Lepton Mixing, Discrete Symmetry Models and Quark Lepton Complementarity

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Certificate of Examination

This is to certify that the dissertation titled **Lepton Mixing, Discrete Symmetry Models and Quark Lepton Complementarity** submitted by **VISHNU PK** (Reg. No. MS12127) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 21,2017

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Ketan Patel at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Ketan Patel
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Abstract

By analyzing the Quark mixing and the Lepton mixing one can find an empirical relation that exists between the solar mixing angle (θ_s) and the Cabibbo angle (θ_c), which is $\theta_s + \theta_c \approx \pi/4$, called Quark-Lepton complementarity (QLC). QLC suggests a possible existence of Quark-Lepton unification. In literature, it has already been shown that such an empirical relation can be obtained from the Grand unified theories.

We discuss an alternative approach in which such a relation emerges only from the group theoretical consideration of the lepton mixing. We assume that the lepton mixing are dominantly given by Bi-maximal mixing and then the corrections from the charged leptons will generate a QLC like relation. Such corrections are also assumed to be fixed by group theoretical constraints. After scanning several discrete subgroups of $SU(3)$ (of order < 2000) we find that the corrections from the charged leptons sector must be in terms of more than one angle to get a viable PMNS matrix. As one of the consequences of the exercise, we find that $\theta_{23}^{pmns} > \pi/4$, which can be confirmed or ruled out from the currently ongoing experiments.

Contents

Chapter 1

Introduction

The fundamental particles of nature interact via four forces, which are strong, weak, electromagnetic and gravitational force. The interaction of elementary particles via former three forces (that mentioned above) can be explained by the Standard Model(SM), which is an extraordinarily succeful theory, based on quantum field theoritical framework.

In the SM, particles are of two types, fermions and bosons. Fermions are the half integer spin particles, which are the constituent of matter. Bosons are the integer spin particle and considered as the force mediators between the particles.
Fermions are of two types, quarks and the leptons. Quarks particles have color charge, so they can interact via strong force. Quarks also participitate in electro weak interaction. But in the case of leptons, which do not posses any color charge, so they cann't participitate in strong interaction. But they participitate in eletroweak interactions.
Bosons also are of two types, gauge bosons and Higgs bosons. Gauge bosons can be considered as the force carrieres between the particles,eg. $W^{\pm}$, Z^0 and photon. Higgs bosons are scalar bosons, which are associated with the fundamental field called Higgs field. These fundamental particles came to have mass via interacting with the Higgs field.
In the SM the Yukawa interaction terms will lead to mass terms for the fermions after the sponatneous symmetry breaking of the gauge group $SU(2)_L \times U(1)_Y$ into elctromagenetic gauge group $U(1)_{em}$. Due to the non equivalence of flavour basis(in which elctroweak interaction terms are diagonal) and mass basis(in which mass matrix of is diagonal) of quarks, a unitary matrix will introduce in charged current weak interaction term for the quarks as a consequence of the basis trasnformation from flavour to

mass basis, this unitary matrix is called CKM matrix. There are four independent parameters in the CKM matrix (three angles and one phase), in which one angle is observed to be comparively larger than other two angles, called the Cabibbo angle θ_c, which is about 13 degrees [1](in standrd parametrized form of the CKM matrix). In the SM framework, there will not be any corresponding mixing matrix in the lepton sector. This is because the neutrinos are remains massless in the SM. But the experimental evidence for the neutrino oscillations suggests that neutrinos are massive. So one need to modify the SM to incorporate the mass terms for the neutrinos. In this extended model, corresponding to CKM matrix there will be a mixing matrix in the lepton sector, called PMNS matrix. If the neutrinos are Majorana type, then the PMNS matrix will have two additional phases compare to the CKM matrix, called the Majorana phases. But in the case of Dirac type of neutrinos, there will be four independent parameters in the PMNS matrix (similar to the case of quarks). The angle parameters of PMNS matrix consists of a large angle θ_{23}(about 41.6^0) which is nearly maximal, called the atmospheric angle, a relatively large angle θ_{12}(about 33.56^0) but not maximal called the solar mixing angle and a small angle $\theta_{13}(8.46^0)$ called the reactor angle[2]. These wide range of angles (from small to nearly maximal) in the case of PMNS matrix are in a sharp contrast to the CKM matrix parameters. An understanding of this sharp contrast between the mixing patterns of the quarks and the leptons considered as a major challenge in physics.

QLC can be considered as an approach to this problem. This is based on the empirical relations that

$$\theta_s + \theta_c \approx \frac{\pi}{4} \tag{1.1}$$

$$\theta_{23}^{pmns} + \theta_{23}^{ckm} \approx \frac{\pi}{4} \tag{1.2}$$

This emirical relations suggest that

$$Lepton \quad mixing = BM \quad mixing - quark \quad mixing \tag{1.3}$$

this is called QLC.

In literature it has already been shown that such relations can emerge from models using quark-lepton unification and discrete groups [8]. They assumed that in the leading order the PMNS matrix is exactly BM mixing and the CKM matrix is equivlent to Identity. In the next leading order, the down quark mass matrix and charged lepton mass matrix are equal(or nearly equal) in such a way that both mixing matrices corrected by $O(\theta_c)$ corrections. In this way, a QLC emrge from quak-lepton unification

at high scale.
In this work we will propose an alternative approach in which QLC emerges only from the group theoretical considerations of the lepton mixing. In this approach we assume that the PMNS matrix is dominantly Bi-maximal mixing (coming from the neutrino sector) and then the corrections of CKM matrix like (coming from charged lepton sector) will generate the lepton mixing matrix. In our approach these corrections also constrained by the group theoretical considerations.

This report is organized in the following way

- In chapter 2 we will discuss the mixing matrices and their properties .
- In chapter 3 we will focus on the relation between the discrete symmetries and the lepton mixing matrix.
- Chapter 4 we will the QLC and our alternative approach towards it. The results of the anlysis and the conclusions also will be covered in that chapter.
- In the Appendix A, we will give a brief introduction to the discrete subgroups of $SU(3)$.

Chapter 2

Quark Mixing and Lepton Mixing

The nonequivalence of flavour basis and mass basis will result in flavour changing charge current terms in the weak interaction. This can be represented by a unitary matrix (called mixing matrices). In this chapter, we will discuss this mixing matrices in the lepton sector as well as in the quark sector.

2.1 Quark Mixing

The Yukawa interaction term in the Standard Model (SM) for the quark sector can be written as

$$-L_y^{quark} = y_{ij}^d \bar{Q}_{Li}^f \phi d_{Rj}^f + y_{ij}^u \bar{Q}_{Li}^f \widetilde{\phi} u_{Rj}^f + h.c \tag{2.1}$$

Where $y^{u,d}$ are the Yukawa coupling constants for the up quark and down quark respectively, Q_L^f is the left-handed(LH) quark doublet in the flavour basis(where $Q = (u,d)^T$), ϕ is the Higgs field doublet ($\phi = \frac{1}{\sqrt{2}}(\phi^+, \phi^0)^T$),$\widetilde{\phi} = i\sigma_2\phi$ (Where σ_2 is the pauli matrix) and d_R ,u_R are the right-handed(RH) down quark and up quark singlets respectively.

After the spontaneous symmetry breaking of electroweak gauge group $SU(2)_L \times U(1)_Y$ into $U(1)_{em}$ (by ϕ taking a vacuum expectation value $\phi_{vev} = \frac{1}{\sqrt{2}}(0,v)^T$),the Yukawa interaction term will lead to the quark mass terms, which can be written as

$$-L_m^{quark} = M_{ij}^d \bar{d}_{Li}^f d_{Rj}^f + M_{ij}^u \bar{u}_{Li}^f u_{Rj}^f + h.c \tag{2.2}$$

where $M^{u,d}$ are the up quark mass matrix and the down quark mass matrix respectively.

The physical basis (mass basis) can be obtained by diagonalyzing $M^{u,d}$ using unitary matrices. Since these mass matrices are complex in nature it will require two unitary matrices to diagonalize each mass matrix.
In the mass basis the corresponding quarks fields can be defined as

$$u^m_{L,R} \equiv V^u_{L,R} u^f_{L,R} \qquad d^m_{L,R} \equiv V^d_{L,R} d^f_{L,R} \tag{2.3}$$

Where $V^{u,d}_{L,R}$ are the unitary matrices which diagonalize $M^{u,d}$. That means

$$V^u_L M^u V^{u\dagger}_R = Diag(m^u_1, m^u_2, m^u_2) \qquad V^d_L M^d V^{d\dagger}_R = Diag(m^d_1, m^d_2, m^d_2) \tag{2.4}$$

Where $m^{u,d}_i$ are the eigen values of $M^{u,d}$ respectively.
This basis transformation from flavour to mass basis will not alter any terms in the SM lagrangian except the ***charged current weak interaction term*** in the quark sector. This charged current weak interaction term will be modified by a unitary matrix. This can be written as

$$-\mathcal{L}^Q_{c.c} = \frac{g}{\sqrt{2}}(\bar{u_L}, \bar{c_L}, \bar{t_L})\gamma^\mu W^+_\mu V_{ckm} \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} + h.c \tag{2.5}$$

Where $V_{ckm} = V^u_L V^{d\dagger}_L$, called **CKM matrix (quark mixing matrix)** .

2.1.1 Standard Parametrization of CKM matrix

The independent parameters in CKM matrix are 4 (three angles + one dirac phase). The standard parametrized form of CKM matrix can be written as:

$$V_{ckm} = R_{23}(\theta_{23}) R_{13}(\theta_{13}, \delta_{CP}) R_{12}(\theta_{12}) \tag{2.6}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{cp}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{cp}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

$$\tag{2.8}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{cp}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{cp}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{cp}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{cp}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{cp}} & c_{23}c_{13} \end{pmatrix} \tag{2.9}$$

Where $c_{ij} = \cos\theta_{ij}, s_{ij} = \sin\theta_{ij}$ and δ_{cp} is the dirac phase (cp phase).

The experimental values of V_{ckm} can be written as [1]

$$= \begin{pmatrix} 0.97434^{0.00011}_{0.00012} & 0.22506 \pm 0.0005 & 0.00357 \pm 0.00015 \\ 0.22492 \pm 0.00050 & 0.97357 \pm 0.00013 & 0.0411 \pm 0.0013 \\ 0.00875^{0.00032}_{0.00033} & 0.0403 \pm 0.0013 & 0.99915 \pm 0.00005 \end{pmatrix} \tag{2.10}$$

2.2 Lepton Mixing

The Yukawa interaction term in the SM for the lepton sector can be written as

$$- L_y^{leptons} = y^l_{ij} \bar{l}^f_{Li} \phi e^f_{Rj} + h.c \tag{2.11}$$

Where y^l is the Yukawa coupling constants for the charged leptons, l^f_L is the LH lepton doublet in the flavour basis (where $l = (\nu, e)^T$), ϕ is Higgs field doublet and e^f_R is the RH SM singlet for the charged lepton in the flavour basis.
Since there is no RH neutrino field in the SM, there will not be any Yukawa term corresponding to this field.
After the Spontaneous symmetry breaking of $SU(2)_L \times U(1)_Y \to U(1)_{em}$, the mass term for the leptons can be written as

$$- \mathcal{L}^l_m = M^l_{ij} \bar{e^f_{Li}} e^f_{Rj} + h.c \tag{2.12}$$

Where M^l is the charged lepton mass matrix, $e^f_{L,R}$ are the charged lepton LH and RH fields in flavour basis respectively. There will not be any mass term corresponding to neutrinos in the SM.

In order to diagonalize M^l one required two unitary matrices. So in the physical basis, the corresponding charged lepton fields can be defined as

$$e^m_{L,R} \equiv V^l_{L,R} e^f_{L,R} \tag{2.13}$$

Where $V^l_{L,R}$ are the unitary matrices which diagonalize M_l.
This transformation from the flavour basis to the mass basis will not affect any other term in SM lagrangian except the charged current weak interaction term for the leptons. The charged current weak interaction term will be modified by a unitry matrix V^l_L . But since the neutrinos are massless in the SM one can absorb this unitary matrix into neutrino field (by redefining the neutrino field). So there will not be any lepton mixing in the SM formalism.

But the evidence for the neutrino oscillations indicates that neutrinos have non-zero mass. In order include this fact, one need to modify the SM. This can be done by extending the fermion sector by including sterile neutrinos and/or extending the Higgs sector. When we consider this extended model one will be able to explain the lepton mixing matrix (similar to quark sector). This lepton mixing matrix is called PMNS matrix and can be defined as

$$U_{pmns} = U_l^\dagger U_\nu \tag{2.14}$$

Where U_l is the unitary matrix which diagonalize $M_l M_l^\dagger$ (where M_l is the charged lepton mass matrix) and U_ν is the unitary matrix which diagonalize M_ν (M_ν is the neutrino mass matrix) in the case of Majorana type neutrinos (because in this case M_ν is a symmetric matrix) and $M_\nu M_\nu^\dagger$ in the case of Dirac type neutrinos.

2.2.1 Standard Parametrization of PMNS matrix

The number of independent parameters in the PMNS matrix will be different for different type of neutrinos.
In the case of Dirac type neutrinos the case will be similar to CKM matrix, that means there will be four independent parameters in the PMNS matrix (3 angles + 1 dirac phase). But in the case of Majorana type neutrinos there will be six independent

parameters in the PMNS matrix (3 angles + 1 dirac phase + 2 majorana phases)

$$U_{pmns} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{cp}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{cp}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.15}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{cp}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{cp}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{cp}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{cp}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{cp}} & c_{23}c_{13} \end{pmatrix} \tag{2.16}$$

Where $c_{ij} = \cos\theta_{ij}, s_{ij} = \sin\theta_{ij}$ and δ_{cp} is the dirac phase. In the case of majorana neutrinos two additional phases also required.

The maximum mixing angle considered to be $\pi/4$, this is because the oscillation probabilities relate to the mixing angle as $\sin^2 2\theta$, so the maximum probability will be in the case where $\theta = \pi/4$ and minimum will be in the case where $\theta = 0$.

Now we will discuss two mixing matrices, which where proposed earlier as a possible candidates for the lepton mixing.

Bi-maximal Mixing (BM)

In the case of Bi-maximal mixing θ_{23}, θ_{12} are maximal and θ_{13} is minimal, that means, $\theta_{23} = \frac{\pi}{4}$, $\theta_{12} = \frac{\pi}{4}$ and $\theta_{13} = 0$. So Bi-maximal mixing can be written as

$$U_{BM} = R_{23}(\frac{\pi}{4})R_{13}(0)R_{12}(\frac{\pi}{4}) \tag{2.17}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{2.18}$$

Tribimaximal Mixing (TB)

In the case of Tribimaximal mixing $\sin^2\theta_{12} = \frac{1}{2}, \sin^2\theta_{23} = \frac{1}{3}$ and $\sin^2(\theta_{13}) = 0$. So tribimaximal mixing can be written as

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{2.19}$$

Now we will discuss the current values of the elements of PMNS matrix.

Current PMNS Matrix within 3σ

The current $|U_{pmns}|$ matrix within 3σ [2] can be written as

$$|U_{pmns}| = \begin{pmatrix} 0.800 \to 0.844 & 0.515 \to 0.581 & 0.139 \to 0.155 \\ 0.229 \to 0.516 & 0.438 \to 0.699 & 0.614 \to 0.790 \\ 0.249 \to 0.528 & 0.462 \to 0.715 & 0.595 \to 0.776 \end{pmatrix} \tag{2.20}$$

Where the standard parameters $(\theta_{12}, \theta_{23}, \theta_{13})^{pmns}$ are $(33.56^{+0.77}_{-0.75}, 41.6^{+1.5}_{-1.2}, 8.46^{+0.15}_{-0.15})$ degrees.

Chapter 3

Lepton Mixing and Discrete Symmetry Models

Discrete groups were used as a tool to study and explain the lepton mixing patterns. In this chapter, we will discuss about how one can use discrete groups to explain the lepton mixing patterns and from the given mixing pattern how one can find the corresponding discrete groups etc...At the end of the chapter we will discuss several examples to clarify the ideas.

3.1 General Formalism

In this section, we will discuss a general formalism [3] of fixing the lepton mixing patterns using discrete groups.

Suppose we have a discrete group G_f (called **flavour group**), under which the underlying theory of the leptons is invariant. And using some symmetry breaking mechanism the flavour group G_f is broke down into G_ν in the neutrino sector and G_l in the charged lepton sector, these groups are called **residual groups**. The generators of the group G_ν (say S_i's) and the generator of the group G_l(say T_l) assume to have the properties that

$$S_i^T M_\nu S_i = M_\nu \qquad T_l^\dagger M_l M_l^\dagger T_l = M_l M_l^\dagger \tag{3.1}$$

Where M_ν is the neutrino mass matrix and M_l is the charged lepton mass matrix. The transpose kind of symmetry transformation in the case of M_ν will preserve the symmetric nature of neutrino mass matrix (in this case neutrinos are assumed to be Majorana type, so the corresponding mass matrix will be symmetric).

In the case of Dirac type neutrinos, the symmetry transformations of neutrino mass matrix will be

$$S^\dagger M_\nu M_\nu^\dagger S = M_\nu M_\nu^\dagger \tag{3.2}$$

It is also assumed that the elements within G_ν and G_l will commute among themselves. So that one can simultaneously diagonalize the elements of the respective residual groups.

Denoting the unitary matrix that diagonalize S_i as V_ν and the unitary matrix that diagonalize T_l as V_l. That means,

$$V_\nu^\dagger S_i V_\nu = d_s \qquad V_l^\dagger T_l V_l = d_l \tag{3.3}$$

Where d_s and d_l are the diagonal matrices.

Since T_l and $M_l M_l^\dagger$ are commuting 3.1, one can simultaneously diagonalize them. Similarly in the case of S_i and M_ν. Then one can write,

$$U_\nu = V_\nu P_\nu \quad and \quad U_l = V_l P_l \tag{3.4}$$

Where U_ν is the unitary matrix that diagonalize M_ν, U_l is the unitary matrix that diagonalize $M_l M_l^\dagger$[that already mentioned in section 2.2] and P_ν and P_l are the diagonal phase matrices (which can't be determined by group theoretical methods).

Then one can write

$$U_{pmns} = P_l^* V_l^\dagger V_\nu P_\nu \Rightarrow |U_{pmns}| = |V_l^\dagger V_\nu| \tag{3.5}$$

Note:

- One will have the freedom in permuting the columns and the rows U_{pmns} matrix **??**. This is because the group theoretical considerations does not fix the positions of eigen values. So one will have the freedom in choosing the position of these eigen values. Permuting the columns and the rows of U_{mixing} matrix corresponds to the changing the position of these eigen values.

- One can consider continuous groups as the flavour G_f, but the problem is that it will create additional complication like Goldstone modes in the theory (because the symmetry breaking of a continuous group will result in massless Goldstone bosons)

3.2 Bottom-Up Approach

In this section, we will discuss bottom-up approach [4], which is used to find the group G_f from the given mixing matrix. In this discussion, we will assume that the neutrinos are Majorana type and there are three of them.

Consider the basis where $M_l M_l^\dagger$ is diagonal, then one write that

$$U_l = I \quad \Rightarrow \quad U_\nu = U_{pmns} \tag{3.6}$$

Then the neutrino mass matrix can be written as

$$M_\nu = U_{pmns}^* diag(m_1, m_2, m_3) U_{pmns}^\dagger \tag{3.7}$$

Where m_1, m_2 and m_3 are the eigen values of M_ν.
Since it is assumed that the neutrinos are Majorana type, then the symmetry transformations of M_ν will obey

$$S^2 = I \tag{3.8}$$

(This is because the neutrino mass terms are in the form $\nu_i^T M_{ij} C \nu_j$, so under the action of $e^{i\alpha}$ on the neutrino field ν, the corresponding mass term will vanish, unless $\alpha = 0, \pi$.)

This kind of symmetry transformation will generate the group $\mathbb{Z}_2$ as G_ν. But this residual group is not sufficient enough to fully determine the $|U_{pmns}|$, because the eigen values of S are degenerate ($pm1$), so the columns of V_ν(unitary matrix which diagonalize S) is not fully determined. So one need to look for the next option, which is $\mathbb{Z}_2 \times \mathbb{Z}_2$, this residual group is sufficient because it has three commuting generators, so that the other columns can be fixed by simultaneously diagonalize them.

From the U_{pmns} matrix one can find the generators of the group G_ν. Which can be written as [4],

$$S_i = -I + 2u_i u_i^\dagger \quad i = 1, 2, 3 \tag{3.9}$$

Where u_i is the i^{th} column of U_{pmns}. These generators will also obey the conditions,

$$S_i^2 = I \qquad S_i S_j = S_k = S_j S_i \tag{3.10}$$

$$S_i^T M_\nu S_i = M_\nu \tag{3.11}$$

For i, j, k all are different and $i, j, k = 1$ to 3.
These properties of S_i will justify that they can act as the generators of the group G_ν. In order to ensure the diagonality of $M_l M_l^\dagger$ imposing a symmetry transformation F on $M_l M_l^\dagger$, such that F is a diagonal matrix and

$$F^n = I \quad n \geq 3 \tag{3.12}$$

These symmetry transformations will generate the group $\mathbb{Z}_n$ as the residual group G_l. Then the flavour group G_f will be generated by F and S_i's, so that it will have both G_ν and G_l as its subgroups.
Note: In the case of Dirac type neutrinos, G_ν can be more general groups like $\mathbb{Z}_n$ (for $n \geq 3$) as well as $\mathbb{Z}_2 \times \mathbb{Z}_2$. Because, in this case there will not be any constrains like $S^2 = I$, on the symmetry transformations of M_ν.

3.3 Examples

In this section, we will give several examples to clarify the ideas that mentioned in last two sections.

Tri-Bimaximal Mixing-Using Bottom-Up approach

TB mixing matrix is given in 2.19. Using 3.9 the generators of G_ν can be written as

$$S_2 = \begin{pmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{pmatrix} \qquad S_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \tag{3.13}$$

And F is taking to be

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{3.14}$$

Where $\omega^3 = 1 \Rightarrow F^3 = I$.
F will generate the group $\mathbb{Z}_3$ and S_i's will generate the group $\mathbb{Z}_2 \times \mathbb{Z}_2$(as we already mentioned earlier).

It is found that[4] the group generated by S_i's and F will be S_4 (which is the permutation group of four objects).

Bi-maximal mixing and Current Values of U_{pmns} within 3σ

Here we discuss a different method, in which determining the flavour group corresponding to a lepton mixing matrix by scanning the discrete groups[3].
The idea is that for each discrete subgroup(DSG) of $SU(3)$ there exist many possible mixing matrices. So by scanning the DSG's of $SU(3)$ it is possible to find out the smallest group, which will give the desirable mixing matrix.
The procedure to find out the possible mixing matrices of a discrete subgroup(say G_f) of $SU(3)$ is given below

1. Using the generators of the discrete subgroups(DSG) of $SU(3)$ [AppendixA], one can numerically generate all the elements of the group G_f.

2. The elements of G_f are divided into two sets. Set H_1 contains all the elements which satisfies the conditions that

$$g_i^2 = I \qquad g_i g_j = g_k = g_j g_i \qquad g_{i,j,k} \in H_1 \tag{3.15}$$

   Set H_2 contains all the elements which have all the three eigen values are distinct.

3. The elements of set H_1 can be considered as the possible generators of the residual group G_ν (as S_i's) and the elements of set H_2 can be considered as the possible generators of the residual group G_l(as T_l).

$$G_l \subset G_f \quad G_\nu \subset G_f \tag{3.16}$$

4. Then using the ideas mentioned in section2(General formalism) one can generate the corresponding mixing matrices for different choices of G_l and G_ν within G_f.

After scanning the DSG's of $SU(3)$ it is found that group S_4(permutation group of 4 objects) is the smallest group which gives Bi-maximal mixing matrix[4].
After scanning several DSG's of $SU(3)$, it is found that [3] in the case of Majorana

type neutrinos, $\Delta(6 \times 18^2)$[Appendix A] will give

$$|U_{mixing}| = \begin{pmatrix} 0.804 & 0.577 & 0.142 \\ 0.279 & 0.577 & 0.767 \\ 0.525 & 0.577 & 0.625 \end{pmatrix} \tag{3.17}$$

Which is within 3σ of the experimental value of U_{pmns} [2].

In the case of Dirac type neutrinos, we need to modify the second step in the procedure. Because in this case there is no constrains like 3.8 on the symmetry transformations of the neutrino mass matrix (Which is mentioned in sec(3.2)). So here in this case the set H_1 and the set H_2 will be same.
After taking these things into considerations, we found that $\Delta(6 \times 9^2)$ will give lepton mixing matrix which is exactly equal to $|U_{mixing}|$ in 3.17.

Note: In both cases the permutation of second and third row of $|U_{mixing}|$ 3.17 is still allowed because experimentally it is still inconclusive that whether the θ_{23}^{pmns} is greater than $\pi/4$ or less than $\pi/4$.

Chapter 4

Quark Lepton Complementarity and Discrete Symmetry Models

Quark Lepton Complementarity (QLC) can be viewed as an alternative description of the fermion mixing[5]. In this chapter we will discuss what is mean by QLC and its implications. In section 2 we will discuss an alternative approach in which QLC emerges only from the group theoretical considerations of the lepton mixing.

4.1 Quark Lepton Complementarity

There exist empirical relations between the quark mixing and the lepton mixing, which are

$$\theta_s + \theta_c \approx \frac{\pi}{4} \tag{4.1}$$

$$\theta_{23}^{pmns} + \theta_{23}^{ckm} \approx \frac{\pi}{4} \tag{4.2}$$

Where θ_s is the solar mixing angle (θ_{12}^{pmns}) and θ_c is the cabibbo angle (θ_{12}^{ckm}).
These empirical relations qualitatively means that[5]

- Because of the relatively large $1-2$ quark mixing, the $1-2$ lepton mixing deviates from the maximal mixing ($\frac{\pi}{4}$) by a significant amount.
- The $2-3$ lepton mixing deviates from the maximal mixing by a small amount because of the corresponding mixing parameter in the quark sector is relatively small.

Which means, one can think of the lepton mixing as

$$Lepton \quad mixing = BM \quad mixing - quark \quad mixing \tag{4.3}$$

This is called QLC.

Now we will discuss two possible scenarios based on the origin of BM mixing.

4.1.1 BM Mixing: From the charged lepton sector

In this case

$$U_l = U_{BM}^\dagger \qquad U_\nu = V_{ckm}^\dagger \tag{4.4}$$

So

$$U_{pmns} = U_{BM} V_{ckm}^\dagger \tag{4.5}$$

In this approach generating the lepton mixing using quark mixing as a correction from the neutrino sector on the BM mixing.

Since in the case of the quark mixing, θ_c is relatively larger than the other two angles in the CKM matrix, one can write quark mixing as

$$V_{ckm} \approx R_{12}(\theta_c) \tag{4.6}$$

$$\approx \begin{pmatrix} \cos\theta_c & \sin\theta_c & 0 \\ -\sin\theta_c & \cos\theta_c & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.7}$$

Then in this case

$$U_{pmns} = U_{BM} V_{ckm}^\dagger \tag{4.8}$$

$$\approx R_{23}(\frac{\pi}{4}) R_{12}(\frac{\pi}{4}) R_{12}(-\theta_c, \alpha) \tag{4.9}$$

$$\approx R_{23}(\frac{\pi}{4}) R_{12}(\frac{\pi}{4} - \theta_c, \alpha) \tag{4.10}$$

Where α is a complex phase.
From the above equation by comparing the elements, one can write the parameters of U_{pmns} as a function of θ_c and α. Which can be written as

$$\sin^2\theta_{12}^{pmns} = \frac{1}{2}(1 - \sin 2\theta_c \cos\alpha) \tag{4.11}$$

$$\theta_{13}^{pmns} = 0 \qquad \theta_{23}^{pmns} = \frac{\pi}{4} \tag{4.12}$$

This approach was a viable one before 2008, because the θ_{13}^{pmns} was observed to be very small at that time[6].
But from the recent neutrino oscillations experiments it is evident that θ_{13}^{pmns} is nonzero, infact it is approximately equal to 8.46 degrees[2]. So from the current data one can rule out this approach.

4.1.2 BM Mixing: From the neutrino sector

In this case

$$U_\nu = U_{BM} \qquad V_l = V_{ckm} \tag{4.13}$$

Then

$$U_{pmns} = V_{ckm}^{\dagger} U_{BM} \tag{4.14}$$

In this approach, generating the lepton mixing using quark mixing as a correction from the charged lepton sector on the BM mixing.

As we mentioned in the first scenario(sec 4.1.1), one can approximate the CKM matrix as a one parameter mixing (θ_c)
Then in this case one can write

$$U_{pmns} \approx R_{12}(-\theta_c, \alpha) R_{23}(\frac{\pi}{4}) R_{12}(\frac{\pi}{4}) \tag{4.15}$$

From the above equation by comparing the elements, one can write the parameters of PMNS matrix as a function of θ_c and α. Which can be written as

$$\sin^2 \theta_{13}^{pmns} = \frac{\sin^2 \theta_c}{2} \tag{4.16}$$

$$\sin^2 \theta_{23}^{pmns} = \frac{\cos^2 \theta_c}{2 - \sin^2 \theta_c} \tag{4.17}$$

$$\sin^2 \theta_{12}^{pmns} = \frac{\frac{1}{2} + \frac{1}{2}\cos^2 \theta_c - \frac{1}{\sqrt{2}} \sin 2\theta_c \cos \alpha}{2 - \sin^2 \theta_c} \tag{4.18}$$

This approach will predict a nonzero value for θ_{13}^{pmns}, $\theta_{13}^{pmns} \approx \dfrac{\theta_c}{\sqrt{2}}$.

In order to check the viability of this approach in generating the correct lepton mixing matrix, one can use χ^2 test.

χ^2 value can be defined as

$$\chi^2 = \Sigma_i (\frac{O_i - Exp_i}{SD_i})^2 \tag{4.19}$$

Where Exp_i is the expected value of the parameter i, O_i is the observed value of the parameter i and SD_i is the standard deviation in the observation of the parameter i. The idea is that

- The independent parameters of PMNS matrix can be written as the functions of θ and α. This relations will be equivalent to 4.16-4.18 by considering θ instead of θ_c there.
- Then using these relations and the observed values of PMNS matrix parameters and their standard deviations, one can write χ^2 as a function of θ and α.
- θ can vary from 0 to 90 degrees and α can vary from 0 to 360 degrees.
- Using this, one can plot the contour diagram for χ^2 with respect to θ and α.
- From this contour diagram, one can observe the allowed range of θ. If the Cabibbo angle within this range then one can conclude that this approach is an efficient way to generate the lepton mixing matrix.

Following the above mentioned procedure, the χ^2 analysis for the PMNS matrix from the 2014 data[6] is given below
The parameters of PMNS matrix[7] can be written as

$$\sin^2\theta_{12} = 0.323 \pm 0.016 \quad \sin^2\theta_{23} = 0.567^{+0.032}_{-0.128} \quad \sin^2\theta_{13} = 0.0234 \pm 0.002 \tag{4.20}$$

Then using these values, one can write the χ^2 as the functions of θ and α. Which is given by

$$\chi^2(\theta, \alpha) = \chi^2_{13} + \chi^2_{23} + \chi^2_{12} \tag{4.21}$$

Where $\chi^2_{13,23,12}$ are the contribution to χ^2 from the parameters θ_{13}, θ_{23} and θ_{12} respectively. Which can be written as

$$\chi^2_{13} = \left(\frac{\frac{\sin^2\theta}{2} - 0.0234}{0.002} \right)^2 \tag{4.22}$$

$$\chi^2_{23} = \left(\frac{\dfrac{\cos^2\theta}{2-\sin^2\theta} - 0.567}{0.08} \right)^2 \tag{4.23}$$

$$\chi^2_{12} = \left(\frac{\dfrac{1/2 + (1/2)\cos^2\theta - (1/\sqrt{2})\sin 2\theta \cos\alpha}{2-\sin^2\theta} - 0.323}{0.016} \right)^2 \tag{4.24}$$

Then using eq.(4.21) one can plot the χ^2 contour diagram. Which is given below

From the 4.2, one can make the following obeservations that

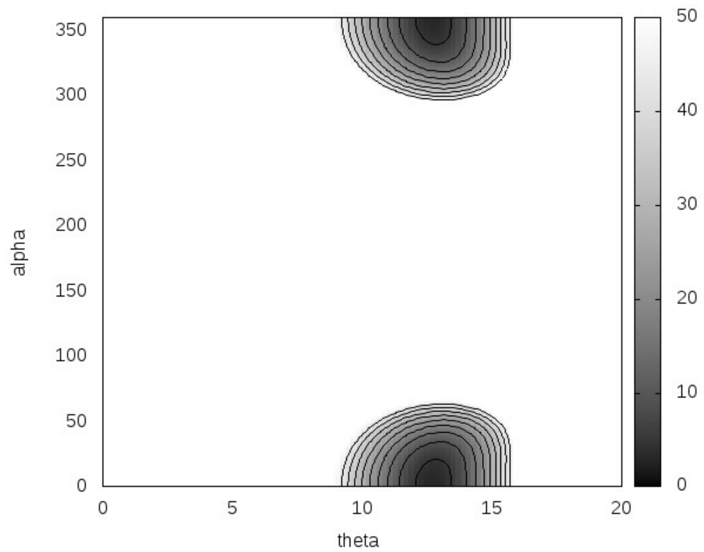

Figure 4.1: χ^2 contour diagram. The grey color scale indiacating the values of the χ^2 function, here it given from 0 to 50. The horizontal axis representing the θ values in degrees and the vertical axis representing the α in degrees.

- The range of θ, which will give the minimum valued contours of χ^2 observed to be [10:15] degrees.
- In the case of α parameter, there is a degeneracy in χ^2 contours, the range observed to [0:60] degrees and [300:360] degrees. This degeneracy can be explained from the 4.24, where χ^2 depents on α in terms of $\cos\alpha$.

The minimum value of χ^2 observed to be 2.93.
χ^2 value minimized for θ in the range [11.8:13.7] degrees (with a confidence of 95.4%).

Since the value of the Cabibbo angle is within this range, this approach is an efficient way to generate the lepton mixing matrix.

The PMNS matrix values were recently updated [2]. For the current values of PMNS matrix parameters, the χ^2 analysis is given below
The current values of the parameters are

$$\sin^2\theta_{12} = 0.306 \pm 0.012 \quad \sin^2\theta_{23} = 0.587^{+0.020}_{-0.024} \quad \sin^2\theta_{13} = 0.02179 \pm 0.00076 \tag{4.25}$$

Similar to last case, one can construct the χ^2 as a function of θ and α. The corresponding contour diagram for χ^2 is given below

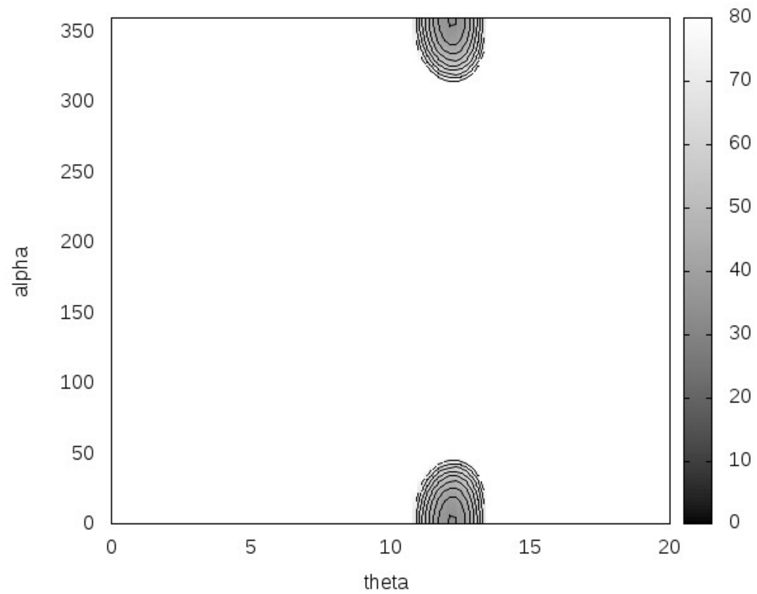

Figure 4.2: χ^2 contour diagram. The grey color scale indiacating the values of the χ^2 function, here it given from 0 to 80. The horizontal axis representing the θ values in degrees and the vertical axis representing the α in degrees.

In this case the minimum value of the χ^2 is about 34.3. Which is quite high. So from the current values of PMNS matrix, this approach is not favourable.

In literature it has already been shown that using this approach (BM mixing from the neutrino sector) in the Grand unified theories framework, one can naturally realize the QLC [8].The basic idea is that in the leading order the PMNS matrix is assumed to be BM mixing and the CKM matrix assumed to be Identity matrix. This can be achieved by using some discrete groups. Then the corrections from the next leading order terms (of $O(\theta_c)$) will correct these mixing matrices. In order to have same amount of corrections from the charged lepton and the down quark, one need to have nearly equal mass matrices for the down quark and the charged lepton. In this case, QLC can naturraly emerge from the quark-lepton unification at a high scale.

4.2 An Alternative Approach

In the last section we discussed the realization of QLC in the GUT frame work using some discrete groups. In this section we will discuss an alternative approach in which quark-lepton unification is not mandatory to realize the QLC. This approach is completely based upon group theoretical methods only.
Here we assumes that the lepton mixing is dominantly given by Bi-maximal mixing (from the neutrino sector) and the CKM matrix like corrections from the charged lepton sector will give QLC like relation. We also assumes that **these corrections are also constrained by the group theoretical considerations** .

For a discrete group G_f, the possible correction matrices can be derived as

- For a flavour group G_f, the possible mixing matrices can be written as 3.5

$$U_{mixing} = P_1 V_l^\dagger V_\nu P_2 \tag{4.26}$$

  Where P_1 and P_2 are the diagonal phase matrices, V_l is the unitary matrix that diagonalize the generator T_l of the residual group G_l and V_ν is the unitary matrix that diagonalize the generators S_1 and S_2 of the residual group G_ν.

- In our approach since we are assuming that Bi-maximal mixing coming from the neutrino sector,then $V_\nu = U_{BM}$.

- Since the correction matrix is assume to be coming from the charged lepton sector, V_l will act as the correction matrix.

- Then one can find the correction matrix V_l from the equ(4.26).

$$U_{mixing} = P_1 V_l^\dagger U_{BM} P_2 \tag{4.27}$$

Then

$$V_l = U_{BM} P_2 U^{\dagger}_{mixing} P_1 \quad \Rightarrow |V_l| = |U_{BM} P_2 U^{\dagger}_{mixing}| \tag{4.28}$$

Where P_2 (diagonal phase matrix) is the freedom that we have from the group theoretical methods.

4.3 Analysis

In order to check the viability of this alternative approach, one can scan the discrete subgroups of $SU(3)$ and check for the CKM matrix like corrections.

Since $\Delta(6 \times n^2)$ type of discrete subgroups of $SU(3)$ can generate the possible mixing matrices that one can get from the most of the discrete subgroups of $SU(3)$ [3], we will give our primary attention to this class of discrete groups in our analysis.

One can follow the following procedure to find the correction matrices corresponding to $\Delta(6 \times n^2)$ type of discrete groups

- Following the procedure mentioned in section(3.3), one can find the possible mixing matrices for the $\Delta(6 \times n^2)$ type of discrete groups.
- The mixing matrix U_{mixing} which obeys the condition that $|U_{mixing}|$ should be within 5σ range of U_{pmns} experimental, would be consider for the further analysis.
- Then using eq.(4.28)4.28 one can find the possible correction matrices for $\Delta(6 \times n^2)$ type of discrete groups.
- The freedom in P_2 matrix and permutations of the second and third column of V_l (this is because the permutations of the 2^{nd} and 3^{rd} columns of V_l will be equivalent to permutations of the 2^{nd} and 3^{rd} rows of U_{mixing} matrix, which is still allowed because the experiments are still inconclusive about the quadrant of θ^{pmns}_{23} angle) can be used to bring V_l in V_{ckm} like form.

4.4 Results

Following the above mentioned procedure, we scanned the $\Delta(6 \times n^2)$ type of discrete groups (of $n < 19$).
We did the analysis for the case in which the neutrinos were assumed to be Majorana type particles and the case in which the neutrinos were assumed to be Dirac type particles. The results of the analysis are given below.

4.4.1 Neutrinos: Majorana type

In this case we assumes that neutrinos are Majorana type particles.

$\Delta(6n^2)$		$\lvert V_l \rvert$	$(\theta_{12}, \theta_{23}, \theta_{13})^l$	$\lvert U_{mixing} \rvert$
$n = 16$	U_{mixing} within 3σ	–	–	–
	U_{mixing} within 5σ	$\begin{pmatrix} 0.974 & 0.223 & 0.032 \\ 0.225 & 0.964 & 0.139 \\ 0.0 & 0.143 & 0.990 \end{pmatrix}$	$(12.89, 7.99, 1.84)$	$\begin{pmatrix} 0.801 & 0.577 & 0.159 \\ 0.262 & 0.577 & 0.773 \\ 0.538 & 0.577 & 0.614 \end{pmatrix}$
$n = 18$	U_{mixing} within 3σ	$\begin{pmatrix} 0.977 & 0.211 & 0.037 \\ 0.214 & 0.971 & 0.109 \\ 0.013 & 0.114 & 0.993 \end{pmatrix}$	$(12.18, 6.25, 2.13)$	$\begin{pmatrix} 0.804 & 0.577 & 0.142 \\ 0.279 & 0.577 & 0.767 \\ 0.525 & 0.577 & 0.625 \end{pmatrix}$
	U_{mixing} within 5σ	–	–	–

Table 4.1: Column1 of the table specifying the group that we are considering, column2 specifying whether the $|U_{mixing}|$ that we are considering within 3σ or 5σ of the $|U_{pmns}|$ experimental, column3 specifying the absolute value of V_l that we are getting from the corresponding groups, column4 specifying the standard parameters of V_l and column5 specifying the U_{mixing} that we got after arranging $|V_l|$ in $|V_{ckm}|$ form.

4.4.2 Neutrinos: Dirac type

In this case we assumes that neutrinos are Dirac type particles.

$\Delta(6n^2)$		$\lvert V_l \rvert$	$(\theta_{12}, \theta_{23}, \theta_{13})^l$	$\lvert U_{mixing} \rvert$
$n = 8$	U_{mixing} within 3σ	–	–	–
	U_{mixing} within 5σ	$\begin{pmatrix} 0.974 & 0.223 & 0.032 \\ 0.225 & 0.964 & 0.139 \\ 0.0 & 0.143 & 0.990 \end{pmatrix}$	$(12.89, 7.99, 1.84)$	$\begin{pmatrix} 0.801 & 0.577 & 0.159 \\ 0.262 & 0.577 & 0.773 \\ 0.538 & 0.577 & 0.614 \end{pmatrix}$
$n = 9$	U_{mixing} within 3σ	$\begin{pmatrix} 0.977 & 0.211 & 0.037 \\ 0.214 & 0.971 & 0.109 \\ 0.013 & 0.114 & 0.993 \end{pmatrix}$	$(12.18, 6.25, 2.13)$	$\begin{pmatrix} 0.804 & 0.577 & 0.142 \\ 0.279 & 0.577 & 0.767 \\ 0.525 & 0.577 & 0.625 \end{pmatrix}$
	U_{mixing} within 5σ	–	–	–
$n = 11$	U_{mixing} within 3σ	$\begin{pmatrix} 0.975 & 0.219 & 0.030 \\ 0.221 & 0.968 & 0.120 \\ 0.003 & 0.123 & 0.992 \end{pmatrix}$	$(12.68, 6.88, 1.73)$	$\begin{pmatrix} 0.802 & 0.577 & 0.154 \\ 0.267 & 0.577 & 0.772 \\ 0.535 & 0.577 & 0.617 \end{pmatrix}$
	U_{pmns} within 5σ	–	–	–
$n = 14$	U_{mixing} within 3σ	$\begin{pmatrix} 0.976 & 0.218 & 0.032 \\ 0.220 & 0.968 & 0.117 \\ 0.005 & 0.121 & 0.992 \end{pmatrix}$	$(12.57, 6.74, 1.81)$	$\begin{pmatrix} 0.802 & 0.577 & 0.152 \\ 0.270 & 0.577 & 0.770 \\ 0.532 & 0.577 & 0.619 \end{pmatrix}$
	U_{mixing} within 5σ	–	–	–
$n = 17$	U_{mixing} within 3σ	$\begin{pmatrix} 0.976 & 0.216 & 0.033 \\ 0.219 & 0.969 & 0.116 \\ 0.006 & 0.120 & 0.993 \end{pmatrix}$	$(12.50, 6.56, 1.87)$	$\begin{pmatrix} 0.802 & 0.577 & 0.150 \\ 0.271 & 0.577 & 0.770 \\ 0.531 & 0.577 & 0.620 \end{pmatrix}$
	U_{mixing} within 5σ	–	–	–

Table 4.2: Column1 of the table specifying the group that we are considering, column2 specifying whether the $|U_{mixing}|$ that we are considering within 3σ or 5σ of the $|U_{pmns}|$ experimental, column3 specifying the absolute value of V_l that we are getting from the corresponding groups, column4 specifying the standard parameters of V_l and column5 specifying the U_{mixing} that we got after arranging $|V_l|$ in $|V_{ckm}|$ form.

From the 4.1 and 4.2, one can observe that the correction matrices are not exactly in CKM form. Infact the contributions from the θ_{23} and θ_{13} parameters of correction matrices are larger than the corresponding parmeters from the CKM matrix[eq2.10]. One can relate the above observation with the χ^2 analysis that we did in section4.1. The large values of χ^2 (> 1) that got from the analysis will implies that one angle correction will not able to generate the lepton mixing matrix completely.

One can also observe that the second and third rows of $|U_{mixing}|$ that we got after arranging $|V_l|$ in $|V_{ckm}|$ kind of form, are fixed. In other words if one interchange the second and the third rows of the lepton mixing matrix (which is possible, because experiments are still inconclusive about the quadrant of θ_{23}^{pmns} angle) then it will will result in the interchange of second and third columns of the correction matrix, this will completely destroy the CKM like structure of the correction matrix.

4.5 Conclusions

From the analysis, we are able to conclude that

- In order to get a viable U_{pmns} in QLC approach, the corrections from the charged lepton sector should be more than one parameter. And the correction matrix $|V_l|$ is not strictly $|V_{ckm}|$ like.
- If $|V_l|$ is assumed to be approximately $|V_{ckm}|$ like and it is fixed by group theoretical methods, then it prefers $\theta_{23}^{pmns} > \dfrac{\pi}{4}$. This can be confirmed or ruled from the ongoing experiments.

4.6 Future Plans

In the conventional approach QLC means generating the lepton mixing pattern using the quark mixing pattern as a correction on the dominant contribution of lepton mixing, which is taken to be Bi-maximal mixing. From our analysis we were able to show that, by group theoretical considerations only one cann't get QLC completely. So we motivated to think that, rather than taking BM mixing as a dominant lepton mixing matrix we will choose some other dominant lepton mixing matrix such that one can still generate the lepton mixing pattern using the quark mixing as a correction on this dominant lepton mixing matrix. One should understand that the essence of the QLC

is still preserved in this different approach also.
In future, we will try this different approach towards QLC by group theoretical considerations only. Here we layout the method that we will implement in future

Since $U_{mixing} = P_1 V_l^\dagger V_\nu P_2$ (eq.2.5) and in our new approach $V_l = V_{ckm}$, then one can write

$$U_{mixing} = P_1 V_{ckm}^\dagger V_\nu P_2 \tag{4.29}$$

So

$$V_\nu = V_{ckm} P_1^* U_{mixing} P_2^* \quad \Rightarrow |V_\nu| = |V_{ckm} P_1^* U_{mixing}| \tag{4.30}$$

Just like in our earlier analysis P_2 is the freedom that we have from the group theoretical methods and we will choose particular type of U_{mixing} which will obey the condition that $|U_{mixing}|$ lies within 5σ of U_{pmns} experimental.
Then we will scan the DSG's of $SU(3)$ to check the form of dominant mixing matrix V_ν.

Appendix A

Finite Subgroups of $SU(3)$

Here we will discuss about the finite subgroups of $SU(3)$.

A.1 Classification of finite subgroups of $SU(3)$

The finite subgroups of $SU(3)$ can be classified into five different classes [9]

Type A

Groups of diagonal matrices. Since diagonal matrix multiplication is abelian in nature these groups will correponds to Abelian groups.

Type B

Groups correspoding to the linear transformations of two variables.
The elements of these groups will have the structure (upto some basis transformations)

$$\begin{pmatrix} (detA)^* & 0_{1\times 2} \\ 0_{2\times 1} & A \end{pmatrix} \quad Where A \in U(2) \tag{A.1}$$

Where the $(detA)^*$ will ensure the determinant of the elements of the groups is 1.

Type C

The groups generated by

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad F(n, a, b) = \begin{pmatrix} \eta^a & 0 & 0 \\ 0 & \eta^b & 0 \\ 0 & 0 & \eta^{-a-b} \end{pmatrix} \tag{A.2}$$

can be represented as $C(n, a, b)$. Where $\eta = e^{i2\pi/n}$ and a, b are integers with $0 \leq a, b \leq n-1$.

Type D

The groups generated by E and $F(n, a, b)$ (given in eq.A2) and

$$G = \begin{pmatrix} \delta^r & 0 & 0 \\ 0 & 0 & \delta^s \\ 0 & -\delta^{-r-s} & 0 \end{pmatrix} \tag{A.3}$$

can be represented as $D(n, a, b; d, r, s)$. Where $\delta = e^{2\pi i/d}$ and r, s are integers with $0 \leq r, s \leq 0$.

Type E

There exist six finite subgroups of $SU(3)$ which do not fall into any of the type that mentioned above. This exceptional groups can be denoted by $\Sigma(60)$, $\Sigma(168)$, $\Sigma(36\times3)$, $\Sigma(72\times3)$, $\Sigma(216\times3)$ and $\Sigma(360\times3)$.

A.2 List of Generators of $SU(3)$

Here we list the all the generators that one required to generate the finite subgroups of $SU(3)$ of order less than 512 [9].

$$H = \frac{1}{2}\begin{pmatrix} -1 & \mu_- & \mu_+ \\ \mu_- & \mu_+ & -1 \\ \mu_+ & -1 & \mu_- \end{pmatrix} \quad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{A.4}$$

$$K = \frac{1}{\sqrt{3}i}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad L = \begin{pmatrix} 1 & 1\omega^2 & 1 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix} \tag{A.5}$$

$$M = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^4 \end{pmatrix} \quad P = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon\omega \end{pmatrix} \tag{A.6}$$

$$N = \frac{i}{\sqrt{7}} \begin{pmatrix} \beta^4 - \beta^3 & \beta^2 - \beta^5 & \beta - \beta^6 \\ \beta^2 - \beta^5 & \beta - \beta^6 & \beta^4 - \beta^3 \\ \beta - \beta^6 & \beta^4 - \beta^3 & \beta^2 - \beta^5 \end{pmatrix} \quad Q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & -\omega^2 & 0 \end{pmatrix} \tag{A.7}$$

Where $\eta = e^{2\pi i/n}$, $\delta = e^{2\pi i/d}$, $\mu_\pm = \frac{1}{2}(-1 \pm \sqrt{5})$, $\omega = e^{2\pi i/n3}$, $\beta = e^{2\pi i/7}$, $\epsilon = e^{4\pi i/9}$
The generators E, $F(n,a,b)$ and $G(d,r,s)$ are given in eq.A2 and eq.A3.

A.3 Non-abelian finite subgroups of $SU(3)$

Here we will list the all the non-abelian finite subgroups of $SU(3)$ which have a faithful three dimensional irreducibel representation [9].

Group	Generators
$C(n,a,b)$	$E,\ F(n,a,b)$
$D(n,a,b;d,r,s)$	$E,\ F(n,a,b),\ G(d,r,s)$
$\Delta(3n^2) = C(n,0,1), n \geq 2$	$E,\ F(n,0,1)$
$\Delta(6n^2) = D(n,0,1;2,1,1), n \geq 2$	$E,\ F(n,0,1),\ G(2,1,1)$
$T_n = C(n,1,a),\ (1+a+a^2)\ mod\ n = 0$	$E,\ F(n,1,a)$
$A_5 = \Sigma(60)$	$E,\ F(2,0,1),\ H$
$\Sigma(168)$	$E,\ M,\ N$
$\Sigma(36)$	$E,\ J,\ K$
$\Sigma(72)$	$E,\ J,\ K,\ L$
$\Sigma(216)$	$E,\ J,\ K,\ P$
$\Sigma(72)$	$E,\ F(2,0,1),\ H,\ Q$

Table A.1: Finite non abelian subgroups of $SU(3)$

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