SUBNORMAL SUBGROUPS

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Certificate of Examination

This is to certify that the dissertation titled "Subnormal Subgroups" submitted by Mr. Shrinit Singh (Reg. No. MS12098) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. I.B.S. Passi at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

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Shrinit Singh

Dedicated to my grandfather

Contents

Introduction

Subnormality is a very natural generalisation of normality. Not much attention was given to subnormal subgroups until Wielandt proved his classic result on join of subnormal subgroups of finite groups in 1939.[3]

In my thesis, I am reviewing the properties of subnormal subgroups and those groups which have every subgroup subnormal.

I have devoted the first chapter to give elementary results on join of subnormal subgroups. In the end of the first chapter, I have given three proofs of Wielandt join theorem.

In the second chapter, I have focused on those groups which have every subgroup subnormal. My main focus is to study non-nilpotent groups with every subgroup subnormal, mainly Heineken-Mohamed groups.

Chapter 1

Subnormal Subgroups

1.1 Definitions

- A subgroup N of a group G is *normal* in G if $g^{-1}Ng = N \ \forall \ g \in G$, or equivalently, A subgroup N of a group G is *normal* in G if N is a kernel of a homomorphism from $\phi : G \to H$ for some group H. Then we write $N \lhd G$.

- A subgroup H of a group G is said to be *subnormal* in G if there exists a non negative integer m and a series

  $$H = H_m \lhd H_{m-1} \lhd ...... \lhd H_0 = G$$

  of subgroups of G such that each H_{i+1} is normal in $H_i \ \forall \ 0 \leq i \leq m-1$. We shall call this series as subnormal series of H in G. If H is subnormal in G, we denote

  $$H \ sn \ G \ \ or \ H \lhd^m G \ \ or \ H \lhd \lhd G$$

  Now for a given subgroup H of a group G, we can have a number of subnormal series with different length m. The smallest such m is called as subnormal defect of H in G. Sometimes we use the notation $H \lhd^m G$, for convinience, to say that the subnormal defect of H in G is at most m.

Examples :

1) Every group has a subnormal subgroup, the group itself, with defect 0.

2) Every normal subgroup of a non trivial group is subnormal subgroup with defect 1.

3) A subgroup of order 2 in the alternating group of degree 4 and the non-central subgroups of order 2 in the dihedral group of order 8 are examples of subnormal subgroups of defect 2.

4) Let M be any non-central subgroups of order 2 in the dihedral group of order 2^m are subnormal of defect $m-1$, for all $m \geq 3$.

5) The subgroups of order 2 in the symmetric group with 3 indices(S_3) are not subnormal in the S_3.

1.2 Fastest decreasing subnormal series

Given a subgroup of a group, how will we determine whether the subgroup is subnormal or not? We will address this problem, which will be sufficient to produce the most efficient subnormal series for subnormal subgroup, in this section.

Take H to be any subgroup of a group G. We define normal closure of a subgroup H in the group G, denoted by H^G, as the smallest normal subgroup of G containing H. So

$$H^G = \langle h^g \mid h \in H, g \in G \rangle.$$

Here $h^g = g^{-1}hg$ and for any subsets X and Y of group G, we write

$$X^y = \langle x^y | \ x \in X, y \in Y \rangle.$$

We keep $H_0 = G$ and $H_1 = H^G$, the smallest normal subgroup of G containing H. Similarly we define H_2 to be the smallest normal subgoup of H_1 containing H. Inductively for all $i \geq 0$,

$$H_{i+1} = H^{H_i}.$$

Hence

$$H \leq ..... \trianglelefteq H_{i+1} \trianglelefteq H_i \trianglelefteq .... \trianglelefteq H_1 \trianglelefteq H_0 = G.$$

Claim:- This is the fastest decreasing subnormal series containing H.

Suppose we have an arbitrary subnormal series of H in G.

$$H \leq ......K_{i+1} \trianglelefteq K_i \trianglelefteq .... \trianglelefteq K_1 \trianglelefteq K_0 = G$$

We will prove the claim by induction

For $i = 0$, $G = H_0 \leq K_0 = G$

Suppose it is true for i, $H_i \leq K_i$

Then for $i + 1$,

$$H_{i+1} = H^{H_i} \leq H^{K_i} \leq K_{i+1}$$

So $H_i \leq K_i$ for all $i \geq 0$.

We proved that the inductively defined subnormal series is the fastest decresing subnormal series. We call this series and H_i as normal closure series of H in G and the i^{th} normal closure of H in G respectively.

Comment: We defined subnormal series for subnormal subgroups but in this section, we defined subnormal series for any subgroup. We can define subnormal series for a subgroup which is not a subnormal subgroup. The difference is that we will not get finite m such that $H_m = H$.

Given two elements x and y in G, we define $[x, y] = x^{-1}y^{-1}xy$, the commutator of x and y. Given any two subsets X and Y in G, we define $[X, Y] = \langle [x, y] \mid x \in X, y \in Y \rangle$ the commutator subgroup of X and Y.

Since $[x, y]^{-1} = [y, x]$, we conclude $[X, Y] = [Y, X]$. For subsets $X_0, X_1, ..., X_n$ of a group G, we define $[X_0] = \langle X_0 \rangle$ and inductively $[X_0, X_1, ..., X_{n-1}, X_n] = [[X_0, X_1, ..., X_{n-1}]X_n] \ \forall \ n \geq 1$ and when $X_1 = X_2 = ... = X_n = X$, we write $[X_{0},_{n} X] = [[X_0, X_1, ..., X_{n-1}]X_n]$

Proposition 1.1[1] Suppose K is a subgroup of a G. Then

(i) i^{th} normal closure of K in G is $K[G,_i K]$;

(ii) $K \lhd^m G$ if and only if K coincides with its m^{th} normal closure in G.

Proof. (i) The proof goes by induction.

For $i = 0$; $K_0 = G$ and $K[G,_0 K] = G$

For $i = 1$; we have to prove $K^G = K[G, K]$

Take a generator element $g^{-1}kg$ from K^G

$$g^{-1}kg = kk^{-1}g^{-1}kg = k[k, g] = k[g, k]^{-1} \in K[G, K]$$

To prove every element of $K[G, K]$ is also in K^G. Clearly $K \subseteq K^G$ and $[G, K] \subseteq K^G$. So $K^G = K[G, K] = K_1$.

Suppose it is true for $i \geq 0$. We have to prove for $i + 1$.

Given $K_i = K[G,_i K]$ and to prove $K_{i+1} = K[G,_{i+1} K]$

Now $K_{i+1} = K^{K_i} = K^{K[G,_i K]} = K^{[G,_i K]} = K[G,_{i+1} K]$

(ii) If $K \lhd^m G$ then K coincides with its m^{th} normal closure in G since normal closure series is the fastest decreasing subnormal series and the converse part is obvious.

Proposition 1.2[1]

(i) Suppose $H \lhd^m G$ and $K \leq G$. Then $H \cap K \lhd^m K$.In particular, H is subnormal in any subgroup L, containing H, of the group G .

(ii) If $H_\lambda \lhd^m G, \forall \ \lambda \in \Lambda$ (an indexing set) where m does not depend on λ, then $\cup_\lambda H_\lambda \lhd^m G$.

Proof. (i) Given $H \lhd^m G$; so

$$H = H_m \trianglelefteq H_{m-1} \trianglelefteq ..... \trianglelefteq H_{i+1} \trianglelefteq H_i \trianglelefteq .... \trianglelefteq H_1 \trianglelefteq H_0 = G$$

It will suffice to prove that $H_{i+1} \cap K \trianglelefteq H_i \cap K$.

Take any $y \in H_{i+1} \cap K$ and any $x \in H_i \cap K$.

See that $y^x \in H_{i+1}$ and $y^x \in K$ so $y^x \in H_{i+1} \cap K$.

So $H_{i+1} \cap K \trianglelefteq H_i \cap K$.

(ii) We have

$$H_\lambda = H_{\lambda,m} \lhd .... \lhd H_{\lambda,1} \lhd H_{\lambda,0} = G \ , \forall \ \lambda \in \Lambda$$

It will suffice to prove

$$\cap_\lambda H_{\lambda,i+1} \lhd \cap_\lambda H_{\lambda,i}$$

This can be proved by the same method as used in (i).

Proposition 1.3[1] If $H \lhd^m K \lhd^n G$, then $H \lhd^{m+n} G$.

Proof. The proof is trivial so we omit the proof.

Proposition 1.4[1] Suppose $H \lhd^m G$ and ϕ is a homomorphism of G, then $\phi(H) \lhd^m \phi(G)$. Thus if $N \lhd G$, then $HN/N \lhd^m G/N$ and $HN \lhd^m G$. In fact the normal closure series of any subgroup M of G is mapped by ϕ onto the normal closure series of $\phi(M)$ of $\phi(G)$.

Proof. Given $H = H_m \lhd H_{m-1} \lhd ..... \lhd H_1 \lhd H_0 = G$

It will be enough to prove that $\phi(H_{i+1}) \lhd \phi(H_i)$.

Now take $h_{i+1} \in H_{i+1}$ and $h_i \in H_i$.

Consider this

$$\phi(h_{i+1})^{\phi(h_i)}$$

$$\phi(h_i)^{-1}\phi(h_{i+1})\phi(h_i)$$

$$\phi({h_i}^{-1}h_{i+1}h_i) \in \phi(H_{i+1})$$

So $\phi(H_{i+1}) \lhd \phi(H_i)$ and homomorphism preserves subnormality.

Given $N \lhd G$ and we have to prove that $H_{i+1}N \lhd H_iN$.

It easily follows from the fact that $H_{i+1} \lhd H_i$ and N normalizes H_i.

$HN/N \lhd^m G/N$ easily follows from the above (Choose suitable ϕ).

1.3 Some Results on Joins

Subnormality is just the transitivization or (say generalization) of normality. Normal subgroups enjoy the property of making lattice that is to say that join and intersection of two normal subgroups are again normal subgroups. The intersection of two subnormal subgroups of a group is again subnormal. Now the question arises for their join. In 1939, Wielandt, in his classic paper[7], gave the affirmative answer for finite groups. Then Zassenhaus, in 1958[8], constructed the first example where join of two subnormal subgroups can fail to be subnormal. In this section, we will make a progress on join of subnormal subgroups.

Theorem 1.5.[1] Suppose H and K are subnormal subgroups of G and $J = \langle H, K \rangle$. If K normalizes H i.e. $H^K = H$, then J is subnormal in G. More precisely, if $H \lhd^m G$ and $K \lhd^n G$, then $J \lhd^{mn} G$.

Proof. H_i is the i^{th} normal closure of H in G. First, we will prove that every K normalizes H_i for all i.

We go by induction to prove this. We will apply induction on i.

For $i = 0$, $H_0 = G$

$$G^K = G$$

Suppose K normalizes H_i for some $i \geq 0$, we have to prove for K normalizes H_{i+1}.

Given $H_i^K = H_i$

To prove $H_{i+1}^K = H_{i+1}$ .... (1)

We know that $H_{i+1} = H^{H_i}$

We take left side of equation (1).

$$H_{i+1}^K = (H^{H_i})^K = H^{(H_i^K)} = H^{H_i} = H_{i+1}$$

So $H_{i+1} \lhd H_i K$.

Observe that $K \lhd^n H_i K$ (using proposition 1.2).

And by proposition 1.4, we see that $H_{i+1}K \lhd^n H_i K$.

Then by proposition 1.3,

$$J = \langle H, K \rangle \ = HK = H_m K \lhd^{mn} H_0 K = G$$

Corollary 1.6.[1] Suppose $H \lhd^2 G$ and K is subnormal in G, then their join $J = \langle H, K \rangle$ is always subnormal in G.

Proof. Since $H \lhd H^G \lhd G$, it follows from Proposition 1.1,

We get $H^g \lhd H^G$ for all $g \in G$ (follows from H is normal in H^G) .

So $H^K \lhd H^G \lhd G$ (Normal subgroups in H^G forms lattice) and since K normalizes H^K , we get $J = \langle H, K \rangle = \langle H^K, K \rangle$ is subnormal in G with subnormal defect $2n$ by previous theorem.

Theorem 1.7.[1] G has two subnormal subgroups H, K and $J = \langle H, K \rangle$. Then the following conditions are equivalent:

(i) $J \lhd \lhd G$.

(ii) $H^K \lhd \lhd G$.

(iii) $[H, K] \lhd \lhd G$.

Proof. $(i) \Longrightarrow (ii)$:

J is subnormal in G. We claim H^K is normal in J then Proposition 1.3 will enable us to prove (ii) from (i).

$J = \langle H, K\rangle$, clearly $(H^K)^K = H^K$ and $H \leq H^K$

So $(H^K)^J = H^K \lhd J \lhd \lhd G$.

$(ii) \implies (iii)$:

Given : H^K sn G.

To prove : $[H, K]$ sn G.

Claim: $[H, K]$ is normal in H^K.

Clearly, H and K both normalizes $[H, K]$. Now see that $h^k = h[h, k]$ (enough to prove the claim).

So by proposition 1.3, we get $[H, K]$ sn G.

$(iii) \implies (i)$

Given: $[H, K]$ sn G.

To prove: $J = \langle H, K\rangle$ sn G.

Since K normalizes $[H, K]$. By theorem 1.5, we get $[H, K]K$ sn G.

However H normalizes $[H, K]K$ and again by the Theorem 1.5 we have

$$J = H[H, K]K \ sn \ G$$

We will be using a result (every subgroup of nilpotent groups are subnormal) which will be proved later in second chapter.

Corollary 1.8.[1]

If H, K are two subnormal subgroups of a group G and the derived subgroup G' is nilpotent then $\langle H, K\rangle$ is subnormal in G.

Proof. Since $[H, K] \leq G'$ and all subgroups of a nilpotent group are subnormal, we have [H,K] sn $G' \lhd G$. So $[H, K] \lhd^m G$ for some m.

By the previous theorem we have J is also subnormal in G.

Proposition 1.9.[1]

Let H and K be two subnormal subgroups of a group G and $J = \langle H, K\rangle = HKH$, then $J = HK$.

Proof. Let $H \lhd^m G$. We will proceed by induction on m. The result is clear for $m = 0$ *or* 1 Assume $m \geq 2$ and the usual induction hypothesis. Then with $H^J = H_1$,

$$H_1 = H(H_1 \cap KH) = H(H_1 \cap K)H = H(H_1 \cap K),$$

By induction , since $J = H_1K = H(H_1 \cap K)K = HK$.

1.4 Wielandt's Join Theorem

Wielandt's Join Theorem states that the join of two subnormal subgroup in a finite group is again subnormal. In this section, we will give three proofs of this theorem.

Theorem 1.10.[1] Let H, K be two subnormal subgroups of a finite group G. Then $J = \langle H, K \rangle$ is subnormal in G.

Definition.
A group G is said to satisfy $Max - sn$ or *maximal condition for subnormal subgroups* if there exist at least one maximal element for any non-empty set of subnormal subgroups of G.
Certainly finite groups satisfy $Max - sn$.

First proof of Theorem 1.10

Theorem 1.11[1] G saisfies $Max - sn$ and H and K are subnormal subgroups of G. Then $J = \langle H, K \rangle$ is also subnormal subgroup in G.

Proof. Given: $H \lhd^n G$ and $K \lhd^m G$.
We will prove by induction on n, the subnormal defect of H in G.
For $n = 1$, $H \lhd G$ and by proposition 1.4 we got $J = \langle H, K \rangle = HK \ sn \ G$
Suppose $n \geq 2$ and assume the induction hypothesis.
We have $H \lhd^{n-1} H^G$ and apply second induction on r, the join of any finite number

r of conjugates of H in G is subnormal in G.
Since G satisfies $Max-sn$, it follows that H^K is generated by finitely many conjugates of H by elements of K and by induction hypothesis, join of finitely many conjugates of H in G is subnormal in G, thus H^K is subnormal in G.

Second proof of Theorem 1.10

Theorem 1.12.[1] Let H, K be subnormal subgroups of G and suppose that the set of subnormal subgroups of G lying between H and $J = \langle H, K\rangle$ contains a maximal member. Then J *sn* G.

First we prove a lemma which will bolster in the proof of theorem.

Lemma 1.13.[1] Let T be a subgroup of G and suppose that H is a maximal member of the set of subnormal subgroups of G lying in T. Then $H \lhd T$.

Proof. We wiil go by the subnormal defect n of H in T. By contradiction; suppose $n \geq 2$

$$H = H_n \lhd H_{n-1} \lhd H_{n-2} \lhd .... \lhd H_1 \lhd H_0 = T$$

H_i is the i^{th} normal closure of H in T.
There exists $x \in H_{n-2}$ such that x does not belong to H_{n-1}.
H^x normalizes H since $H \lhd H_1 = H^{H_{n-2}}$
Realize that $H^x \neq H$. So

$$H < HH^x = H^*$$

Then $H^* \leq T$ and by Theorem 1.5, H^* *sn* G. Contradiction to H being maximal member of the set of subnormal subgroups of G lying in T.
Hence $H \lhd G$.

Proof of Theorem 1.12 Suppose M is the maximal member of the set of subnormal subgroups of G lying between H and $J = \langle H, K\rangle$. Then by previous lemma, we see that $M \lhd J$ and $H \leq M$ so $J = \langle M, K\rangle$ and $M^K = M$. Both M and K are subnormal in G. Then by Theorem 1.5, we conclude that $J = \langle M, K\rangle = \langle H, K\rangle$ is subnormal in G.

Corollary 1.14.[1] Let H, K be two subnormal subgroups of G and $J = \langle H, K\rangle$. Suppose the index $|J : H|$ is finite then J is subnormal in G.

Proof. The proof is the same as the proof of Theorem 1.12.

Theorem 1.15.[1] Let H and K be subnormal subgroups of G and $J = \langle H, K\rangle$. Suppose that the set

$$\{\langle H, H^{x_1}, H^{x_2}, ..., H^{x_s}\rangle | x_1, x_2, ...x_s \in J, s \geq 0\}$$

satisfies the maximal condition. Then J *sn* G.

Proof. Suppose M is the maximal member of the set of subnormal subgroups of G which belong to above mentioned set. Then $M \lhd J$ and Theorem 1.5 establishes the result.

Theorem 1.16.[1] Let $\{H_\lambda | \lambda \in \Lambda\}$ be a set of subnormal subgroups of G and let J be their join. Then J is subnormal in G if and only if the set of subnormal subgroups of G lying in J contains a maximal member.

Proof. If J is subnormal in G, J is the maximal member lying in J.
Conversely, suppose K is the subnormal subgroup of G maximal with respect to lying in J. By Lemma 1.13, $K \lhd J$ and so $H_\lambda K \lhd\lhd G$ by Theorem 1.5.
So $H_\lambda \leq M$ for all $\lambda \in \Lambda$ amd hence $J = M \lhd\lhd G$.

Theorem 1.17[1] Let N be minimal normal subgroup of a finite group G. Then N normalizes every subnormal subgroup of G.

Proof. A normal subgroup N is said to be *minimal normal* if N does not have any non-trivial normal subgroup in G.

We will go by induction on $|G|$. Suppose K is subnormal in G, $K \neq G$ and put $K_1 = K^G$. So $K_1 < G$.

Now if $N \nleq K_1$, then $N \cap K_1 = 1$ and thus $[N, K_1] = 1$.

Hence $[N, K] \leq [N, K_1] = 1$ implies $[N, K] = 1$.

Now if $N \lhd K_1$, then $N = {N_1}^G$ for some minimal normal subgroup N_1 of K_1 (N_1 could be N also). So each conjugate ${N_1}^g$, for $g \in G$, will be a minimal normal subgroup of K_1 and so be a minimal normal subgroup of H_1 and so will normalize H, by induction. Hence N normalizes H.

Third proof of Theorem 1.10[1] We proceed by induction on $|G|$ and let N be a minimal normal subgroup of G. By induction JN/N *sn* GN/N and so, by Theorem 1.17, $J \lhd \lhd G$.

Theorem 1.18[1] Suppose H and K are subnormal subgroups of G and G' satisfies $Max - sn$. Then $J = \langle H, K \rangle$ *sn* G.

Proof. Suppose n is the subnormal defect of H in G. Let $H_1 = H^G$ so that ${H_1}' = G'$ and therefore ${H_1}'$ satisfies $Max - sn$. Therefore,by induction on n and second induction on r,

$$\mathcal{X} = \{\langle H^{x_1}, H^{x_2}, ..., H^{x_r} \rangle | x_1, x_2, ..., x_r \in J, r \geq 1\}$$

is a set of subnormal subgroups of G. We show that $H^K \in \mathcal{X}$ and then Theorem 1.7 applies.

Suppose $H^K \notin \mathcal{X}$. Then there exists an infinite strictly ascending chain of subgroups

$$X_1 < X_2 < ......$$

with $X_i = \langle H, H^{x_1}, H^{x_2}, ..., H^{x_i} \rangle$ where $x_1, ..., x_i \in K$. Suppose $M_i = \langle [H, x_1], [H, x_2], ..., [H, x_i] \rangle$, then

$$M_i \lhd \langle H, M_i \rangle = X_i \ sn \ G$$

and so M_i sn G'. From the hypothesis, it follows that there exists $s \geq 1$ such that $M_s = M_{s+1} = M_{s+2} = ...$ and then $X_s = X_{s+1} = X_{s+2} = ...$, contardicting the assumption.

Chapter 2

Groups With All Subgroup Subnormal

We will make a progress studying all those groups which have every subgroup subnormal. We will denote the class of groups which have every subgroup subnormal by $\mathcal{N}_1$. Our aim is to construct an example of non-nilpotent $\mathcal{N}_1$ groups.

2.1 Classes of groups

Class of Groups- A *class of groups* is a family of groups which is closed under isomorphism and contains trivial group.

Examples - Trivial group, Class of abelian groups, Class of nilpotent groups, Class of soluble groups, Class of groups in which every subgroup has every subgroup subnormal etc.

Countable recognition

A class of groups $\mathcal{S}$ is *countably recognizable* if a group G belongs to $\mathcal{S}$ provided that all countable subgroups of G belong to $\mathcal{S}$.

2.2 Groups having every subgroup subnormal

Clearly, the class of abelian groups (denoted by $\mathcal{R}$) is contained in the class $\mathcal{N}_1$. A group where every subgroup is normal also belongs to $\mathcal{N}_1$. We will denote this class by $\mathcal{C}$. This result is due to Dedekind and Baer.

Dedekind and Baer[3]
All the subgroups of a group G are normal if and only if G is abelian or the direct product of a quaternion group of order 8, an elementary abelian 2-group and an abelian group with all elements of odd order.

2.3 Nilpotent Groups

First we will define a series of normal subgroups

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq ...... \supseteq G_m \supseteq .....$$

$$G_0 = G, G_{n+1} = [G, G_n]$$

We write $\gamma_i(G) = G_{i-1}$

Definition
A group G is said to be nilpotent if $G_n = 1$ for some n. The smallest n such that $G_n = 1$ is called the nilpotency class of G.

Theorem 2.1. Every subgroup of a nilpotent group is subnormal.

Proof. Let G be a nilpotent group and H be its subgroup.
ith normal closure of H in G is $H[G,_i H]$. (by Proposition 1.1)
But $[G,_n H] \leq [G,_n G] = 1$ for some n.
So H coincides with its n^{th} normal closure which means to say that H is subnormal in G.

The subnormal defect of any subgroup of a nilpotent group does not exceed the nilpotency class of the nilpotent group. Much more can be said when the group is finite.

Theorem 2.2. Let G be a finite group. All of its subgroups are subnormal if and only if G is nilpotent.

Proof. Proof immidiately follows from the following result:
A finite group $G$is nilpotent if and only if for every proper subgroup of H of G, we have $N_G(H) > H$.

Theorem 2.3.[2] Let $1 \leq c \in \mathbb{N}$. The following classes of groups are countably reognizable: nilpotent groups, nilpotent groups of class atmost c.

Now we have seen that all finite $\mathcal{N}_1$ groups are nilpotent. So the natural question arises whether the converse is true or not. We will answer it in the following section.

2.4 Non-Nilpotent $\mathcal{N}_1$ Groups

Two completely disparate methods have been developed for establishing non-nilpotent $\mathcal{N}_1$ groups. The first method goes to a celebrated 1968 paper[4] by H. Heineken and I.J. Mohamed while the second was discovered by H. Smith[9] in 1982.[2]

2.4.1 Remarks

H. Heineken and I.J. Mohamed provided the first example of non-nilpotent $\mathcal{N}_1$ groups[4]. The groups they constructed are p-groups and every proper subgroup is nilpotent and obviously subnormal. We call non-nilpotent groups which have every proper subgroup subnormal and nilpotent Heineken- Mohamed groups (in short H-M groups).

Heineken and Mohamed construction was studied by many authors. Here, we will present a construction of H-M groups by F. Menegazzo [5] but before that we will prove the following fact.

Proposition 2.4.[4] Suppose p is a prime and G is a $p-group$ of Heineken-Mohamed type such that $G \neq G^{'}$. Then
(i) G is countable;
(ii) $G/G^{'} \simeq C_{p^{\infty}}$ and $(G^{'})^{p} \neq G^{'} = \gamma_3(G)$;
(iii) for every $H \leq G$, $G^{'}H = G$ implies $H = G$.

Conversely, if G is a non-nilpotent p-group with a normal nilpotent subgroup N of finite exponent such that $G/N \simeq C_{p^{\infty}}$ and $NK \neq G$ for every proper subgroup K of G, then G is a group of Heineken- Mohamed type.

Lemma2.5.[6] Suppose G is a non-trivial group such that $ST \neq G$ for every pair of proper normal subgroups S and T of G. Then there is a prime number p such that $G/G^{'}$ is locally cyclic p-group and $G^{'} = \gamma_3(G)$.

Proof. We know that $G/G^{'}$ is abelian. Suppose $G/G^{'}$ is generated by two proper normal subgroups then G is also generated by two proper normal subgroups (not the case).
We start with the case when G is abelian. Let $1 \neq x \in G$; then there is a prime p such that $< x^p > \neq < x >$.
Consider

$$\mathcal{B} = \{K \ s.t. \ K < G, < x^p > \subseteq \ K \ but \ < x > \nsubseteq K\}$$

Certainly $< x^p > \in \mathcal{B}$, so $\mathcal{B}$ is non-empty. It is partially ordered set by inclusion so it contains at least one maximal element S by Zorn's Lemma. Then S is a maximal subgroup of G such that $x^p \in S$ but x does not belong to S.

Then every non-trivial subgroups of G/S contain xS. Since G/S is abelian, we have that G/S is either a non-trivial cyclic p-group or isomorphic to C_{p^∞}. If G is not a p-group there exists $s \in G$ and prime $q \neq p$ such that $< s^q > \neq < s >$. Similarly we get a proper subgroup T of G such that G/T is a q-group. But, then clearly, $G = ST$ (not the case).

So G is a p-group with either being cyclic or of type C_{p^∞}.

For general case, we have to check $G^{'} = \gamma_3(G)$. But this follows from $T = G/\gamma_3(G)$ being nilpotent group and $T/T^{'} \simeq G/G^{'}$ is cyclic or Prufer group, and so T is abelian.

Lemma 2.6.[1] Let M and N be two normal nilpotent subgroup of a group G. Then their join $\langle M, N\rangle = MN$ is also nilpotent.

Proof. The commutator identity $[mn, l] = [m, l]^n[n, l]$ shows that if L is also a normal subgroup of G then $[MN, L] = [M, L][N, L]$. For any group Y, and for $i \geq 1$, we write

$$\gamma_i(Y) = [Y,_i Y]$$

the i^{th} term of lower central series.

Suppose M and N have nilpotency class c and d respectively. By induction on j we get

$$\gamma_j(MN) = \Pi[Y_1, Y_2, ...Y_{j+1}]$$

where Y_i, for each i, is either from M or from N and all 2^{j+1} possible factors in products are included. Taking $j = c + d$, each factor must contain $c + 1$ M's or $d + 1$ N's among Y_i. Since $\gamma_c(M) = \gamma_d(N) = 1$ and $M, N \lhd G$ so is each $\gamma_i(M)$ and $\gamma_i(N)$. So $\gamma_{c+d}(MN) = 1$.

Lemma 2.7. Let G be a $\mathcal{N}_1$-group, and M a normal nilpotent subgroup of G. If G/M is finitely generated, then G is nilpotent.

Proof. Let G be a $\mathcal{N}_1$-group, and M be a normal nilpotent subgroup such that

G/M is finitely generated. Let $y_1M, y_2M, ..., y_nM$ be a set of generators of G/M. Then, since G is a $\mathcal{N}_1$-group, $H = \langle y_1, y_2, ...y_n \rangle$ is a nilpotent subnormal subgroup of G. So $G = MH$ is nilpotent (by lemma 2.6).

Proof of Proposition 2.4.

(i) If G satisfies the condition given in proposition 2.4, then G can not be finite. If G is countable then we are done. Suppose G is uncountable and all of its countable subgroups are nilpotent then by Theorem 2.3, G is nilpotent (not the case). So G must be countable.

(iii) Since $G \neq G'$, then ,by lemma 2.6, for every $H \leq G$, $G'H = G$ implies $H = G$.

(ii) By Lemma 2.7 , G/G' is not finitely generated and also by Lemma 2.6, G is not the product of two proper normal subgroups and therefore $G/G' \simeq C_{p^\infty}$ by Lemma 2.5. Suppose ${G'}^p = G'$. Then, by Lemma 2.5, G' is an abelian divisible group. Every cyclic subgroup Y of G is subnormal and so G' is centralized by Y. We conclude that $G' \leq Z(G)$ which implies G is nilpotent. So ${G'}^p \neq G'$.

Conversely, suppose K is any proper subgroup of G. Then $NK \neq G$ and it is also given that $G/K \simeq C_{p^\infty}$, NK/N is finite. Therefore NK is nilpotent which implies K is also nilpotent and subnormal in KN. Since KN is normal in G, so K is subnormal in G. So G is a Heineken-Mohamed group.

2.5 Construction[2]

We will follow Menegazoo's approach [5].

First we choose a prime p and denote the Prufer group C_{p^∞} by T. We fix standard generators $t_1, t_2, t_3, ...$ for T. So

$$T = \langle t_1, t_2, t_3, ... \mid {t_1}^p = 1,\ {t_{i+1}}^p = t_i\ \forall\ i \geq 1 \rangle$$

We write $T_i = \langle t_i \rangle$ for every $i \geq 1$. $R = \mathbb{F}_p[T]$ is the group algebra of T over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and we denote its augmentation ideal by $\mathfrak{T}$.

Augmentation ideal is the kernel of the map $\phi : R \to \mathbb{F}_p$ defined by

$$\phi(\sum_{t \in T} s_t t) = \sum_{t \in T} s_t$$

Claim: $\mathfrak{T}$ is generated by elements of type $t - 1$ for $t \in T$.

Clearly $\langle t - 1 \mid t \in T \rangle \subseteq \mathfrak{T}$

Take any element $s = \sum_{t \in T} s_t t \in \mathfrak{T}$, so $\sum_{t \in T} s_t = 0$

Now

$$s = \sum_{t \in T} s_t t$$

$$= \sum_{t \in T} s_t t + \sum_{t \in T} s_t - \sum_{t \in T} s_t$$

$$= \sum_{t \in T} s_t (t - 1)$$

For every $i \geq 1$, we keep $R_i = \mathbb{F}_p[T_i]$ and we reserve $\mathfrak{T}_i$ for augmentation ideal of R_i.

Then the followings are easy to verify

(i) $R = \cup_{i \geq 1} R_i$

(ii) $\mathfrak{T} = \cup_{i \geq 1} \mathfrak{T}_i$

(iii) $\mathfrak{T}_i = (t_i - 1) R_i$ for $i \geq 1$

Every elements of $\mathfrak{T}$ are nilpotent (follows from $(t_i - 1)^{p^i} = 0$). Any element in R which is not in $\mathfrak{T}$ is invertible.

Take $s = \sum_{t \in T} s_t t \in R$ but $s \notin \mathfrak{T}$ then

$$\sum_{t \in T} s_t \neq 0$$

$$s = \sum_{t \in T} s_t t$$

$$= \sum_{t \in T} s_t t + \sum_{t \in T} s_t - \sum_{t \in T} s_t$$

$$= \sum_{t \in T} s_t (t - 1) + \sum_{t \in T} s_t$$

So any element s can be written in the form of $r + b$ where r is nilpotent and b is invertible. The rest work is elementary ring theory.

Lemma 2.8.[2] The ideals of R_i are principal ideals of the form

$$(t_i - 1)^m \ for \ 0 \leq m \leq p^i.$$

All these ideals are distinct and with respect to inclusion, they form a totally ordered set.

Proof. The proof follows from the fact that $R_i \simeq \frac{\mathbb{F}_p[x]}{\langle x^{p^i} - 1 \rangle}$.

Lemma 2.9.[2] The set of ideals in R is totally ordered set.

Proof. Take $a, b \in R$. There exists an $i \geq 1$ such that $a, b \in R_i$. But by previous Lemma either $aR_i \leq bR_i$ or $aR_i \leq bR_i$. Thus the Lemma is proved.

We can parametrize the set of all ideals of R with the help of Lemma 2.8. Let $\mathcal{J}$ be any ideal of R then for each $i \geq 1$, there is a unique $0 \leq m_i \leq p^i$ such that

$$\mathcal{J} \cup R_i = (t_i - 1)^{m_i} R_i$$

So, we associate the sequence $(m_1, m_2, m_3, ...)$ to $\mathcal{J}$. This sequence uniquely determines $\mathcal{J}$. Observe that, since $(\mathcal{J} \cap R_{i+1}) \cap R_i = \mathcal{J} \cap R_i$, the sequence is such that,

$$p(m_i - 1) < m_{i+1} \leq pm_i \ for \ all \ i \geq 1.(*)$$

Conversely, a sequence $(m_1, m_2, m_3, ...)$ of integers $0 \leq m_i \leq p^i$ satisfying (*) is the sequence associated to the ideal $\sum_{i \geq 1} (t_i - 1)^{m_i} R$ of R.

Lemma 2.10.[2] Suppose $\mathfrak{J}$ is a non zero ideal of R and let $(s_1, s_2, s_3, ...)$ be the sequence associated to $\mathfrak{J}$. Then $\mathfrak{J}\mathfrak{T} = \mathfrak{J}$ iff for each $i \geq 1$, there exists a $j \geq i$ such that $ps_j > s_{j+1}$.

Proof. Suppose that the sequence given for $\mathfrak{J}$ satisfies the condition in the statement and let $i \geq 1$. Now choose such $j \geq i$ for that $ps_j > s_{j+1}$. Then,

$$(t_j - 1)^{s_j} = (t_{j+1} - 1)^{ps_j} = (t_{j+1} - 1)^{s_{j+1}}(t_{j+1-1})^t$$

for some $t > 0$.

We conclude that $(t_j - 1)^{s_j} \in \mathfrak{J}\mathfrak{T}$ and consequently, $(t_i - 1)^{s_i} \in \mathfrak{J}\mathfrak{T}$. Therefore $\mathfrak{J}\mathfrak{T}$ has the same sequence as $\mathfrak{J}$ and so $\mathfrak{J}\mathfrak{T} = \mathfrak{J}$.

Conversely, assume $\mathfrak{J}\mathfrak{T} = \mathfrak{J}$ and let $i \geq 1$ with $(t_i - 1)^{s_i} \neq 0$. Then there exists $m \geq i$ such that $(t_i - 1)^{s_i} \in (\mathfrak{J} \cap R_m)\mathfrak{T}_t$. Since $\mathfrak{J} \cap R_m = (t_m - 1)^{s_m} R_m$ and $\mathfrak{T}_m = (t_m - 1)R_m$, we have that $(t_i - 1)^{s_i} = (t_m - 1)^{s_m+1}v$ for some $v \in R_m$. Hence $(t_i - 1)^{s_i} R < (t_m - 1)^{s_m} R$, and so in the chain

$$(t_i - 1)^{s_i} R \leq (t_{i+1} - 1)^{s_{i+1}} R \leq .... \leq (t_m - 1)^{s_m} R$$

at least one of the inclusion is proper $((t_m - 1)^{s_m} R < (t_{j+1} - 1)^{s_{j+1}} R)$, which means $k_{j+1} < pk_j$.

Now we will define an HM-system. Take M to be a right R-module which is generated by a sequence $\mathbf{x} = (x_i)_{i \geq l}$ for some positive integer l. For any sequence $\mathbf{m} = (m_i)_{i \geq l}$ of elements of M, we set

$$\tau_{i,k}(\mathbf{m}) = -m_i + \sum_{s=0}^{k} x_{i+s}(t_{i+s} - 1)^{p^s - 1} + m_{i+k+1}(u_{i+k+1} - 1)^{p^{k+1} - 1}$$

for all $i \geq l, k \geq 0$.

Then we say that $\mathbf{x}$ is a HM-system in M is

$$M = \langle \tau_{i,k}(\mathbf{m}) \mid i \geq l, k \geq 0 \rangle$$

for every sequence $\mathbf{m} = (m_i)_{i \geq l}$.

Proposition 2.11[5] Suppose G is a $p-group$ with a normal elementary abelian subgroup $N \neq 1$ such that $[G, N] = N$ and $G/N \simeq C_{p^\infty} = T$. Let $\xi : T \to G/N$ be an isomorphism, and make N into a $R-module$ in the usual way. For every $i \geq 1$, let $g_i \in R_i$ such that $g_iN = t_i^{\xi}$, and let $x_i = {g_i}^{-1}{g_{i+1}}^p \in N$. Suppose that $G = \langle g_i \mid i \geq l \rangle$ for some $l \geq 1$. If the sequence $\mathbf{x} = (x_i)_{i \geq l}$ is a $HM - sequence$ for N then G is a Heineken-Mohamed group.

Proof. Since $[G, N] = N \neq 1$, then G is not nilpotent. So, by Proposition 2.4, it suffices to show that $KN = G$ forces K to be equal to G for every $K \leq G$. For $n \in N$ and $t \in T$, we denote $n^t = n^{t^{\xi}}$, and $\forall\ i \geq l, k \geq 0$, we set

$$\chi_{i,k} = \prod_{s=0}^{k} {x_{i+s}}^{(t_{i+s}-1)^{p^s-1}}.$$

We show by induction on $k \geq 0$, that ${g_{i+k+1}}^{p^{k+1}} = g_i\chi_{i,k}$ for all $i \geq l$.
For $k = 0$, $\chi_{i,0} = x_i$ [Obvious from the fact $x_i = {g_i}^{-1}{g_{i+1}}^p$]
Let $k \geq 1$ and assume ${g_{i+k}}^{p^k} = g_i\chi_{i,k-1}$.
Then

$${g_{i+k+1}}^{p^{k+1}} = ({g_{i+k+1}}^p)^{p^k} = (g_{i+k}x_{i+k})^{p^k}$$

$$= {g_{i+k}}^{p^k}{x_{i+k}}^{{t_{i+k}}^{p^k-1}+....+t_{i+k}+1}$$

$$= g_i\chi_{i,k-1}{x_{i+k}}^{{t_{i+k}-1}^{p^k-1}} = g_i\chi_{i,k}.$$

Now, let $K \leq G$ with $NK = G$. Then $\forall\ i \geq l$, K contains an element of the form g_im_i with $m_i \in N$. Let $\mathbf{m}$ be the sequence $(m_i)_{i \geq l}$. For every $i \geq l, k \geq 0$, denoting $\tau_{i,k} = \tau_{i,k}(m)$ and using the identities established above, we get

$$(g_{i+k+1}m_{i+k+1})^{p^{k+1}} = {g_{i+k+1}}^{p^{k+1}}{m_{i+k+1}}^{(t_{i+k+1}-1)^{p^{k+1}-1}} = g_i\chi_{i,k}{m_{i+k+1}}^{(t_{i+k+1}-1)^{p^{k+1}-1}}$$

$$= g_im_i({m_i}^{-1}\chi_{i,k}{m_{i+k+1}}^{(t_{i+k+1}-1)^{p^{k+1}-1}}) = g_im_i\tau_{i,k}$$

As a consequence, $\tau_{i,k} \in K$ for every $i \geq l$ *and* $k \geq 0$, and therefore K contains the subgroup generated by the elements $\tau_{i,k}$, which is N and since $\mathbf{x}$ is a $HM - system$. Therefore $K \geq NK = G$, and so $K = G$ as wanted.

Proposition 2.12.[2] Suppose $\mathfrak{J}$ is a non-zero ideal of R such that $\mathfrak{J}\mathfrak{T} = \mathfrak{J} < \mathfrak{T}$, and let $(s_1, s_2, s_3, ...)$ be the sequence associated to $\mathfrak{J}$. Keep $l \geq 1$ with $0 < s_l < p^l$, and for each $i \geq l$, set

$$c_i = \begin{matrix} (t_i - 1)^{s_i} & if s_{i+1} = ps_i \\ (t_{i+1} - 1)^{ps_i - 1} & if s_{i+1} < ps_i \end{matrix}$$

Then $\mathbf{c} = (c_i)_{i \geq l}$ is a $HM - system$ for $\mathfrak{J}$ as a $R - module.$

Proof. We will prove that $\mathbf{c}$ generates $\mathfrak{J}$. Let $\mathfrak{I}$ be the ideal generated by $\mathbf{c}$. Then $\mathfrak{I} \leq \mathfrak{J}$. By definition if $s_{i+1} = ps_i$, then $c_i \in \mathfrak{J}$ and, if $s_{i+1} < ps_i$, then $c_i = (t_{i+1} - 1)^{ps_i - 1} \in (t_{i+1} - 1)^{s_{i+1}} R \leq \mathfrak{J}$.
For reverse inclusion, consider first $i \geq l$. If $s_{i+1} = ps_i$, then $R_i \cap \mathfrak{J} = c_i R_i \leq \mathfrak{I}$; if $s_{i+1} < ps_i$,

$$R_i \cap \mathfrak{J} = (t_i - 1)^{s_i} R_i = (t_{i+1} - 1)^{ps_i} R_i = c_i (t_{i+1} - 1) R_i \leq \mathfrak{I}$$

If $1 \leq i \leq l$, then $(t_i - 1)^{s_i} \in (t_l - 1)^{s_l} R \leq \mathfrak{I}$.
Therfore $\mathfrak{J} = \mathfrak{I}$.
Now we have to make sure that $\mathbf{c}$ satisfies the conditions of a $HM - system$ for $\mathfrak{J}$ as a $R - module$. Let $m = (m_i)_{i \geq l}$ be a sequence of elements of $\mathfrak{J}$, and for each $i > l$, $\exists$ $d \geq 0$ such that

$$(t_{i+1} - 1)^{s_i - 1} \in \tau_{i,d} R.$$

This implies that $\mathfrak{J}$ is generated by the set $\{\tau_{i,d} \mid i \geq l, d \geq 0\}$. So $\mathbf{c}$ is a $HM - system$ for $\mathfrak{J}$.
Therefore, suppose $i \geq l$. If $s_i = ps_{i-1}$ then by Lemma 2.10, there exists $j \geq i$ such that $(t_{i-1} - 1)^{s_{i-1}} = (t_{j-1} - 1)^{s_{j-1}}$ and $s_j < ps_{j-1}$. So we may assume $s_i < ps_{i-1}$. Then there exists $h > 0$ such that $m_i \in \mathfrak{J} \cap R_{i+h}$, and there exists $k \geq h$ such that $s_{i+k+1} < ps_{i+k}$. Then $c_{i+k} = (t_{i+k+1} - 1)^{ps_{i+k} - 1}$, and

$$\tau_{i,k} = -m_i + c_i + ... + c_{i+k+1}(t_{i+k+1} - 1)^{p^{k-1} - 1} + y....(**)$$

where $y = c_{i+k}(t_{i+k} - 1)^{p^k - 1} + m_{i+k+1}(t_{i+k+1} - 1)^{p^{k-1} - 1}$
Then

$$y = (t_{i+k+1} - 1)^{ps_{i+k} - 1}(t_{i+k} - 1)^{p^k - 1} + m_{i+k+1}(t_{i+k+1} - 1)^{p^{k+1} - 1}$$

$$= (t_{i+k+1}-1)^{ps_{i+k}-1+p^{k+1}-p} + m_{i+k+1}(t_{i+k+1}-1)^{p^{k+1}-1}$$

$$= (t_{i+k+1}-1)^{p^{k+1}-1}(m_{i+k+1} + (t_{i+k+1}-1)^{p(s_{i+k}-1)})$$

$$= (t_{i+k+1}-1)^{p^{k+1}-1}(m_{i+k+1} + (t_{i+k}-1)^{(s_{i+k}-1)}).$$

Now $m_{i+k+1} \in \mathfrak{J}$ and $(t_{i+k}-1)^{s_{i+k}-1} \notin \mathfrak{J}$, and so it follows from total orderness of R that $(t_{i+k}-1)^{s_{i+k}-1}$ *and* $(t_{i+k}-1)^{s_{i+k}-1} + m_{i+k+1}$ generate the same ideal of R. Hence, there exists an invertible element $\mu \in R$ such that

$$(t_{i+k}-1)^{s_{i+k}-1} + m_{i+k+1} = (t_{i+k}-1)^{s_{i+k}-1}\mu.$$

Therefore, $y = (t_{i+k+1}-1)^{p^{k+1}-1+p(s_{i+k}-1)}\mu$. Rest of the summands in right term of $(**)$ belongs to $\mathfrak{J} \cap R_{i+k}$; hence writing by y' their sum, we get $y' = (t_{i+k}-1)^f\epsilon = (t_{i+k+1}-1)^{pf}\epsilon$ for some $f \geq s_{i+k}$ and some invertible elements $\epsilon \in R_{i+k}$. The exponents of $t_{i+k+1}-1$ in y and in y' are not congruent modulo p, we conclude that the two ideals $y'R$ and yR are distinct. Therefore $\tau_{i,k} = y + y'$ generates the largest of the ideals yR and $y'R$. Particularly,

$$(t_{i+k+1}-1)^{p^{k+1}-1+p(s_{i+k}-1)} = y\mu^{-1} \in \tau_{i,k}R....(***)$$

Now, we have (from $ps_{i-1} \geq s_i + 1$)

$$p^{k+2}s_{i-1} \geq p^{k+1}(s_i+1) \geq ps^{i+k} + p^{k+1} > p^{k+1} - 1 + p(s_{i+k}-1),$$

and by $(***)$, $(t_{i-1}-1)^{s_{i-1}} = (t_{i+k+1}-1)^{p^{k+2}s_{i-1}} \in \tau_{i,k}R$. This proves $(*)$ and the proposition.

Now we proceed to construction of Heineken-Mohamed groups.

Theorem 2.13.[5] For every non-zero ideal $\mathfrak{J}$ of R such that $\mathfrak{J} = \mathfrak{J}\mathfrak{T} < \mathfrak{T}$, there corresponds to a Heineken-Mohamed group $G = G(\mathfrak{J})$ such that $G/G' \simeq T$ and $G' \simeq \mathfrak{J}$ (as R-modules). Moreover, if $\mathfrak{I}$ is another ideal of R with $\mathfrak{I} = \mathfrak{I}\mathfrak{T} < \mathfrak{T}$ and $\mathfrak{J} \neq \mathfrak{I}$, then $G(\mathfrak{J})$ and $G(\mathfrak{I})$ are not isomorphic.

Proof. Suppose $(s_1, s_2, s_3, ...)$ is the associated sequence of $\mathfrak{J}$. Select $l \geq 1$ such that $1 < s_l < p^l$ and for every $i \geq l$ define the element c_i as in Proposition 2.12. Now we will define a sequence $(x_i)_{i\geq l}$ of elements of R inductively such that

$$x_i \in (t_i - 1)R \text{ and } x_{i+1}(t_{i+1} - 1)^{p-1} = x_i + c_i .....(\star)$$

for each $i \geq l$. Set $x_l = 0$, and assume $x_l, x_{l+1}, ...x_i$ have the desired properties. Now $s_i \geq s_l > 1$ and c_i is either $(t_i - 1)^{s_i}$ or $(t_{i+1} - 1)^{ps_i - 1}$; then conclude that $c_i \in (t_i - 1)R$ in any case and so there exists $d \in R$ such that $c_i + x_i = (t_i - 1)d = (t_{i+1} - 1)^p d$. By setting $x_{i+1} = (t_{i+1} - 1)d$, a new element has been introduced in the sequence which satisfies $(\star)$.

Now consider the semidirect product $V = R \rtimes T$, where R is the additive group of the ring and for each $i \geq l$, let $g_i = (x_i, t_i)$. Let $G = G(\mathfrak{J})$ be the subgroup of V generated by all the g_i's:

$$G = \langle (x_i, t_i) \in V \mid i \geq l \rangle.$$

So for each $i \geq l$,

$${g_{i+1}}^p = (x_{i+1}(t_{i+1} - 1)^{p-1}, {t_{i+1}}^p) = (x_i + c_i, t_i) = g_i(c_i, 1)$$

and hence $G \cap (R \times 1)$ contains the T-invariant subgroup N generated by the set $\{(c_i, 1) \mid i \geq l\}$ which is isomorphic to $\mathfrak{J}$. Clearly $G/N = \langle g_i N \mid i \geq l \rangle \simeq T$. Since $\mathfrak{J}\mathfrak{T} = \mathfrak{J}$, we get $N = [N, T] = [N, G]$. Finally the sequence $({g_i}^{-1} g_{i+1} = (c_i, 1))\ for\ i \geq l$ is a HM-system for $N \simeq_T \mathfrak{J}$, and we apply proposition 2.11 to say that G is a Heineken-Mohamed group.

Let $\mathfrak{I}$ be another ideal with $\mathfrak{I} = \mathfrak{I}\mathfrak{T} < \mathfrak{T}$. Denote $G_1 = G(\mathfrak{J}), G_2 = G(\mathfrak{I})$. Suppose there exists an isomoprphism $\iota : G_1 \to G_2$. By construction ${G_1}' \simeq_R \mathfrak{J}$ and ${G_2}' \simeq_R \mathfrak{I}$. Now ι induces an isomorphism between $G_1/{G_1}'$ and $G_2/{G_2}'$, which together with the natural isomorphisms with T, gives an isomorphism of T, which we extend by linearity to an isomorphism ϑ of R. Then, for each $x \in \mathfrak{J} = {G_1}'$ and $t \in R$:

$$\iota(xt) = \iota(x)\vartheta(t)$$

. It follows that $Ann_R(\iota(x)) = Ann_R(x)$ for each $x \in \mathfrak{J}$. If $x = (t_i - 1)^{mp^{i-k}}$ with $1 \leq k \leq i$ and $(m, p) = 1$, it is clear that

$$Ann_R(x) = (t_k - 1)^{p^k - m} R.$$

Hence for all $i \geq l$, $Ann_R(\iota(c_i)) = Ann_R(c_i)$ implies $\iota(c_i)R = c_iR$. So we conclude that $\mathfrak{J} = \iota(\mathfrak{J}) = \mathfrak{I}$.

Corollary 2.14. [2]

For every prime p there are 2^{N_0} non-isomorphic group G of Heineken-Mohamed type such that $G/G^{'} \simeq C_{p^\infty}$ and $G^{'}$ elementary abelian.

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