Gauge-hierarchy Problem, Seesaw Mechanisms and Discrete symmetries

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. **Certificate of Examination**

This is to certify that the dissertation titled '**Gauge-hierarchy problem, Seesaw mechanisms and Discrete Symmetries**' submitted by Mr.Pratik Chattopadhyay (Reg.No.MP14005) for the partial fulfilment of MS degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute.The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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. Dated:April 20,2017

Declaration

I hereby declare that the work presented in this thesis entitled "**Gauge-hierarchy problem, Seesaw mechanisms and Discrete symmetries**" is carried out by me under the supervision of Dr.Ketan Patel at the Indian Institute of Science Education and Research, Mohali.

This is a presentation of my original research work.Every effort is made to indicate clearly the contributions of others wherever they are used, with due reference to the literature.The interpretations are based on my readings and the understanding acquired from original texts and research articles. This work has not been submitted in part or in full to any other university or institute for a degree, a diploma or any other publication.This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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. Dated: April 20, 2017

In my capacity as a supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

. Dr Ketan Patel
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Contents

Abstract

The physical mass of the Higgs particle is approximately 126 GeV, which also sets the electroweak scale.But the expected mass of the Higgs due to quantum corrections from heavier particles should be much higher, unless there is an unnatural fine-tuning cancellation in the parameters.Seesaw models of neutrino mass generation require extra heavy particles beyond the standard model which interact with the Higgs and contribute to the correction of its mass.We derive the loop amplitudes responsible for the mass correction of the Higgs due to these particles.Then we discuss about the naturalness criteria and find relations between the coupling constants and the mass scales of heavy particles.Next we motivate a framework which incorporates the natural electroweak seesaw.The naturalness of the electroweak scale in the light of type-I seesaw model with Yukawas of order unity leads to TeV scale masses for the extra heavy fermion singlets which play the role of right handed neutrinos.This requires the mechanism of seesaw cancellation through special correlations among the $\mathcal{O}(1)$Yukawa couplings which can be motivated through discrete flavor symmetries.We provide a candidate model based on the discrete group $\Sigma(81)$ and illustrate on the generic perturbations that lead to viable neutrino masses.We give phenomenological implications like flavour violating processes and analyse their branching ratios.We then briefly discuss the scenarios of leptogenesis and baryogenesis and elaborate on resonant leptogenesis as a viable process in the context of our model.

Chapter 1

Introduction

Since the time of Galileo, modern science has come a long way through its evolution in a dynamic fashion.The process of intuition, analysis and empirical tests have been more or less the guiding framework in this journey and hopefully will continue to be so. Over the ages, one of our incessant curiosities has been to understand the fundamental building blocks of nature. Ironically, in its most modern scientific form, the legitimacy of this question is not very clear but it nevertheless assisted a great deal to unravel some of the deep aspects of nature and has been prone to aesthetically appeal to the ideology of reductionism. We have understood till date that there exists some basic physical laws and fundamental interactions which underlie the structure of more complicated phenomena.There exists four fundamental interactions in nature,viz.electromagnetic, gravitational,weak and strong interaction.It is stunning to realize that at the elementary level, only these four basic interactions govern almost all the breathtaking complexities which surround us. People have been involved in experimentation and theoretical research to understand these basic interactions and their roles in our day-to-day life processes.In the modern context, they came up with precise theoretical frameworks which enunciates the understandings of those.While the Physics at length scales of daily life objects like pistons and pulleys is adequately described by classical mechanics(based on Newton's laws), it hopelessly fails to explain the Physics of very short scales like electrons, atoms and molecules. Quantum mechanics came into existence hence when it was realized that several discrepancies exist in the classical framework.This realization has been a result of numerous experiments, strategic guesses and wrong attempts. One important aspect that this process taught us is the way science progresses and develops itself under critical scrutiny and analysis.Quantum mechanics is a framework which governs the microscopic dynamics of one-particle system or system of finitely multiple number of particles(finite degrees of freedom).On the other hand, Maxwell has given the correct theoretical framework to understand classical electromagnetism quite a long time ago in terms of an object which in principle consists of infinite degrees of freedom, i.e a field.However, problems still lasted to understand how electromagnetic waves(propagating disturbances in the field), viz.radiation travels in space without necessarily requiring a medium. This very simple question led to the theoretical structure of what is known as special relativity(formulated by Einstein), which showed that there exists an upper limit to the velocity of any propagating information and as a result leads to the principle of causality. Further, it produced other interesting results of which a significant one being energy and mass are equivalent and as a result there can exist physical processes which allow non-conservation of particle number(i.e creation and destruction of particles).While quantum mechanics works well for microscopic particles, one inevitably confronts a problem when the microscopic particles have very high velocities and relativistic effects starts to take over.Firstly, the idea of non-conservation of particle number in special relativity is inconsistent with quantum mechanics and the causality principle could not be maintained in the context where quantum mechanical laws overlap, since it can be predicted from quantum mechanics that there is a finite probability amplitude of finding a particle in a space-like separated region.Moreover, quantum mechanics cannot adequately describe fields and their dynamics. To merge the principles of quantum mechanics with special relativity, it again needed a humongous amount of analysis and attempts.Nevertheless, this led to the formulation of quantum field theory, a consistent framework to describe physical processes at short scales and high energies. The beauty of this formulation lies in the fact that it revived the old notion of particles as fundamental objects and set a stage where more abstract quantities called fields(with infinite degrees of freedom) are ingrained in the framework as fundamental objects.Elementary particles are then understood as physical excitations of their corresponding fields.

Quantum field theory as a concrete framework revolutionized our understanding of elementary particles to a significant extent.It also relates crucially to statistical physics and condensed matter physics where the ideas of renormalization method and the renormalization group finds an eloquent place.The important understanding that the Physics of diverse phenomena differ across length scales and that many microscopic theories can possibly give rise to the same macroscopic theory as a virtue of integrating out microscopic degrees of freedom(this is equivalent to coarse-graining in statistical physics) has its roots in the structure of quantum field theory.Several successful results in the framework of relativistic quantum field theories include the theory of quantum electrodynamics, where theoretically predicted quantities, like the electron g-factor were experimentally verified to an accuracy to the tenth place of decimal; quantum chromodynamics, which describes the working of the strong interaction and predicted the existence of asymptotic freedom and quark confinement and quantum flavour-dynamics, which provides the understanding of the flavour structure of elementary particles like quarks and leptons.Perhaps the most beautiful physical structure which resulted from quantum field theory is the standard model of particle physics.

The standard model of particle physics is a theoretical model which has many other extra features inbuilt into it.Firstly, it is a special type of quantum field theory, colloquially known as a gauge theory.Therefore this kind of a model incorporates gauge symmetries into it which play vital roles.Specifically, the standard model has a product of three gauge groups,viz.$SU(3), SU(2)$ and $U(1)$.The lagrangian of the standard model has a symmetry under the product of these gauge groups.It further includes all quarks, leptons and gauge fields in the model.One essential component in it is the mechanism of spontaneous symmetry breaking.This mechanism is possible if one includes a scalar field in the theory.Such a field is named after its founder, Sir Peter Higgs. When this field takes its vacuum expectation value, the gauge group is broken to a smaller group, generating mass for the elementary particles. The exception here is the neutrino.Neutrinos remain massless in the standard model because they do not have a right-handed partner.On the other hand, neutrino oscillations confirm that neutrinos must possess mass, albeit a tiny one.This clearly is a signature of Physics beyond the standard model and hence requires extra mechanisms and/or particles to understand the required dynamics.The seesaw mechanisms are one of the prominent ones which propose to generate tiny left handed neutrinos masses as observed in experiments.They are well motivated from the perspective of Grand Unified Theories(GUTs) and hence are viable candidates. These mechanisms introduce new heavy particles which interact with the standard model Higgs particle and the left neutrinos.We elaborate on this in the later sections.

A puzzling issue within the standard model framework is that the Higgs vacuum is not stable and undergoes significant perturbations from quantum fluctuations due to interactions with other particles.While the physical Higgs mass is ≈ 126 GeV, its theoretical expected mass should be far higher in scale if standard model is not the only fundamental theory which can be extended to any arbitrary scale.The Higgs vev determines the electroweak scale by generating masses for W and Z bosons.On the other hand we know that gravitational interactions become important at the Planck scale.So it naturally bothers us to ponder on the question as in why the electroweak scale is so far lower than the Planck scale when the natural expectation is to have the Higgs mass to be much higher due to new Physics.This problem is known in the literature as the hierarchy problem.

Since seesaw mechanisms, particularly the type-I seesaw predict new heavy particles(right handed fermion singlets) and hence new Physics at a higher scale, one finds similar issues with the mass correction of the Higgs due to its interaction with these particles.We discuss this part in the later sections elaborately.This issue will lead to constraints on the mass scale of the right handed fermion singlets on one hand and naturalness of the Higgs mass on the other.If the right handed particles are close to the upper bound set by electroweak naturalness(Higgs naturalness) then it requires the Yukawas couplings of $\mathcal{O}(10^{-4})$ in order to produce viable light neutrino masses.When the mass scale of right handed heavy particles are further lowered down, the Yukawa couplings are smaller.We argue in this project that type-I seesaw mechanism loses its inherent naturalness when the criteria of electroweak naturalness is imposed to it.The fundamental Yukawa couplings of order unity when put together with electroweak naturalness requires the masses of fermion singlets as light as $\mathcal{O}(TeV)$.Thus seesaw mechanism can no longer naturally account for tiny neutrino masses and it is to be replaced with some alternative mechanism which ensures small neutrino masses.

We propose a suitable framework based on discrete symmetries which offers such a replacement.Discrete symmetries are often used in particle physics to predict flavor mixing patterns in the lepton sector.We assume that the standard model leptons possess global $Z_n \times Z_m \times Z_p$ symmetry with $n, p, m \geq 3$.This is a residual symmetry of standard model neutrinos and their Majorana nature implies all of them to be massless.The right handed heavy neutrinos are assigned appropriate discrete symmetries in a way such that there exist three massive states and atleast one non-vanishing Dirac Yukawa coupling.As we show in this project, this necessarily leads to two degenerate right handed neutrinos and one massive right neutrino which completely decouples from the standard model.The symmetries of leptons and heavy right neutrinos can be combined to a discrete group G_f which is a symmetry of the leptons in the underlying theory.We provide a model of this class of symmetries and discuss the generic perturbations and their phenomenology.

The thesis is organized as the following.

- In the next section we provide some experimental facts on neutrinos.
- In section 3, we discuss the seesaw models of mass generation.
- In section 4, we elaborate on the hierarchy problem and the one-loop effects on the Higgs μ^2 parameter from different seesaw models.
- In section 5, we discuss about the naturalness of type-I seesaw
- In section 6, we do a comparative study of electroweak naturalness and seesaw naturalness and arrive at plausible symmetries.
- In section 7, we provide a description of finite discrete groups.
- In section 8, we propose a $\Sigma(81)$ model to illustrate the electroweak natural seesaw.
- In section 9, we provide the phenomenological implications of our model.
- In section 10, we give a brief review of cosmology and discuss leptogenesis in the light of our model.
- In section 11, we give numerical reports for masses and mixing angles of left neutrinos in accordance with our symmetry model.
- Finally in section 12, we summarise the work.

Chapter 2

Some experimental facts about neutrinos

2.0.1 Neutrino detectors

a)**Liquid Scintillators**: Collection of light released by charged particles propagating in scintillators is amplified by the photomultiplier.The Liquid Scintillator Neutrino Detector (LSND) was a scintillation counter at Los Alamos National Laboratory that measured the number of neutrinos being produced by an accelerator neutrino source. The LSND project was created to look for evidence of neutrino oscillation, and its results conflict with the standard model expectation of only three neutrino flavors, when considered in the context of other solar and atmospheric neutrino oscillation experiments.

b)**Cherenkov Detectors**: Cherenkov detectors take advantage of a phenomenon called Cherenkov light. Cherenkov radiation is produced whenever charged particles such as electrons or muons are moving through a given detector medium somewhat faster than the speed of light in that medium. In a Cherenkov detector, a large volume of clear material such as water or ice is surrounded by light-sensitive photomultiplier tubes. A charged lepton produced with sufficient energy and moving through such a detector does travel somewhat faster than the speed of light in the detector medium (although somewhat slower than the speed of light in a vacuum). The charged lepton generates a visible "optical shockwave" of Cherenkov radiation. This radiation is detected by the photomultiplier tubes and shows up as a characteristic ring-like pattern of activity in the array of photomultiplier tubes. As neutrinos can interact with atomic nuclei to produce charged leptons which emit Cherenkov radiation, this pattern can be used to infer direction, energy, and (sometimes) flavor information about incident neutrinos.

c)**Tracking calorimeters**:Tracking calorimeters such as the MINOS detectors use alternating planes of absorber material and detector material. The absorber planes provide detector mass while the detector planes provide the tracking information. Steel is a popular absorber choice, being relatively dense and inexpensive and having the advantage that it can be magnetised. The NOA proposal suggests eliminating the absorber planes in favor of using a very large active detector volume. The active detector is often liquid or plastic scintillator, read out with photomultiplier tubes.

d)**Heavy water detection**: The charged current interaction can produce reactions like : ν_e+d$\rightarrow$ $p + p$+e$^-$.Resulting electron gets detected via Cherenkov radiation.This gives the information about neutrino energy and direction.The neutral current interaction can have the following reaction:
ν_μ, ν_τ+d$\rightarrow$$\nu_\mu, \nu_\tau$+n+p.Neutral current cross sections are same irrespective of neutrino flavour.

e)**Super Kamiokande Detector**: The Super-Kamiokande detector is a 50,000 ton tank of water, located approximately 1 km underground. The water in the tank acts as both the target for neutrinos, and the detecting medium for the by-products of neutrino interactions.The inside surface of the tank is lined with 11,146 50-cm diameter light collectors called photo-multiplier tubes. In addition to the inner detector, which is used for physics studies, an additional layer of water called the outer detector is also instrumented light sensors to detect any charged particles entering the central volume, and to shield it by absorbing any neutrons produced in the nearby rock. In addition to the light collectors and water, a forest of electronics, computers, calibration devices, and water purification equipment is installed in or near the detector cavity.This detector essentially uses the process of Cherenkov radiation to detect atmospheric and solar neutrinos.

2.0.2 Major sources of neutrino production

a)**Solar neutrino and its anomaly**: The energy of Sun comes from the nuclear fusion in its core where a helium atom and an electron neutrino are generated by 4 protons. These neutrinos emitted from this reaction are called solar neutrinos. Photons, created by the nuclear fusion in the center of the Sun, take millions of years to reach the surface; on the other hand, solar neutrinos arrive at the earth in eight minutes due to their lack of interactions with matter. Hence, solar neutrinos make it possible for us to observe the inner Sun in formidable amount of time.
Flux of neutrinos from Sun is approximately $6 \times 10^{10}\ cm^{-2}s^{-1}$
The different cycles that lead to neutrino ejection and heavy metal formation on Sun are as follows:
pp-I
p+p$\rightarrow$d+e$^+$+ν_e
$p + e^- + p$$\rightarrow$d+$\nu_e$
$d + p$$\rightarrow$ γ+He^3
$He^3 + He^3$$\rightarrow$He$^4 + p + p$

$pp - II$
$He^3 + He^3$$\rightarrow$He$^4 + e^+$+$\nu$

$He^3 + He^4 \rightarrow Be^7 + \gamma$
$Be^7 + e^- \rightarrow Li^7 + \nu_e$
$Li^7 + p \rightarrow He^4 + He^4$

b)**Atmospheric neutrino and its anomaly**: Cosmic rays are a radiation of high energy particles arriving at the Earth from the Universe. In the GeV/nucleon energy region, these cosmic-ray particles are mostly protons, about 5 percent are Helium nuclei and a still smaller fraction of heavier nuclei. Electrons and photons also compose a part of the cosmic rays. However, since these components are nothing to do with the neutrino production, these particles will not be mentioned later. The energy spectrum of these particles extends to very high energies, although the flux of these particles decreases rapidly with the increasing energy. These particles, once enter into the Earth's atmosphere, interact with the nuclei in the high altitude atmosphere. Typically, in these high-energy nuclear interactions, many π mesons, and less abundantly K mesons, are produced. Since these mesons are unstable, they decay to other particles. For example, a $\pi+$ decays to a muon ($\mu+$) and a ν_μ,which is also unstable and decays to a positron e^+,ν^-, μ and a ν_e.

A similar decay process occur for π^- and K mesons. In this manner, neutrinos are produced when a cosmic-ray particle enters an atmosphere.These neutrinos are called atmospheric neutrinos. The primary cosmic-ray flux decreases rapidly with the energy, approximately $E^{2.7}$ in the GeV to TeV energy region. Therefore, the calculated neutrino flux rapidly decreases with the increasing energy.

2.0.3 Neutrino Oscillations

Neutrino oscillation arises from a mixture between the flavour and mass eigenstates of neutrinos.The three neutrino states that interact with the charged leptons in weak interactions are each a different superposition of the three neutrino states of definite mass. Neutrinos are created in weak processes in their flavour eigenstates. As a neutrino propagates through space, the quantum mechanical phases of the three mass states advance at slightly different rates due to the slight differences in the neutrino masses. This results in a changing mixture of mass states as the neutrino travels, but a different mixture of mass states corresponds to a different mixture of flavour states.

So if we initially start with an electron neutrino, it will be some mixture of electron, mu, and tau neutrino after traveling some distance. Since the quantum mechanical phase advances in a periodic fashion, after some distance the state will nearly return to the original mixture, and the neutrino will be again mostly electron neutrino. The electron flavour content of the neutrino will then continue to oscillate as long as the quantum mechanical state maintains coherence. Since mass differences between neutrino flavours are small in comparison with long coherence length for neutrino oscillations this microscopic quantum effect becomes observable over macroscopic distances.

Chapter 3

Neutrino Mass Generation Models

The most general $SU(3)_c \otimes SU(2)_L \otimes U(1)_y$ gauge invariant renormalizable lagrangian with Higgs doublet H,three lepton doublets L=$(\nu_L$,l$_L)^T$ and three lepton singlets e_R, etc beyond minimal gauge interaction terms is:

$$\begin{aligned}\mathcal{L}_{SM} = \mathcal{L}_{minimal} + (\lambda_e^{ij} E^i L^j H^* + \lambda_d^{ij} D^i Q^j H + \lambda^{ij} U^i Q^j H + h.c) \\ +m^2|H|^2 - \frac{1}{4}\lambda_H |H|^4\end{aligned} \tag{3.1}$$

Baryon number and Lepton number are natural symmetries in this scenario.The Higgs vacuum expectation value(vev) breaks the $SU(2)_L \otimes$U$(1)_y \rightarrow U(1)_{em}$ with the vev of Higgs being 174 GeV.
But the neutrinos remain massless in this picture because they do not have a right handed partner in the standard model.So to generate masses for neutrino sector, one invokes for non-renormalizable operators from which an effective lagrangian can be constructed by integrating out the heavy degrees of freedom.One such primary operator is the 5-dimensional operator LLHH/λ introduced first by Weinberg, where λ is the suppression factor.

3.0.1 Different schemes of realization of 5-dimensional operators: Seesaw Mechanisms

Type-I Seesaw

The five-dimensional Weinberg operator can be recovered in the limit where the external energy is very low as compared to the BSM heavy particle and hence this extra heavy particle is integrated out from the theory, which finally results in an effective theory with the aforementioned operator.The seesaw models are the prominent ones which exhibit this feature.So we now describe these models:

In Type-I Seesaw model, we add new extra singlet fermions with no gauge interactions which play the role of right handed neutrinos.They can have Yukawa interaction and in general, they can be Majorana particles because no standard model symmetry is going

to be violated(except B-L symmetry which is not a necessary requirement).
So we can add a Majorana mass term for these right handed fermion singlets in the extended Lagrangian along with Yukawa couplings with the leptonic sector.

The extended Lagrangian is given as follows

$$\mathcal{L} = \mathcal{L}_{\mathcal{SM}} + \bar{N_i^c}\partial\!\!\!/ N_i + (\lambda_{ijN} N_i L_j H + \frac{1}{2} M_N^{ij} \bar{N_i^c} N_j + h.c) \tag{3.2}$$

If we consider just one generation for simplicity, then the mass is generated in the following way: When the Higgs field takes a vacuum expectation value(vev) which is given by $\frac{v}{\sqrt{2}}$, then the Lagrangian becomes

$$\mathcal{L} = \lambda \bar{e_L} H^+ N_R + \frac{v}{\sqrt{2}} \bar{\nu_L} N_R + \frac{1}{2} M (N_R)^2 \tag{3.3}$$

The mass matrix of the neutrinos is given by:

$$K = \begin{pmatrix} 0 & m_d \\ m_d & M_N \end{pmatrix} \tag{3.4}$$

where m$_d$ =λv/$\sqrt{2}$ is the Dirac mass of the neutrinos

The mass matrix in this form is not diagonal.So to extract the mass eigenstates and the eigen-values, we diagonalise this mass matrix. The diagonal mass matrix is given by:

$$K' = \begin{pmatrix} m_d^2/M & 0 \\ 0 & M \end{pmatrix} \tag{3.5}$$

The mass basis is given by:

$$n_1 = (m_d/M)\nu_L + (M/m_d) N_R \tag{3.6}$$

$$n_2 = \nu_L + (m_d/M) N_R \tag{3.7}$$

If we take the right handed fermion singlet mass M to be very heavy compared to Dirac mass m_d, then in this limit we obtain:

$$n_1 \approx N_R, \quad n_2 \approx \nu_L \tag{3.8}$$

We also obtain the respective mass of the left handed neutrino to be:

$$m \approx \frac{(m_d)^2}{M} << 1. \tag{3.9}$$

Particularly with M of the order of 10^3 TeV and y≈ $O(10^{-4})$, m turns out to be consistent with experimental results.

Now considering the general framework of three generations of leptons, we would get a Dirac mass matrix $\bar{m_d}$ whose elements will be:

$$m_{ij} = \frac{\lambda_{ij} v}{\sqrt{2}} \tag{3.10}$$

The Majorana mass matrix $\bar{M}$ for right handed fermion singlets will have elements:M_{kl}

So the total mass matrix will be:

$$J = \begin{pmatrix} 0 & \bar{m_d} \\ (\bar{m_d})^T & \bar{M} \end{pmatrix} \tag{3.11}$$

This matrix is a 6×6 matrix with $\bar{m_d}$ and $\bar{M}$ being each 3×3 matrices.
We block diagonalize this matrix by a standard mathematical technique which goes as follows:
For some general block matrix: X=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ there would be an equivalent matrix Y such that Y=$R^{-1}XS$ is block diagonal, where

$$R^{-1} = \begin{pmatrix} I & -bd^{-1} \\ 0 & I \end{pmatrix} \tag{3.12}$$

$$S = \begin{pmatrix} I & 0 \\ -dc^{-1} & I \end{pmatrix} \tag{3.13}$$

And Y is given by:

$$Y = \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \tag{3.14}$$

This is an equivalent process of finding the Jordan Canonical form for matrices.
Using the above solution, we find that the block diagonal mass matrix for the neutrino sector is given by:

$$J' = \begin{pmatrix} -\bar{m_d}\bar{M}^{-1}\bar{m_d}^T & 0 \\ 0 & \bar{M} \end{pmatrix} \tag{3.15}$$

Therefore the mass matrix of left handed neutrino sector in the mass basis mentioned in equation.(8) is given by

$$m_v = -\bar{m_d}\bar{M}^{-1}\bar{m_d}^T \tag{3.16}$$

Type-II Seesaw

In Type-II Seesaw, we have scalar triplets, coined as beyond the standard model(BSM) Higgs particles.

They have a triplet representation as

$$\begin{pmatrix} \delta++ \\ \delta+ \\ \delta_0 \end{pmatrix} \tag{3.17}$$

The hypercharge of the triplet is Y=2

The gauge invariant extended Lagrangian in this case is as follows:

$$\begin{aligned} \mathcal{L} = \mathcal{L}_{\mathcal{SM}} + g_{ij}\bar{L}_i(\tau.\delta)L_j + \lambda_H\tilde{H}(\tau.\delta)H - M^2(\delta^\dagger\delta) \\ +\lambda(H^\dagger H)(\delta^\dagger\delta) - \mu^2 H^\dagger H + \lambda_\phi(H^\dagger H)^2 + \lambda_\delta(\delta^\dagger\delta) \end{aligned} \tag{3.18}$$

When the standard model Higgs and the BSM scalar triplet(Extra Higgs) simultaneously take vacuum expectation values, then this process yields the left neutrinos to become massive.

The term in the lagrangian which contributes to the mass of the left handed neutrinos is given in the matrix form as follows:

$$L' = \begin{bmatrix} \bar{e_L} \\ \bar{\nu_L} \end{bmatrix}^T \begin{bmatrix} \delta_0 & \delta_{++} - i\delta_+ \\ \delta_{++} + i\delta_+ & -\delta_0 \end{bmatrix} \begin{bmatrix} e_L \\ \nu_L \end{bmatrix} \tag{3.19}$$

When the Standard model Higgs develop the vev $\frac{v}{\sqrt{2}}$ and the BSM scalar triplet develop vev $\frac{v'}{\sqrt{2}}$, the mass term for the left handed neutrinos appear as:

$$\frac{v'}{\sqrt{2}} g_{ij}\bar{\nu_{iL}}\nu_{jL} \tag{3.20}$$

So the mass of the neutrinos is given by $\frac{v'}{\sqrt{2}}g_{ij}$

The vev of BSM scalar triplet is derived from the potential in the extended Lagrangian which is given by:

$$\begin{aligned} V = \lambda_H\tilde{H}(\tau.\delta)H - M^2(\delta^\dagger\delta) + \lambda(H^\dagger H)(\delta^\dagger\delta) \\ -\mu^2(HH) + \lambda_\phi(H^\dagger H)^2 + \lambda_\delta(\delta^\dagger\delta) \end{aligned} \tag{3.21}$$

Minimizing this potential w.r.t δ_0 and putting the condition that the Higgs vacuum expectation value is v, we obtain that the vacuum expectation value of the neutral component of the scalar triplet obeys this quadratic equation:

$$-\lambda_H v^2 - 2M^2 v' + 4\lambda_\delta v'^3 + 2\lambda v^2 v' = 0 \tag{3.22}$$

Using the approximation that the BSM scalar triplet is too heavy M$\gg v, v'$ and -$\lambda_H \approx M$ we obtain

$$v' \approx \frac{v^2}{2M} \tag{3.23}$$

Type-III Seesaw

In Type-III Seesaw, we have BSM right handed fermion triplets(charged fermions).

They have the three-dimensional representation under the Lorentz group:

$$X = \begin{pmatrix} X_+ \\ X_- \\ X_0 \end{pmatrix} \tag{3.24}$$

Hypercharge of X=0.

The electromagnetic charged states are

$$\begin{aligned} X_+ &= \frac{(X_1 + iX_2)}{\sqrt{2}} \\ X_- &= \frac{(X_1 - iX_2)}{\sqrt{2}} \\ X_0 &= X_3 \end{aligned} \tag{3.25}$$

The extended Lagrangian is given by:

$$\mathcal{L} = \mathcal{L}_{SM} + i\bar{X}_i \not{D} X_j - M_{ij}\bar{X}_i X_j + y_{ij}\bar{L}_i(\tau.X_j)\tilde{H} \tag{3.26}$$

When the Higgs takes the vacuum expectation value, the left neutrino sector acquires a mass term similar to the seesaw-I model.

The mass acquired by the left handed neutrinos is given by:

$$m \approx -v^2 y M^{-1} y^T \tag{3.27}$$

where y is the Yukawa coupling matrix of the interaction between the fermion triplets, the Higgs and the leptons,M is the right triplet mass matrix and v is the Higgs vev.

The simplest version of type-III seesaw does not take into account the masses of the superheavy extra particles in the light neutrino mass formula.This is an interesting fact and has the consequence that it opens the possibility to predict quark and lepton masses in a reasonable way.Recent investigations also show that such a model can be used in understanding leptogenesis scenarios, specifically resonant leptogenesis where there can be resonant enhancement without fine-tuning in the neutrino masses.

After having discussed the three seesaw models, we now move to the problem of hierarchy and discuss the issues underlying it.We first try to motivate the puzzle from a scaling scenario and then bridge it to the idea of naturalness.This would, in turn, lead to the stability of the Higgs vacuum configuration and its possible relations with the mass scale of beyond the standard model particles which are actively involved in the seesaw mechanisms.

Chapter 4

Hierarchy Problem

One of the interesting aspects of the physical world is that there are different scales associated to different phenomena which are observed in nature.Particle physics is too not an exception.In the domain of high energy particle physics, by scale, one conventionally refers to the energy scale or the length scale since these are equivalent in the natural system of units.It turns out that the model which describes elementary particle interactions upto a certain scale,i.e the standard model of particle physics, has several limitations and puzzling issues ingrained in its framework.One such puzzle is the enigma of the hierarchy of scales of different fundamental interactions.While the Planck scale$\approx 10^{19}$ GeV is an absolute scale in nature which is also thought to be the regime where gravitational effects become equally dominant in comparison to other fundamental forces, the huge difference it and the electroweak scale(where electromagnetism and weak forces unify) is a question which one ponders because such a huge numerical difference in scales is expected to have some interesting reason(s).In the standard model, the mass scales of different fundamental particles are generally set by the Higgs vacuum configuration through a process known as spontaneous symmetry breaking.Therefore it also naturally sets the energy scale of the fundamental processes that occur in nature. The weak scale is around 100 GeV which in turn is the mass scale of W and Z bosons which are the mediators of the weak force.But the non-zero vacuum configuration of the Higgs is unstable and prone to quantum fluctuations due to interactions with other particles. In fact, since the Planck scale is so high, one expects that the natural value for its vacuum configuration to be of the order of $\approx 10^{19}$ GeV.The miraculously low value of the Higgs vacuum inspite of the Planck scale sets the weak scale to be at such a small energy. This particular puzzle of the value of the Higgs vacuum which results in the huge hierarchy of the scales between the weak interaction and the Planck energy is traditionally referred to as the gauge-hierarchy problem.

There have been many proposed models and extended theoretical frameworks to resolve this puzzle.Some of them are supersymmetry, extra dimensions and composite Higgs models.However, most of these solutions require extra particles as additional ingredients which need to be validated from experiments.But so far, no experimental verification has come about to test the above models.Nevertheless, it is hoped that as the particle accelerators(like LHC) initiate running at higher energy scales, some of the predicted beyond the standard model particles will be observed.

4.0.1 Naturalness problem of the Higgs particle as a Hierarchy problem

The Higgs potential is parametrized by a dimensional mass-squared parameter μ^2 and a dimensionless Higgs self-coupling λ. Together they set the Higgs vacuum expectation value (vev) v $=\sqrt{\mu^2/\lambda} = 246$ GeV, which ultimately controls the masses of the W and Z bosons as well as that of the standard model fermions (except the neutrinos). These parameters also set the mass of the physical Higgs boson (the Higgs particle), $M_h^2 = \lambda v^2$.Other than a more complicated Higgs sector, the LHC measurement of the Higgs particle mass allows one to completely reconstruct the Higgs potential at the weak scale.

The hierarchy, fine-tuning or naturalness problem refers to quantum corrections to the Higgs mass-squared parameter μ^2 or, equivalently, the Higgs vacuum expectation value. Corrections to a scalar mass-squared are quadratically divergent and hence loops of Standard Model particles induce quantum corrections proportional to the unknown cuto scale. The top quark, as the most strongly coupled SM particle to the Higgs field, will induce the most relevant such correction.These corrections can be tackled by renormalization procedures since the standard model is a renormalizable theory.But this would lead to cancellation of correction terms by the parameters in the tree level contribution and if the corrections are very high, then unnatural fine-tuning is required in the tree level parameters to arrive at the observed mass of the Higgs particle.The naturalness condition imposes the constraint that the loop corrections would at most be of the order of 100 GeV,i.e, the mass scale of the Higgs(and hence the electroweak theory).

It is known that the Standard Model is not the complete theory of nature.It is missing descriptions of gravitation, dark matter, neutrino masses, and inflation.Also, the fact that several of its couplings have Landau poles due to renormalization flow indicates that the physics must change at very short distances for internal consistency.However, Landau poles occur around or above the Planck scale, where the Standard Model must be supplemented by a theory of quantum gravity.Although we don't have any consistent theory of quantum gravity at present and standard model is not UV complete, we follow Kenneth Wilson's idea of effective field theory, where one deals with an effective description of the high energy physics at low energy regimes by integrating out heavy particles.These heavy particles can add quantum loop corrections to the standard model Higgs mass and shift it to arbitrarily large scales.It would be unnatural if one requires an extremely large correction and hence a fine-tuning to set the observed mass scale of the Higgs in the process.We thus explicitly state a constraint which would avoid such unnatural fine-tuning cancellations: the naturalness condition.As a necessary consequence of the naturalness condition, we try to find out relations between the heavy particle masses and couplings when they interact with the standard model Higgs.

We have earlier explained about the seesaw mechanisms that generate tiny neutrino masses.Essentially we put forward the three different types of seesaw, each with different beyond the standard model(BSM) particles with heavy mass scales.Now it is time to investigate how do they couple to the Higgs and what are their effects on the mass correction of the Higgs.We would seek to derive naturalness condition through the bounds on the heavy mass scales of the respective BSM particles.

4.0.2 The Seesaw models and the Naturalness problem of the Higgs particle

One loop effect on Higgs mass from Seesaw-I

We once again state for brevity that the type-I seesaw mechanism is the simplest model to realize how tiny masses are generated for left neutrinos.This particular mechanism invokes for certain beyond the standard model fermionic particles which are gauge singlets and are right-handed in nature.They interact with the Higgs and the left neutrinos through the Yukawa coupling.This suggests that there should be corrections to the self-interaction of Higgs through the mediating right handed fermionic particles and the left neutrinos in higher orders of perturbation theory.There would thus be amplitudes at different orders which will contribute to the μ^2 parameter of the Higgs corresponding to the self interaction $\mid H \mid^2$. Here we consider the one-loop correction to the Higgs mass squared parameter, state the naturalness criteria and study different plausible cases.

The relevant part of the Lagrangian of Seesaw-I in this context is:

$$\mathcal{L} = \mu^2|H|^2 + \lambda|H|^4 + y_{ij}\bar{l_i}N_{jR}H \tag{4.1}$$

If we take just one generation, then

$$\mathcal{L} = \mu^2|H|^2 + \lambda|H|^4 + ylN_RH \tag{4.2}$$

where l_i is the left handed doublet (e_i, ν_i) and N_R is the heavy fermion singlet.

At one loop level the Higgs mass correction due to right handed neutrino goes as

$$\delta\mu^2 = y^2 \int \frac{d^4l}{(2\pi)^4}.Tr\Big[\frac{(\not{l} + \not{q} + m)(\not{l} + M)}{((l+q)^2 - m^2)(l^2 - M^2)}\Big] \tag{4.3}$$

where 'm' is the left handed neutrino mass and 'M' is the right fermion singlet mass, l is the momenta flowing in one side of the loop and q is the external momenta.

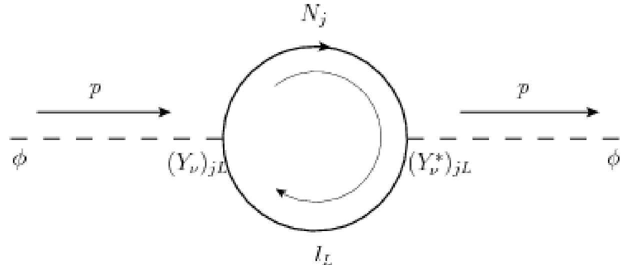

Figure 4.1: One-loop Feynman diagram corresponding to Higgs μ^2 correction

By solving this integral using Feynman parametrization technique and putting the cutoff equal to the mass scale of the right fermion singlet, we found that the dominant term contributing is:

$$\delta\mu^2 \approx \frac{1}{4\pi^2} y^2 M^2 \tag{4.4}$$

The idea of the naturalness problem of Higgs leads us to the condition that the corrections to its mass can be at most of the order of its vacuum expectation value, otherwise one would require fine tuning of the parameters in order to arrive at the desired value of its physical mass.This kind of fine-tuning has no good physical reasoning and hence we try to avoid it.The naturalness condition thus limits any higher order correction to that of the order of vev of the Higgs field.In this case, such a constraint is given by

$$\delta\mu < 100GeV \Rightarrow \delta\mu^2 < 10^4 Gev^2 \tag{4.5}$$

Therefore this would imply that

$$y^2 M^2 < 10^4 \times (4\pi^2) Gev^2 \Rightarrow yM < 440GeV \approx 0.5TeV \tag{4.6}$$

Now we consider the light neutrino masses from the Yukawa coupling which we derived to be

$$\frac{(m_d)^2}{M} \tag{4.7}$$

where

$$m_d = \frac{yv}{\sqrt{2}} \tag{4.8}$$

From neutrino experiments, we have the solar and atmospheric mass squared differences of neutrinos and the bound on the total mass.

Cosmological data provides the bound on the total mass

$$\sum m < 0.7eV \tag{4.9}$$

whereas Kamiokande and Chooz experimental data provide their mass squared differences as:

$$m_2^2 - m_1^2 \approx 10^{-5} eV^2 \tag{4.10}$$

$$m_3^2 - m_1^2 \approx 10^{-3} eV^2 \tag{4.11}$$

If we take $m_1 \approx 0$,then $0.05 < m_3 < 0.1eV$

Thus we have

$$0.05 < \frac{y^2 v^2}{(2M)} < 0.1eV \tag{4.12}$$

This implies that if M $\approx 1TeV$, then $10^{-5} > y > 0.3 \times 10^{-5}$

From equation (4.6) we observe that if M is of the order of 0.5TeV, then y can have any value of order less than 1.

If we impose the constraint of equation(4.21),we find that to account for both naturalness criteria of the Higgs particle and the seesaw mechanism, if M is of the order of 1 TeV, the only consistent value of y should lie in the following range:

$$10^{-5} > y > 0.3 \times 10^{-5} \tag{4.13}$$

On the other hand, if y$\approx \mathcal{O}(1)$ coupling, then seesaw condition imposes the constrain $10^{13} < M < 10^{14} GeV$ which contradicts the naturalness criteria of equation(4.6).

Thus we conclude from one generation calculations that y cannot be $\mathcal{O}(1)$ coupling in general to accommodate both seesaw mechanism and satisfy Higgs naturalness.

Now we proceed to calculate the naturalness constraints from the full lepton sector with all three generations:

The extended Lagrangian with the Yukawa term is given in this case by:

$$\mathcal{L} = \mathcal{L}_{SM} + \bar{N_i^c} \not{\partial} N_j + (y_{ij} N_i \bar{L}_j H + \frac{1}{2} M_{ij} \bar{N_i^c} N_j + h.c) \tag{4.14}$$

The Lagrangian is expressed in terms of the weak eigenbasis.We write it in the mass eigenstates to by transforming the fields to the mass basis as follows:
We transform the fermion singlets:

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = U \begin{pmatrix} N_e \\ N_\mu \\ N_\tau \end{pmatrix} \tag{4.15}$$

where U is the transformation matrix.

Similarly for the left handed neutrinos we perform the following:

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = V^{-1} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \tag{4.16}$$

In this mass basis, the Lagrangian is expressed as:

$$\mathcal{L} = \mathcal{L}_{sm} + \bar{N_i^c} \not{\partial} N_i + y_{ij} N_j (\alpha_i \nu_1 + \beta_i \nu_2 + \gamma_i \nu_3) H + \frac{1}{2} M_i \bar{N^c}_i N_i + h.c \tag{4.17}$$

The Yukawa term can be expanded in the following way:

$$\begin{aligned} L' = &(y'_{11} \nu_1 N_1 H_0 + y'_{21} \nu_2 N_1 H_0 + y'_{31} \nu_3 N_1 H_0) \\ &+(y'_{12} \nu_1 N_1 H_0 + y'_{22} \nu_2 N_1 H_0 + y'_{32} \nu_3 N_1 H_0) \\ &+(y'_{13} \nu_1 N_1 H_0 + y'_{23} \nu_2 N_1 H_0 + y'_{33} \nu_3 N_1 H_0) \end{aligned} \tag{4.18}$$

where

$$y'_{11} = y_{11}\alpha_1 + y_{21}\alpha_2 + y_{31}\alpha_3 \tag{4.19}$$

$$y'_{21} = y_{11}\beta_1 + y_{21}\beta_2 + y_{31}\beta_3 \tag{4.20}$$

$$y'_{31} = y_{11}\gamma_1 + y_{21}\gamma_2 + y_{31}\gamma_3 \tag{4.21}$$

and so on....

The one loop mass correction due to these couplings goes as follows:

For the coupling of ν_1 with N_1

$$\begin{aligned}\delta\mu^2 &= -{y'_{11}}^2 \int \frac{(dl)^4}{(2\pi)^4}.Tr\Big[\frac{(\not{l} + \not{q} + m_1)(\not{l} + M_1)}{((l+q)^2 - m_1^2)(l^2 - M_1^2)}\Big] \\ &= -{y'_{11}}^2 M_1^2/4\pi^2 + \text{logarithmic corrections}\end{aligned} \tag{4.22}$$

We get contributions from other couplings in a similar way.The total one loop correction is:

$$\delta\mu^2 = -1/4\pi^2 \sum_{i=1}^{3}\sum_{j=1}^{3} |y'_{ij}|^2 M_j^2 \tag{4.23}$$

Now we consider a particular parametrization for the Yukawa couplings of the neutrinos, known as the Casas-Ibbara Parametrization and is the following:

$$y' \approx iV\sqrt{M_\nu}O\sqrt{M_N} \tag{4.24}$$

where m_d is the dirac mass matrix,V is the CKM matrix which is unitary,Mν is the left handed neutrino mass matrix,M_N is the right handed fermion singlet mass matrix and O is an arbitrary complex orthogonal matrix.

Since we know that the Dirac mass is related to the Yukawa coupling as:

$$m_d = \frac{y'v}{\sqrt{2}} \tag{4.25}$$

where y' is Yukawa coupling matrix.

So

$$m_d = iVv\sqrt{2M_\nu}O\sqrt{M_N} \tag{4.26}$$

Now if the Yukawa couplings are real, then

$$\sum_{i=1}^{3}\sum_{j=1}^{3} |y_{ij}|^2 M_j^2 = Tr[y'{y'}^T M_N] \tag{4.27}$$

Hence

$$\sum_{i=1}^{3}\sum_{j=1}^{3}|y_{ij}|^2 M_j^2 = -Tr[2VV^T M_\nu M_N/v^2], \tag{4.28}$$

which is independent of the O matrix.

But if the Yukawa couplings are complex in general,then it follows that

$$\sum_{i=1}^{3}\sum_{j=1}^{3}|y_{ij}|^2 M_j^2 = Tr[y' y'^{\dagger} M_N] = -Tr[OO^{\dagger} 2M_\nu M_N/v^2] \tag{4.29}$$

which depends on the O matrix.

We denote R=$OO^{\dagger}$ This implies $R^{\dagger} = R$ and $R^T = R^{-1}$. So R is a complex orthogonal hermitian matrix.

After using the hermiticity condition,the R matrix is as follows:

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12}^* & R_{22} & R_{23} \\ R_{13}^* & R_{23}^* & R_{33} \end{bmatrix} \tag{4.30}$$

Now using the orthogonality condition, we have the following constraints with the elements of R:

$$R_{11}^2 + R_{12}^2 + R_{13}^2 = 1 \tag{4.31}$$

$$(R_{12}^*)^2 + R_{22}^2 + R_{23}^2 = 1 \tag{4.32}$$

$$(R_{13}^*)^2 + (R_{23}^*)^2 + R_{33}^2 = 1 \tag{4.33}$$

$$R_{11}R_{12}^* + R_{12}R_{22} + R_{13}R_{23} = 0 \tag{4.34}$$

$$R_{11}R_{13}^* + R_{12}R_{23}^* + R_{13}R_{33} = 0 \tag{4.35}$$

$$R_{12}^*R_{13}^* + R_{22}R_{23}^* + R_{23}R_{33} = 0 \tag{4.36}$$

From the first three constraints,R can be parametrized by three angles and one phase as follows:

$$R_{11} = \cos\alpha \tag{4.37}$$

$$R_{12} = \cos\theta e^{i\phi} \tag{4.38}$$

$$R_{13} = \sqrt{1-\cos^2\alpha - \cos^2\theta e^{2i\phi}} \tag{4.39}$$

$$R_{22} = \cos\beta \tag{4.40}$$

$$R_{23} = \sqrt{1-\cos^2\beta - \cos^2\theta e^{-2i\phi}} \tag{4.41}$$

$$R_{33} = \sqrt{\cos^2\alpha + \cos^2\beta + 2cos^2\theta\cos 2\phi - 1} \tag{4.42}$$

When we impose the other three constraints, we find that there is just one independent constraint and all others are redundant.The independent constraint is given by:

$$\cos^2\alpha + \cos^2\beta + 2cos^2\theta\cos 2\phi - 1 = \cos^2\alpha\cos^2\beta + cos^4\theta \\ -2\cos\alpha\cos\beta\cos^2\theta \tag{4.43}$$

Thus we infer that there are three free parameters.

The R matrix in its parametrized form is given as follows:

$$R = \begin{bmatrix} c(\alpha) & c(\theta)e^{if(\theta,\alpha,\beta)} & \sqrt{1-c^2(\alpha)-c^2(\theta)e^{2if(\theta,\alpha,\beta)}} \\ c(\theta)e^{-if(\theta,\alpha,\beta)} & c(\beta) & \sqrt{1-c^2(\beta)-c^2(\theta)e^{-2if(\theta,\alpha,\beta)}} \\ \sqrt{1-c^2(\alpha)-c^2(\theta)e^{-2if(\theta,\alpha,\beta)}} & \sqrt{1-c^2(\beta)-c^2(\theta)e^{2if(\theta,\alpha,\beta)}} & c^2(\alpha)+c^2(\beta)+2c^2(\theta)c(2\phi)-1 \end{bmatrix} \tag{4.44}$$

where c(.) denotes cos(.) and

$$f(\theta,\alpha,\beta) = \frac{1}{4\cos^2\theta} \times \arccos\left(1+\cos^2\alpha+\cos^2\beta+\cos^4\theta \right. \\ \left. -2\cos\alpha\cos\beta\cos^2\theta - \cos^2\alpha - \cos^2\beta\right) \tag{4.45}$$

We try to deduce the naturalness condition as follows:
From equations(4.28) and (4.29),we have:

$$\delta\mu^2 = 4\pi^2 Tr[y'y'^{\dagger}M] = Tr[RM_\nu M_N/v^2] < 10^4 GeV^2 \tag{4.46}$$

$$\Rightarrow M_1^3 m_1 R_{11} + M_2^3 m_2 R_{22} + M_3^3 m_3 R_{33} < 5\times 10^3 v^2 GeV^4. \tag{4.47}$$

Now we consider some specific cases where we try to relax the bound using the free parameters of the R matrix as follows:

Case 1:

From the mass squared differences of light neutrinos,
we consider a viable case where $m_1 = 0, m_2 \approx 0.003eV$ and m$_3 \approx 0.03eV$.

Then for degenerate mass of BSM fermion singlets, we obtain the condition:

$$M < \frac{3.17\times 10^6}{R_{33}^{1/3}} GeV \tag{4.48}$$

So if we choose $R_{33}^{1/3} = 0.0001$, then $M < 3.17\times 10^{10}$ GeV which is relaxed than the previous results and satisfies the bound given in all versions of leptogenesis.

Case 2:

Again for degenerate mass of BSM fermion singlets, we have

$$M < \frac{2.1 \times 10^7}{(0.003 R_{22} + 0.03 R_{33})^{1/3}} GeV \tag{4.49}$$

then adjusting the value of $(0.003R_{22} + 0.03R_{33})^{1/3} \approx 0.001$

we can have the bound on $M < 2.1 \times 10^{10}$GeV which is again relaxed than the previous bounds and is in agreement with leptogenesis results.

Case 3:

Now we consider non-degenerate masses of BSM fermion singlets:

Then the bound condition on the sum of mass cubes is as follows:

$$(0.003 R_{22} M_2^3 + 0.03 R_{33} M_3^3) < 1.5 \times 10^8 GeV^3 \tag{4.50}$$

Thus one of the fermion singlet with mass M_1 can be arbitrarily high value because it will be suppressed by a very very small neutrino mass(m_1)≈ 0

On the other hand, since we can choose R_{22} and R_{33} freely owing to free parameters, we can in principal make the upper-bound on some of the masses to be very high by choosing the matrix elements to be very small.

This would agree with the leptogenesis results regarding the bounds on their masses.

One loop effect on Higgs mass from Seesaw-II

As we have mentioned earlier, we have extra scalar triplets of hypercharge 2 in this scenario which couples to the leptonic sector and the Higgs.

Their representation is given by:

$$\begin{pmatrix} \delta ++ \\ \delta + \\ \delta_0 \end{pmatrix} \tag{4.51}$$

The gauge invariant extended Lagrangian in this case is as follows:

$$\begin{aligned} \mathcal{L} = \mathcal{L}_{SM} + g_{ij} \bar{L}_i (\tau.\delta) L_j + \lambda_H \tilde{H} (\tau.\delta) H - M^2 (\delta^\dagger \delta) \\ + \lambda (H^\dagger H)(\delta^\dagger \delta) - \mu^2 H^\dagger H + \lambda_\phi (H^\dagger H)^2 + \lambda_\delta (\delta^\dagger \delta) \end{aligned} \tag{4.52}$$

The mass of the neutrinos is given by $g_{ij}v'$
The vev of BSM scalar triplet is derived from the potential in the extended Lagrangian which is given by:

$$\begin{aligned} V = \lambda_H \tilde{H}(\tau.\delta)H - M^2(\delta^\dagger\delta) + \lambda(H^\dagger H)(\delta^\dagger\delta) \\ -\mu^2(HH) + \lambda_\phi(H^\dagger H)^2 + \lambda_\delta(\delta^\dagger\delta) \end{aligned} \tag{4.53}$$

Minimizing this potential w.r.t δ_0 and putting the condition that the Higgs vacuum expectation value is v, we obtain that the vacuum expectation value of the neutral component of the scalar triplet through this equation:

$$-\lambda_H v^2 - 2M^2 v' + 4\lambda_\delta v'^3 + 2\lambda v^2 v' = 0 \tag{4.54}$$

Now the one loop correction to the Higgs mass will come from the coupling term:

$$\lambda(H^\dagger H)(\delta^\dagger\delta) \tag{4.55}$$

The one loop integral corresponding to a total mass correction is given by

$$\delta\mu^2 = \frac{\lambda}{\pi^4}\sum_{i=1}^{3}\int \frac{d^4k}{(k^2 + M_i^2)} \tag{4.56}$$

Taking the cutoff to be the mass scale of the scalar triplet components, we have:

$$\delta\mu^2 = \frac{3\lambda}{8\pi^2}\sum_{i=1}^{3} M_i^2 \tag{4.57}$$

Taking the naturalness constraint, $\delta\mu^2 < v^2$ we see that:

$$\frac{3\lambda}{8\pi^2}\sum_{i=1}^{3} M_i^2 < v^2 \tag{4.58}$$

If we consider one generation, mass of the neutrino is given by:

$$m \approx gv' \tag{4.59}$$

From the mass squared difference, we have,

$$0.05eV < m < 0.1eV \tag{4.60}$$

$$\Rightarrow 0.05eV < gv' < 0.1eV \tag{4.61}$$

Now from equation (4.54) taking the approximations $M >> v, v'$ we deduce

$$v' \approx \lambda_H v^2/2M^2 \tag{4.62}$$

$$\Rightarrow 0.05eV < g\lambda_H v^2/2M^2 < 0.1eV \tag{4.63}$$

$$\Rightarrow 10^7 GeV < M < \sqrt{2} \times 10^7 GeV \tag{4.64}$$

If λ_H and g are both O(1) couplings, this implies that the scalar triplet strongly interacts with the neutrino sector. Assuming degenerate scalar triplet masses,from the mass correction we have:

$$\delta\mu^2 = \frac{9\lambda M^2}{8\pi^2} \tag{4.65}$$

Imposing the naturalness constraint, we find:

$$\Rightarrow \lambda M^2 < 8 \times 10^4 GeV^2 \Rightarrow \lambda < 10^{-24} \tag{4.66}$$

which indicates that the scalar triplet would interact very weakly with the Standard Model Higgs. On the other hand, a strongly interacting Yukawa coupling would result in a weakly coupled Higgs-scalar triplet which would shift the mass of the scalar triplet to a high scale Therefore, naturalness constraint along with seesaw condition implies that if Yukawa(g)$\approx O(1)$ coupling, then the mass scale of the extra scalar triplet is around $10^7 GeV$ which is far higher than the seesaw-I scale for one generation.

One loop effect on Higgs mass from Seesaw-III

As we have mentioned earlier, in Type-III Seesaw we include right handed fermion triplets with hypercharge 0.

The extended Lagrangian is given by:

$$\mathcal{L} = \mathcal{L}_{SM} + i\bar{X}_i \not{D} X_j - \frac{1}{2} M_{ij} \bar{X}_i X_j + y_{ij} \bar{L}_i (\tau . X_j) \tilde{H} \tag{4.67}$$

We calculate the one loop corrections for a single generation at first:

The Yukawa coupling terms which contribute to the loop correction are:

$$\mathcal{L}' = iy(X_0 e^- H_0 + X_+ H_0 \bar{\nu_L} + X_- H_0 \bar{\nu_L}) + h.c \tag{4.68}$$

The loop integrals will give the dominant contribution to be:

$$\delta\mu^2 = \frac{3y^2 M^2}{8\pi^2} \tag{4.69}$$

where M is the mass of the fermion triplet

For naturalness constraint, $\delta\mu^2 < v^2$

$$\Rightarrow M < 2v\sqrt{2}\pi/\sqrt{3}|y| \tag{4.70}$$

$$\Rightarrow M < 640/|y| GeV \tag{4.71}$$

Now from seesaw condition,we have the mass of the left handed neutrinos to be:

$$m \approx y^2 v^2 / M \tag{4.72}$$

From neutrino experiments,we have

$$m_2^2 - m_1^2 \approx 10^{-5} eV^2 \tag{4.73}$$

$$m_3^2 - m_1^2 \approx 10^{-3} eV^2 \tag{4.74}$$

If we take $m_1 \approx 0$,then $0.05 < m_3 < 0.1 eV$

Then we have

$$0.05 eV < y^2 v^2/(2M) < 0.1 eV \tag{4.75}$$

$\Rightarrow$ if M$\approx 1 TeV$ then

$$10^{-5} > y > 0.3 \times 10^{-5} \tag{4.76}$$

From equation (4.71) we observe that if M is of the order of 640 GeV, then y can have any value of order less than 1.
We find that to account for both naturalness and seesaw criteria, if M is of the order of 1 TeV, the only consistent value of y should lie in the following range :

$$10^{-5} > y > 0.3 \times 10^{-5} \tag{4.77}$$

On the other hand,if y $\approx \mathcal{O}(1)$ coupling, then seesaw criteria imposes the constrain $0.3 \times 10^6 < M < 10^7 GeV$ which contradicts the naturalness criteria of eq.(4.70).

Having discussed the effects of various beyond the standard model particles on the Higgs naturalness, we now proceed to understand the framework of incorporating seesaw-1 model with the hierarchy problem in the most natural setting.While we have till now devised general conditions for the naturalness of the Higgs mass as a result of its interaction with extra heavy particles, the next task would be to set a framework where the naturalness of the seesaw can also be invoked. Although an assumption of having very small Yukawa couplings in the theory do not imply any inconsistency problem in general,it confronts a conventional idea of the naturalness of type-1 seesaw mechanism.

Chapter 5

Naturalness of Type-I Seesaw Mechanism

We once again state the lagrangian for an extension of the Standard Model by n number of gauge singlet Majorana fermions N_α.We can write their complete renormalizable interaction as

$$\mathcal{L} = \mathcal{L}_{SM} + \bar{N_\alpha}\sigma_\mu\partial_\mu N_\alpha - \frac{1}{2}(M_N)_{\alpha\beta}\bar{N}^c_\alpha N_\beta - y_{i\alpha}\bar{L}_i N_\alpha \tilde{H} + h.c \tag{5.1}$$

where $i = 1,2,3$ and $\alpha, \beta = 1,2,3....n$ are flavour indices,$L = (\nu_L, e_L)^T$, $\tilde{H} = i\tau_2 H^*$ where H is the Standard Model Higgs(SM Higgs)with vacuum expectation value $v \equiv H = 174GeV.y_{i\alpha}$ are the Dirac Yukawa couplings and M_N is the Majorana fermion mass matrix.Without loss of generality, we consider a basis in which the 3×3 charged lepton Yukawa matrix Y_l and $n \times n$ matrix M_N are diagonal with real and positive elements. We denote such diagonal elements in M_N as $M_{N\alpha}$. When the electroweak symmetry is broken and if $M_{N\alpha} \gg v$, the Standard Model neutrinos acquire masses which can be expressed as

$$M_\nu = v^2 y M_N^{-1} y^T \tag{5.2}$$

The symmetric matrix M_ν is diagonalized by a unitary matrix such that

$$U^\dagger M_\nu U^* = Diag.(m1, m2, m3) \tag{5.3}$$

where U is the leptonic mixing matrix, known as the PMNS matrix and m_i are the mass eigenvalues of light neutrinos.

As mentioned earlier, we once again write the naturalness(of the Higgs mass) criteria to illustrate the constraint it puts on the electroweak naturalness. The criteria is given by

$$\mid y_{i\alpha} \mid M_{N\alpha} < O(TeV) \tag{5.4}$$

It requires the singlet fermions at TeV scale if the neutrino Yukawa couplings are of O(1) or extremely small couplings if the mass scale of singlet fermion is heavier than TeV. The latter possibility is actually constrained by the observed light neutrino masses and it can be seen that one cannot consider arbitrarily small $y_{i\alpha}$ and arbitrarily large $M_{N\alpha}$.

In the simplest case of single generation of light neutrino and fermion singlet, the above equation implies $y^2 \approx M_N m_\nu / v^2$ which then leads to a generic bound from the criteria of naturalness:

$$\frac{M_N^3 m_\nu}{4\pi^2 v^2} \leq (TeV)^2 \quad \Rightarrow \quad M_N \leq 2.9 \times 10^7 \times \left(\frac{\sqrt{m_{atm}^2}}{m_\nu}\right)^{\frac{1}{3}} GeV. \tag{5.5}$$

The above bound on the mass of the singlet neutrino do not get modified significantly if three generations of light and heavy neutrinos are considered. In a special case when the lightest neutrino is massless, one of the three fermion singlets can have arbitrarily large mass unconstrained by the electroweak naturalness. This can be seen from the fact that if we take a particular linear combination of states N_α, it completely decouples from the Standard Model. This particular linear combination has only self-interaction giving no contribution in the Higgs mass correction.

Chapter 6

Electroweak naturalness vs Seesaw naturalness

The electroweak naturalness demands the scale of fermion singlets below 10^7 GeV as explained in the previous section. A generic observation from eq.(5.4) then implies that the Dirac type Yukawa couplings are required to be small to account for light neutrino masses. If $m_i \leq 0.1eV$ then

$$\mid y_{i\alpha} \mid \leq O(10^{-4}) \tag{6.1}$$

Further, if $M_{N\alpha} \approx 1TeV$ then $\mid y_{i\alpha} \mid$ are typically required to be $\leq 10^{-6}$. Although an assumption of having such small Yukawa couplings in the theory do not pose any consistency in general but it confronts a conventional idea of the naturalness of Type-I seesaw mechanism itself. In the Type-I seesaw mechanism, the natural assumption is to consider the Dirac type interactions of the neutrinos to be of the same order as other fermionic interactions.In such a scenario, the light neutrino masses are generated by introducing a heavy scale in the extra right neutrino sector, possibly heavier than 10^9 GeV. One other instance for this kind of an assumption of the order of Dirac type interactions is in the framework of quark-lepton unification. There, the Dirac Yukawa couplings and up-type quark Yukawa couplings unify at some heavy scale. In this class of theories, it is typically expected that at least one of the $y_{i\alpha}$ is as large as the top quark Yukawa coupling, $y_t \approx 1$. It is now clear from the discussion above that M_N does not provide sufficient suppression to generate tiny mass for the neutrinos if the mass scale for the fermion singlets is $\leq 10^9$. Some other source of suppression must be arranged. This can be achieved either by taking $y_{i\alpha}$ small or by arranging some cancellation within the product in the right hand side of the seesaw formula, eq.(5.2). For the case of natural seesaw, we discuss here about the second arrangement, which then has to be justified from a symmetry principle. We first derive the general condition that has to be followed by $y_{i\alpha}$ in order to reproduce the correct light neutrino masses being consistent with electroweak naturalness.

We consider three generations of light neutrinos and n generations of heavy fermion singlets. Using eq. (5.2), we find

$$Tr(M_\nu) = v^2 \sum_{\alpha,i} \frac{y_{i\alpha}^2}{M_{N\alpha}} = \sum_i m_i \equiv m_{sum} \tag{6.2}$$

where m_{sum} is the sum of three neutrino masses.An upper bound on m_{sum} is derived from the cosmological observation. The current limit by Planck data implies m$_{sum}\leq 0.7eV$. m_{sum} is also bounded from below. In the case of lightest neutrino being massless, $m_{sum}\geq \sqrt{m^2_{atm}}$. Redefining $M_{N\alpha}=r_\alpha\tilde{M_N}$ where $\tilde{M_N}$ is mass of the heaviest singlet fermion, i.e. $r_\alpha<1$, one gets

$$1.7\times 10^{-12}\times\left(\frac{\tilde{M_N}}{TeV}\right)<\sum_{\alpha,i}\frac{y^2_{i\alpha}}{r_\alpha}\leq 2.3\times 10^{-11}\times\left(\frac{\tilde{M_N}}{TeV}\right) \tag{6.3}$$

In the case of three generations of SM neutrinos, it is possible to consider $y_{i\alpha}$are of order unity and still to satisfy the above constraint together with electroweak naturalness constraint,eq.(5.4). It however requires a special structure in Yukawa coupling matrix y and fermion singlet mass matrix MN such that it leads to tiny neutrino mass despite of having low seesaw scale. Our aim now is to check a viability of $O(1)$ Yukawa couplings with the constraints imposed by electroweak naturalness, eq.(5.4), and neutrino masses, eq.(5.2). Clearly, this necessarily requires atleast some the couplings in y to be complex. If all the singlet fermions are degenerate ($r_\alpha=1$) and have mass 1 TeV then the bound in eq.(9) simplifies to

$$1.7\times 10^{-12}<<\sum_{\alpha,i}y^2_{i\alpha}<2.3\times 10^{-11} \tag{6.4}$$

In the simplest example, if we consider two generations of degenerate fermion singlets then this can be achieved with the following ansatz of the matrix y:

$$y=\begin{pmatrix} y_1 & iy_1(1+\epsilon_1)\\ y_2 & iy_2(1+\epsilon_2)\\ y_3 & iy_3(1+\epsilon_3)\end{pmatrix} \tag{6.5}$$

where y_is are O(1) complex parameters and ϵ_i are small parameters. Replacement of the above y and $M_N=diag.(\tilde{M_N},\tilde{M_N})$ with $\tilde{M_N}=1$TeV in eq. (5.2) leads to a massless and two massive neutrinos. By varying randomly $\mid y_i\mid\in[0.2,1]$ and $Arg.(y_i)\in[0,2\pi]$, we obtain the range in $\mid\epsilon_i\mid$ which can reproduce the solar and atmospheric mass squared differences withing the 3σ range of their global fit values. The results are displayed in section(10). We also show the distributions in three mixing angles in this case.

6.0.1 Symmetry structures

We now discuss how to understand the features that come about from the contexts of electroweak and seesaw naturalness. We have seen that in order to ensure that the seesaw scale is low(and possibly within the current reach of LHC) and that the naturalness criterion of the Higgs mass is satisfied, we need to resort to some special structure in the Yukawa coupling matrix.Unless this kind of a structure is unconvincingly random, it invokes the need to understand it from a point which captures the broader symmetry(s) in the problem and what further connections may it have from other candidate models which resolve some of the issues of the Standard Model.In principle, this kind of a motivation has been useful from a historical perspective and seeds the rationale behind looking for symmetry structures in a broader and deeper way.We have understood the importance of continuous symmetries reasonably well since the last few decades and the structures that are explored in this context ubiquitously fall in the domain of Lie groups and its representations.While on the other hand, the domain of discrete finite groups was not at the forefront of modeling in the arena of particle physics but they have proved to be quite appropriate in condensed matter systems which involve lattice structures. But recent investigations show that they are equally proving to be useful and interesting in the particle sector where one deals with flavor physics.Since the Standard Model has three generations of fermions with each generation following a particular pattern of masses and mixing angles(in the quark sector) and its extended versions(including lepton sector) following some different patterns, the question of how and why these patterns exist and differ from one another naturally needed to be addressed. Thus model building with finite discrete groups was a sensible idea to be dealt with. We now briefly address the problem of masses and mixings in the leptonic sector for the sake of completeness and will then move to our original problem.

The observed leptonic mixing angles are known to be close to special values. The atmospheric mixing angle is close to maximal with $\sin^2\theta_{23} \equiv 0.44$,the solar angle θ_{12} and the reactor angle θ_{13} satisfy $\sin^2\theta_{13} \equiv 0.023 \equiv 0$ and $\sin^2\theta_{12} \equiv 0.21$.It is natural to look for group theoretical explanations for such special values as has been extensively done earlier. In this approach, it is assumed that the underlying theory of leptonic flavor possesses some discrete symmetry G.The group G breaks to= smaller non-commuting subgroups G and G_l which correspond to unbroken symmetries respectively of the neutrino and the charged lepton mass matrices M and M_l, more precisely of $M_l M_l^\dagger$.While possible choices of G are a priori unknown and numerous, one can relate G and G_l to the known structure of the mixing matrix. Thus it becomes more profitable to start with possible choices of G and G_l dictated from physical considerations and search for groups which contain them as subgroups.

In all these analyses, the basic but implicit assumption is that neutrinos are Majorana particles and all three of them are massive.The present neutrino data are however quite consistent with one of the neutrinos being exactly massless both in the case of normal and inverted hierarchy for neutrino masses. The underlying symmetry G and hence possible choice of G become quite different in this case. In this note, we discuss possible symmetry groups G and embedding of G and G_l into some bigger group G assuming that one of the three neutrinos is massless.

A schematic mathematical outline about how mixing angles can be related to symmetry groups G is as follows.Let U_v and U_l diagonalize M_v and $M_l M_l^\dagger$ respectively.

$$U_v^T M_v U = Diag.(m_{v_1}, m_{v_2}, m_{v_3}), \tag{6.6}$$

$$U_l^T M_l M_l^\dagger U_l = Diag.(m_e^2, m_\mu^2, m_\tau^2). \tag{6.7}$$

We now consider that M_v and $M_l M_l^\dagger$ are invariant under some set of discrete group elements B_i and D_i respectively

$$B_i^T M_v B_i = M_v \tag{6.8}$$

and

$$D_i^T M_l M_l^\dagger D_i = M_l M_l^\dagger \tag{6.9}$$

Without loss of generality, one can consider that the elements B_i and D_i commute amongst themselves and hence can be diagonalized simultaneously. Let the diagonalizing matrices for B_i and D_i are X_v and Y_l respectively. Then it can deduced that

$$U_v = X_v P_v \tag{6.10}$$

and

$$U_l = Y_l P_l \tag{6.11}$$

where P_v and P_l are diagonal phase matrices.
Therefore we can write the leptonic mixing matrix

$$U_{PMNS} = U_l^\dagger U_v = P_l^* Y_l^\dagger X_v P_v \tag{6.12}$$

Thus it can be seen that mixing patterns in the flavor sector(here, the leptonic part) can be related to symmetry structures from a group theoretic perspective.

In a bottom up approach, one tries to determine groups of B_i and D_i and then uses them to find the larger group G whose subgroup is generated by the elements. A complete set of B_i and D_i may depend on the underlying dynamics but one can define a minimal set which can always be taken as symmetries of mass matrices.

In our problem as discussed earlier, we need to address the specific Yukawa structure so that in the exact symmetric case, the neutrinos are massless and Higgs naturalness is satisfied.We thus try to deduce this result by identifying the possible symmetry which might exist in the Yukawa sector and the right handed fermion sector so that all the light neutrinos turn out to be massless.A viable scenario which can provide us with the requisite understanding of the Yukawa structure is the following symmetry:

$$S^\dagger Y S' = Y, \quad S'^T M_N S' = M_N \tag{6.13}$$

where $S' \in SU(3)$ and $S \in U(3)$ with Eigenvalues of $S \neq \pm 1$. This implies that the effective neutrino mass matrix would obey the following symmetry condition

$$S^{\dagger} M_v S^{*} = M_v \tag{6.14}$$

where M_v is the effective neutrino mass matrix. This symmetry condition particularly ensures that the effective neutrino mass matrix is identically zero, rendering all left neutrinos to be massless.

In a suitable basis where the M_N is diagonal, the Yukawas have the form described in the earlier section and the light neutrinos are massless. We illustrate this with an example:

$$Y = \begin{pmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \\ y_7 & y_8 & y_9 \end{pmatrix}, \quad M_N = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_2 & m_4 & m_5 \\ m_3 & m_5 & m_6 \end{pmatrix} \tag{6.15}$$

where M_N has the Majorana structure(symmetric).

We use elements S and S' from SU(3) and U(3) respectively as

$$S = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, \quad S' = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.16}$$

Imposing the symmetry condition as mentioned in equation (6.13), we get the forms of Y and M_N

$$Y = \begin{pmatrix} 0 & 0 & y_3 \\ 0 & 0 & y_6 \\ 0 & 0 & y_9 \end{pmatrix}, \quad M_N = \begin{pmatrix} 0 & m_2 & 0 \\ m_2 & 0 & 0 \\ 0 & 0 & m_6 \end{pmatrix} \tag{6.17}$$

Transforming to the basis where M_N is diagonal, we get the form of Y as

$$Y = \begin{pmatrix} 0 & iy_3 & y_3 \\ 0 & iy_6 & y_6 \\ 0 & iy_9 & y_9 \end{pmatrix} \tag{6.18}$$

Thus using certain symmetry structures we arrive at the desired form of the Yukawa matrix of equation(6.5) upto the perturbations in the parameter ϵs which will be used to generate the tiny mass of light neutrinos.

6.0.2 Use of group theoretic models to motivate the residual symmetries

While the gauge interactions are flavour-blind, the flavour dependence of the Yukawa couplings is crucial because this is, in conjunction with the vacuum expectation values determined by the minimum of the scalar potential, the origin of fermion masses and the mixing matrices. In the general case, the fermion masses and the parameters of the mixing matrices are completely free. Up to now, we do not know of any fundamental

principle, comparable to the importance of the gauge principle, which would allow us to constrain the Yukawa sector of the Lagrangian L_Y and the scalar potential such that the fermion mass spectra and the entries of the mixing matrices find a satisfactory explanation.
Since symmetries in fundamental interactions have been proven to be successful, we resort hence to flavor symmetries to put constraints on the Lagrangian and the corresponding mixing matrices.
If we want that spontaneous symmetry breaking in the flavor sector is exhibited in a way similar to the gauge sector, then the only way to avoid Goldstone modes in the problem is to deal with finite groups. In finite groups, if we choose the simplest possible groups which are abelian, it turns out that they cannot satisfactorily reproduce the correct mixing structure in the flavor sector. Therefore we resort to non-abelian finite discrete groups to invoke model building in the flavor sector.A_4, the smallest group with an irreducible three-dimensional representation, has become very popular for its capacity to enforce tri-bimaximal mixing, provided one finds a solution to the vacuum alignment problem.

In the problem concerning the naturalness of the Higgs mass, it turns out that even though we start modelling it with a group theoretic structure, i.e building a Lagrangian which is invariant under the group concerned, we need to break this symmetry at some specific energy scale to a smaller residual symmetry of the Yukawas and the right handed neutrino masses. This smaller symmetry structure consisting of some of the group elements need not themselves form a subgroup of the original group since the left handed neutrinos and the right neutrinos are in general not unified under a single representation in viable models. But as a motivating idea to persuade such a unified representation scheme in certain beyond the standard model theories like Grand Unified theories, this can be a starting point. Hence, we try to propose some specific group which accounts for the property that the residual symmetries themselves form a subgroup of it.

The residual symmetries in our problem are explicitly described by two elements of the group, i.e, a unitary matrix $S \in U(3)$ with all eigenvalues to be pure phases and another special unitary matrix $S' \in SU(3)$ with two eigenvalues to be pure phases and one of them is 1. More schematically, the structure of the residual symmetries consists of the following constraints:

$$S^{\dagger} Y S' = Y \tag{6.19}$$

and

$$S'^{T} M S' = M \tag{6.20}$$

where Y is the Yukawa coupling matrix and M is the right handed neutrino mass matrix.

The explicit form of S' is as follows:

$$S' = Diag(\eta, \eta^{*}, 1) \tag{6.21}$$

where η is some nontrivial phase.
The form of S can have two possibilities which we list below:

$$S = Diag(\eta, \eta, \eta) \tag{6.22}$$

or

$$S = Diag(\eta^*, \eta^*, \eta^*) \tag{6.23}$$

If the above structures are satisfied by the elements of the symmetry group we started with, mathematical consistency forces us to conclude that two Majorana right handed neutrino masses must be degenerate and the third would not have any bounds and is decoupled from the rest of the interactions. This can also be understood by a basis transformation on the fundamental fields and end up with an interaction where one of the right handed fermion singlets decouples from the Yukawa interactions and only has self- interactions.The underlying mechanism governing this can be motivated from the perspective of lepton number conservation.But here we take a different approach and find that we can design a model based on some appropriate subgroup of U(3). The subgroup U(3) is required to have an element in it of the form of S which would serve as a residual symmetry in the broken sector. This would then depict a natural electroweak seesaw with $\mathcal{O}(1)$ Yukawa couplings and satisfies the naturalness condition of the Higgs simultaneously. This would give massless neutrinos even after seesaw mechanism and thus extends the idea of seesaw and introduces additional structure. In order to generate tiny neutrino mass, we further require breaking this residual symmetry with generic perturbations. We do not discuss any particular model for the perturbative breaking of symmetry but we give the plausible phenomenological implication.

Chapter 7

Finite discrete groups and representation theory

In this section, we present a collection of properties of finite groups that may satisfy the basic needs of model building in particle physics. The striking feature of finite groups, which has no counterpart in infinite groups, is that many of their properties are expressed in terms of the integers associated with the group. Such integers are, for instance, the order of a group, the number of conjugacy classes, the dimensions of its irreducible representations, etc.

A set of generators or generating set of a group G is a subset S of G such that every element of G can be written as a finite product of elements of S and their inverses. We have earlier introduced symmetries of the Lagrangian; these symmetries can be regarded as representations of the set of group generators on the field multiplets. A group is called finitely generated if there is a finite set S of generators. Since we will be dealing with finite groups, all our groups will be finitely generated. The precise definition of a presentation of a group G is complicated. A presentation consists of a set S of generators and a set R of relations among the generators which completely characterize the group. This means that writing strings of the generators and by using R to shorten the strings one obtains all group elements. A presentation of a group is by no means unique. It is often useful to choose different presentations for different purposes. The simplest example of a presentation is that of the cyclic group Z_n. It has one generator a and one relation $a^n = e$, which completely characterizes the group.

One of the most important group-theoretic application to physics is the theory of group representations. Unless otherwise mentioned, we generally deal with group representations in complex spaces. In other words, a reasonable definition of a group representation is a homomorphism of the group to the general linear group $GL(n, C)$ of n-dimensions which can act on n-dimensional complex vector spaces which in a physical perspective might refer to the state-space of the system in the problem.The following theorem tells us that for finite groups, without loss of generality, we can confine ourselves to unitary representation matrices:

Theorem: Every finite-dimensional representation D of a finite group G is equivalent to a unitary representation, i.e.

$$\exists S : S^{-1}DS = D',\ D(a)^{\dagger} = D(a)^{-1}\ \forall\ a \in G \tag{7.1}$$

We list (without rigorous proofs) some of the important theorems in representation theory which will be useful in the applications to particle physics and specifically to our problem:

Theorem 1: Every finite group has faithful finite-dimensional representations.

Theorem 2: The number of irreducible representations of a finite group is equal to the number of its conjugacy classes.

Theorem 3: The dimension of an irreducible representation of a finite group is a divisor of the order of the group.

Theorem 4: The regular representation of a finite group G contains each of its inequivalent irreducible representations $D^{(\alpha)}$ with the multiplicity of its dimension.

Theorem 5: No abelian subgroup of a finite group can have any non-trivial faithful three-dimensional representation.

Schur's Lemma: Let $D^{(1)}$ and $D^{(2)}$ be finite-dimensional irreducible representations of a finite group G on the linear spaces V_1 and V_2 respectively, and let $S : V_1 \to V_2$ be a linear operator such that

$$D^{(2)}(a)S = SD^{(1)}(a)\ \forall\ a \in G \tag{7.2}$$

then S is either zero or invertible. If S is invertible, then the representations are equivalent.

7.0.1 Discrete subgroups of U(3) and their representation

The interesting problem of lepton masses and mixings motivated the study of discrete symmetries in the problem under which the lagrangian remains invariant.To generate mixing matrices of the form which closely resembles experimental bounds several such discrete groups are being considered in many contexts. On the other hand, in our problem concerning the naturalness of the Higgs mass and the naturalness of the seesaw mechanism, discrete symmetries are seen to play a vital role in order to explain them coherently. As mentioned earlier, the symmetry structure of the light neutrino mass matrix as a result of the symmetry structures of Yukawa matrix and the right neutrino mass matrix necessarily implies that there must be at least one group element whose representation matrix must be unitary but not special unitary. This motivates us to find a suitable discrete group which would be a subgroup of U(3) and whose three-dimensional representations should contain elements, some of which are only unitary and some special unitary.Earlier works have been done to classify the subgroups of U(3) till order 512.

Using the minimal prescription in terms of smallest order group, we list here two such possible groups which can be used to build a viable model in our problem.

[[27,4]]

The group $[[27, 4]]$ is the smallest non-trivial subgroup of U(3).It is a group of order twenty seven and has eleven conjugacy classes. Due to eleven conjugacy classes, it has eleven inequivalent irreducible representations with nine one dimensional and two three dimensional representations.The generators for three dimensional irreducible representations are given by

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \omega^2 & 0 & 0 \end{pmatrix} \tag{7.3}$$

We list the character table for one dimensional and three dimensional representations as follows

Conjugacy classes	h	$\chi 1_{(0,0)}$	$\chi 1_{(0,1)}$	$\chi 1_{(0,2)}$	$\chi 1_{(1,0)}$	$\chi 1_{(1,1)}$	$\chi 1_{(1,2)}$	$\chi 1_{(2,0)}$	$\chi 1_{(2,1)}$	$\chi 1_{(2,2)}$	χ_3
C_1	1	1	1	1	1	1	1	1	1	1	3
C_2	3	1	1	1	1	1	1	1	1	1	3ω
C_3	3	1	1	1	1	1	1	1	1	1	$3\omega^2$
C_4	9	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2	0
C_5	9	1	1	1	ω^2	ω^2	ω^2	ω	ω	ω	0
C_6	3	1	ω	ω^2	1	ω	ω^2	1	ω	ω^2	0
C_7	3	1	ω^2	ω	1	ω^2	ω	1	ω^2	ω	0
C_8	9	1	ω	ω^2	ω	ω^2	1	ω^2	1	ω	0
C_9	9	1	ω^2	ω	ω	1	ω^2	ω	ω	1	0
C_{10}	9	1	ω	ω^2	ω^2	1	ω	ω	ω^2	1	0
C_{11}	9	1	ω^2	ω	ω^2	ω	1	ω	ω^2	1	0

The generators for the defining representations are:

$$1_{(i,j)} : R \to w^i, S \to w^j \quad (i, j = 0, 1, 2), \tag{7.4}$$

$$3 : R \to R, S \to S, \tag{7.5}$$

$$3^* : R \to R^*, S \to S^* \tag{7.6}$$

We list the tensor product decompositions for this group in the index section.

$\Sigma(81)$

$\Sigma(81)$ is another discrete subgroup of U(3) which has eighty-one elements. It consists of seventeen conjugacy classes. Therefore there are seventeen irreducible representations of this group; nine one-dimensional and eight three dimensional.The nine one-dimensional representations are represented by $\mathbf{1}_l^k$ where $k, l = 0, 1, 2$ and eight three-dimensional respresentations are represented by $\mathbf{3}_A, \mathbf{3}_B, \mathbf{3}_C, \mathbf{3}_D, \overline{\mathbf{3}}_A,$
$\overline{\mathbf{3}}_B, \overline{\mathbf{3}}_C, \overline{\mathbf{3}}_D$. There are four generators,i.e, b, a, a', a''.

We list the character table for all one dimensional representations as follows:

Conjugacy classes	h	$\chi_{1_0^0}$	$\chi_{1_1^0}$	$\chi_{1_2^0}$	$\chi_{1_0^1}$	$\chi_{1_1^1}$	$\chi_{1_2^1}$	$\chi_{1_0^2}$	$\chi_{1_1^2}$	$\chi_{1_2^2}$
C_1	1	1	1	1	1	1	1	1	1	1
$C_1^{(1)}$	1	1	1	1	1	1	1	1	1	1
$C_2^{(2)}$	1	1	1	1	1	1	1	1	1	1
$C_3^{(0)}$	3	1	1	1	1	1	1	1	1	1
$C_3^{'(0)}$	3	1	1	1	1	1	1	1	1	1
$C_3^{(1)}$	3	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2
$C_3^{'(1)}$	3	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2
$C_3^{"(1)}$	3	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2
$C_3^{(2)}$	3	1	1	1	ω^2	ω^2	ω^2	ω	ω	ω
$C_3^{'(2)}$	3	1	1	1	ω^2	ω^2	ω^2	ω	ω	ω
$C_3^{"(2)}$	3	1	1	1	ω^2	ω^2	ω^2	ω	ω	ω
$C_9^{(0)}$	3	1	ω	ω^2	1	ω	ω^2	1	ω	ω^2
$C_9^{(1)}$	9	1	ω	ω^2	ω	ω^2	1	ω^2	ω^2	ω
$C_9^{(2)}$	9	1	ω	ω^2	ω^2	1	ω	ω	ω^2	1
$C_9^{'(0)}$	3	1	ω^2	ω	1	ω^2	ω	1	ω^2	ω
$C_9^{'(1)}$	9	1	ω^2	ω	ω^2	ω	1	ω	1	ω^2
$C_9^{'(2)}$	9	1	ω^2	ω	ω	1	ω^2	ω^2	ω	1

We list the set of four generators for each of the 3-dimensional representations as follows:

The generator b is common for all the triplets and is represented as

$$b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{7.7}$$

.

The generators a, a', a'' are represented on each of the triplets as follows:

$$a = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}, a^{''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix} \tag{7.8}$$

on $\mathbf{3}_A$,

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, a' = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, a^{''} = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.9}$$

on $\mathbf{3}_B$,

$$a = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}, a' = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, a^{''} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{7.10}$$

on $\mathbf{3}_C$,

$$a = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, a' = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a^{''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{7.11}$$

on $\mathbf{3}_D$.

The representations on $\overline{\mathbf{3}}_A$,$\overline{\mathbf{3}}_B$,$\overline{\mathbf{3}}_C$,$\overline{\mathbf{3}}_D$ are taken to be complex conjugates of the representations on $\mathbf{3}_A, \mathbf{3}_B, \mathbf{3}_C, \mathbf{3}_D$. These generators are represented on the singlets $\mathbf{1}_l^k$ as $b = \omega^l, a = a' = a'' = \omega^k$.

We list the tensor product decompositions for this group in the index.

Chapter 8

A $\Sigma(81)$ model for electroweak natural seesaw

We propose a model for the electroweak natural seesaw by using the group $\Sigma(81)$.In this model, we assign different particles with distinct representations of the group and provide the relevant lagrangian. The assignment of different representations to the particles are as follows:

$$N : \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}_{3_D} \quad , \quad \phi : \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_{3_D} \tag{8.1}$$

$$l : \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}_{3_C} \quad , \quad \psi_A : \begin{pmatrix} \psi_{A1} \\ \psi_{A2} \\ \psi_{A3} \end{pmatrix}_{3_A} \tag{8.2}$$

$$\psi_B : \begin{pmatrix} \psi_{B1} \\ \psi_{B2} \\ \psi_{B3} \end{pmatrix}_{3_B} \quad , \quad \psi_C : \begin{pmatrix} \psi_{C1} \\ \psi_{C2} \\ \psi_{C3} \end{pmatrix}_{3_C} \tag{8.3}$$

We require four flavon fields to completely account for the residual symmetry structure as explained earlier in the context of natural seesaw symmetries. The relevant part of the Lagrangian in addition to the Standard Model part is given as:

$$\begin{aligned} -\mathcal{L}_N = \frac{1}{\Lambda} y (\bar{L}N)_{\bar{3}_A} \psi_A \tilde{H} + \frac{1}{\Lambda} y' (\bar{L}N)_{\bar{3}_B} \psi_B \tilde{H} + \frac{1}{\Lambda} y'' (\bar{L}N)_{\bar{3}_C} \psi_C \tilde{H} \\ + \frac{1}{2} \lambda (\bar{N}^c N)_{\bar{3}_D} \phi + \frac{1}{2} \lambda' (\bar{N}^c N)_{\bar{3}_D} \phi + h.c \end{aligned} \tag{8.4}$$

where $(\bar{N}^c N)_{\bar{3}D}$ represent two different invariant combinations corresponding to the tensor product decomposition eq.(14.29).

In the right handed neutrino mass sector, we expand the mass terms and obtain the following:

$$\mathcal{L}_{RH} = \frac{1}{2}\lambda\Big(\bar{N}_1 N_1 \phi_1 + \bar{N}_2 N_2 \phi_2 + \bar{N}_3 N_3 \phi_3\Big) + \frac{1}{2}\lambda'\Big((\bar{N}_2 N_3 + \bar{N}_3 N_2)\phi_1 + (\bar{N}_3 N_1 + \bar{N}_1 N_3)\phi_2 + (\bar{N}_2 N_1 + \bar{N}_1 N_2)\phi_3\Big) \tag{8.5}$$

Demanding invariance of ϕ under a'^2 of 3_D representation, we get the VEV alignment of ϕ as:

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ 0 \\ v_\phi \end{pmatrix} \tag{8.6}$$

and the corresponding right handed neutrino mass matrix as

$$M_N = v_\phi \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & \lambda' \end{pmatrix} \tag{8.7}$$

Now we come to the Dirac sector.Here the relevant part of the lagrangian is extended as:

$$\begin{aligned} L_{Dirac} = y\Big(\bar{\nu}_2 N_2 \psi_{A1} + \bar{\nu}_3 N_3 \psi_{A2} + \bar{\nu}_1 N_1 \psi_{A3}\Big)\tilde{H} \\ +y'\Big(\bar{\nu}_2 N_3 \psi_{B1} + \bar{\nu}_3 N_1 \psi_{B2} + \bar{\nu}_1 N_2 \psi_{B3}\Big)\tilde{H} \\ +y''\Big(\bar{\nu}_2 N_1 \psi_{C1} + \bar{\nu}_3 N_2 \psi_{C2} + \bar{\nu}_1 N_3 \psi_{C3}\Big)\tilde{H} \end{aligned} \tag{8.8}$$

Demanding invariance conditions of the flavon fields as below:

$$a(a')^2\psi_A = \psi_a,\ a^2 a''\psi_B = \psi_B,\ \ a'(a'')^2\psi_C = \psi_C \tag{8.9}$$

we obtain the following VEVs

$$\langle\psi_A\rangle = \begin{pmatrix} 0 \\ 0 \\ v_{\psi_A} \end{pmatrix} \tag{8.10}$$

$$\langle\psi_B\rangle = \begin{pmatrix} 0 \\ v_{\psi_B} \\ 0 \end{pmatrix} \tag{8.11}$$

$$\langle\psi_C\rangle = \begin{pmatrix} v_{\psi_C} \\ 0 \\ 0 \end{pmatrix} \tag{8.12}$$

We therefore have the Dirac Yukawa matrix as follows:

$$Y_D = \frac{1}{\Lambda}\begin{pmatrix} y''v_{\psi_C} & 0 & 0 \\ yv_{\psi_A} & 0 & 0 \\ y'v_{\psi_B} & 0 & 0 \end{pmatrix} \tag{8.13}$$

The structures of Y_D and M_N can be brought to the usual forms eq.(6.17) when the right handed heavy neutrinos are in their mass eigenstates(M_N is diagonal).This is done by the basis transformation

$$M_N \to U^T M_N U, \quad Y_D \to Y_D U \tag{8.14}$$

with

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8.15}$$

which results into

$$Y = \begin{pmatrix} y_1 & iy_1 & 0 \\ y_2 & iy_2 & 0 \\ y_3 & iy_3 & 0 \end{pmatrix}, \quad M_N = \begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M_3 \end{pmatrix} \tag{8.16}$$

with $y_1 = \frac{y''}{\sqrt{2}}\frac{v_{\psi C}}{\Lambda}$,$y_2 = \frac{y}{\sqrt{2}}\frac{v_{\psi A}}{\Lambda}$,$y_3 = \frac{y'}{\sqrt{2}}\frac{v_{\psi B}}{\Lambda}$,$M = \lambda v_\phi$ and $M_3 = \lambda' v_\phi$.

Thus we see that this particular group- theoretic model captures the essential symmetry structure of our problem.We now briefly discuss the implications of our symmetry model from a phenomenological perspective and give a suitable example to illustrate it.

Chapter 9

Phenomenological implications

9.0.1 Flavour violating processes

In the lepton sector, flavour violation is a consequence of neutrino oscillations.The PMNS matrix gives the correct mixing angles for the leptons of different flavours and the mass eigenstates of leptons are in general a superposition of their flavour eigenstates.Therefore there exists theoretical predictions for phenomenological interactions like decay processes which involve violation of the flavour quantum number.But due to the GIM mechanism, lepton flavour violating processes are strongly suppressed in general.Therefore they are not observed till date experimentally(because of extremely small amplitudes).Here we provide a phenomenological account on the branching ratio of $\mu \to e\gamma$ decay process and illustrate on its significant enhancement in the light of our symmetry model.

9.0.2 Branching ratio for $\mu \to e\gamma$

The relevant part of the interaction lagrangian is:

$$\begin{aligned} L_{int} = \bar{l}_{i_L}\gamma^{\mu}W_{\mu}^{\dagger}\nu_{j_L} + \bar{l}_{i_R}\gamma^{\mu}W_{\mu}^{\dagger}N_{j_R} + y_{ij}\bar{\nu}_{iL}N_{Rj} \\ +eA^{\mu}(\partial_{\mu}W_{\nu}^{-} - \partial_{\nu}W_{\mu}^{-})W^{+\nu} + h.c \end{aligned} \tag{9.1}$$

The amplitude for the process $\mu \to e\gamma$ mediated by ν_{iL} and N_{jR} is given by:

$$A = \frac{g^2}{8M_W^2}\frac{em_{\mu}}{32\pi^2}T_i \tag{9.2}$$

Here

$$T_i = U_{\mu i}^{*}U_{ei}\cos^2\theta_i F\Big(\frac{m_i^2}{M_W^2}\Big) + U_{\mu i}^{*}U_{ei}\sin^2\theta_i F\Big(\frac{M_i^2}{M_W^2}\Big) \tag{9.3}$$

where U is the PMNS matrix, θ_i are the mixing angles, m_i are the left neutrino masses and M_i are the right heavy neutrino masses. The functional F(x) is given by:

$$F(x) = 2(x+2)I^{(3)}(x) - 2(2x-1)I^{(2)}(x) + 2xI^{(1)}(x) + 1, \tag{9.4}$$

where

$$I^{(n)}(x) = \int_0^1 \frac{z^n}{z + (1-z)x} dz \tag{9.5}$$

The decay rate is given by

$$\Gamma(\mu \to e\gamma) = \frac{2m_\mu^3}{8\pi} \mid A \mid^2 \tag{9.6}$$

Taking the approximation where the right neutrino masses are much much greater than the masses of left neutrinos and W-bosons, we obtain the following branching ratio:

$$B(\mu \to e\gamma) = \frac{3\alpha}{8\pi} \mid U_{\mu i}^* U_{ei} \theta_i^2 \mid^2 \tag{9.7}$$

where

$$\theta_i = \frac{y_i v}{2M_i} \tag{9.8}$$

Using $M_i \approx 1\text{TeV}$, $v \approx 174$ GeV, we consider the ratio of two branching ratios corresponding to Yukawa couplings of $y \approx O(1)$ and $y' \approx O(10^{-5})$. We find,

$$\frac{B(\mu \to e\gamma)_y}{B(\mu \to e\gamma)_{y'}} \approx 10^{20} \tag{9.9}$$

Therefore, there is a significant increase in the branching ratio of the respective process if one considers the natural electroweak seesaw where the Yukawa couplings are order one numbers. Our symmetry which invokes such a Yukawa structure satisfactorily predicts this enhancement.

We next discuss about the cosmological front and address the issue of leptogenesis.

Chapter 10

Cosmological perspectives

Given that our group theoretic model establishes a concrete framework of natural electroweak seesaw incorporating the naturalness problem of the Higgs, the next question we wish to ponder is how it affects the scenarios in cosmology,viz.the viable models of leptogenesis and baryogenesis.This question is important to address since it accounts for the long-standing problem in cosmology, i.e matter-antimatter asymmetry in the universe.We first give some brief review on standard cosmology and then discuss about leptogenesis and the possible link with our model.

In the hot big bang model, the history of the universe has two quite well distinguished stages. In a first hot phase, the early universe, matter and radiation were coupled and the growth of baryonic matter perturbations was inhibited. In this stage matter was in the form of a plasma and properties of elementary particles were crucial in determining the evolution of the universe.After matter-radiation decoupling occurring during the so-called re-combination epoch, when electrons combined with protons and Helium-4 nuclei to form atoms, baryonic matter perturbations could grow quite quickly under the action of dark matter inhomogeneities, forming the large-scale structure that we observe.With the astonishing progress in observational cosmology during the last fifteen years, we have today quite a robust minimal cosmological model, the so-called 'Λ-Cold Dark Matter' (ΛCDM)model sometime popularly dubbed as the 'vanilla model'. The ΛCDM model is very successful in explaining all current cosmological observations and its parameters are currently measured with a precision better than $\approx 10\%$.
The ΛCDM model relies on general relativity, Einstein's theory of gravity, for a description of the gravitational interactions on cosmic scales. It belongs to the class of Friedmann cosmological models based on the assumption of the homogeneity and isotropy of the Universe.In this case the space-time geometry is conveniently described in the comoving system by the Friedmann-Robertson-Walker metric in terms of just one time-dependent parameter: the scale factor. The distances among objects at rest with respect to the comoving system are just all proportional to the scale factor, as the distances between points on the surface of an inflating balloon. The expansion of the universe is then described in terms of the scale factor time dependence that can be worked out as a solution of the Friedmann equations.
However, a solution of the Friedmann equations also require s the knowledge of the

energy- matter content of the Universe. In this respect, the ΛCDM model also belongs to a sub-class of Friedmann models, the Lemaitre models, where the energy- matter content is described by an admixture of three different forms of fluids: i) matter, the non-relativistic component where energy is dominated by the mass term; ii) radiation, the ultra-relativistic component where energy is dominated by the kinetic energy term (in the case of photons this is exactly true); iii) vacuum energy density and/or a cosmological constant term.
The current cosmological observations are also able to determine quite precisely the values of the different contributions to the total energy density from the three different fluids in the ΛCDM model: radiation contributes at the present time only with a tiny 0.1%, matter gives a more significant 26.5% contribution, while the dominant remaining 73.5% contribution is in the form of a cosmological constant and/or vacuum energy density. This particular combination of values, a sort of cosmological recipe, reproduces very well all cosmological observations and in particular the above-mentioned acceleration of the Universe at the present time. A fundamental ingredient of the ΛCDM model is the existence of a very early stage in the history of the universe called inflation, characterised by a super-luminal expansion, that was able to bring a microscopic sub-atomic portion of space to have a macroscopic size corresponding today to our observable Universe where a homogeneous, isotropic and a flat space-time geometry holds with very good approximation. In this way homogeneity, isotropy and flatness of the Universe has not to be postulated but are instead a natural result of the inflationary stage. However, inflation also predicts, at the end of the inflationary stage, the presence of primordial perturbations that acted as seeds for the formation of galaxies, clusters of galaxies and super- clusters of galaxies: the so called large-scale structure of the universe. The same primordial perturbations are also responsible for the observed cosmic microwave background (CMB) temperature anisotropies and for their properties. In particular for the explanation of the so called acoustic peaks in the angular power spectrum of the CMB temperature anisotropies. The acoustic peaks originate from the compression and rarefaction of the coupled baryon-photon fluid at the time of recombination.

The existence of a first hot stage is not only fundamental to understand the existence and the properties of the cosmic microwave background radiation (CMB) and the nuclear composition of the Universe with big bang nucleosynthesis (BBN) but it also seems to enclose the secrets for the solution of those cosmological puzzles of the ΛCDM model that strongly hint at new Physics. These include: i) the existence of Dark Matter, that has a crucial role in making possible the quick formation of galaxies after the matter-radiation decoupling; ii) the observed matter-antimatter asymmetry of the Universe; iii) the necessity of an inflationary stage and iv) the presence of the mysterious form of energy, currently indistinguishable from a cosmological constant, that is driving the acceleration of the universe at the present time.

In the present context, we are going to discuss the problem of matter-antimatter asymmetry of the universe.The first insightful idea to understand the asymmetry was given by Sakharov, a Russian physicist.Sakharov identified, three necessary conditions, bearing now his name, for a successful model of baryogenesis, which can explain the observed

asymmetry between matter and anti-matter: (i) the existence of an elementary process that violates the baryon number, (ii) violation of charge conjugation C and of CP, where P indicates parity transformation, (iii) a departure from thermal equilibrium during baryogenesis. This departure from thermal equilibrium has to be permanent, since otherwise, the baryon asymmetry would be subsequently washed-out.
It became apparent that the Sakharov conditions can be fulfilled within the standard Model.The most efficient model was realised by electroweak baryogenesis.The departure from thermal equilibrium is strong if there occurs a strong first order phase transition at the electroweak symmetry breaking phase. But to its disadvantage, this requires a stringent bound on the mass of the standard model Higgs $m_H \leq 40$ GeV which is at odds with the experimental results and also the LEP lower bound $m_H \geq 114$ GeV. No other plausible arrangement is found within the Standard Model to infer this particular asymmetry.

Therefore, the asymmetry in matter-antimatter compels us to look for new physics one of which is evidently that of extra right handed fermions which exhibit the seesaw mechanism to generate tiny neutrino masses.We now elaborate on the possibility of leptogenesis as a viable model to explain baryogenesis(hence, baryon asymmetry).

10.0.1 Thermal leptogenesis

Leptogenesis belongs to a class of models of baryogenesis where the asymmetry is generated from the out-of-equilibrium decays of very heavy particles, quite interestingly the same class as the first model proposed by Sakharov belongs. These class of models became very popular with the advent of grand-unified theories (GUTs) that provided a specific well definite and motivated framework.In GUT baryogenesis models the very heavy particles are the same new gauge bosons predicted by GUTs. However, the final asymmetry depends on too many untestable parameters, so that imposing successful baryogenesis does not lead to compelling experimental predictions. This lack of predictability is made even stronger considering that the decaying particles are too heavy to be produced thermally and one has therefore to invoke a non-thermal production mechanism of the gauge bosons. This is because while the mass of the gauge bosons is about the grand-unification scale,$\approx 10^{15-16}$ GeV,the reheating temperature at the end of inflation T_{RH} cannot be higher than$\approx 10^{15}$ GeV from CMB observations.The reheating temperature is the initial value of the temperature at the beginning of the radiation dominated regime after inflation.Below this temperature, the inflationary stage can, therefore, be considered concluded. The minimal (and original) version of leptogenesis is based on the type I see-saw mechanism and the asymmetry is produced by the three heavy right handed neutrinos.

We call 'minimal leptogenesis scenarios' those scenarios where we have a type I see-saw mechanism and a thermal production of the right handed neutrinos (thermal leptogenesis), implying that T_{RH} is comparable at least to the lightest RH neutrino mass M_1, are assumed. At these high temperatures, the right handed neutrinos can be produced by the Yukawa interactions of leptons and Higgs bosons in the thermal bath.

When the right handed neutrinos are produced, they decay either into leptons,$N_i \rightarrow l_i + \phi^\dagger$ or antileptons,$N_i \rightarrow \bar{l_i} + \phi$ with decay rates Γ_i and $\bar{\Gamma_i}$ respectively, where ϕ is the standard model Higgs. Since both the Higgs particle and RH neutrinos do not have lepton number, both these inverse processes and decays violate lepton number conservation($\Delta L = 1$) and also charge-parity(CP) symmetry.They as well violate B-L symmetry($\Delta(B-L)$=1).At temperatures T≫100 GeV there exists some non-perturbative processes called sphalerons(static saddle-point solutions of the electroweak field equations) which are in equilibrium. They violate both lepton and baryon number while they still conserve the BL symmetry.In this way, the lepton asymmetry produced in the elementary processes is reprocessed in a way that at the end approximate 1/3 of B-L asymmetry is in the form of baryon asymmetry and 2/3 of it is in the form of lepton number asymmetry. This immediately implies that two out of three of the Sakharov conditions are satisfied. The condition of departure from thermal equilibrium is also satisfied since some of the decays occur out-of-equilibrium and hence show the departure.The asymmetry survives and does not get washed out from inverse processes.

10.0.2 Resonant leptogenesis

In resonant leptogenesis, the usual process of generating asymmetry in the leptonic sector gets enhanced due to the mixings of nearly degenerate heavy neutrinos that have their difference of masses comparable to their decay widths.The motivation to incorporate degenerate heavy neutrinos as candidates for leptogenesis comes from the understanding of mass scales in grand unified theories(GUTs).For a consistent seesaw model within GUTs, the mass scale of the heavy neutrinos has to be of the order of the GUT scale$\approx 10^{16}$GeV.But certain inflationary models predict a reheating temperature(T_{RH}) of the order of 10^9GeV.This low re-heating temperature gives rise to the constraint on the order of mass scale in thermal leptogenesis. The Majorana neutrinos which are responsible for predicting the baryon asymmetry need to have a mass scale lower than the reheating temperature for the abundant production in the early universe.Such a mass scale is deemed to be unnaturally low in regard to the GUT scale.

In order to avoid this problem, one looks for low scale thermal leptogenesis which relies on a dynamical mechanism, in which heavy-neutrino self- energy effects on the leptonic asymmetry become dominant and get resonantly enhanced, when a pair of heavy Majorana neutrinos have a mass difference comparable to their decay widths.In this case, the masses of heavy neutrinos can be as low as 1 TeV without any further problems.

In the particular symmetry model which we have dealt with, corresponds to degenerate Majorana neutrino masses as a result of the symmetry structure.We, therefore, can link our viable model of electroweak natural seesaw with resonant leptogenesis in order to explain the observed baryonic asymmetry.

Chapter 11

Numerical results

We present here the numerically obtained distributions for the perturbation parameters to generate the mass squared differences of the light neutrinos to 3σ accuracy according to the latest experimental data. We also plot the distribution of the corresponding mixing angles by producing the results numerically with the obtained perturbation parameters.In the plots of the mixing angles, the shaded regions indicate the 3σ accuracy corresponding to the latest experimental data. In the numerical computation, we took our residual symmetry structure for two generations of neutrinos,used degenerate right handed neutrino masses each of 1TeV and varied the complex Yukawa couplings randomly in the range $\mid y_i \mid \in [0.2, 1]$ and $\text{Arg.}(y_i) \in [0, 2\pi]$ to generate the perturbation parameters which reproduces the solar and atmospheric mass squared differences. We then use these parameters to produce the distribution of mixing angles in the leptonic sector.In the simplest example, we consider two generations of degenerate fermion singlets and this can be achieved with the following ansatz of the matrix y:

$$y = \begin{pmatrix} y_1 & iy_1(1+\epsilon_1) \\ y_2 & iy_2(1+\epsilon_2) \\ y_3 & iy_3(1+\epsilon_3) \end{pmatrix} \tag{11.1}$$

11.0.1 Masses and mixing angles for neutrinos

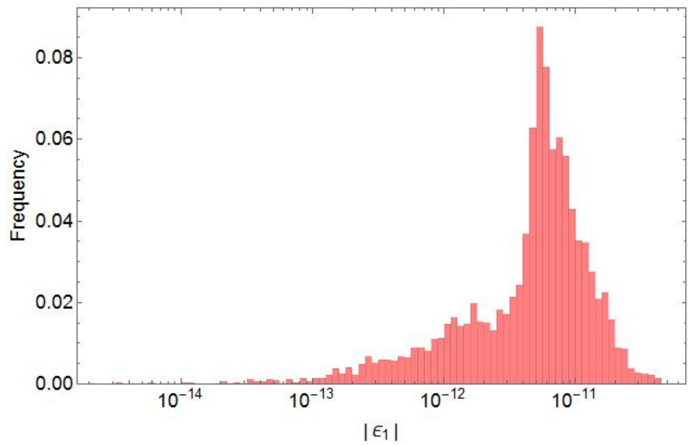

Figure 11.1: Distribution of parameter $|\ \epsilon_1\ |$

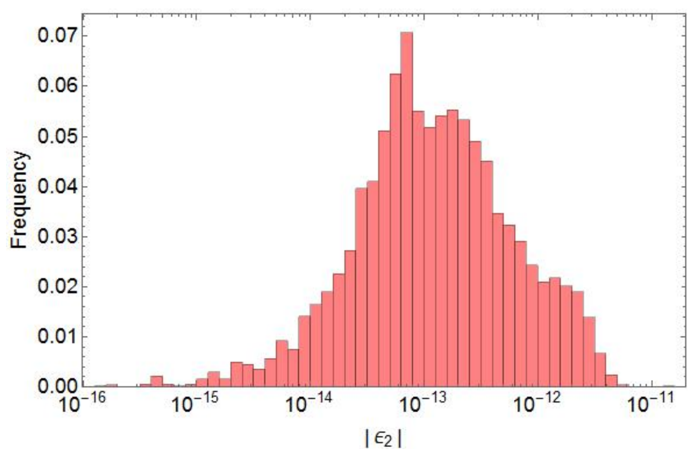

Figure 11.2: Distribution of parameter $|\ \epsilon_2\ |$

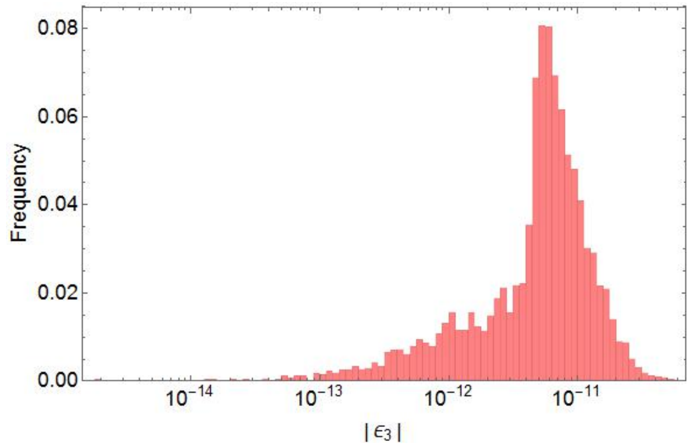

Figure 11.3: Distribution of parameter $|\ \epsilon_3\ |$

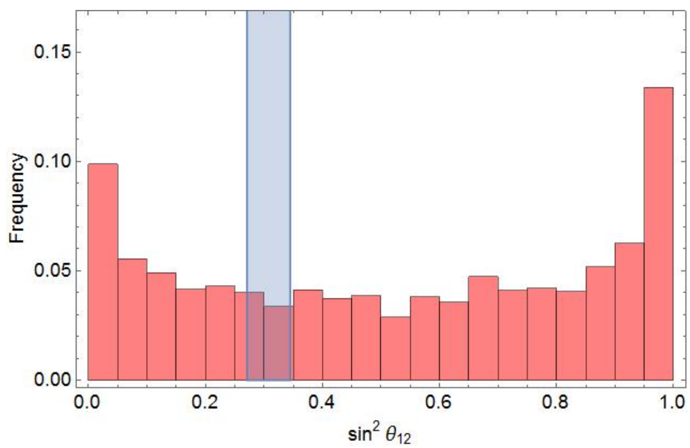

Figure 11.4: Distribution of mixing angle $\sin^2\theta_{12}$

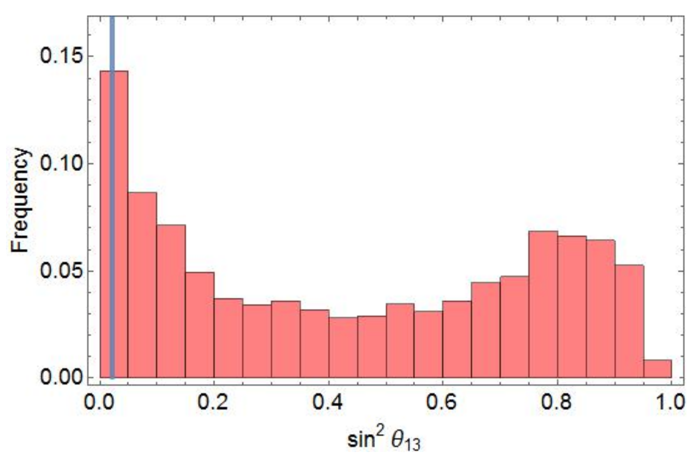

Figure 11.5: Distribution of mixing angle $\sin^2\theta_{13}$

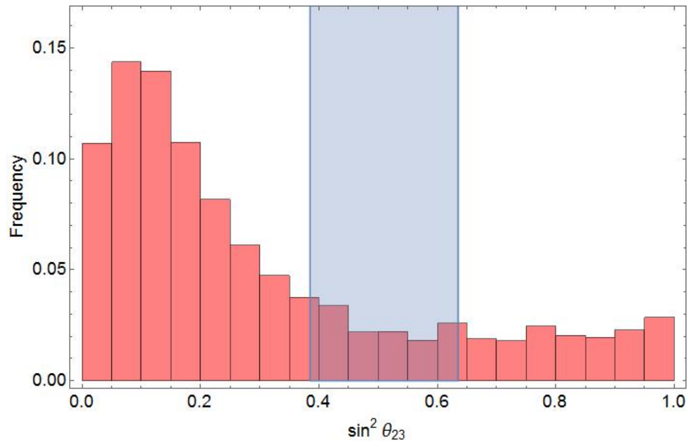

Figure 11.6: Distribution of mixing angle $\sin^2\theta_{23}$

Chapter 12

Summary and outlook

Seesaw mechanisms are promising candidates for neutrino mass generation.They are well motivated from some specific Grand Unified Theories(GUTs).There have been different types of seesaw depending on the property of beyond the standard model particles.We have elaborated on the three basic types which have been our focus throughout this work.The hierarchy problem is one of the long-standing issues in particle physics and is yet unresolved. Several models indicating new Physics at high energy scales are proposed but almost none of them have passed adequate experimental tests.There has not been any noteworthy signature of beyond the standard model physics at the energies where LHC is running currently. But the hope still remains as it reaches higher energies in the future runs.We have discussed how the hierarchy problem in particle physics is related to the problem of the naturalness of Higgs boson's vacuum configuration.The naturalness criteria imposes that corrections to the Higgs μ^2 parameter should be at most of the order of its vacuum expectation value in order to preserve the electroweak naturalness.In the type-I seesaw model of neutrino masses, the naturalness of the electroweak scale restricts the masses of heavy right neutrinos to be $\leq 10^7$ GeV and their Yukawa couplings to be of $\mathcal{O}(10^{-4})$.If the couplings are taken to be of order unity then the right neutrinos turn out to be as light as few TeV.In this case the seesaw mechanism cannot be considered as the correct framework to generate small neutrino masses.To produce viable neutrino mass spectrum which is consistent with solar and atmospheric mass squared differences, one needs to arrange for finely tuned correlations among the Yukawa couplings and right neutrino masses.Unless these correlations are random, they must be motivated by specific symmetry considerations.We, therefore, motivate such fine-tuning through finite discrete flavour symmetry under which all the standard model leptons and fermion singlets transform non-trivially.

We discussed the basics of finite discrete groups and their representations.Some useful theorems from representation theory are studied for the sake of completeness of the analysis.Two interesting groups and their representations, viz.$[[27, 4]]$ and $\Sigma(81)$ are studied vividly which are fruitful in modelling the current problem.The group $\Sigma(81)$ has been chosen to build the corresponding model of flavor symmetry because it is found to be the smallest group where one can assign 3-dimensional irreducible representations to the three generations of right neutrinos and the leptons.Such an assignment is necessary in order to account for the residual symmetries in the leptonic sector and the right neutrino sector.This underlying symmetry yields the notion of seesaw cancellation and leads to massless neutrinos at the leading order.Generic perturbations to the symmetry then produce tiny neutrino masses. We then study the phenomenology of flavour violating processes in the light of our proposed symmetry model.We explicitly calculated the branching ratio concerning one such process($\mu \rightarrow e\gamma$) and compared it with the earlier results in the literature.There is a significant enhancement observed in the branching ratio based on our framework.Finally, we discussed the mechanisms of leptogenesis and baryogenesis in the context of cosmology and elaborated on two viable ways of leptogenesis.We find that in our model, small deviations from degeneracy in the masses of right heavy fermions are compatible with the data and resonant leptogenesis mechanism hence may naturally emerge in such class of models.Such a framework can then successfully account for baryon asymmetry of the universe.

Chapter 13

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Chapter 14

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$[[27, 4]]$

The tensor product decompositions of the group $[[27, 4]]$ and the invariant subspaces are given as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_3 \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_3 = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}_{3^{\bar{*}}} \oplus \begin{pmatrix} x_3y_2 \\ \omega x_1y_3 \\ \omega^2 x_2y_1 \end{pmatrix}_{3^{\bar{*}}} \oplus \begin{pmatrix} x_2y_3 \\ \omega x_3y_1 \\ \omega^2 x_1y_2 \end{pmatrix}_{3^{\bar{*}}} \tag{14.1}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_3 \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{3^*} = \left(x_1y_1 + x_2y_2 + x_3y_3 \right)_{1_{(0,0)}} \tag{14.2}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_3 \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{3^*} = \left(x_1y_2 + x_2y_3 + \omega^2 x_3y_1 \right)_{1_{(0,1)}} \tag{14.3}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_3 \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{3^*} = \left(x_1y_3 + \omega^2 x_2y_1 + x_3y_2 \right)_{1_{(0,2)}} \tag{14.4}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_3 \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{3^*} = \left(x_1y_1 + \omega x_2y_2 + \omega^2 x_3y_3 \right)_{1_{(1,0)}} \tag{14.5}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_3 \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{3^*} = \Big(x_1y_2 + \omega x_2y_3 + \omega x_3y_1 \Big)_{1_{(1,1)}} \tag{14.6}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_3 \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{3^*} = \Big(x_1y_3 + x_2y_1 + \omega x_3y_2 \Big)_{1_{(1,2)}} \tag{14.7}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_3 \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{3^*} = \Big(x_1y_1 + \omega x_2y_2 + \omega x_3y_3 \Big)_{1_{(2,0)}} \tag{14.8}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_3 \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{3^*} = \Big(x_1y_2 + \omega^2 x_2y_3 + x_3y_1 \Big)_{1_{(2,1)}} \tag{14.9}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_3 \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{3^*} = \Big(x_1y_3 + \omega^2 x_2y_1 + x_3y_2 \Big)_{1_{(2,2)}} \tag{14.10}$$

$\Sigma(81)$

The tensor product decompositions of the triplets in the group $\Sigma(81)$ are as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_A} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_A} = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}_{\bar{\mathbf{3}}_A} \oplus \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\bar{\mathbf{3}}_B} \oplus \begin{pmatrix} x_3y_2 \\ x_1y_3 \\ x_2y_1 \end{pmatrix}_{\bar{\mathbf{3}}_B} \tag{14.11}$$

,

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_A} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\bar{\mathbf{3}}_A} &= \Big(\sum_{l=0,1,2} (x_1y_1 + \omega^{2l} x_2y_2 + \omega^l x_3y_3)_{\mathbf{1}_l^0} \Big) \\ &\oplus \begin{pmatrix} x_2y_1 \\ x_3y_2 \\ x_1y_3 \end{pmatrix}_{\mathbf{3}_D} \oplus \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \end{pmatrix}_{\bar{\mathbf{3}}_D} \end{aligned} \tag{14.12}$$

,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_A} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_B} = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}_{\bar{\mathbf{3}}_C} \oplus \begin{pmatrix} x_3y_2 \\ x_1y_2 \\ x_2y_1 \end{pmatrix}_{\bar{\mathbf{3}}_A} \oplus \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\bar{\mathbf{3}}_A} \tag{14.13}$$

,

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_A} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\bar{\mathbf{3}}_B} &= \Big(\sum_{l=0,1,2} (x_1y_1 + \omega^{2l} x_2y_2 + \omega^l x_3y_3)_{\mathbf{1}_l^2} \Big) \\ &\oplus \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\mathbf{3}_D} \oplus \begin{pmatrix} x_3y_2 \\ x_1y_3 \\ x_2y_1 \end{pmatrix}_{\bar{\mathbf{3}}_D} \end{aligned} \tag{14.14}$$

,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_A} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_C} = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}_{\bar{\mathbf{3}}_B} \oplus \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\bar{\mathbf{3}}_C} \oplus \begin{pmatrix} x_3y_2 \\ x_1y_3 \\ x_2y_1 \end{pmatrix}_{\bar{\mathbf{3}}_C} \tag{14.15}$$

,

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\bar{\mathbf{3}}_A} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_C} &= \Big(\sum_{l=0,1,2} (x_1y_1 + \omega^{2l} x_2y_2 + \omega^l x_3y_3)_{\mathbf{1}_l^2} \Big) \\ &\oplus \begin{pmatrix} x_3y_1 \\ x_1y_2 \\ x_2y_3 \end{pmatrix}_{\mathbf{3}_D} \oplus \begin{pmatrix} x_3y_2 \\ x_1y_3 \\ x_2y_1 \end{pmatrix}_{\bar{\mathbf{3}}_D} \end{aligned} \tag{14.16}$$

,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_A} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_D} = \begin{pmatrix} x_3y_3 \\ x_1y_1 \\ x_2y_2 \end{pmatrix}_{\mathbf{3}_A} \oplus \begin{pmatrix} x_3y_1 \\ x_1y_2 \\ x_2y_3 \end{pmatrix}_{\mathbf{3}_B} \oplus \begin{pmatrix} x_3y_2 \\ x_1y_3 \\ x_2y_1 \end{pmatrix}_{\mathbf{3}_C} \tag{14.17}$$

,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\bar{\mathbf{3}}_A} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_D} = \begin{pmatrix} x_2y_1 \\ x_3y_2 \\ x_1y_3 \end{pmatrix}_{\bar{\mathbf{3}}_A} \oplus \begin{pmatrix} x_2y_2 \\ x_3y_3 \\ x_1y_1 \end{pmatrix}_{\bar{\mathbf{3}}_B} \oplus \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\bar{\mathbf{3}}_C} \tag{14.18}$$

,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_B} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_B} = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}_{\bar{\mathbf{3}}_B} \oplus \begin{pmatrix} x_3y_2 \\ x_2y_1 \\ x_1y_2 \end{pmatrix}_{\bar{\mathbf{3}}_C} \oplus \begin{pmatrix} x_2y_3 \\ x_1y_2 \\ x_2y_1 \end{pmatrix}_{\bar{\mathbf{3}}_C} \tag{14.19}$$

,

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_B} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\bar{\mathbf{3}}_B} = &\Big(\sum_{l=0,1,2} (x_1y_1 + \omega^{2l} x_2y_2 + \omega^l x_3y_3)_{\mathbf{1}_l^0} \Big) \\ &\oplus \begin{pmatrix} x_2y_1 \\ x_3y_2 \\ x_1y_3 \end{pmatrix}_{\mathbf{3}_D} \oplus \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \end{pmatrix}_{\bar{\mathbf{3}}_D} \end{aligned} \tag{14.20}$$

,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_B} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_C} = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}_{\bar{\mathbf{3}}_A} \oplus \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\bar{\mathbf{3}}_B} \oplus \begin{pmatrix} x_3y_2 \\ x_1y_3 \\ x_2y_1 \end{pmatrix}_{\bar{\mathbf{3}}_B} \tag{14.21}$$

,

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\bar{\mathbf{3}}_B} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_C} = &\Big(\sum_{l=0,1,2} (x_1y_1 + \omega^{2l} x_2y_2 + \omega^l x_3y_3)_{\mathbf{1}_l^1} \Big) \\ &\oplus \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\mathbf{3}_D} \oplus \begin{pmatrix} x_1y_3 \\ x_2y_1 \\ x_3y_2 \end{pmatrix}_{\bar{\mathbf{3}}_D} \end{aligned} \tag{14.22}$$

,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_B} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_D} = \begin{pmatrix} x_3y_1 \\ x_1y_2 \\ x_2y_3 \end{pmatrix}_{\mathbf{3}_A} \oplus \begin{pmatrix} x_3y_3 \\ x_1y_1 \\ x_2y_2 \end{pmatrix}_{\mathbf{3}_B} \oplus \begin{pmatrix} x_3y_2 \\ x_1y_3 \\ x_2y_1 \end{pmatrix}_{\mathbf{3}_C} \tag{14.23}$$

,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\bar{\mathbf{3}}_B} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_D} = \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\bar{\mathbf{3}}_A} \oplus \begin{pmatrix} x_2y_1 \\ x_3y_2 \\ x_1y_3 \end{pmatrix}_{\bar{\mathbf{3}}_B} \oplus \begin{pmatrix} x_2y_2 \\ x_3y_3 \\ x_1y_1 \end{pmatrix}_{\bar{\mathbf{3}}_B} \tag{14.24}$$

,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_C} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_C} = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}_{\bar{\mathbf{3}}_C} \oplus \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\bar{\mathbf{3}}_A} \oplus \begin{pmatrix} x_3y_2 \\ x_1y_3 \\ x_2y_1 \end{pmatrix}_{\bar{\mathbf{3}}_A} \tag{14.25}$$

,

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_C} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\bar{\mathbf{3}}_C} &= \Big(\sum_{l=0,1,2} (x_1y_1 + \omega^{2l} x_2y_2 + \omega^l x_3y_3)_{\mathbf{1}_l^0} \Big) \\ &\oplus \begin{pmatrix} x_2y_1 \\ x_3y_2 \\ x_1y_3 \end{pmatrix}_{\mathbf{3}_D} \oplus \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \end{pmatrix}_{\bar{\mathbf{3}}_D} \end{aligned} \tag{14.26}$$

,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_C} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_D} = \begin{pmatrix} x_3y_2 \\ x_1y_3 \\ x_2y_1 \end{pmatrix}_{\mathbf{3}_A} \oplus \begin{pmatrix} x_3y_1 \\ x_1y_2 \\ x_2y_3 \end{pmatrix}_{\mathbf{3}_B} \oplus \begin{pmatrix} x_3y_3 \\ x_1y_1 \\ x_2y_2 \end{pmatrix}_{\mathbf{3}_C} \tag{14.27}$$

,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\bar{\mathbf{3}}_C} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_D} = \begin{pmatrix} x_2y_2 \\ x_3y_3 \\ x_1y_1 \end{pmatrix}_{\bar{\mathbf{3}}_A} \oplus \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\bar{\mathbf{3}}_B} \oplus \begin{pmatrix} x_2y_1 \\ x_3y_2 \\ x_1y_3 \end{pmatrix}_{\bar{\mathbf{3}}_C} \tag{14.28}$$

,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_D} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\mathbf{3}_D} = \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}_{\bar{\mathbf{3}}_D} \oplus \begin{pmatrix} x_2y_3 \\ x_3y_1 \\ x_1y_2 \end{pmatrix}_{\bar{\mathbf{3}}_D} \oplus \begin{pmatrix} x_3y_2 \\ x_1y_3 \\ x_2y_1 \end{pmatrix}_{\bar{\mathbf{3}}_D} \tag{14.29}$$

,

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathbf{3}_D} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_{\bar{\mathbf{3}}_D} &= \Big(\sum_{l=0,1,2} (x_1y_1 + \omega^{2l} x_2y_2 + \omega^l x_3y_3)_{\mathbf{1}_l^0} \Big) \\ &\oplus \Big(\sum_{l=0,1,2} (x_1y_1 + \omega^{2l} x_2y_2 + \omega^l x_3y_3)_{\mathbf{1}_l^1} \Big) \\ &\oplus \Big(\sum_{l=0,1,2} (x_1y_1 + \omega^{2l} x_2y_2 + \omega^l x_3y_3)_{\mathbf{1}_l^2} \Big) \end{aligned} \tag{14.30}$$

.

The tensor products of singlets are given by

$$\mathbf{1}_l^k \otimes \mathbf{1}_{l'}^{k'} = \mathbf{1}_{l+l'(mod3)}^{k+k'(mod3)} \tag{14.31}$$

.