{\text{va}} \langle K, 1 \rangle = K$. There exists a subgroup $G_0 \leq_f G$ such that $G_0' \leq 1$. Thus we get $G_0' = 1$ and $G_0$ is abelian; this is a contradiction to Proposition 3.2.6. Hence $[1] = \{1\}$.

We will now recollect some definition from [CD73] for the lattice $\mathcal{L}$. Let $1, 0$ are the greatest and least elements of the lattice $\mathcal{L}$. A *complement* to an element $a \in \mathcal{L}$ is an element $b \in \mathcal{L}$ such that $a \vee b = 1$ and $a \wedge b = 0$. If every element of $\mathcal{L}$ has a unique complement, then $\mathcal{L}$ is said to be *uniquely complemented.* For any elements $a, b, c \in \mathcal{L}$ with $a \leq c$, if the following condition is satisfied

$$a \vee (b \wedge c) = (a \vee b) \wedge c)$$

then $\mathcal{L}$ is called a *modular lattice*. The lattice $\mathcal{L}$ is said to be *distributive* if the operators $\vee$ and $\wedge$ satisfy the distributive laws. A lattice is called *Boolean* if it is uniquely complemented and distributive.

**Lemma 3.2.15** *The lattice $\mathcal{L}$ is uniquely complemented.*

*Proof:* Choose an element $[K] \in \mathcal{L}$, where $1 \neq K \in \mathbb{L}$. By Proposition 3.2.10 $K$ has a pseudo-complement in $\mathbb{L}$. Let $D$ be a pseudo-complement for $K$ in $\mathbb{L}$. We have $\langle K, D \rangle = K \times D$ and $K \times D \leq_{\text{va}} G$. Then,

$$[K] \vee [D] = [K \vee D] = [K \times D] = [G]$$

and

$$[K] \wedge [D] = [K \wedge D] = [K \cap D] = [1].$$

Thus $[D]$ is a complement to $[K]$ in $\mathcal{L}$.

Now, suppose that $[B]$ is also a complement to $[K]$ in $\mathcal{L}$. Then by definition we have

$$[B] \vee [K] = [B \vee K] = [B \times K] = [G]$$

and

$$[B] \wedge [K] = [B \wedge K] = [B \cap K] = [1].$$

Thus $\langle K, B\rangle = K \times B \leq_{\text{va}} G$, so that $K$ is pseudo-complement for $B$ in $\mathbb{L}$. Since $B$ and $D$ have a common pseudo-complement $K$ in $\mathbb{L}$, $[B] = [D] \in \mathcal{L}$. Hence $\mathcal{L}$ is uniquely complemented. $\square$

**Lemma 3.2.16** *$L$ is a modular lattice.*

*Proof:* Suppose that $a, b, c \in \mathcal{L}$ and $a \leq c$. Let $a = [A], b = [B], c = [C]$ where $A, B, C$ are non-trivial basal subgroups in $\mathbb{L}$. As $a \leq c$, we have $A \cap C \sim A$. Thus we may assume that $A \leq C$. Then Dedekind's modular [CD73] law implies that $A(B \cap C) = AB \cap C$. Now, taking equivalence classes of subgroups on the each side of the equality, we get the result. $\square$

**Theorem 3.2.17** *$\mathcal{L}$ is a Boolean lattice.*

*Proof:* Uniquely complemented modular lattice are distributive [CD73]. Thus the result follows from Lemma 3.2.15 and Lemma 3.2.16. $\square$

Now, we define a finite sublattice $\mathcal{L}_N$ of the lattice $\mathcal{L}$, as

$$\mathcal{L}_N = \{[H] | H \trianglelefteq N\}, \quad \text{where } N \trianglelefteq_f G.$$

It is showed in Proposition (10.1.2), [Har02] that the sublattice $\mathcal{L}_N$ is a finite Boolean sublattice of $\mathcal{L}$.

### 3.2.4 Basal subgroups

Let $M$ be a non-trivial subgroup of $G$ and $1 \neq g \in G$. The subgroup $M$ is said to be *basal* if and only if $M$ has only finitely many conjugates in $G$ and $M \cap M^g = 1$ whenever $M \neq M^g$. That is, $M$ is basal if and only if $M^G$ is the direct product of finitely many conjugates of $M$. If $M \in \mathbb{L}$ then $M$ said to be is basal if and only if distinct conjugates of $M$ intersect trivially [refer [Wil00], [Har02]]. We defines a subcollection $\mathcal{M}$ of $\mathcal{L}$ as,

$$\mathcal{M} = \mathcal{M}(G) = \{[M] \in \mathcal{L} | 1 \neq M \text{ is basal}\}.$$

**Lemma 3.2.18** *If $1 \neq K \in \mathbb{L}$ then $K$ contains a non-trivial basal subgroup $M$ satisfying $K \sim \langle M^g | g \in G, M^g \leq K\rangle$.*

*Proof:* Observe that normal subgroups of $G$ are trivially basal. Suppose that $core_G(K)$ is non-trivial. Then $core_G(K)$ is a basal subgroup that is contained in $K$. Since $K \geq core_G(K) \trianglelefteq_{\text{va}} G$, $core_G(K) \sim G$.
Let $core_G(K) = 1$. We can find a subgroup $M$ which in an intersection of conjugates of $K$ such that $M$ is maximal subject to the condition $1 < M \leq K$. Clearly, $M \in \mathbb{L}$. If we choose any conjugate $M^g$ of $M$ such that $M^g \neq M$, then the subgroups $M$ and $M^g$ intersect trivially; otherwise, it is a contradiction to the maximality of $M$. Thus distinct conjugates of $M$ intersect trivially, and $M$ is basal.
Let $M^g$ be a conjugate of $M$ and $M^g \nleq K$. Then $M^g \cap K = 1$ by the same argument as above. By Corollary 3.2.9 the intersection of the product of all these conjugates, say $B$, with $K$ is trivial. Let $C$ denotes the direct product of all conjugates of $M$ which are contained in $K$, i.e., $C = \langle M^g | g \in G, M^g \leq K \rangle$. Then $BC = M^G \trianglelefteq_{\text{va}} G$. The quotient map from $G$ to $G/BC$ induces a homomorphism from $K$ to $G/BC$ with $K \cap BC$ as the kernel. But $K \cap BC = C(K \cap B) = C$. Thus $K/C \cong G/(BC)$, so that $K/C$ is virtually abelian and $K \sim C$. $\square$

**Lemma 3.2.19** *Let $M, N$ be two basal subgroups of $G$.*

1. *$M \cap N$ is basal.*
2. *If $1 \neq M \leq N$ then $N_G(M) \leq N_G(N)$.*
3. *If $M \sim N$ then $N_G(M) = N_G(N)$.*
4. *If $N_G(M) \leq L \leq G$ then $M^L$ is basal and $N_G(M^L) = L$.*

*Proof:*

(1) Assume that $(M \cap N)^g \neq M \cap N$ for some $g \in G$. Thus either $M^g \neq M$ or $N^g \neq N$. Since $M$ and $N$ are basal we have $M \cap M^g = 1$ or $N \cap N^g = 1$. We get $(M \cap N) \cap (M \cap N)^g = 1$, so that $M \cap N$ is basal.

(2) Choose an element $g \in G$ which is not contained in $N_G(N)$. Thus $N^g \neq N$, so that $N^g \cap N = 1$, since $N$ is basal. As $M \leq N$, we get $M^g \cap M = 1$. Hence $g \notin N_G(M)$.

(3) We may suppose that $M$ and $N$ are non-trivial basal subgroups. Since $M \sim N$ we have $M \cap N \sim M, N$. Furthermore, $M \cap N$ is basal from (1) above. Thus we may assume that $M \leq N$. From (2) above we have $N_G(M) \leq N_G(N)$, so that it suffices

to show that $N_G(N) \leq N_G(M)$. Let $g \in N_G(N)$. Then $M^g \leq_{\text{va}} N$ as $M \leq_{\text{va}} N$. Consequently $M \cap M^g \leq_{\text{va}} N$. Hence $M^g \cap M$ is non-trivial otherwise, this is a contradiction to Proposition 3.2.6. As $M$ is basal, we get $M^g = M$ and $g \in N_G(M)$. We conclude that $N_G(M) = N_G(N)$.

(4) The statement is trivial for $M = 1$. Let $M$ be a non-trivial subgroup of $G$. Suppose that $(M^L)^g \cap M^L$ is non-trivial. The subgroups $M^L$ and $(M^L)^g$ are the direct product of subgroups of the form $M^l$ and $M^{lg}$ for $l \in L$, as $M$ is basal. Thus $(M^L)^g \cap M^L \neq 1$ implies $M^{l_1 g} = M^{l_2}$ for some $l_1, l_2 \in L$,so that $l_1 g l_2^{-1} \in N_G(M)$. But $N_G(M) \leq L$, hence $g \in L$ and we get $M^L$ is basal. Now, let $g \in N_G(M^L)$. Then $g \in L$ as $(M^L)^g \cap M^L = M^L \neq 1$. Hence we get the equality $N_G(M^L) = L$, since $L \leq N_G(M^L)$. □

**Corollary 3.2.20** *$\mathcal{M}$ satisfies the maximal condition.*

*Proof:* Consider an ascending chain $([Mi])_{i\geq 1}$ of elements in $\mathcal{M}$, where $M_i$ are non-trivial basal subgroups of $G$. For each $i$, $M_{i-1} \sim (M_{i-1} \cap M_i) \leq M_i$. By Lemma 3.2.19 we have $N_G(M_{i-1}) = N_G(M_{i-1} \cap M_i) \leq N_G(M_i)$, so that $N_G(M_1)$ normalizes $M_i$ for all $i$. Therefore we can find a normal subgroup of finite index in $G$ which normalizes each $M_i$. Let $N \trianglelefteq_f G$. Then $M_i \sim (M_i \cap N) \trianglelefteq N$, and $[M_i] = [M_i \cap N] \in \mathcal{L}_N$. Since $\mathcal{L}_N$ is finite the chain $([M_i])_{i\geq 1}$ get stabilized after a finite length. Therefore $\mathcal{M}$ satisfies the maximal condition. □

Now we will put one more assumption on $G$. We assume that $G$ is residually finite. Then we obtain an important theorem on the cardinality of $\mathcal{L}$ and $\mathcal{M}$. The proof of the theorem uses the following results.

**Lemma 3.2.21 (refer Lemma** (10.3.2),**[Har02])** *Let $r$ be a positive integer and $K$ be a residually finite group having no non-trivial normal subgroups. If every finite quotient of $K$ has an abelian normal subgroup of index at most $r$ then $K$ is finite of order at most $r$.*

**Proposition 3.2.22 (refer Proposition** (10.3.3), **[Har02])** *Suppose that $G$ is residually finite and consider infinite descending chains $(B_i)$ of normal subgroups of finite index with the following properties: $B_0 = G$, and $B_i/B_{i+1}$ has no non-trivial abelian normal subgroup of finite index less than $i$ for each $i$.*

*1. Such chain exists.*

*2. The intersection of the terms of any such chain is trivial.*

**Theorem 3.2.23** *If $G$ is residually finite then $\mathcal{M}$ and $\mathcal{L}$ are countable.*

*Proof:* By Proposition 3.2.22 we can choose a chain $(B_i)$ of normal subgroups of finite index in $G$. Suppose that there exists a non-trivial basal subgroup $M$ of $G$ such that $B_i \nleq N_G(M)$ for each $i$. Let $b_i \in B_i \setminus N_G(M)$. Then for each $i$,

$$1 = [M, M^{b_i}] \equiv [M, M] \mod B_i,$$

so that $[M, M] \leq \bigcap_i B_i = 1$. This is a contradiction to Proposition 3.2.6. Thus for each $i$ we have $B_i \leq N_G(M)$. Therefore $M \cap B_i \sim M$ where $M \cap B_i$ is a normal subgroup of $B_i$. This implies $M \in \mathcal{L}_{B_i}$. Hence $\mathcal{M} \subseteq \bigcup_i \mathcal{L}_{B_i}$ and since $\mathcal{L}_{B_i}$ is finite for each $i$, $\mathcal{M}$ is countable. Then Lemma 3.2.18 implies that $\mathcal{L}$ is countable. □

**Theorem 3.2.24** *$\mathcal{L}$ is finite if and only if the subgroup $N = \bigcap(N_G(M)|M$ basal) has finite index in $G$.*

*Proof:* Suppose that $\mathcal{L}$ is finite. Then $\mathcal{L}$ contains finitely many equivalence classes of members of $\mathbb{L}$. In particular, there are only finitely many equivalence classes of basal subgroups. By Lemma 3.2.19 the number of distinct normalizers of basal subgroups is also finite. Thus $N \leq_f G$ as each normalizer has finite index in $G$.
Now, assume that $N$ has finite index in $G$. As $N \trianglelefteq G$, we get $N \trianglelefteq_{\text{va}} G$. Thus for any non-trivial basal subgroup $M$ we have $M \cap N \leq_{\text{va}} M$. But $M \cap N \trianglelefteq N$, so that $[M] \in \mathcal{L}_N$. Therefore $\mathcal{M} \leq \mathcal{L}_N$. Then by Lemma 3.2.18, $\mathcal{L} \leq \mathcal{L}_N$ and hence $\mathcal{L}$ is finite □

This is an important result on the characterization of the group $G$ with finite structure lattice. Let $a \in \mathcal{L}$ is called an *atom* if a is minimal non-zero element. It is proved in Proposition (11.1.1) of [Har02] that $\mathcal{L}$ is finite if and only if $\mathcal{L}$ has an atom. We now concentrate on the study of the group $G$ with infinite structure lattice. One can refer to [Har02] for details of groups with finite structure lattices.

### 3.2.5 Structure theory

We continue with the assumption on $G$. We further assume that $\mathcal{L}$ is infinite. The results are based on the action of the group $G$ on the structure lattice $\mathcal{L}$ of $G$. For any $1 \neq K \in \mathbb{L}$, and $g \in G$ we have $[K]^g = [K^g]$. Thus the action of $G$ on $\mathcal{L}$ is induced by the action of $G$ on itself by conjugation. We start the discuss with the observation that rigid vertex stabilizer $RiSt_G(u)$ for any vertex $u$ of the rooted tree $T_{\bar{m}}$ is a basal subgroup of $G$ (see Proposition 1.2.2).

**Proposition 3.2.25** *Suppose that $G$ acts faithfully on a rooted tree $T = T_{\bar{m}}$. If $1 \neq K \in \mathbb{L}$ then there is a vertex $u$ of $T$ such that $[RiSt_G(u)] \leq [K]$.*

*Proof:* Let $k$ be a non-trivial element in $K$. We may choose a vertex $u$ in $T$ such that $k(u) \neq u$ and set $v = k(u)$. As $K \in \mathbb{L}$, $N_G(K) \leq_f G$. Thus we can find a normal subgroup $H$ of finite index in $G$ such that $H$ normalizes $K$. Let $f, g \in H \cap RiSt_G(u)$. Then,

$$(g^{-1})^k \in (RiSt_G(u))^k = RiSt_G(k(u)) = RiSt_G(v).$$

The elements $(g^{-1})^k$ and $f$ commute as $RiSt_G(u)$ is basal and $RiSt_G(v)$ is a conjugate of $RiSt_G(u)$. Thus we get,

$$[[k, g], f] = [(g^{-1})^k g, f] = [(g^{-1})^k, f]^g [g, f] = [g, f].$$

But $[[k, g], f] \in K$ as $H$ normalizes $K$, so that $[g, f] \in K$. Since $g, f$ were arbitrary we have $(H \cap RiSt_G(u))' \leq K$. Hence

$$[RiSt_G(u)] = [H \cap RiSt_G(u)] = [(H \cap RiSt_G(u))'] \leq [K].$$

$\square$

**Lemma 3.2.26** *Let $M$ be a basal subgroup of $G$ then $stab_G([M]) = N_G(M)$.*

*Proof:* Assume that $M$ is a non-trivial basal subgroup. Let $g \in N_G(M)$. Then $[M]^g = [M^g] = [M]$, so that $g \in stab_G([M])$. Hence $N_G(M) \leq stab_G([M])$. Now, suppose that $g \in stab_G([M])$. That is $[M]^g = [M^g] = [M]$ which implies $M^g \sim M$. Thus $M^g \cap M \leq_{\text{va}} M^g, M$. Therefore $M^g \cap M$ is non-trivial otherwise, this is a contradiction to Proposition 3.2.6. As $M$ is basal, we get $M^g = M$ and $g \in N_G(M)$ and the result follows. $\square$

We will state the following result without proof. One can refer to Lemma (7.4) of [Wil00] and Lemma (12.1.2) of [Har02] for details.

**Lemma 3.2.27** *Let $N$ be a normal subgroup of finite index in $G$.*

1. *Let $M_1$ be a non-trivial subgroup of $\mathbb{L}$ and $N \leq N_G(M_1)$. Then there is a non-trivial basal subgroup $M \leq M_1$ such that $[M]$ is minimal in $\mathcal{L} \setminus \{1\}$ with respect to being stabilized by $N$.*

2. *Suppose that $M$ is a non-trivial basal subgroup and that $[M]$ is minimal in $\mathcal{L} \setminus \{1\}$ with respect to being stabilized by $N$. If $N < K \trianglelefteq G$ then $M^K$ is basal and $[M^K]$ is minimal in $\mathcal{L} \setminus \{1\}$ with respect to being stabilized by $K$.*

**Lemma 3.2.28** *Suppose that $\mathcal{L}$ is infinite.*

1. *The partially ordered set $\mathcal{M}$ has no minimal elements.*

2. *$G$ has a chain*
$$G = G_0 > G_1 > G_2 \cdots \tag{3.14}$$
*of normal subgroups of finite index.*

*Proof:*

(1) Let $m \in \mathcal{M}$. Clearly, $m$ is not an atom in $\mathcal{L}$ as $\mathcal{L}$ is infinite. Thus we can find a non-trivial element $k \in \mathcal{L}$ such that $k < m$. Suppose that $k = [K]$ for $K \in \mathbb{L}$. Then by Lemma 3.2.18, $K$ contains a basal subgroup $M_1$. If $m_1 = [M_1]$ then $m_1 \in \mathcal{M}$ and $m_1 < m$. Hence $\mathcal{M}$ has no minimal elements.

(2) Suppose that $G$ does not contain a chain of the form (3.14). Any chain of normal subgroups of finite index in $G$ get stabilized after finite length. Thus $G$ satisfies minimal condition on these subgroups. There exists a normal subgroup $N$ of finite index such that $N$ is contained in all other subgroups of finite index. Hence $\bigcap(N_G(M)|M \text{ basal}) \geq N \leq_f G$, which is a contradiction to the assumption on $\mathcal{L}$ by Theorem 3.2.24. $\square$

**Structure tree**

Let $G$ be a just non-(virtually abelian) group having no non-trivial virtually abelian normal subgroup with infinite structure lattice $\mathcal{L}$. Set $\mathcal{F} = (G_i)_{i\geq 0}$ as a strictly decreasing infinite chain of the form (3.14). We now develop a *structure tree* from $\mathcal{L}$ and the characterization of $G$ is based on the action of $G$ on the structure tree. The descriptions are as similar to the papers [Wil00] and [Har02]

Let $m \in \mathcal{M}$ is called a *vertex* if there is an integer $i$ such that $m$ is a minimal element of $\mathcal{L} \setminus \{1\}$ stabilized by $G_i$. The smallest such $i$ is called the *depth* of the element $m$. Let $\mathcal{T} = \mathcal{T}(\mathcal{F})$ denotes the set of all vertices. Clearly, $\mathcal{T}$ is non-empty as $[G] \in \mathcal{T}$.

Let $x, y \in \mathcal{T}$ be two vertices in $\mathcal{T}$ with $\text{depth}(x) = i$, $\text{depth}(y) = j$ and $y < x$. Observe that $i < j$. Suppose to the contrary that $i \geq j$ thus $G_i \leq G_j$, which implies $G_i$ stabilizes $y$ and that is a contradiction to the minimality of $x$. We define a graph structure on $\mathcal{T}$ by introducing edges between vertices. For any pair $(x, y)$ of vertices in $\mathcal{T}$, $x$ is connected to $y$ by an *edge* if $y$ is maximal in the set $\{u \in \mathcal{T} | u < x\}$. Clearly, $x$ is connected to vertex which is strictly smaller than $x$ in $\mathcal{L}$. The set of vertices $\mathcal{T}(\mathcal{F})$ together with these set of edges form a directed graph and is called the *structure tree* of $G$ and is denoted by $\mathcal{T}$.

Let $g \in G$ and $x \in \mathcal{M}$. Suppose that there is an integer $i$ such that $\text{depth}(x) = i$, then $x^g$ is normalized by $G_i$ and $x^g$ forces to be minimal with respect to being normalized by $G_i$. Therefore $x$ is a vertex if and only if distinct conjugates of $x$ are also vertices having depth equal to that of $x$.

The action of $G$ on the structure tree $\mathcal{T}$ is induced by the action of $G$ on the structure lattice $\mathcal{L}$. We will now show that the structure tree $\mathcal{T}$ is a tree.

**Lemma 3.2.29** *Let $x, y \in \mathcal{T}$ with $y < x$. The following statements are equivalent:*

1. *$(x, y)$ is an edge;*
2. *$x = [M_i^G]$ and $y = [M]$ for some basal subgroup $M$ and an integer $i \geq 0$, where $i + 1$ is the depth of $y$.*

*Proof:* Suppose that (1) holds. Set $x = [X]$ and $y = [Y]$, where $X, Y$ are non-trivial basal subgroups in $\mathbb{L}$. As $(x, y)$ is an edge, $y$ is maximal in the set $\{u \in \mathcal{T} | u < a\}$. Thus $1 \neq y < a$, which implies $Y \cap X \sim Y \neq 1$. By Lemma 3.2.19 $Y \cap X$ is a non-trivial basal subgroup, so that we may assume $Y \leq X$. Since $y < [G]$, we have $0 = \text{depth}([G]) < \text{depth}(y)$. Thus we may assume that $\text{depth}(y) = i+1$ for some $i \geq 0$. It follows from Lemma 3.2.27 that $Y_i^G$ is basal and $[Y^{G_i}]$ is minimal with respect to being normalized by $G_i$, as $G_i, G_{i+1} \trianglelefteq G$ and $G_{i+1} < G_i$. Now, let $\mathrm{depth}(x) = j$. Then $j < i+1$ as $y < x$, so that $G_i \leq G_j$. Therefore $x$ is stabilized by $G_i$ and hence $X$ is normalized by $G_i$. We have,

$$y = [Y] < [Y^{G_i}] \leq [X^{G_i}] = [X] = x.$$

The maximality of $y$ implies that $[Y^{G_i}] = x$ and hence (1) implies (2).
Now suppose that $y = [Y]$ and $x = [Y^{G_i}]$ for a non-trivial basal subgroup $Y$ and $\mathrm{depth}(y) = i+1$, $i \geq 0$. By Lemma 3.2.27, $[Y^{G_i}]$ is a vertex and depth of $[Y^{G_i}]$ is at most $i$, so that $y < x$. Let $z = [Z]$, with $1 \neq Z \in \mathbb{L}$, be a vertex in $\mathcal{T}$ and $y < z \leq x$. Suppose that $\mathrm{depth}(z) = j < i+1$. Thus $G_i \leq G_j$ and $Z$ is normalized by $G_i$. Since $[Y] < [Z]$, we have

$$x = [Y^{G_i}] \leq [Z^{G_i}] = [Z] = z.$$

Therefore $x \leq z$ so that $x = z$. Hence $y$ is maximal in the set $\{u \in \mathcal{T} | u < x\}$ so $(x, y)$ is an edge. □

**Corollary 3.2.30** *The vertex $[G]$ has in-degree 0, whereas all other vertices have in-degree 1 (here in-degree stands for the number of edges coming into a particular vertex).*

*Proof:* By the definition of edge and maximality of $G$ implies $G$ has in-degree zero. Suppose that $y = [Y] \in \mathcal{M}$ is a vertex of $\mathcal{L}$ with $\mathrm{depth}(y) = i+1$, $i \geq 0$. Let $(x, y)$ and $(x_1, y)$ be two edges coming into the vertex $y$. Then by Lemma 3.2.29 $x = [Y^{G_i}] = x_1$. Hence the proof is completed. □

**Theorem 3.2.31**

1. *$\mathcal{T}$ has no cycles.*

2. *Let $x, y$ be vertices of $\mathcal{T}$ with depths $i, j$ respectively. If $y < x$ in $\mathcal{L}$, then there is a unique path in $\mathcal{T}$ from $x$ to $y$. This path has length at most $j - i$. In particular, there is a unique path from $[G]$ to $y$ of length at most $j$.*

*Hence $\mathcal{T}$ is a tree.*

*Proof:*

(1) Suppose to the contrary that $\mathcal{T}$ has a cycle. Let $y$ be a minimal vertex in this cycle. Then $y$ is connected to two distinct vertices $x$ and $x_1$ by the edges $(x, y)$ and $(x_1, y)$. But this is a contradiction to Corollary 3.2.30, so that $\mathcal{T}$ has no cycles.

(2) Set $x = [X]$ and $Y = [Y]$ where $X$ and $Y$ are non-trivial basal subgroups in $\mathbb{L}$. As $y < x$, we have $X \cap Y \sim Y$ and $X \cap Y$ is a non-trivial basal subgroup. Thus we assume without loss of generality that $Y \leq X$. We have

$$y < [Y^{G_i}] \leq [X^{G_i}] = x.$$

Clearly, $[Y^{G_i}]$ is stabilized by $G_i$, then the minimality of $x$ implies that $[Y^{G_i}] = x$. Since $j = \mathrm{depth}(y) > \mathrm{depth}(x) = i$, there is a finite chain $G_j < G_{j-1} < \cdots < G_{i+1} < G$ of normal subgroups of the chain $\mathcal{F}$ and we get

$$[Y] = [Y^{G_j}] \leq [Y^{G_{j-1}}] \leq \cdots \leq [Y^{G_{i+1}}] \leq [Y^{G_i}] = [X]. \tag{3.15}$$

Each term of the chain (3.15) is a vertex in $\mathcal{T}$ by Lemma 3.2.27. Let $[Y^{G_{k+1}}] < [Y^{G_k}]$ be a part of the chain (3.15). Suppose that $\mathrm{depth}([Y^{G_{k+1}}]) = l + 1 \leq k + 1$. Then $([Y^{G_{k+1}}]^{G_l}, [Y^{G_{k+1}}])$ is an edge by Lemma 3.2.28. Observe that $([Y^{G_{k+1}}]^{G_l}, ([Y^{G_{k+1}}]) = ([Y^{G_l}], [Y^{G_{k+1}}])$, so that $[Y^{G_{k+1}}]$ is maximal in the set $\{u \in \mathcal{L} | u < [Y^{G_l}]\}$. As $l \leq k$, we have $[Y^{G_{k+1}}] < [Y^{G_k}] \leq [Y^{G_l}]$. Then the maximality of $[Y^{G_{k+1}}]$ implies that $[Y^{G_k}] = [Y^{G_l}]$. Thus $([Y^{G_k}], [Y^{G_{k+1}}])$ is an edge. Each pair of distinct vertices in the chain (3.15) is connected by an edge. Therefore we get a path from $x$ to $y$ of length at most $j - i$ and the uniqueness is followed from the part (1) of this theorem. Consequently, $[G]$ is connected to all vertices of $\mathcal{T}$. Then (1) and (2) together implies that $\mathcal{T}$ is a tree. $\square$

**Proposition 3.2.32** *Let $x, y$ be vertices of $\mathcal{T}$, of depths $i, j$ with $i \leq j$. Write $x = [X], y = [Y]$ where $X$ and $Y$ are non-trivial basal subgroups in $\mathbb{L}$. The following are equivalent:*

1. *$y \leq x$ in $\mathcal{L}$;*
2. *The path in $\mathcal{T}$ from $[G]$ to $y$ passes through $x$;*
3. *$X$ and $Y$ do not centralize each other.*

*Proof:* Assume that (1) holds. There are paths in $\mathcal{T}$ from $[G]$ to $x$, $[G]$ to $y$ and $x$ to $y$ (by Theorem 3.2.31). These paths form a cycle in $\mathcal{T}$. As $\mathcal{T}$ is a tree the path from $[G]$ to $x$ and the path from $x$ to $y$ concatenate to the path from $[G]$ to $y$. Thus (2) holds.

Suppose that (2) holds. Since edges pass through vertices of smaller size in $\mathcal{L}$, it is clear that $y \leq x$. Thus (1) and (2) are equivalent statements.

Again assume that (1) holds. Then $X \cap Y$ is a non-trivial basal subgroup of $\mathbb{L}$. If $(c)$ holds then $X \cap Y$ is an abelian element of $\mathbb{L}$, which is a contradiction to Proposition 3.2.6. Thus (1) implies (2).

Finally, suppose that (3) holds. Then $[X, Y] \neq 1$, so that $X \cap Y \neq 1$ by Corollary 3.2.8. Thus $X \cap Y$ is a non-trivial basal subgroup of $\mathbb{L}$, and is normalized by $G_j$ (as $X$ and $Y$ are normalized by $G_j$). But $[1] < [X \cap Y] \leq [Y]$, then the minimality of $[Y]$ with respect to being stabilized by $G_j$ implies that $[X \cap Y] = [Y]$. Therefore $[Y] = [X \cap Y] \leq [X]$, i.e., $y \leq x$ hence completes the proof. $\square$

We will now consider the action of $G$ on the tree $\mathcal{T}$.

**Proposition 3.2.33**

1. *For each vertex $x$ of $\mathcal{T}$ write $S(x) = \{y \in \mathcal{T} | (x, y) \text{ is an edge}\}$. Then either $S(x)$ is empty or there exists an integer $j \geq 0$ such that the elements of $S(x)$ have depth $j+1$ and are permuted transitively and non-trivially by $G_j$. In the latter case $|S(x)|$ is a divisor greater than 1 of $|G_j/G_{j+l}|$.*

2. *For each $i \geq 0$, $G$ acts on the set of vertices of depth $i$ with at most one orbit. Thus the orbits of $G$ in its action on $\mathcal{T}$ are the non-empty sets of elements of equal depth.*

3. *For each $i \geq 0$, $\mathcal{T}$ has only finitely many vertices of depth $i$.*

*Proof:*

(1) Let $x$ be a vertex of $\mathcal{T}$ with depth $i \geq 0$. Assume that $S(x)$ is non-empty. We choose a vertex $y \in S(x)$ with the smallest depth. Set $y = [Y]$ with $Y$ is a non-trivial basal subgroup in $\mathbb{L}$ and depth$(y) = j+1$, for some $j \geq 0$. Let $g \in G_j$. Then $(x^g, y^g) = (x, y^g)$ is an edge, where $x^g = x$ as $G_j \leq G_i$. Suppose that $y^g = y$ for all $g \in G_j$. Then $Y$ is normalized by $G_j$, so that $x = [Y^{G_j}] = y$ (where the first equality follows from Lemma 3.2.29). This is a contradiction since $(x, y)$ is an edge. Therefore

$(x, y^g)$ is an edge distinct from $(x, y)$ and so the cardinality of $S(x)$ is greater than or equal to two.

Now assume that $(x, z)$ is an edge with $z = [Z]$, where $Z$ is a non-trivial basal subgroup in $\mathbb{L}$. Suppose that depth$(z) = k + 1$, which is greater than $j + 1 =$depth$(y)$ by the choice of $y$. Observe that $[Z] < [Y^{G_j}]$, as $z < a = [Y^{G_j}]$. Thus $Z \cap Y^{G_j} \sim Z \neq 1$. Since $Y^{G_j}$ is basal, there exists some $g \in G_j$ such that $Z \cap Y^g \neq 1$. Clearly, $Y^g$ and $[Y]^g$ have depth $j + 1$. Then $Z \cap Y^g$ is normalized by $G_{k+1}$. But $[Z \cap Y^g] \leq [Z]$, and so the minimality of $Z$ implies $[Z \cap Y^g] = [Z]$. Therefore

$$[Z] = [Z \cap Y^g] \leq [Y^g] = [Y]^g,$$

so that $z \leq y^g$. Suppose that $z < y^g$. Then the vertices $x, y^g, z$ form a cycle in $\mathcal{T}$ as $(x, z)$ and $(x, y^g)$ are edges. Hence $z = y^g$, and the set $S(x)$ is exactly the orbit of $y$ under the action of $G_j$. Observe that the action of $G_j$ on $S(x)$ is transitive and non-trivial. Therefore

$$|S(x)| = |G_j : \mathrm{stab}_{G_j}(y)|,$$

so that $|S(x)|$ divides $|G_j : G_{j+1}|$ as $G_{j+1} \leq \mathrm{stab}_{G_j}(y)$.

(2) The action of $G$ on the tree $\mathcal{T}$ is induced by the action of $G$ on $\mathcal{L}$. Thus $G$ acts on the set of vertices of equal depth. Assume that $i \geq 0$ and the set of vertices of depth $i$ is non-empty. Let $x = [X], y = [Y]$ be two vertices of depth $i$ with $X, Y$ are non-trivial basal subgroups in $\mathbb{L}$. Then $[1] < [Y] \leq [G] = [X^G]$, so that $Y \cap X^G \neq 1$. We can find $g \in G$ such that $Y \cap X^g \neq 1$. Clearly, $[Y \cap X^g]$ is normalized by $i$. But $[Y \cap X^g] \leq y, x^g$ then the minimality of $y, x^g$ implies that $y = [Y \cap X^g] = x^g$. Hence the set of vertices of equal depth forms an orbit under the action of $G$.

(3) Let $S_i$ denotes the set of vertices of depth $i$ for each $i \geq 1$. We have seen that the action of $G$ on the set $S_i$ is transitive. This induces a homomorphism from $G$ to $S_i$. Clearly, the index of the kernel of this map in $G$ is equal to the cardinality of the set $S_i$. Therefore $|S_i|$ is finite as the kernel of the above homomorphism contains the subgroup $G_i \trianglelefteq_f G$. $\square$

Till now we have assumed that $G$ is a just non-(virtually abelian) group having no non-trivial virtually abelian normal subgroups, and the structure lattice $\mathcal{L}$ of $G$ is infinite. We further assume that $G$ satisfies the following restriction on the chain $\mathcal{F}$ and get an important result about the structure of the tree $\mathcal{T}$:

(F): Each basal subgroup of $G$ is normalized by a member of $\mathcal{F}$.

**Lemma 3.2.34** *If $G$ is residually finite then $G$ has infinite descending chains of normal subgroups of finite index which satisfy the condition (F).*

*Proof:* It follows from Theorem 3.2.23 that $\mathcal{M}$ is countable as $G$ is residually finite. Then we can index the set $\mathcal{M}$ as

$$\mathcal{M} = \{[M_i] | i \in \mathbb{N}\}.$$

As $N_G(M_i)$ has finite index in $G$ for each $i$, we can choose a normal subgroup $N_i$ of finite index in $G$ such that $M_i$ is normalized by $N_i$. Set

$$G_n := \bigcap_{i=1}^{n} N_i,$$

then $G_n$ is a normal subgroup of finite index in $G$. Thus we get a chain of normal subgroups of finite index in $G$ of the form $G_1 \geq G_2 \geq \cdots$ Suppose that this chain get stabilized after finite length. Then for some integer $n$, we have

$$G_n = \bigcap_{i=1}^{\infty} G_i = \bigcap_{i=1}^{\infty} N_i.$$

But $\bigcap_{i=1}^{\infty} N_i$ is contained in the subgroup $N = \bigcap(N_G(M) | M \text{ is basal})$, as $N_i \leq N_G(M_i)$ for each $i$. Hence we get $N \trianglelefteq_f G$, and by Theorem 3.2.24 $\mathcal{L}$ is finite. This is a contradiction to the assumption on $\mathcal{L}$. Therefore the chain $G_1 \geq G_2 \geq \cdots$ is infinite. By avoiding the repeated terms, we get the required series. By construction $\mathcal{F}$ satisfies the condition (F). □

**Proposition 3.2.35** *If $\mathcal{F}$ satisfies the condition (F) then $\mathcal{T}$ has no minimal vertices.*

*Proof:* Let $x = [X]$ be a vertex of $\mathcal{T}$, with $X$ is a non-trivial basal subgroup in $\mathbb{L}$. It follows from Lemma 3.2.28 that there is a non-trivial basal subgroup $M_1$ that is contained in $M$ and $[M_1] < [M]$. As $\mathcal{F}$ satisfies the condition (F), $G_i \leq N_G(M_1)$ for some integer $i$. By Lemma 3.2.27 we can find a non-trivial basal subgroup $M_2 \leq M_1$ such that $[M_2]$ is a vertex in $\mathcal{T}$. Thus $[M_2] < [M]$ and $\mathcal{T}$ has no minimal elements.

□

**Theorem 3.2.36** *Suppose $\mathcal{F}$ satisfies the condition (F) then $\mathcal{T}$ is a rooted tree and $G$ acts transitivity on each layer of its levels.*

*Proof:* The Proposition 3.2.35 implies that $\mathcal{T}$ is infinite. We have seen that $\mathcal{T}$ is a tree and the vertex $[G]$ has in-degree zero. Furthermore, for any vertex $x \in \mathcal{T}$ there is a unique path from $[G]$ to $x$. The sets $S_i$ of vertices of depth $i$ constitute distinct levels of $\mathcal{T}$. Clearly, $[G]$ is the only vertex in the level zero of $\mathcal{T}$. We label the vertex $[G]$ as the root. Then the tree $\mathcal{T}$ becomes a rooted tree and $G$ acts transitively on each level of $\mathcal{T}$ by Proposition 3.2.33. □

**Lemma 3.2.37** *Let $N$ be the subgroup $\bigcap(N_G(M)|M$ is basal). Suppose that $\mathcal{F}$ satisfies condition (F). Then $N$ is the kernel of the action of $G$ on $\mathcal{T}$ , and if $N = 1$ then the rigid stabiliser of each vertex of $\mathcal{T}$ is non-trivial.*

*Proof:* Let $E$ denotes the kernel of the action of $G$ on $\mathcal{T}$. Clearly, $E = \bigcap(\mathrm{stab}_G(x)|x \in \mathcal{T})$. Let $x = [X]$, with $X$ is a non-trivial basal subgroup in $\mathbb{L}$. Then by Lemma 3.2.26 we have $\mathrm{stab}_G(x) = N_G(X)$. Therefore $N \leq E$.
Let $x \in E$ and $M$ be a non-trivial basal subgroup in $\mathbb{L}$. As $\mathcal{F}$ satisfies condition (F), there is an integer $i$ such that $M$ is normalized by $G_i$. It follows from Lemma 3.2.27 that $M$ contains a non-trivial basal subgroup $M_1$ such that $[M_1]$ is a vertex of $\mathcal{T}$. Then $x \in N_G(M_1) \leq N_G(M)$, so that $E \leq N$. Hence we get the equality.
Assume that $N = 1$. Let $x = [X]$ be a vertex of $\mathcal{T}$ with $X$ is a non-trivial basal subgroup of $\mathbb{L}$. Choose a vertex $y = [Y]$ with $Y$ is a non-trivial basal subgroup in $\mathbb{L}$ such that $y$ does not belong to the subtree $\mathcal{T}_x$ hanging below the vertex $x$. There are two possibilities either $x \leq y$ or $x \nleq y$. If $x \nleq y$, then Proposition 3.2.32 implies that $X$ and $Y$ centralize each other. If $x \leq y$ then $X \leq N_G(X) \leq N_G(Y)$. Hence $X$ normalizes each vertex of $\mathcal{T} \setminus \mathcal{T}_x$ so that $Y \leq RiSt_G(x) \neq 1$. □

As defined in [Har02], a *rigid normalizer* of any non-trivial basal subgroup $C$ of $G$ is the subgroup $R_G(C) = \bigcap(N_G(M)|M \text{ basal}, M \cap C = 1)$.

**Lemma 3.2.38** *Let $C, C_1, C_2$ be non-trivial basal subgroups of $G$.*

1. $C \leq R_G(C) \leq N_G(C)$.

2. *If* $C_1 \sim C_2$ *then* $R_G(C_1) = R_G(C_2)$.

3. *If* $[C_1] \leq [C_2]$ *then* $R_G(C_1) \leq R_G(C_2)$.

*4. If $R = R_G(C)$ then $R \trianglelefteq R^G \trianglelefteq G$ and $R$ has only finitely many conjugates; thus $R \in \mathbb{L}$.*

*Proof:*

(1) Let $M$ be a non-trivial basal subgroup and $M \cap C = 1$. By Corollary 3.2.9 $[M, C] = 1$ and $C \leq N_G(M)$ so that $C \leq R_G(C)$. Since $C$ is basal, $C^G \trianglelefteq G$ is the direct product of finitely many conjugates of $C$ in $G$. As distinct conjugates of $C$ has trivial intersection with $C$, $R_G(C)$ normalizes each of them. Thus $C$ has to be normalized by $R_G(C)$.

(2) $C_1 \sim C_2$ implies that $C_1 \cap C_2 \leq_{\text{va}} C_1, C_2$. By Corollary 3.2.12 $C_G(C_1) = C_G(C_1 \cap C_2) = C_G(C_2)$. For any two non-trivial basal subgroups $M$ and $C$ we have $[M, C] = 1$ if and only if $M \cap C = 1$, thus the result follows.

(3) Since $[C_1] \leq [C_2]$, $C_1 \cap C_2 \sim C_1$. Thus $C_1 \cap C_2$ is a non-trivial basal subgroup of $G$ and by (2) above $R_G(C_1 \cap C_2) = R_G(C_1)$. Hence we may assume that $C_1 \leq C_2$. Now the result is a clear consequence of the definition of rigid normalizer.

(4) Let $g \in G$. We have

$$R^g = \bigcap (N_G(M^g) | M \text{ basal}, M \cap C = 1) = \bigcap (N_G(M) | M \text{ basal}, M \cap C^g = 1) = R_G(C^g).$$

Since $C$ is basal, $C$ has only finitely many conjugates in $G$, so is for $R$ and thus $R \in \mathbb{L}$. From the definition of $R$ and part (1) of this lemma we get $R \leq N_G(R_G(C^g))$. Hence $R \trianglelefteq R^G \trianglelefteq G$. □

**Proposition 3.2.39** *Suppose that $G$ acts faithfully on a rooted tree $\mathcal{T}$ and that the rigid stabilizer of each vertex is non-trivial. Then $R_G(RiSt_G(u)) = RiSt_G(u)$ for each $u \in \mathcal{T}$.*

*Proof:* By the first part of Lemma 3.2.38, we have $RiSt_G(u) \leq R_G(RiSt_G(u))$. Let $u \in \mathcal{T}$. Suppose that $h \in R_G(RiSt_G(u))$ we will show that $h \in RiSt_G(u)$. We have seen that $N_G(RiSt_G(u)) = \text{stab}_G(u)$, so that $h$ fixes $u$. We choose a vertex $v \in \mathcal{T} \setminus \mathcal{T}_u$. Then either $u \leq v$ or $u \not\leq v$. If $u \leq v$, the $h$ must fix $v$ as $h$ fixes $u$. If $u \not\leq v$ then $RiSt_G(u) \cap RiSt_G(v) = 1$. As $h \in RiSt_G(u)$, $h$ normalizes $RiSt_G(v)$. Thus $h \in \text{stab}_G(v)$. Therefore $h$ fixes all $u \in \mathcal{T} \setminus \mathcal{T}_u$, so that $h \in RiSt_G(u)$. Hence we get the equality. □

**Theorem 3.2.40** *Let $G$ be an abstract group. Then $G$ is a branch group if and only if each of the following conditions hold:*

1. *$G$ is just non-(virtually abelian) with no non-trivial virtually abelian normal subgroups;*

2. $\bigcap(N_G(M)|M\, basal) = 1$;

3. *For each non-trivial basal subgroup $C$, the normal closure in $G$ of the subgroup $\bigcap(N_G(M)|M\, basal, M \cap C = 1)$ has finite index in $G$.*

*Proof:* Let $G$ be a branch group. As a consequence of Theorem 3.2.1 and Lemma 3.2.2 we have seen that $G$ is a just non-(virtually abelian) group having no non-trivial virtually abelian normal subgroups. We will now show that $G$ also satisfies (2) and (3). As $G$ is a branch group, $G$ acts faithfully on a rooted tree $T_{\bar{m}} = T$. Let $u$ be a vertex of $T$. Then $RiSt_G(u)$ is a non-trivial basal subgroup of $G$. But $\bigcap(N_G(M)|M \text{ basal}) \leq \bigcap(N_G(RiSt_G(u))|u \in T) = \bigcap(\text{stab}_G(u)|u \in T) = 1$. Thus (2) holds.
Suppose that $M$ is a non-trivial basal subgroup of $G$. By Proposition 3.2.25 we can find a vertex $u \in T$ such that $[RiSt_G(u)] \leq [M]$. Now it follows from Lemma 3.2.38 and Proposition 3.2.39 that

$$RiSt_G(u) = R_G(RiSt_G(u)) \leq R_G(M).$$

Clearly, $(R_G(M))^G$ has finite index in $G$.
Now assume that $G$ satisfies the conditions (1) and (2). This implies $\mathcal{L}$ is infinite and $G$ is residually finite. Then it follows from the results 3.2.34, 3.2.35, 3.2.36 and 3.2.37 that $G$ acts faithfully on the rooted tree $\mathcal{T}$, the action of $G$ on $\mathcal{T}$ is transitive on each level of $\mathcal{L}$ and all rigid vertex stabilizers are non-trivial. If $G$ satisfies (1) and (2) then $G$ is said to be a *generalized branch group*. Now suppose that condition (3) holds. Let $u$ be a vertex of $\mathcal{T}$. By Proposition 3.2.39 $R_G(RiSt_G(u)) = RiSt_G(u)$. Thus $R_G(RiSt_G(u))^G$ has finite index in $G$ by the condition (3). But $(R_G(RiSt_G(u)))^G = RiSt(n) \trianglelefteq_f G$, for $|u| = n$. Hence $G$ is a branch group by Definition 3.0.2. $\square$

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