Uncertainty Principles

Shikha Nagal

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Certificate of Examination

This is to certify that the dissertation titled ”**Uncertainty Principles**” submitted by **Ms. Shikha Nagal (Reg. No. MS12029)** for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Amit Kulshrestha

Dr. Lingaraj Sahu

Prof. Shobha Madan
(Supervisor)

Dated: April 21, 2017

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. Shobha Madan at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Shikha Nagal

Dated: April 21, 2017

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Shobha Madan

(Supervisor)

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Notation

$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{C}$	set of complex numbers
L^p	Lebesgue measurable space
$\frac{\partial}{\partial z}$	partial derivative with respect to z
$\hat{f}$	Fourier transform of a function f
$\check{f}$	inverse Fourier transform of f
$\mathcal{F}$	Fourier transform operator
$S_N(f)$	partial sum of a Fourier series
$\mathcal{C}^k$	k times differentiable functions
$f * g$	convolution
$\chi_{[a,b]}$	characteristic function of the interval $[a, b]$
$\|f\|_p$	p-norm of f
$dim(G)$	dimension of G
$Ker(f)$	kernel of f
$Im(f)$	image of f
$Re(z)$	real part of z
$Im(z)$	imaginary part of z

Abstract

Most of life is uncertain, no one with 100 percent accuracy can tell what's going to happen next in their life. In 1927 Prof. Heisenberg claimed that it is not possible to simultaneously measure the complementary pairs; in his case momentum and position of a particle with 100 percent accuracy. In 1928 Kennard and Weyl separately gave the detailed proof of the claim. From there several questions came into existence regarding under what different conditions can you see uncertainties and how these uncertainties help us to make optimised decisions. Taking inspiration from the classical uncertainty principle: the Heisenberg uncertainty principle; I tried to analyse the simultaneous behaviour of a function and its Fourier transform under different notions of concentration.

Contents

Chapter 1

Fourier Analysis

1.1 Basic Preliminaries

Definition 1.1. *A function f is said to be continuous at x if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

Definition 1.2. *Let X be a topological space.For a continuous function,$f : X \to \mathbb{R}(or\ \mathbb{C})$ support of a function f is defined as*

$$supp(f) = \overline{\{x \in X | f(x) \neq 0\}}.$$

Definition 1.3. *A topological space,X is compact if every open cover of X has finite subcover.*

Definition 1.4. *A function is said to be holomorphic if it is complex differentiable in a neighbourhood of every point in its domain.*

Definition 1.5. *A complex function is called an entire function if it is analytic at all finite points of the complex plane.*

Most common example of entire functions are polynomials.

Theorem 1.6. *(Dominated Convergence theorem)*
Let $\{f_n\}$ be a sequence of measurable functions that converges point-wise to f, i.e.$f_n \to f\ a.e.\ as\ n \to \infty$. Let g be an integrable function i.e. $g \in L^1$ and $|f_n| \leq g\ \forall\ n$ then f is integrable and limit and integral can be exchanged i.e.

$$\int f d\mu = \int lim_{n\to\infty} f_n d\mu = lim_{n\to\infty} \int f_n d\mu.$$

Theorem 1.7. *(Fubini's Theorem) Let* A, B *be complete measure spaces and* $f(x,y)$ *be* $A \times B$ *measurable. If* $\int_{A\times B} |f(x,y)|d(x,y) < \infty$ *then*

$$\int_A \Big(\int_B f(x,y)dy \Big) dx = \int_B \Big(\int_A f(x,y)dx \Big) dy = \int_{A\times B} |f(x,y)|d(x,y).$$

Theorem 1.8. *(Cauchy-Schwarz Inequality)*

If $f, g \in L^2$ *then*

$$| < f, g > | \leq \|f\|_2 \|g\|_2.$$

Theorem 1.9. *(Morera's Theorem)*

Let f *be a continuous function in the open disc* D*. If for every triangle in* D *we have*

$$\int_T f(z)dz = 0$$

then f *is holomorphic.*

Theorem 1.10. *(Liouville's Theorem)*

If $f(z)$ *is entire and bounded then* $f(z)$ *is constant.*

Theorem 1.11. *(Leibniz Rule)*

For all $\mathcal{C}^m$ *functions* f, g *on* $\mathbb{R}$*, Leibniz rule is*

$$\frac{d^m}{dt^m}(fg) = \sum_{k=0}^{m} \binom{m}{k} \frac{d^k f}{dt^k} \frac{d^{m-k} g}{dt^{m-k}}.$$

Theorem 1.12. *(Maximum Modulus Principle)*

Let f *be a non-constant holomorphic function in region* D *then* $|f|$ *cannot attain maximum in* D*.*

Theorem 1.13. *(Hőlders inequality)*

Let $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^q(\mathbb{R}^n)$ *and* p, q *be such that* $1/p + 1/q = 1$ *then*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Theorem 1.14. *(Minkowski's Integral Inequality)*

Let f *be Lebesgue measurable and* $p \in [1, \infty)$ *then*

$$\Big(\int | \int h(x,y)dy|^p dx \Big)^{1/p} \leq \int \Big(\int |h(x,y)|^p dx \Big)^{1/p} dy.$$

Lemma 1.15. *(Fatou's lemma)*

Let $\{f_n\}$ *be sequence of non-negative measurable functions converging to the function* f*, then*

$$\int lim_{n\to\infty} f_n d\mu \leq lim_{n\to\infty} \int f_n d\mu.$$

1.2 Fourier Series

1.2.1 Definitions

Fourier series of an integrable function f on $[-\pi, \pi]$ is defined as

$$\sum_{n \in \mathbb{N}} \hat{f}(n) e^{inx}$$

where $\hat{f}(n)$ is Fourier coefficient of f given by

$$\hat{f}(n) = \frac{1}{2\pi} \int_a^b f(x) e^{-2inx} dx$$

Partial sums, S_N are defined as $S_N(f) = \sum_{-N}^{N} \hat{f}(n) e^{inx}$.

Theorem 1.16. *Let f be a twice continuously differentiable function on the circle, then we have* $\hat{f}(n) \to 0$ *as* $|n| \to \infty$.

Proof It is obtained by using integration by parts twice. For $n \neq 0$, we get

$$\begin{aligned}
2\pi \hat{f}(n) &= \int_{-\pi}^{\pi} f(m) e^{-inm} dm \\
&= \left[\frac{-f(m) e^{-inm}}{in} \right]_{-\pi}^{\pi} + \frac{1}{in} \int_{-\pi}^{\pi} f'(m) e^{-inm} dm \\
&= \frac{1}{in} \int_{-\pi}^{\pi} f'(m) e^{-inm} dm \; (as \; f \; is \; 2\pi \; periodic \; function) \\
&= \frac{1}{in} \left[\frac{-f'(m) e^{-inm}}{in} \right]_{-\pi}^{\pi} + \frac{1}{(in)^2} \int_{-\pi}^{\pi} f''(m) e^{-inm} dm \\
&= \frac{-1}{n^2} \int_{-\pi}^{\pi} f''(m) e^{-inm} dm \; (as \; f' \; is \; also \; 2\pi \; periodic \; function)
\end{aligned}$$

Now,

$$|2\pi n^2 \hat{f}(n)| = 2\pi |n^2| |\hat{f}(n)| = |-\int_{-\pi}^{\pi} f''(m) e^{-inm} dm| \leq \int_{-\pi}^{\pi} |f''(m)| |e^{-inm}| dm \leq M$$

As $1/n^2$ converges to 0, so does $\hat{f}(n)$.

Also, it can be seen that $\hat{f'} = in\hat{f} \;\; \forall \, n \in \mathbb{Z}$.

1.2.2 Convolution

Convolution of 2π functions $f \; and \; g$ on $\mathbb{R}$ is defined as

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x - y) dy.$$

Partial sums are important while considering convergence of Fourier series, to see that, let us first see different ways in which partial sums can be written(here f is 2π periodic function)

$$\begin{aligned} S_N(f)(x) &= \sum_{n=-N}^{N} \hat{f}(n)e^{inx} \\ &= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)e^{-iny}dy \right) e^{inx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \Big(\sum_{n=-N}^{N} e^{in(x-y)} \Big) dy \\ &= (f * D_N)(x) \end{aligned}$$

where D_N is N^{th} Dirichlet kernel defined as

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}.$$

Proposition 1.17. *For 2π periodic functions f, g and h, we have*

(i) $f * (g + h) = f * g + f * h$

(ii) $(cf) * g = c(f * g) = f * (cg)$ *for some* $c \in \mathbb{C}$.

(iii) $f * g = g * f$

(iv) $(f * g) * h = f * (g * h)$

(v) $f * g$ *is continuous.*

(vi) $\widehat{f * g} = \hat{f}\hat{g}$

1.2.3 Good Kernels

The family $\{K_n\}_{n\in\mathbb{N}}$ is a family of good kernels on circle if it satisfies three conditions:

(a) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x)dx = 1 \ \ \forall\, n \geq 1.$

(b) for all $n \geq 1$, there exists $M > 0$ such that

$$\int_{-\pi}^{\pi} |K_n(x)|dx \leq M.$$

(c) for each $\delta > 0$ as $n \to \infty$,

$$\int_{\delta \leq |x| \leq \pi} |K_n(x)| dx \to 0.$$

Good kernels along with convolutions, gives the important result

Theorem 1.18. *Suppose f be an integrable function on circle and $\{K_n\}_{n \in \mathbb{N}}$ be family of good kernels, then whenever f is continuous at x we have,*

$$lim_{n \to \infty}(f * K_n)(x) = f(x).$$

The limit will be uniform be if f is continuous everywhere.

Proof As f is continuous at x, then for every $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(y) - f(x)| < \epsilon$.

Then,

$$\begin{aligned}
|(f * K_n)(x) - f(x)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y) f(x-y) dy - f(x) \right| \\
&= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y)[f(x-y) - f(x)] dy \right| (from\ condition(a)) \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |K_n(y)||f(x-y) - f(x)| dy \\
&= \frac{1}{2\pi} \int_{|y| \leq \delta} |K_n(y)||f(x-y) - f(x)| dy \\
&\quad + \frac{1}{2\pi} \int_{\delta \leq |y| \leq \pi} |K_n(y)||f(x - y - f(x)| dy \\
&\leq \frac{\epsilon}{2\pi} \int_{|y| \leq \delta} |K_n(y)| dy + \frac{2B}{2\pi} \int_{\delta \leq |y| \leq \pi} |K_n(y)| dy \\
&\leq \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} |K_n(y)| dy + \frac{2B}{2\pi} \int_{\delta \leq |y| \leq \pi} |K_n(y)| dy
\end{aligned}$$

where f is bounded by B. Form condition(c), second term is less than ϵ for large values of n and from condition(b), first term is bounded by $\epsilon M / 2\pi$. Therefore, for all large n, there exists $C > 0$ such that

$$|(f * K_n)(x) - f(x)| \leq \epsilon.$$

Now if f is everywhere continuous, then we can choose δ independent of x. Therefore $(f * K_n)(x) \to f(x)$ uniformly.

1.3 Fourier transform

Fourier transform of a function, $f \in L^1(\mathbb{R}^n)$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\xi}dx \ \ \forall \ \xi \ \in \ \mathbb{R}^n.$$

Lemma 1.19. *(Riemann-Lebesgue lemma)*

If $\hat{f} \in L^1(\mathbb{R}^n)$ *then* $\hat{f}(\xi) \to 0$ *as* $|\xi| \to \infty$*. Thus,* $\hat{f} \in \mathcal{C}_0$*.*

Theorem 1.20. *Let* $f \in L^p(\mathbb{R}^n)$ *where* $p \in [1, \infty)$ *and let* $g \in L^1(\mathbb{R}^n)$ *then* $h = f * g$ *is well defined and* $h \in L^p(\mathbb{R}^n)$*. In fact* $\|h\|_p \leq \|f\|_p \|g\|_1$*.*

Proof As, $|h(x)| = |(f * g)(x)| \leq \int_{\mathbb{R}^n} |f(x-y)||g(y)|dy$

then by Minkowski's integral inequality, we get

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |h(x)|^p dx \right)^{1/p} &= \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y)g(y)dy \right|^p dx \right)^{1/p} \\
&\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|^p |g(y)|^p dx \right)^{1/p} dy \\
&= \int_{\mathbb{R}^n} \left(| \int_{\mathbb{R}^n} |f(x-y)|^p dx \right)^{1/p} |g(y)| dy \\
&= \|f\|_p \|g\|_1.
\end{aligned}$$

Proposition 1.21. *Fourier transforms of a function,* $f \in L^1(\mathbb{R}^n)$ *under different conditions, where* $h, \xi, x \in \mathbb{R}^n; \delta > 0; g \in L^1(\mathbb{R}^n)$*:*

(i) $f(x+h) \to \hat{f}(\xi)e^{2\pi i h\xi}$

(ii) $f(x)e^{-2\pi i x h} \to \hat{f}(\xi + h)$

(iii) $f(\delta x) \to \delta^{-1}\hat{f}(\delta^{-1}\xi)$

(iv) $\dfrac{\partial}{\partial x_k} f(x) \to 2\pi i \xi_k \hat{f}(\xi)$

(v) $-2\pi i x f(x) \to \frac{d}{d\xi}\hat{f}(\xi)$

(vi) $\widehat{f * g} = \hat{f}\hat{g}$

In order to see formula for inverse of a Fourier transform, we first define Abel mean and Gauss mean.

Abel mean of a function, f is defined as

$$A_\epsilon = A_\epsilon(f) = \int_{\mathbb{R}^n} f(x)e^{-\epsilon|x|}dx$$

$\int_{\mathbb{R}^n} f(x)dx$ is said to be Abel summable if

$$lim_{\epsilon\to 0}A_\epsilon(f) = lim_{\epsilon\to 0}\int_{\mathbb{R}^n} f(x)e^{-\epsilon|x|}dx$$

exists and is finite.

Gauss mean of a function, f is defined as

$$G_\epsilon = G_\epsilon(f) = \int_{\mathbb{R}^n} f(x)e^{-\epsilon|x|^2}dx$$

$\int_{\mathbb{R}^n} f(x)dx$ is said to be Gauss summable if

$$lim_{\epsilon\to 0}G_\epsilon(f) = lim_{\epsilon\to 0}\int_{\mathbb{R}^n} f(x)e^{-\epsilon|x|^2}dx$$

exists and is finite.

Let us collectively, define $M_{\epsilon,\phi}$ such that

$$M_{\epsilon,\phi}(f) = \int_{\mathbb{R}^n} \phi(\epsilon x)f(x)dx$$

where $\phi(t) > 0, \phi \in (C)_0$ and $\phi(0) = 1$.

We will also require the following theorem,

Theorem 1.22. *(Multiplication Formula)*

For $f, g \in L^1(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \hat{f}(x)g(x)dx = \int_{\mathbb{R}^n} f(x)\hat{g}(x)dx.$$

Proof From Fubini's theorem we get,

$$\begin{aligned}\int_{\mathbb{R}^n} \hat{f}(x)g(x)dx &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(t)e^{-2\pi ixt}dt\right)g(x)dx \\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} g(x)e^{-2\pi ixt}dx\right)f(t)dt \\ &= \int_{\mathbb{R}^n} f(t)\hat{g}(t)dt\end{aligned}$$

Let Φ be an integrable function and ϕ denotes its Fourier transform. Also, for $\epsilon > 0$ let us define $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$. Then we have

$$\widehat{(\delta_\epsilon \Phi)}(x) = \epsilon^{-n}\phi(x/\epsilon) = \phi_\epsilon(x).$$

On applying the multiplication formula to $e^{2\pi itx}\delta_\epsilon\Phi(x)$ and $f(x)$ we get the following result

Theorem 1.23. *Let $f, \Phi \in L^1(\mathbb{R}^n)$ and ϕ be Fourier transform of Φ, then*

$$\int_{\mathbb{R}^n} \hat{f}(x) e^{2\pi i t x} \Phi(\epsilon x) dx = \int_{\mathbb{R}^n} f(x) \phi_\epsilon(x - t) dx.$$

Theorem 1.24. *Let $\phi \in L^1(\mathbb{R}^n)$ such that $\phi(x) > 0$ for $x \in \mathbb{R}^n$ and $\int_{\mathbb{R}^n} \phi(x) dx = 1$. For $\delta > 0$ let $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$, then for $f \in L^p(\mathbb{R}^n)$ where $p \in [1, \infty)$ or $f \in \mathcal{C}_0 \subset L^\infty(\mathbb{R}^n)$ we get*

$$\|f * \phi_\epsilon - f\|_p \to 0 \ as\ \epsilon \to 0.$$

Theorem 1.25. *(Inversion theorem)*
If Φ and its Fourier transform ϕ are integrable such that $\int_{\mathbb{R}^n} \phi(x) dx = 1$ then Φ means of the integral $\int_{\mathbb{R}^n} \hat{f}(t) e^{2\pi i x t} dt$ converge to $f(x)$ in L^1 norm.

Corollary 1.26. *Let $f, \hat{f} \in L^1(\mathbb{R}^n)$ then*

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(t) e^{2\pi i x t} dt \ \ a.e.$$

Therefore, if $\hat{f}(t) = 0 \ \forall\, t \in \mathbb{R}^n$ then $f(x) = 0$ $a.e.$
Applying the result to the function $f = f_1 - f_2$ we get uniqueness result for the Fourier transform.

Theorem 1.27. *Let $f \in L^1 \cap L^2$ then $\|\hat{f}\|_2 = \|f\|_2$.*

Proof Let $g(x) = \overline{f(-x)}$ then $\hat{g} = \overline{\hat{f}}$. Define $h = f * g \in L^1$, from the property of Fourier transform on convolution we get that $\hat{h} = \widehat{f * g} = \hat{f}\hat{g}$, so $\hat{h} = |\hat{f}|^2$. Now, we have

$$\int |\hat{f}(x)|^2 dx = \int \hat{h}(x) dx = h(0) = \int f(x) g(0 - x) dx = \int |f(x)|^2 dx.$$

Thus, Fourier transform is a bounded operator on $L^1 \cap L^2 \subset L^2$ with the L^2 norm, in fact its an isometry. Therefore, there exists a unique extension $\mathcal{F}$, known as Fourier transform operator, on all of L^2.

Theorem 1.28. *(Plancherel Theorem)*
Fourier transform, $\mathcal{F}$ is an isometry on L^2 and Inverse of Fourier transform is defined as $(\mathcal{F}^{-1}g)(x) = (\mathcal{F}g)(-x)$ for all $g \in L^2$.

1.4 Special functions

1.4.1 Hermite Functions

Definition 1.29. *Hermite functions are defined as*

$$H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} D^n(e^{-x^2})$$

where D is differential operator.

Proposition 1.30. *Hermite functions satisfy the following recursion formulas*

(i) $H'_n(x) - xH_n(x) = -(n+1)H_{n+1}(x)\ for x \geq 0.$

(ii) $\widehat{H_n}(x) = (-i)^n H_n(x).$

Proof

(i) The following recursion formula can be easily obtained by differentiating $H_n(x)$,

$$\begin{aligned}
H'_n(x) &= \frac{d}{dx}(H_n(x)) \\
&= \frac{(-1)^n}{n!}\left[(D^n(e^{-x^2})\frac{d}{dx}(e^{x^2/2}) + (e^{x^2/2})\frac{d}{dx}D^n(e^{-x^2})\right] \\
&= \frac{(-1)^n}{n!}\left[(D^n(e^{-x^2})xe^{x^2/2} + (e^{x^2/2})D^{n+1}(e^{-x^2})\right] \\
&= xH_n(x) + \frac{(-1)^n}{n!}(e^{x^2/2})D^{n+1}(e^{-x^2}) \\
&= xH_n(x) - (n+1)H_{n+1}(x)
\end{aligned}$$

(ii) Let us prove that $\hat{H}_0 = H_0$ i.e. recursion formula is true for $n = 0$. From the definition of Hermite functions we get that $H_0(x) = e^{-x^2/2}$ and derivative of its Fourier transform will be

$$\begin{aligned}
\widehat{H_0}'(y) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2/2}e^{-ixy}(-ix)dx \\
&= \frac{i}{\sqrt{2\pi}}\int_{\mathbb{R}} (-x)e^{-x^2/2}e^{-ixy}dx \\
&= \frac{i}{\sqrt{2\pi}}\int_{\mathbb{R}} (e^{-x^2/2})'e^{-ixy}dx \\
&= \frac{i}{\sqrt{2\pi}}\left[e^{-x^2/2}e^{-ixy}\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}}(-iy)e^{-ixy}e^{-x^2/2}dx\right] \\
&= \frac{i}{\sqrt{2\pi}}\int_{\mathbb{R}}(iy)e^{-ixy}e^{-x^2/2}dx \\
&= -y\widehat{H_0}(y)
\end{aligned}$$

from above calculation we can see that

$$\Rightarrow \frac{\hat{H_0}'(y)}{\hat{H_0}(y)} = -y \Rightarrow \hat{H_0}(y) = \hat{H_0}(0)e^{-y^2/2}$$

as $\widehat{H_0}(0) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2/2}dx = 1 \Rightarrow \widehat{H_0}(y) = e^{-y^2/2}$.

Since, Hermite functions satisfy the recursion formula

$$H'_n(x) - xH_n(x) = -(n+1)H_{n+1}(x) \; for \; x \geq 0,$$

and multiplying both sides with $(i)^{n+1}$ we get that

$$\begin{aligned}
-(n+1)(i)^{n+1}\widehat{H_{n+1}}(x) &= (i)^{n+1}(H'_n(x) - xH_n(x))\hat{} \\
&= (i)^{n+1}\widehat{H'_n}(x) - (i)^{n+1}(xH_n(x))\hat{} \\
&= (i)^{n+1}iy\widehat{H_n}(x) + (i)^n(-ixH_n(x))\hat{} \\
&= -yi^n\widehat{H_n}(x) + i^n(\widehat{H_n}(x))'
\end{aligned}$$

since $i^n\widehat{H_n}$ satisfy same recursion formula as H_n, we get that

$$i^n\widehat{H_n} = H_n \Rightarrow \widehat{H_n} = (-i)^n H_n.$$

Remark

- Hermite functions are eigenfunctions of the Fourier transform operator and they have eigenvalues as some power of $-i$.
- Hermite functions,H_n are n-degree polynomials multiplied with $e^{-x^2/2}$.

1.4.2 Gaussian Functions

Definition 1.31. *Functions of the form $g(x) = e^{-ax^2}$ where $a > 0$ are called Gaussian functions.*

Here we will compute an important integral for the Gaussian function $e^{-\pi x^2}$.

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-\pi x^2}dx\right)^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)}dxdy \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-\pi r^2}rdrd\theta \quad (changing\ to\ polar\ coordinates) \\
&= \int_0^{\infty} 2\pi re^{-\pi r^2}dr \\
&= \Big[-e^{-\pi r^2}\Big]_0^{\infty} \\
&= 1
\end{aligned}$$

Theorem 1.32. *Fourier transform of Gaussian functions are Gaussian.*

Proof Let us first prove it for $a = \pi$. As $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}dx$, we first calculate $\hat{f}(0)$.

$$\begin{aligned}\hat{f}(0) &= \int_{\mathbb{R}} e^{-\pi x^2} dx \\ &= 1.\end{aligned}$$

$$\begin{aligned}\frac{d}{d\xi}\hat{f}(\xi) &= \int_{\mathbb{R}} (-2\pi i x) f(x) e^{-2\pi i x\xi} dx \;\; (from\ Prop.\ 1.21(iv)) \\ &= i\int_{\mathbb{R}} f'(x)e^{-2\pi i x\xi}dx \;\; (as\ f'(x) = \frac{d}{dx}(e^{-\pi x^2}) = -2\pi x e^{-\pi x^2}) \\ &= i(2\pi i\xi)\hat{f}(\xi) \;\; (from\ Prop.\ 1.21(v)) \\ &= -2\pi\xi\hat{f}(\xi) \\ &\Rightarrow \frac{\hat{f}'(\xi)}{\hat{f}(\xi)} = -2\pi\xi\end{aligned}$$

taking the integral on both sides gives us $\hat{f}(\xi) = e^{-\pi\xi^2} + c\,(c : constant\ of\ integration)$. But from Riemann-Lebesgue lemma, $|\hat{f}(\xi)| \to 0\ as\ |\xi| \to \infty$ therefore $c = 0$. For $a \neq \pi$, result can be obtained by suitable change of variables.

Chapter 2

Uncertainty Principles

We will try to see behaviour of a function,f and its Fourier transform, $\hat{f}$ under different notions of concentration.

2.1 Concentration: Support

Here we try to analyse the behaviour of f and $\hat{f}$ while considering support of a function. Let us first start by considering a function $f \in \mathbb{R}$ which has a compact support, say $A \subseteq [a, b]$. Suppose Fourier transform of f also has compact support, say $C \subseteq [c, d]$ which is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi ix\xi}dx$$

Now, by the Inversion formula we get that f is equal to some continuous function. Let us now define a function g as

$$g(z) = \int_a^b f(x)e^{-2\pi ixz}dx \; \forall \; z \in \mathbb{C}$$

the integral is well-defined as for $z = p + iq$ we have
$\int_a^b |f(x)e^{-2\pi ixz}|dx = \int_a^b |f(x)|e^{2\pi xq}dx < \infty$ as f is continuous function with compact support so it will be bounded, say the bound is M and $e^{2\pi bq} - e^{2\pi aq} < \infty$.
Since we know that $e^{-2\pi ixz}$ is an entire function so by Cauchy's its integral over a closed

curve γ will be zero i.e. $\int_\gamma e^{-2\pi ixz}dz = 0$. Therefore, we have

$$\begin{aligned}\int_\gamma F(z)dz &= \int_\gamma \bigg(\int_a^b f(x)e^{-2\pi ixz}dx \bigg) dz \\ &= \int_a^b f(x) \bigg(\int_\gamma e^{-2\pi ixz}dz \bigg) dx \ (Fubini's\ theorem) \\ &= 0\end{aligned}$$

Thus, by Morera's theorem F is an entire function.

Since $F|_{\mathbb{R}} = \hat{f} \Rightarrow zeros\ of\ \hat{f} \subseteq zeros\ of\ F$. But zeros of an analytic function are isolated thus $\hat{f}$ cannot have compact support.

Now, we look at a stronger condition where support of a function is finite(need not be compact).

2.1.1 Benedicks Theorem

Theorem 2.1. ***Benedicks Theorem***

Let $f \in L^1(\mathbb{R})$ such that supports of f and $\hat{f}$ are of finite measure i.e

$$|suppf||supp\hat{f}| < \infty$$

then $f = 0$ a.e.

Proof $f \in L^1$ (given),so $\hat{f} \in C_0(\mathbb{R})$ (Riemann-Lebesgue lemma)

To see $\hat{f} \in L^1$ $(\mathbb{R})$

Define $A = \{x \in \mathbb{R} : f(x) \neq 0\}$ and $B = \{\xi \in \mathbb{R} : \hat{f}(\xi) \neq 0\}$

Let $supp\hat{f} = \overline{B}$ be finite then

$$\|\hat{f}\|_1 \leqslant \|\hat{f}\|_\infty |\overline{B}| \Rightarrow \hat{f} \in L^1(\mathbb{R})$$

$$\begin{aligned}f(x) &= \int_{\mathbb{R}} \hat{f}(t)e^{2\pi ixt}dt\ a.e\ (by\ Inversion\ formula) \\ &= \int_{\mathbb{R}} g(t)e^{2\pi ixt}dt\ (let\ \hat{f}(t) = g(t)) \\ &= \hat{g}(-x) \in C_0(\mathbb{R}) \\ &\Rightarrow f(x) \in C_0(\mathbb{R})\end{aligned}$$

Therefore $f, \hat{f} \in L^1 \cap C_0(\mathbb{R})$

For $\delta > 0$, define $f_\delta(x) = f(\delta x) \, \forall \, x \in \mathbb{R}$

$$\begin{aligned} suppf_\delta &= \overline{\{x \in \mathbb{R} : f_\delta(x) \neq 0\}} \\ &= \overline{\{x \in \mathbb{R} : f(\delta x) \neq 0\}} \\ &\Rightarrow \overline{\{x \in \mathbb{R} : \delta x \in suppf\}} \\ &\Rightarrow \overline{\{x \in \mathbb{R} : x \in \frac{suppf}{\delta}\}} \\ &\Rightarrow suppf_\delta = \frac{suppf}{\delta} \end{aligned}$$

Therefore by choosing δ as $\alpha|suppf|$ where $\alpha > 1$, we can get $|suppf_\delta| < 1$.

Hence by dilation we assume $|suppf| < 1$.

Assume $|A| > 0$ and $|B| > 0$

Define 1-periodic function,

$$G(\xi) = \sum_{n \in \mathbb{Z}} \chi_B(\xi + n) \, \forall \, \xi \in \mathbb{R}$$

where χ_B is the indicator function.

$$0 < |B| = \int_{\mathbb{R}} \chi_B(t) dt$$

Also,by change of variables

$$\begin{aligned} \int_0^1 \sum_{n \in \mathbb{Z}} \chi_B(\xi + n) d\xi &= \sum_{n \in \mathbb{Z}} \int_n^{n+1} \chi_B(\xi) d(\xi) \\ &= \int_{\mathbb{R}} \chi_B(\xi) d(\xi) = |B| \end{aligned}$$

Since $supp\hat{f}$ has finite measure and $B \subset_{open} \overline{B} \Rightarrow \int_0^1 G(\xi) d\xi < \infty$.

Since $\int_0^1 G(\xi) d\xi < \infty$ and $G(\xi) \geq 0 \Rightarrow G(\xi) < \infty \; a.e.$

So $\exists \, E \subseteq [0,1], |E| = 1$ such that $G(\xi) < \infty \, \forall \, \xi \in E$.

Therefore, there can be only finitely many non-zero terms in the definition of $G(\xi)$ for $\xi \in E$. Hence for $\xi \in E$,$(\xi + \mathbb{Z}) \cap B$ is finite.

For each $\eta \in E$, let us now define a 1-periodic function ϕ_η such that

$$\phi_\eta(x) = \sum_{n \in \mathbb{Z}} f(x+n) e^{-2\pi i \eta (x+n)} \, \forall \, x \in \mathbb{R}$$

Claim $\phi_\eta \in L^1[0,1]$ and $\hat{\phi_\eta}(k) = \hat{f}(\eta + k)$ where $k \in \mathbb{Z}$

(i) $\phi_\eta \in L^1[0,1]$

$$\begin{aligned}
\int_0^1 |\phi_\eta(x)|dt &= \int_0^1 |\sum_{n\in\mathbb{Z}} f(x+n)e^{-2\pi i\eta(x+n)}|dx \\
&\leqslant \int_0^1 \sum_{n\in\mathbb{Z}} |f(x+n)|dx \\
&= \sum_{n\in\mathbb{Z}} \int_n^{n+1} |f(x)|dx \\
&= \int_{\mathbb{R}} |f(x)|dx \; < \infty \; (\because f \in L^1(\mathbb{R}))
\end{aligned}$$

(ii) $\hat{\phi_\eta}(k) = \hat{f}(\eta + k)$ where $k \in \mathbb{Z}$

$$\begin{aligned}
\hat{\phi_\eta}(k) &= \int_0^1 \phi_\eta(x)e^{-2\pi i x k}dx \\
&= \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)e^{-2\pi i\eta(x+n)}e^{-2\pi i x k}dx \\
&= \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)e^{-2\pi i\eta(x+n)}e^{-2\pi i(x+n)k}dx \; (\because e^{-2\pi i n k} = 1) \\
&= \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)e^{-2\pi i(\eta+k)(x+n)}dx \\
&= \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n)e^{-2\pi i(\eta+k)(x+n)}dx \; (Fubini's\ theorem) \\
&= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(x)e^{-2\pi i(\eta+k)(x)}dx \; (change\ of\ variables) \\
&= \int_{\mathbb{R}} f(x)e^{-2\pi i x(\eta+k)}dx \\
&= \hat{f}(\eta+k) \\
&\Rightarrow \hat{\phi_\eta}(k) = \hat{f}(\eta+k)
\end{aligned}$$

Since $(\xi + \mathbb{Z}) \cap B$ is finite for $\xi \in E \subseteq [0,1]$.
Therefore, for $\eta \in E, \hat{f}(\eta + k) \neq 0$ for finitely many k. So for $\eta \in E, \phi_\eta$ has only finitely many non-zero Fourier coefficients,therefore

$$\phi_\eta(t) = \sum_{j=1}^{m} \hat{\phi_\eta}(k_j)e^{2\pi i k_j t}$$

let us now substitute $\hat{\phi}_\eta(k_j)$ by c_j and $e^{2\pi it}$ by z

For $z \in \mathbb{C} \setminus \{0\}$ and fix η , define

$$\begin{aligned}
F(z) &= \sum_{j=1}^{m} c_j z^{k_j} \\
&= z^{-N} \sum_{j=1}^{m} c_j z^{l_j} \; (with\; l_j \geq 0 \; and \; N \; sufficiently \; large) \\
&= z^{-N} Q(z) \; (Q(z) \; is \; polynomial \; in \; z)
\end{aligned}$$

Since $Q(z)$ have finitely many zeros in $\mathbb{C}$ unless it is identically zero,therefore it will have finitely many zeros in [0,1].So, ϕ_η is a trigonometric polynomial which has finitely many zeros in [0,1] unless it is identically zero.

Also,

$$\begin{aligned}
|\phi_\eta(x)| &\leq \sum_n \chi_A(x+n)|f(x+n)| \\
&\leq \|f\|_\infty \sum_n \chi_A(x+n)
\end{aligned}$$

But,

$$\int_0^1 \sum_n \chi_A(x+n)dx = |A| < 1$$

and since χ_A is either 0 or 1.

$\Rightarrow \sum_n \chi_A(x+n)$ must vanish on set of positive measure, so also $\phi_\eta \; \forall \; \eta \in E \subseteq [0,1]$

$\Rightarrow$ For almost all $\eta \in [0,1], \phi_\eta = 0$

$\Rightarrow \hat{f}(\eta + k) = 0 \; a.e. \; \eta \in [0,1]$

$\Rightarrow \hat{f} = 0 \; a.e.$

2.1.2 Amrein-Berthier Inequality

Amrein-Berthier inequality is the quantitative version of Benedicks Theorem.

Theorem 2.2. ***Amrein-Berthier Inequality***

Let E, Σ be two subsets of finite measure in $\mathbb{R}$ then there exists a positive constant $C_{E,\Sigma}$ such that $\forall \; f \in L^2(\mathbb{R})$

$$\|f\|_2^2 \leq C(\int_{\mathbb{R}\setminus E} |f(x)|^2 dx + \int_{\mathbb{R}\setminus\Sigma} |\hat{f}(\xi)|^2 d\xi).$$

Proof [1]:We divide the proof in two steps:

Step 1: $f \in L^2(\mathbb{R})$ (given).

Assume that $supp\hat{f} \subseteq \Sigma \Rightarrow \int_{\mathbb{R}\setminus\Sigma} |\hat{f}(\xi)|^2 d\xi = 0$.

Suppose above inequality does not hold true, then for each n, we can find a function $f_n \in L^2(\mathbb{R}), \|f_n\|_2 = 1$ with $supp\hat{f_n} \subseteq \Sigma$ such that

$$1 > n \int_{\mathbb{R}\setminus E} |f_n(x)|^2 dx$$

Since Fourier transform is an isometry on $L^2(\mathbb{R})$, so unit ball is mapped to unit ball and since unit ball is weakly compact, we can get a subsequence $\{f_n\}$ such that $\{\hat{f_n}\}$ converges weakly to $g \in L^2(\mathbb{R})$. Now to check that $supp\ g \subseteq \Sigma$.

Let $F \subset \Sigma^c$ such that $|F| < \infty$, then

$$0 = \int \hat{f_n}(\xi)\chi_F(\xi)d\xi \longrightarrow \int g(\xi)\chi_F(\xi)d\xi = 0 \ \ (as\ \hat{f_n} \underset{\rightarrow}{w}\ g)$$

$$then\ \forall\ F \subset \Sigma^c, \int_F g(\xi)d\xi = 0 \Rightarrow supp\ g \subseteq \Sigma.$$

Since $\hat{f} \in L^2(\mathbb{R})$ and $supp\hat{f} \subseteq \Sigma\ and\ |\Sigma| < \infty$, therefore $\hat{f} \in L^1(\mathbb{R}) \ \Rightarrow\ \hat{f} \in L^1 \cap L^2$.

Let $T = (\mathcal{F})^{-1}$ be Inverse Fourier operator defined on $L^2(\mathbb{R})$ as

$$T : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$
$$\hat{f_n} \longrightarrow f_n$$

Also,

$$T\Big|_{L^1(\mathbb{R})} : \hat{f_n} \longrightarrow h_n$$

Since, Inverse Fourier operator on L^2 is an extension of Inverse Fourier transform on L^1, so by uniqueness property of Fourier transform $f_n = h_n$. Therefore, we can write the inversion formula,

$$f_n(x) = \int_{\mathbb{R}} \hat{f_n}(\xi)\chi_\Sigma(\xi)e^{2\pi i\xi x}d\xi\ a.e.$$

We may assume that each f_n is continuous. Also,

let $\phi_x(\xi) = \chi_\Sigma(\xi)e^{2\pi ix\xi}$,then

$$\int \hat{f_n}(\xi)\phi_x(\xi)d\xi \longrightarrow \int g(\xi)\phi_x(\xi)d\xi\ (as\ \hat{f_n} \underset{\rightarrow}{w}\ g)$$

[1]This proof was thought by Prof. Aline Bonami over a cup of morning coffee and was neatly and systematically written by Prof. Shobha Madan

$$\Rightarrow lim_{n\to\infty} f_n = f \ where \ f = \check{g}$$

To show: $supp f \subseteq E$

$$\begin{aligned}
\int_{\mathbb{R}\setminus E} |f(x)|^2 dx &= \int_{\mathbb{R}\setminus E} lim_{n\to\infty} |f_n(x)|^2 dx \\
&\leq lim_{n\to\infty} \int_{\mathbb{R}\setminus E} |f_n(x)|^2 dx \ (Fatou's \ lemma) \\
&\leq lim_{n\to\infty} 1/n = 0 \\
&\Rightarrow supp f \subseteq E
\end{aligned}$$

Now, by Hőlder's inequality

$$\begin{aligned}
|f_n(x)|^2 \chi_E(x) &= |\int_{\mathbb{R}} \hat{f}_n(\xi) \chi_\Sigma(\xi) e^{2\pi i \xi x} d\xi|^2 \chi_E(x) \\
&\leq \|\hat{f}_n\|_2^2 |\Sigma| \chi_E(x) \\
&= |\Sigma| \chi_E(x) \ \ (as \ \|\hat{f}_n\|_2 = 1)
\end{aligned}$$

Also,

$$\int_{\mathbb{R}} |\Sigma| \chi_E(x) dx = |\Sigma||E| < \infty$$

Since,

$$\begin{aligned}
\|f\|_2^2 &= \int_{\mathbb{R}} |f(x)|^2 dx \\
&= \int_{\mathbb{R}} |f(x)|^2 \chi_E(x) dx \ \ (as \ supp f \subseteq E) \\
&= \int_{\mathbb{R}} |lim_{n\to\infty} f_n(x)|^2 \chi_E(x) dx \\
&= \int_{\mathbb{R}} lim_{n\to\infty} |f_n(x)|^2 \chi_E(x) dx \\
&= lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x)|^2 \chi_E(x) dx \ (by \ DCT) \\
&= lim_{n\to\infty} \int_{\mathbb{R}} (|f_n(x)|^2 - |f_n(x)|^2 \chi_{\mathbb{R}\setminus E}) dx \\
&= lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x)|^2 dx - lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x)|^2 \chi_{\mathbb{R}\setminus E} dx \\
&= 1 \ \ \ (as \ \|f_n\|_2 = 1 \ and \ lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x)|^2 \chi_{\mathbb{R}\setminus E} dx = 0) \\
&\Rightarrow \|f\|_2^2 = 1
\end{aligned}$$

Step 2: Let $f \in L^2(\mathbb{R})$, and we can write $\hat{f} = \hat{f}\chi_\Sigma + \hat{f}\chi_{\mathbb{R}\setminus\Sigma}$.

$$\begin{aligned}\|f\|_2^2 &= \|\hat{f}\|_2^2 \ (Plancherel\ theorem)\\ &= \|\hat{f}\chi_\Sigma\|_2^2 + \|\hat{f}\chi_{\mathbb{R}\setminus\Sigma}\|_2^2 \ (Pythagoras\ theorem)\\ &= \|(\hat{f}\chi_\Sigma)\check{}\|_2^2 + \|\hat{f}\chi_{\mathbb{R}\setminus\Sigma}\|_2^2 \ (Plancherel\ theorem)\\ &\leq C(\int_{\mathbb{R}\setminus E} |(\hat{f}\chi_\Sigma)\check{}(x)|^2 dx) + \int_{\mathbb{R}\setminus\Sigma} |\hat{f}(\xi)|^2 d\xi \ \ (first\ term\ from\ Step\ 1)\ldots(*)\end{aligned}$$

Using the fact that $|a+b|^2 \leq 2(|a|^2 + |b|^2)$ and decomposition for first term as $(\hat{f}\chi_\Sigma)\check{}(x) = (\hat{f} - \hat{f}\chi_{\mathbb{R}\setminus\Sigma})\check{}(x)$, we get

$$\int_{\mathbb{R}\setminus E} |(\hat{f}\chi_\Sigma)\check{}(x)|^2 dx \leq 2(\int_{\mathbb{R}\setminus E} |f(x)|^2 dx + \int_{\mathbb{R}\setminus E} |(\hat{f}\chi_\Sigma)\check{}(x)|^2 dx$$

Substituting above inequality in the expression of $(*)$, we get

$$\|f\|_2^2 \leq C(2\int_{\mathbb{R}\setminus E} |f(x)|^2 dx + 2\int_{\mathbb{R}\setminus E} |(\hat{f}\chi_\Sigma)\check{}(x)|^2 dx) + \int_{\mathbb{R}\setminus\Sigma} |\hat{f}(\xi)|^2 d\xi$$

Now since,

$$\int_{\mathbb{R}\setminus E} |(\hat{f}\chi_\Sigma)\check{}(x)|^2 dx = \int_{\mathbb{R}\setminus E} |(\hat{f}\chi_\Sigma)(x)|^2 dx \leq \int_{\mathbb{R}} |(\hat{f}\chi_\Sigma)(x)|^2 dx = \int_{\mathbb{R}\setminus\Sigma} |\hat{f}(\xi)|^2 d\xi$$

Finally we get ,

$$\|f\|_2^2 \leq C_{E,\Sigma}(\int_{\mathbb{R}\setminus E} |f(x)|^2 dx + \int_{\mathbb{R}\setminus\Sigma} |\hat{f}(\xi)|^2 d\xi).$$

Another proof of Amrein-Berthier inequality is given by using concept of Hilbert-Schmidt operator.

Let us first see some pre-requisites needed for the proof:-

- A linear operator T on $L^2(X, d\mu)$ is Hilbert-Schmidt operator if there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ such that $\sum_n \|Te_n\|^2 < \infty$.
- A linear operator T on $L^2(X, d\mu)$ is Hilbert-Schmidt operator iff there is a function $k \in L^2(X \times X, d\mu \times d\mu)$ such that

$$Tf(x) = \int_X k(x,y)f(y)d\mu(y)\ \forall f \in L^2(X, d\mu).$$

In fact $\|T\|_{HS} = \|k\|_2$ where $\|T\|_{HS} = (\sum_n \|Te_n\|^2)^{\frac{1}{2}}$ and $\|k\|_2\ is\ L^2\ norm.$

Proof Let T be Hilbert-Schmidt operator and $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis on

$L^2(X, d\mu)$.Then we know that

$$\sum_n \|Te_n\| = \sum_{n,m=1}^{\infty} |\langle Te_n, e_m\rangle| < \infty$$

and the sum is independent of orthonormal basis. For $n, m \geq 1$ let $\sigma_{n,m}$ be the function on $X \times X$ defined as

$$\sigma_{n,m}(x, y) = e_n(x)e_m(y)$$

then $\sigma_{n,m}$ is an orthonormal basis for $L^2(X \times X, d\mu \times d\mu)$.Define

$$k(x, y) = \sum_{n,m} \langle Te_n, e_m\rangle \sigma_{n,m}(x, y) \quad for\ x, y \in X.$$

Square summability of the coefficients of above series implies that the series converges in $L^2(X \times X, d\mu \times d\mu)$ and hence the resulting function k is in $L^2(X \times X, d\mu \times d\mu)$. Moreover, if $k \in L^2(X \times X, d\mu \times d\mu)$ then it induces an bounded operator S on Hilbert space $L^2(X, d\mu)$ for $f \in L^2(X, d\mu)$ as:

$$Sf(x) = \int_X k(x, y)f(y)d\mu(y) \quad for\ f\ \in L^2(X, d\mu).$$

To see the boundedness of the operator,

$$\begin{aligned}
\|Sf\|_2^2 &= \int_X |Sf(x)|^2 d\mu(x) \\
&= \int_X |\int_X k(x, y)f(y)d\mu(y)|^2 d\mu(x) \\
&\leq \int_X (\int_X |k(x, y)|^2 d\mu(y))(\int_X |f(y)|^2 d\mu(y))d\mu(x)\ (Cauchy - Schwarz\ inequality) \\
&\leq \int_X \int_X |k(x, y)|^2 d\mu(y)\|f\|_2^2 d\mu(x) \\
&\leq \|f\|_2^2 \int_X \int_X |k(x, y)|^2 d\mu(y)d\mu(x) \\
&\leq \|f\|_2^2 \|k\|_2^2 < \infty
\end{aligned}$$

Now, if we can prove that the operator S and T are in fact same then we are done.

To see this, let us calculate

$$\begin{aligned}\langle Se_n, e_m\rangle &= \int_X Se_n(x)\overline{e_m(x)}d\mu(x)\\ &= \int_X\int_X k(x,y)e_n(y)d\mu(y)\overline{e_m(x)}d\mu(x)\\ &= \int_X\int_X (\sum_{j,k}\langle Te_j, e_k\rangle\sigma_{j,k}(x,y))e_n(y)d\mu(y)\overline{e_m(x)}d\mu(x)\\ &= \int_X\int_X (\sum_{j,k}\langle Te_j, e_k\rangle e_j(x)e_k(y))e_n(y)d\mu(y)\overline{e_m(x)}d\mu(x)\\ &= \int_X\int_X \sum_{j,k}\langle Te_j, e_k\rangle e_j(x)e_k(y))e_n(y)\overline{e_m(x)}d\mu(y)d\mu(x)\end{aligned}$$

Conversely if $k \in L^2(X\times X, d\mu\times d\mu)$ and T is operator given by

$$Tf(x) = \int_X k(x,y)f(y)d\mu(y)\ \forall f \in L^2(X, d\mu),$$

we have to show that T is Hilbert-Schmidt operator.

Let e_n be an orthonormal basis of $L^2(X, d\mu)$, then for any $n \geq 1$ and $x \in X, Te_n(x) = \langle e_n, k_x\rangle$, where $k_x(y) = \overline{k(x,y)}$.

Then from Phythagorean theorem,

$$\sum_{n=1}^{\infty}|Te_n(x)|^2 = \|k_x\|^2 = \int_X |k(x,y)|^2 d\mu(y)$$

Now,

$$\begin{aligned}\sum_{n=1}^{\infty}\|Te_n(x)\|^2 &= \sum_{n=1}^{\infty}\int_X |Te_n(x)|^2 d\mu(x)\\ &= \int_X\sum_{n=1}^{\infty}|Te_n(x)|^2 d\mu(x)\ \ (Fubini's\ theorem)\\ &= \int_X\int_X |k(x,y)|^2 d\mu(x)d\mu(y)\\ &= \|k(x,y)\|_2 < \infty\end{aligned}$$

- Hilbert-Schmidt integral operators are compact.

  Proof Let T be Hilbert-Schmidt operator and $\{e_n\}$ be set of orthonormal basis. The idea is to show that T is compact by expressing it as a norm limit of finite rank operators. Define $T_k : E \longrightarrow F$ (where $k \in \mathbb{N}$ and E, F are Hilbert spaces) given by

$$T_k(\sum_1^{\infty} x_n e_n) = \sum_1^{k} x_n Te_n$$

where $x = \sum_1^\infty x_n e_n$ is an arbitrary element of E i.e. T_k agrees with T in the span of $e_1, e_2, \cdots\cdots, e_k$ and is zero on the span of remaining e_ns. The rank of T_k is atmost k, so T_k is compact. Now for $x = \sum_1^\infty x_n e_n \in E$,

$$(T - T_k)x = \sum_1^\infty x_n Te_n - \sum_1^k x_n Te_n = \sum_{k+1}^\infty x_n Te_n$$

Hence,

$$\begin{aligned}
\|(T - T_k)x\| &= \|\sum_{k+1}^\infty x_n Te_n\| \\
&\leq \sum_{k+1}^\infty |x_n| \|Te_n\| \\
&\leq (\sum_{k+1}^\infty |x|^2)^{1/2} (\sum_{k+1}^\infty \|Te_n\|^2)^{1/2} \\
&\leq \|x\| (\sum_{k+1}^\infty \|Te_n\|^2)^{1/2} \\
&\Rightarrow \|(T - T_k)\| \leq (\sum_{k+1}^\infty \|Te_n\|^2)^{1/2}
\end{aligned}$$

Now, choose k sufficiently large such that $(\sum_{k+1}^\infty \|Te_n\|^2)^{1/2} < \epsilon$

Thus $T \longrightarrow T_k$ in the operator norm, so T is compact.

- A Hilbert-Schmidt integral operator T with $k \in L^2(X \times X)$ such that $k(x, y) = \overline{k(y, x)}$ is self-adjoint and therefore also normal $\Rightarrow$ spectral theorem can applied.

Let us now start with the proof.

Proof 2: Let $F_{E,\Sigma} = \{f \in L^2(\mathbb{R}) : supp f \subset E, supp \hat{f} \subset \Sigma\}$

Claim 1:$dim(F_{E,\Sigma}) = 0$.

Consider the operators defined by $P_E(f) = f\chi_E \; and \; \hat{P}_\Sigma(f) = (\hat{f}\chi_\Sigma)\check{}$.

Claim 2:$P_E \circ \hat{P}_\Sigma$ is Hilbert-Schmidt operator with kernel $k(x, y) = \chi_E(x)\hat{\chi}_\Sigma(x - y)$.

Proof of the claim 2:

$$\begin{aligned}
P_E \circ \hat{P}_\Sigma(f)(x) &= P_E(\hat{P}_\Sigma(f)(x)) \\
&= P_E(\hat{f}(x)\chi_\Sigma(x)\check{)}) \\
&= (\hat{f}(x)\chi_\Sigma(x)\check{)}\chi_E(x) \\
&= \int_{\mathbb{R}} \hat{f}(\xi)\chi_\Sigma(\xi)e^{2\pi i x\xi}d\xi.\chi_E(x) \\
&= \int_{\mathbb{R}} (\int_{\mathbb{R}} f(y)e^{-2\pi i y\xi}dy)\chi_\Sigma(\xi)e^{2\pi i x\xi}d\xi.\chi_E(x) \\
&= \int_{\mathbb{R}} f(y)\int_{\mathbb{R}} \chi_\Sigma(\xi)e^{2\pi i\xi(x-y)}d\xi\chi_E(x)dy \\
&= \int_{\mathbb{R}} f(y)\chi_\Sigma(\hat{x-}y)\chi_E(x)dy
\end{aligned}$$

Therefore $P_E \circ \hat{P}_\Sigma$ is Hilbert-Schmidt operator with kernel $k(x,y) = \chi_E(x)\hat{\chi}_\Sigma(x-y)$. Also,

$$\|P_E \circ \hat{P}_\Sigma\|_{HS}^2 = \|k\|_2^2 = |\Sigma||E|$$

Assume that there exists orthonormal basis $\{e_n\}$ adapted to the decomposition

$$L^2(\mathbb{R}) = Im(P_E \circ \hat{P}_\Sigma) \oplus Ker(P_E \circ \hat{P}_\Sigma)$$

So, $\|P_E \circ \hat{P}_\Sigma\|_{HS}^2 = |E||\Sigma| = \Sigma_n\|(P_E \circ \hat{P}_\Sigma)e_n\|^2 \geq dim(F_{E,\Sigma})$
Since $F_{E,\Sigma} \subseteq Im(P_E \circ \hat{P}_\Sigma)$ and elements of $F_{E,\Sigma}$ are eigenvectors of the operator $P_E \circ \hat{P}_\Sigma$ with eigenvalue equal to 1.
Now to proceed further let us first prove the lemma.

Lemma 2.3. *If $E \subseteq \mathbb{R}$ is a subset with finite measure and $\epsilon > 0$ then there exists $a \neq 0$ such that*

$$|E| < |E \cup (E+a)| < |E| + \epsilon.$$

Proof of the lemma: As $\chi_E * \widetilde{\chi_E}(x) = |E \cap (E+a)|$ and $\chi_E * \widetilde{\chi_E}(0) = |E|$. So, there exists an $a \neq 0$ and $\epsilon_0 > 0$ such that

$$|E| - \epsilon_0 < |E \cap (E+a)| < |E|$$

Now, as $E \cup (E+a) = [E \cap (E+a)] \cup [E \setminus (E+a)] \cup [(E+a) \setminus E]$
we have $|E \cup (E+a)| = |E \cap (E+a)| + |E \setminus (E+a)| + |(E+a) \setminus E|$
As,

$$|E| = |E\setminus(E+a)| + |E\cap(E+a)| \Rightarrow |E| > |E\setminus(E+a)| + |E| - \epsilon_0 \Rightarrow |E\setminus(E+a)| < \epsilon_0$$

Similarly,

$$|(E+a)\setminus E| < \epsilon_0$$

Therefore $|E\cup(E+a)| < |E|+\epsilon_0+\epsilon_0 \Rightarrow |E\cup(E+a)| < |E|+2\epsilon_0 \Rightarrow |E\cup(E+a)| < |E|+\epsilon$ where $2\epsilon_0 = \epsilon$.

Now, for any N , let $M = max\{dim(F_{E,\Sigma}) : |E||\Sigma| < N\}$.

Suppose E, Σ are such that $dim(F_{E,\Sigma}) = M$. Let us choose a basis for this space, say $\phi_1, \phi_2, \cdots\cdots, \phi_M$. Choose an $\epsilon > 0$ such that $(|E|+\epsilon)|\Sigma| < N$. From above lemma 2.2 , we can get a set $E' = E\cup(E+a)$. Then $|E| < |E'| < |E|+\epsilon$ and the function $\phi_{M+1}(x) = \phi_1(x-a)$ lies in $F_{E',\Sigma}$ and the elements $\phi_1, \phi_2, \cdots, \phi_{M+1}$ are linearly independent. Hence $dim(F_{E',\Sigma}) = M+1$, which contradicts the definition of M,unless $M = 0$.

2.1.3 Uncertainty Principle for Finite Fourier Series

In this section we are going to look at Fourier analysis for functions on finite sets, more specifically, on finite abelian groups. Here infinite sums are replaced by finite sums so the issue of convergence disappears.

We are going to start with group $\mathbb{Z}(N)$ which is group of N^{th} roots of unity and proceed by showing that same group can be identified as $\mathbb{Z}/N\mathbb{Z}$ which is equivalence classes of integers modulo N. As $N\to\infty$, group $\mathbb{Z}(N)$ approximates circle. Also we are going to see Vandermonde matrix and proof Chebotarev's theorem and finally we will prove Uncertainty Principle for $\mathbb{Z}/p\mathbb{Z}$.

The Group $\mathbb{Z}(N)$

A complex number z is N^{th} root of unity if $z^N = 1$ where N is a positive integer. N^{th} roots of unity are precisely the set, $\mathbb{Z}(N)$ where

$$\mathbb{Z}(N) = \{1, e^{2\pi i/N}, \cdots, e^{2\pi i(N-1)/N}\}.$$

Then

$$z^N = 1 \Rightarrow re^{iN\theta} = 1 \Rightarrow |z| = |re^{iN\theta}| = |r| = 1 \Rightarrow r = 1.$$

Therefore $e^{iN\theta} = 1 \Rightarrow N\theta = 2\pi k$ where $k\in\mathbb{Z}$. Now, let $\zeta = e^{2\pi i/N}$ then ζ^k achieves all the N^{th} roots of unity and $\zeta^N = 1$. Then

$$\zeta^n = \zeta^m \ \ if\ and\ only\ if\ \ (n-m)\ is\ divisible\ by\ N.$$

It is easy to see that $\mathbb{Z}(N)$ satisfies following properties:

1. If $z, w \in \mathbb{Z}(N)$, then $zw \in \mathbb{Z}(N)$ and $zw = wz$.

2. $1 \in \mathbb{Z}(N)$.

3. If $z \in \mathbb{Z}(N)$,then $z^{-1} = 1/z \in \mathbb{Z}(N)$ and $zz^{-1} = 1$.

So, it can be seen that $\mathbb{Z}(N)$ is an abelian group under multiplication.
Now, let us visualise the group $\mathbb{Z}(N)$ in terms of integer power of ζ. Since $\zeta^n = \zeta^m$ whenever n and m differ by integer multiple of N. Therefore we can choose n such that $0 \leq n \leq N-1$. Since $\zeta^n\zeta^m = \zeta^{n+m}$ and $n+m$ will not necessarily lie in the interval [0,N], so we can choose $\zeta^n\zeta^m = \zeta^k$ such that $(n+m)-k$ is an integer multiple of N. Therefore, this group can be seen as integers modulo N denoted by $\mathbb{Z}/N\mathbb{Z}$. The association

$$R(k) \longleftrightarrow e^{2\pi ik/N}$$

where $R(k)$ denotes the equivalence class or residue class of integer k modulo N.
On $\mathbb{Z}(N)$ consider the N functions $\{e_0, e_1, \cdots, e_{N-1}\}$ defined by

$$e_l(k) = \zeta^{lk} = e^{2\pi ilk/N} \;\; for \;\; l, k = 0, 1, \cdots, N-1 \;\; where \;\; \zeta = e^{2\pi i/N}.$$

Consider complex-valued functions on $\mathbb{Z}(N)$ as a vector space V, with the Hermitian inner product

$$(F, G) = \sum_{k=0}^{N-1} F(k)\overline{G(k)}$$

and the associated norm

$$\|F\|^2 = \sum_{k=0}^{N-1} |F(k)|^2.$$

Lemma 2.4. *The family $\{e_0, \cdots, e_{N-1}\}$ is orthogonal. In fact,*

$$(e_n, e_m) = \begin{cases} N & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$$

Proof Since we have

$$(e_m, e_n) = \sum_{k=0}^{N-1} \zeta^{mk}\zeta^{-nk} = \sum_{k=0}^{N-1} \zeta^{(m-n)k}.$$

If $m = n$ then each term is equal to 1 and the resulting sum is N. If $m \neq n$, then $q = \zeta^{(m-n)} \neq 1$ and the sum will correspond to

$$1 + q + q^2 + \cdots + q^{N-1} = \frac{1 - q^N}{1 - q} = 0 \;\; (as\; q^N = 1).$$

This proves the lemma.

Since the N functions $\{e_0, e_1, \cdots, e_{N-1}\}$ are orthogonal hence they are linearly independent, and since the vector space V is of N dimension, we can conclude that $\{e_0, e_1, \cdots, e_{N-1}\}$ is an orthogonal basis for V. By the lemma each vector e_n has the norm $\sqrt{N}$, so if we define

$$e_n^* = \frac{1}{\sqrt{N}} e_l,$$

$\{e_0^*, e_1^*, \cdots, e_{N-1}^*\}$ is an orthonormal basis for V. Hence for any $F \in V$ we have

$$F = \sum_{k=0}^{N-1} (F, e_n^*) e_n^* \quad as\; well\; as \quad \|F\|^2 = \sum_{k=0}^{N-1} |(F, e_n^*)|^2.$$

We define the n^{th} Fourier coefficient of F by

$$a_n = \frac{1}{N} \sum_{k=0}^{N-1} F(k) e^{-2\pi i k n / N}.$$

Theorem 2.5. *If F is a function on* $\mathbb{Z}(N)$*, then*

$$F(k) = \sum_{n=0}^{N-1} a_n e^{2\pi i k n / N}$$

Also,

$$\sum_{n=0}^{N-1} |a_n|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |F(k)|^2.$$

Vandermonde Matrix

Let $z_0, z_1, \cdots, z_{N-1}$ be complex numbers, the associated Vandermonde matrix would be

$$V_N = \begin{pmatrix} 1 & z_0 & z_0^2 & \cdots & z_0^{N-1} \\ 1 & z_1 & z_1^2 & \cdots & z_1^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z_{N-1} & z_{N-1}^2 & \cdots & z_{N-1}^{N-1} \end{pmatrix}$$

And its determinant is given by

$$det(V_N) = \prod_{0 \leq j < k \leq (N-1)} (z_j - z_k).$$

To see this, we consider a polynomial $P(z)$ given by

$$P(z) = \begin{vmatrix} 1 & z_0 & z_0^2 & \cdots & z_0^{N-1} \\ 1 & z_1 & z_1^2 & \cdots & z_1^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z & z^2 & \cdots & z^{N-1} \end{vmatrix}$$

Note that $P(z)$ is of degree $N-1$ and has $z_0, z_1, \cdots, z_{N-2}$ as its roots. Therefore we can write

$$P(z) = C \prod_{0 \leq k \leq (N-2)} (z - z_k)$$

where C is the coefficient of z^{N-1} and can be determined using Vandermonde matrix for V_{N-1}.Applying induction on N, we get the desired result.

Now let us see the Vandermonde matrix for complex numbers $\{z_0, z_1, \cdots, z_{N-1}\}$ where $z_0 = 1, z_1 = \omega = e^{2\pi i/N}$ and $z_j = \omega^j$ for $j = 2, 3, \cdots, N-1$. We then have

$$V_N = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix}$$

is non-singular.

Since z_j are all different where $z_j = \omega^j$ for $j \in \{0, 1, 2, \cdots, N-1\}$ and $\omega = e^{2\pi i/N}$, it is easy to see that

$$det(V_N) = \prod_{0 \leq j < k \leq (N-1)} (z_j - z_k) \neq 0.$$

Let us now consider a minor $W_k(a\ k \times k\ matrix)$ of V_N with $\{n_1, n_2, \cdots, n_k\}$ as row indices and $\{m, m+1, \cdots, m+k-1\}$ are consecutive k columns. We will now try to see that whether every minor of the Vandermonde matrix is also non-singular. In general, it is not necessarily true but if $N = p$ where $p\ is\ prime$, Chebotarev's theorem shows that every minor of the Vandermonde matrix is also non-singular. But to prove the Chebotarev's theorem we will require a lemma.

Let us first consider an arbitrary $k \times k$ minor of the a Vandermonde matrix,

$$W_k = \begin{pmatrix} 1 & z_{n_1}^{m_1} & z_{n_1}^{m_2} & \cdots & z_{n_1}^{m_k} \\ 1 & z_{n_2}^{m_1} & z_{n_2}^{m_2} & \cdots & z_{n_2}^{m_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z_{n_k}^{m_1} & z_{n_k}^{m_2} & \cdots & z_{n_k}^{m_k} \end{pmatrix}$$

As earlier consider a polynomial $P(z)$ defined as :

$$P(z) = \begin{pmatrix} 1 & z_{n_1}^{m_1} & z_{n_1}^{m_2} & \cdots & z_{n_1}^{m_k} \\ 1 & z_{n_2}^{m_1} & z_{n_2}^{m_2} & \cdots & z_{n_2}^{m_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z^{m_1} & z^{m_2} & \cdots & z^{m_k} \end{pmatrix}$$

Now P(z) is a polynomial of degree $m_k \geq k$ and only k roots of the polynomial P are visible, so we can write

$$P(z_{n_k}) = det(W_k) = \prod_{j<j'}(z_{n_j} - z_{n_{j'}})Q(z_{n_k})$$

but we don't know whether $Q(z_{n_k})$ is zero or not.
To see this we first prove the lemma.

Lemma 2.6. *For p(prime) and n(integer) , let $P(z_1, z_2, \cdots, z_n)$ be polynomial with integer coefficients. Let $\xi_1, \xi_2, \cdots, \xi_n$ be p^{th} roots of unity (not necessarily distinct) such that $P(\xi_1, \xi_2, \cdots, \xi_n) = 0$ then $P(1, 1, \cdots, 1) = 0(mod\, p)$.*

Proof Let ξ denote primitive p^{th} root of unity and $\xi_j = \xi^{k_j}$ for $k_j = 1, 2, \cdots, p$. We then define a polynomial $Q(z)$ such that

$$Q(z) = P(z^{k_1}, z^{k_2}, \cdots, z^{k_n})mod(z^p - 1)$$

Then

$$Q(\xi) = P(\xi^{k_1}, \xi^{k_2}, \cdots, \xi^{k_n}) = P(\xi_1, \xi_2, \cdots, \xi_n) = 0 \ (given)$$

and $Q(1) = P(1^{k_1}, 1^{k_2}, \cdots, 1^{k_n}) = P(1, 1, \cdots, 1)$. As $Q(z)$ is of degree at most (p-1) degree with integer coefficients and thus should be an integer multiple of minimal polynomial $1 + z + z^2 + \cdots + z^{p-1}$ of ξ.

Theorem 2.7. ***Chebotarev's Theorem***
If p is a prime and ξ is a primitive p^{th} root of unity then every minor of the Vandermonde matrix $V = (\xi^{jk})_{j,k=0}^{p-1}$ is non-singular.

Proof [2]Let $1 \leq n \leq p$ and let $k_1, k_2, \cdots, k_n$ and $l_1, l_2, \cdots, l_n$ denote the row and column indices of the minor matrix $W = (\xi^{k_i l_j})_{i,j}$. Also let $\omega_i = \xi^{k_i}$ then each ω_i is

[2]This proof of Cheboratev's theorem was given by Terence Tao

different.

Now consider the polynomial

$$D(z_1, z_2, \cdots, z_n) = det((z_i^{l_j})_{i,j})$$

It is easy to see that $D(1, 1, \cdots, 1) = 0$ but we need to show that $D(\omega_1, \omega_2, \cdots, \omega_n) \neq 0$.
It can be seen that $D = 0$ whenever $z_i = z_{i'}$ for some $1 \leq i < i' \leq n$, so we can write

$$D(z_1, z_2, \cdots, z_n) = \prod_{i<i'}(z_i - z_{i'})P(z_1, z_2, \cdots, z_n)$$

where P is a polynomial with integer coefficients.
We need to show that $P(1, 1, \cdots, 1) \neq 0(mod\, p)$.
To see this we apply the differential operator

$$(z_1 \frac{d}{dz_1})^0 (z_2 \frac{d}{dz_2})^1 \cdots (z_n \frac{d}{dz_n})^{n-1}$$

on $D(z_1, z_2, \cdots, z_n) = \prod_{i<i'}(z_i - z_{i'})P(z_1, z_2, \cdots, z_n)$ and compute the result at $z_1 = z_2 = \cdots = z_n = 1$.
Total number of differential operators applied here are $1 + 2 + \cdots + n - 1 = \frac{n(n-1)}{2}$ which is equal to the total number of linear factors $(z_i - z_{i'})$ in the expression of $D(z_1, z_2, \cdots, z_n)$. By Leibniz rule, each operator $(z_i \frac{d}{dz_i})$ either differentiates one of these linear factors(and reduces them to z_i or differentiate the polynomial $P(z_1, z_2, \cdots, z_n)$. But, in Leibniz expansion only terms which survive when $z_1 = z_2 = \cdots = z_n = 1$ are the linear terms which get differentiated by the operator. Therefore we never have to actually differentiate $P(z_1, z_2, \cdots, z_n)$. Since $(n-1)$ copies of the differential operator $(z_n \frac{d}{dz_n})$ can eliminate only $(n-1)$ copies of the linear factor $(z_i - z_n)$ and it can be done in $(n-1)!$ ways. Similarly, $(n-2)$ copies of the differential operator $(z_{n-1} \frac{d}{dz_{n-1}})$ can eliminate only $(n-2)$ copies of the linear factor $(z_i - z_{n-1})$ and it can be done in $(n-2)!$ ways and so on. Therefore, we get

$$(z_1 \frac{d}{dz_1})^0 (z_2 \frac{d}{dz_2})^1 \cdots (z_n \frac{d}{dz_n})^{n-1} D(z_1, z_2, \cdots, z_n)|_{z_1=z_2=\cdots=z_n=1}$$

$$= (n-1)!(n-2)! \cdots 1! P(1, 1, \cdots, 1)$$

Since,$(n-1)!(n-2)! \cdots 1!$ is not a multiple of p it remains to show that $P(1, 1, \cdots, 1)$ is not a multiple of p.
Another way of differentiating $D(z_1, z_2, \cdots, z_n) = det((z_i^{l_j})_{i,j})$ with respect to the

operator $(z_1\frac{d}{dz_1})^0(z_2\frac{d}{dz_2})^1\cdots(z_n\frac{d}{dz_n})^{n-1}$ is by using multi-linearity of the determinant and the fact that $(z_i\frac{d}{dz_i})z_i^l = lz_i^l$ we get that

$$(z_1\frac{d}{dz_1})^0(z_2\frac{d}{dz_2})^1\cdots(z_n\frac{d}{dz_n})^{n-1}D(z_1,z_2,\cdots,z_n)|_{z_1=z_2=\cdots=z_n=1} = det(l_k^{i-1})$$

Since $det(l_k^{i-1})$ is Vandermonde determinant and each l_k is different modulo p for $k = 1, 2, \cdots, n$, we get that

$$det(l_k^{i-1}) = \prod_{k<k'}(l_k - l_{k'}) \neq 0$$

So we get that $P(1,1,\cdots,1) \neq 0$ so from the lemma $P(\omega_1,\omega_2,\cdots,\omega_n) \neq 0$. Hence $D(\omega_1,\omega_2,\cdots,\omega_n) \neq 0$.

From above theorem we get that

Corollary 2.8. *Let p be a prime and A, B be non-empty subsets of $\mathbb{Z}/p\mathbb{Z}$ such that $|A| = |B|$, then the linear transformation $l^2(A) \longrightarrow l^2(B)$ given as $Tf = \hat{f}|_B$ is invertible(here $l^2(A)$ denotes the set of functions which are zero outside A).*

Uncertainty Principle for $\mathbb{Z}/p\mathbb{Z}$

Theorem 2.9. ***Uncertainty Principle for $\mathbb{Z}/p\mathbb{Z}$***

Let p be prime number and let $f : \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{C}$ be a non-zero function then

$$|supp(f)| + |supp(\hat{f})| \geq p + 1.$$

Conversely, if there are non-zero subsets A and B of $\mathbb{Z}/p\mathbb{Z}$ such that $|A| + |B| \geq p + 1$ then there exists a non-zero function such that $supp(f) = A$ and $supp(\hat{f}) = B$.

Proof On contrary let us assume that there exists a non-zero function f such that

$$|supp(f)| + |supp(\hat{f})| \leq p.$$

Let us write $supp(f) = A$, then we can find a subset B of $\mathbb{Z}/p\mathbb{Z}$ such that $|A| = |B|$ and $B \cap supp(\hat{f}) = 0$. But from above corollary $T : l^2(A) \longrightarrow l^2(B)$ should be invertible but we get $Tf = 0$ for $f \neq 0$.

To prove the converse we first prove it for $|A|+|B| = p+1$. Let us choose a subset F of $\mathbb{Z}/p\mathbb{Z}$ such that $|A| = |F|$ and $F \cap B = (\xi)$. By corollary the map $T : l^2(A) \longrightarrow l^2(F)$ is invertible and therefore we can find a non-zero function $f \in l^2(A)$ such that $\hat{f}$ is zero

on the set $F \setminus \xi$ and non-zero on ξ. Now, such a function f has to be non-zero on all of A and all of B otherwise first statement of uncertainty principle will get violated. Therefore $supp(f) = A$ and $supp(\hat{f}) = B$.
Now, for $|A| + |B| > p + 1$ we can consider the subsets $A' \subset A$ and $B' \subset B$ such that $|A'| + |B'| = p + 1$ and the claim follows by taking generic linear combinations of the two.

2.2 Concentration: Deviation from a point

2.2.1 Heisenberg's Uncertainty Principle

This section is denoted to classical Heisenberg's uncertainty Principle. To prove the classical result we will require a lemma.

Lemma 2.10. *Let $r, s, t \in \mathbb{R}_+$ and $j \in \{1, 2, \cdots, n\}$.If $f \in L^r(\mathbb{R}^n)$ with partial derivative $\partial_j f \in L^s(\mathbb{R}^n)$ and $x_j f \in L^t(\mathbb{R}^n)$ then there exists a sequence of functions $g_n \in C_c^\infty(\mathbb{R}^n)$ such that*

$$\|g_n - f\|_r + \|\partial_j g_n - \partial_j f\|_s + \|x_j g_n - x_j f\|_t \longrightarrow 0 \quad for\ n \longrightarrow \infty.$$

Proof We divide the proof in three steps. The idea is to approximate f with a sequence f_p of functions in $L^r(\mathbb{R}^n)$ with compact support :

$$f_p(x) = k_p(x) f(x) = k(x/p) f(x)$$

where $k : \mathbb{R}^n \longrightarrow [0, 1]$ is in $C_c^\infty(\mathbb{R}^n)$ and defined by

$$k(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 \leq k(x) \leq 1 & \text{for } 1 < |x| < 2 \\ 0 & \text{for } |x| \geq 2. \end{cases}$$

Then for each p approximate f_p with a sequence $g_{p,q} \in C_c^\infty(\mathbb{R}^n)$

$$g_{p,q}(x) = h_q * f_p(x) \ \ where\ h_q(x) = q^{-1} h(qx)$$

with $h \in C_c^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}} h(x) dx = 1$. Proving the convergence for each approximation yields the desired result.

Step 1: Since $|f_p(x)| \leq |f(x)| \; \forall \; x \in \mathbb{R}^n$ and $f_p(x) \longrightarrow f(x)$ pointwise $\Rightarrow f_p \longrightarrow f$ in $L^r(\mathbb{R}^n)$.Similarly $x_j f_p \longrightarrow x_j f$ in $L^t(\mathbb{R}^n)$. Since $\partial_j f \in L^s(\mathbb{R}^n)$ it can be seen that $k(x/p)(\partial_j f) \longleftrightarrow (\partial_j f)$. Also, $\partial_j k_p(x) = \frac{1}{p}\partial_j k$ so $f\partial_j k_p \to 0$ in $L^s(\mathbb{R}^n)$. Using Leibniz's rule and triangle inequality, we get

$$\|\partial_j f_p - \partial_j f\|_s \leq \|(\partial_j f)k_p - \partial_j f\|_s + \|f\partial_j k_p\|_s \longrightarrow 0 \;\; for \; p \to \infty, .$$

Step 2: Now above lemma has been proved for f_p, but the sequence f_p may not be in $C_c^\infty(\mathbb{R}^n)$.The convolution $h_q * f_p$ is in $C^\infty(\mathbb{R}^n)$. Since f_p 's has compact support thus its convolution with f_p will also have compact support. Now, we have sequence $g_{p,q} = h_q * f_p$ that approximates f_p for $q \to \infty$. Since $\partial_j f_p$ is in $L^s(\mathbb{R}^n)$, so we can write

$$\partial_j g_{p,q} = h_q * \partial_j f_p \to \partial_j f_p \;\;\; in \; L^s \; for \; q \longrightarrow \infty.$$

The convolution $h_q * f_p$ has compact support independent of q because

$$supp(h_q * f_p) \subseteq supp(h_q) + supp(f_p) \subseteq supp(h) + supp(f_p)$$

The sets $supp(h)$ and $supp(f_p)$ are compact and therefore the sum is compact. On this set multiplication with x_j is a bounded operator on $L^t(\mathbb{R}^n)$ and is continuous. This gives the last required convergence $x_j g_{p,q} \to x_j f_p$ in $L^t(\mathbb{R}^n)$ as $q \to \infty$.

Step 3: For each k $\in \mathbb{N}$, I can choose p and q such that

$$\|g_{p,q} - f_p\|_r + \|\partial_j g_{p,q} - \partial_j f_p\|_s + \|x_j g_{p,q} - x_j f_p\|_t \leq \frac{1}{2k}$$

$$\|f_p - f\|_r + \|\partial_j f_p - \partial_j f\|_s + \|x_j f_p - x_j f\|_t \leq \frac{1}{2k}$$

setting $g_k = g_{p,q}$ and using Schwarz inequality proves that the sequence $g_k \in C_c^\infty(\mathbb{R}^n)$ satisfies above lemma.

Theorem 2.11. ***Heisenberg's Uncertainty Principle***

Let $f \in L^2(\mathbb{R}_n)$, then for all $j \in \{1, 2, \cdots, n\}$

$$\int_{\mathbb{R}_n} x_j^2 |f(x)|^2 dx \int_{\mathbb{R}_n} y_j^2 |\hat{f}(y)|^2 dy \geq \frac{1}{4}(\int_{\mathbb{R}_n} |f(x)|^2)^2$$

Proof The inequality is obvious if $f(x) = 0$ almost everywhere. So, we will assume that $f(x)$ is non-zero in $L^2(\mathbb{R}_n)$ then neither $x_j f(x)$ nor $y_j \hat{f}(y)$ is zero. Now, if either of them has infinite L^2-norm the inequality is obvious. So, let us now assume that $x_j f(x)$

and $y_j\hat{f}(y)$ are in $L^2(\mathbb{R}^n)$.

We will first prove the inequality for $f \in C_c^\infty(\mathbb{R}^n)$ and then use the above lemma.

For $f \in C_c^\infty(\mathbb{R}^n)$ we have,

$$\begin{aligned}
\int_{\mathbb{R}^n} x_j^2|f(x)|^2dx \int_{\mathbb{R}^n} y_j^2|\hat{f}(y)|^2dy &= \int_{\mathbb{R}^n} x_j^2|f(x)|^2dx \int_{\mathbb{R}^n} |(\partial_j f)(x)|^2dx \\
&\geq (\int_{\mathbb{R}^n} |x_j(\partial_j f)(x)\overline{f(x)}|dx)^2 \\
&\geq (\int_{\mathbb{R}^n} x_j Re((\partial_j f)(x)\overline{f(x)}\ )dx)^2 \ (as\ |z| \geq Re(z)\ ) \\
&= \frac{1}{4}(\int_{\mathbb{R}^n} x_j((\partial_j f)(x)\overline{f(x)} + \overline{(\partial_j f)(x)}f(x))dx)^2 \\
&= \frac{1}{4}(\int_{\mathbb{R}^n} x_j(\partial_j|f|^2)(x)dx)^2 \\
&= \frac{1}{4}(\int_{\mathbb{R}^n} |f(x)|^2)^2 \ (Integration\ by\ parts)
\end{aligned}$$

here first inequality ids due to Cauchy-Schwarz inequality.

Using above lemma with $r = s = t = 2$, proves the Heisenberg's inequality in general, i.e.

$$\int_{\mathbb{R}_n} x_j^2|f(x)|^2dx \int_{\mathbb{R}_n} y_j^2|\hat{f}(y)|^2dy \geq \frac{1}{4}(\int_{\mathbb{R}_n} |f(x)|^2)^2$$

Remark (The case of equality) If $n = 1$ and if $f(x), xf(x)$ and $y\hat{f}(y)$ are in $L^2(\mathbb{R})$ then the equality holds for Gaussian functions.

First let us observe that if $xf(x) \in L^2(\mathbb{R})$ then $\sqrt{|x|}f(x) \in L^2(\mathbb{R})$. To see this let us define a function $g \in L^2(\mathbb{R})$ such that

$$g(x) = \begin{cases} |f(x)| & \text{for } |x| \leq 1 \\ x|f(x)| & \text{for } |x| > 1 \end{cases}$$

and $|x||f(x)|^2 \leq |g(x)|^2\ \forall\ x \in \mathbb{R} \Rightarrow \sqrt{|x|}f(x) \in L^2(\mathbb{R})$. Then $(1+\sqrt{x})f(x) \in L^2(\mathbb{R})$ and since $(1+\sqrt{x})^{-1} \in L^2(\mathbb{R})$, using Hőlder's inequality we get that $f(x) \in L^1(\mathbb{R})$. Similarly, it can be shown that $\hat{f}(x) \in L^1(\mathbb{R})(as\ \|f\|_2 = \|\hat{f}\|_2\ by\ Plancherel\ theorem)$ and now using Inverse transform formula it can be shown that f is equivalent to a continuous function.

Now, assume the equality holds in Heisenberg's inequality, then the equality holds for Cauchy-Schwarz inequality i.e. for the expression

$$\int_{\mathbb{R}^n} x_j^2|f(x)|^2dx \int_{\mathbb{R}^n} |(\partial_j f)(x)|^2dx \geq (\int_{\mathbb{R}^n} |x_j(\partial_j f)(x)\overline{f(x)}|dx)^2$$

which holds for any $f \in L^2(\mathbb{R})$. But it holds only if $kx\overline{f(x)} = \partial f(x)$ for some complex k.Then $\partial f(x)$ is also continuous. Now it can be seen that $\partial f(x)$ is actually $f'(x)$ from

$$\begin{aligned}\int_0^x \partial f(t)dt &= lim_{n\to\infty} \int_0^x g_n'(t)dt \\ &= lim_{n\to\infty}[g_n(x)]_0^x \\ &= lim_{n\to\infty}(g_n(x) - g_n(0)) \\ &= f(x) - f(0)\ (as\ f\ is\ continuous).\end{aligned}$$

Here the sequence g_n is chosen as in above lemma. Now, we have ordinary differential equation of the form

$$f'(x) = kx\overline{f(x)}$$

which when solved by separation of variables, shows that f is a Gaussian function.

2.3 Concentration: Rate of decay

2.3.1 Hardy's Theorem

To prove the Hardy's theorem we will first prove the Phragmén-Lindélőf theorem for a cone.

Theorem 2.12. ***(Phragmén-Lindélőf)*** *Given $a \in (1/2, \infty)$ and $2a\phi < \pi$ define*

$$D = \{z \in \mathbb{C} | -\phi \leq arg(z) \leq \phi\}$$

Let f be a function which is holomorphic in the interior D^o of D and continuous on the boundary ∂D of D and there exists a constants b and **C** *such that*

$$|f(z)| \leq Ce^{b|z|^a}\ for\ z \in D.$$

If there exists a constant M such that $|f(z)| \leq M$ for $z \in \partial D$ then $|f(z)| \leq M$ for all $z \in D$.

Proof As $2a\phi < \pi$, we can choose $s > a$ such that $2s\phi < \pi$. Now, for $A > 0$, let us define

$$h(z) = \frac{f(z)}{exp(Az^s)}, z \in D$$

The function $z \mapsto Az^s$ is holomorphic in D^o and continuous on ∂D thus same holds for $h(z)$.

Now for $z = re^{i\phi}$, we have

$$|h(z)| = \frac{|f(z)|}{|exp(Az^s)|} = \frac{|f(z)|}{|exp(A(r^s e^{i\phi s}))|} \leq \frac{M}{exp(Ar^s cos(s\phi))} \leq M$$

as $s\phi < \frac{\pi}{2}$ so, $cos(s\phi) \geq 0$ and $exp(Ar^s cos(s\phi)) \geq 1$.

Similarly it is also the case with $z = re^{-i\phi}$. So, $|h(z)| \leq M$ on ∂D.

Let us now define

$$D_r = \{z \in D||z| \leq r\} \; for \;\; r \geq 0.$$

We have shown that it is true for $z = re^{i\phi}$ and $z = re^{-i\phi}$ for any $r \geq 0$. We will now show that there will exist $r \geq R_0$ for which $|h(z)| \leq M$ for $z = re^{i\theta}$ where $-\phi < \theta < \phi$ and $r \geq R_0$.

To see this we define $m = inf_{-\phi<\theta<\phi} cos(s\theta)$ then as $s\phi < \frac{\pi}{2}$ and $-\phi < \theta < \phi$ therefore $m > 0$.

Now,

$$|exp(Ar^s e^{is\theta})| = exp(Ar^s cos(s\theta)) = exp(Ar^s m)$$

For $|z| = r$ and $s > a$, we have

$$|h(z)| = \frac{|f(z)|}{|exp(Az^s)|} \leq \frac{Cexp(br^a)}{exp(Ar^s m)} = Cexp(br^a - Ar^s m) \to 0 \; as \; r \to \infty$$

So there exists $r \geq R_0$ for which we have $|h(z)| \leq M$ for $z \in \partial D_r$, by Maximum Modulus theorem we have $|h(z)| \leq M$ for $z \in D_r$ when $r \geq R_0$.

Therefore again by Maximum Modulus theorem we have $|h(z)| \leq M$ for $z \in D$ i.e.

$$|f(z)| \leq Mexp(Az^s) \; for \; z \in D \; and \; A > 0$$

As $A \to 0$ we get $|f(z)| \leq M$ for $z \in D$.

We will also prove a lemma

Lemma 2.13. *For a given function f, let us assume there exists $a \geq 0$ and $C \geq 0$ such that $|f(z)| \leq Ce^{-ax^2}$, then $\hat{f}(z)$ defined as*

$$\hat{f}(z) = \int_{\mathbb{R}} f(x) e^{2\pi ixz} dx \; for \; all \; z \in \mathbb{C}$$

is well-defined and entire.

Proof   The integral given above is well defined as for $z \in \mathbb{C}$

$$\int_{\mathbb{R}} |f(x)||e^{ixz}|dx = \int_{\mathbb{R}} |f(x)|e^{Im(z)x}dx \leq Ce^{Im(z)x-ax^2} < \infty$$

Now to show continuity of $\hat{f}$ let us consider a sequence $\{(z_n)_{n\in\mathbb{N}}\} \in \mathbb{C}$ which converges to $z \in \mathbb{C}$ i.e. $z_n \to z$.

Therefore

$$\begin{aligned}
|\hat{f}(z_n) - \hat{f}(z)| &= \frac{1}{\sqrt{2\pi}}|\int_{\mathbb{R}} f(x)e^{2\pi ixz_n} - f(x)e^{2\pi ixz}dx| \\
&= \frac{1}{\sqrt{2\pi}}|\int_{\mathbb{R}} f(x)(e^{2\pi ixz_n} - e^{2\pi ixz})dx| \\
&\leq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} |f(x)||(e^{2\pi ixz_n} - e^{2\pi ixz})|dx
\end{aligned}$$

Now,since $z_n \to z$ as $n \to \infty$ and the mapping $z \mapsto e^{ixz}$ is continuous for $x \in \mathbb{R}i$ so

$$|\hat{f}(z_n) - \hat{f}(z)| \leq \int_{\mathbb{R}} |f(x)||(e^{2\pi ixz_n} - e^{2\pi ixz})|dx \to 0 \; as \; n \to \infty.$$

In order to prove the that the function $\hat{f}$ is entire, we try to calculate its integral over the loop $\gamma : [0, 1] \to \mathbb{C}$.Since the function e^{-izx} is entire then its closed integral over γ will be zero by Cauchy integral theorem i.e. $\int_\gamma e^{-izx}dz = 0$.

$$\begin{aligned}
\int_\gamma \hat{f}(z) &= \int_\gamma \int_{\mathbb{R}} f(x)e^{-izx}dxdz \\
&= \int_0^1 \int_{\mathbb{R}} f(x)e^{-i\gamma(s)x}\gamma'(s)dxds \\
&= \int_{\mathbb{R}} f(x) \int_\gamma e^{-izx}dzdx \\
&= 0
\end{aligned}$$

By Morera's theorem, we get that $\hat{f}$ is entire function.

Theorem 2.14. ***Hardy's Inequality***

Let f be a function which satisfies

$$|f(x)| \leq Ce^{-ax^2} and |\hat{f}(\xi)| \leq De^{-b\xi^2}$$

where $a, b, C, D \in \mathbb{R}^+$ and $x, \xi \in \mathbb{R}$, then we have following three properties:

(i) If $ab = 0$ then f is a Gaussian function.

(ii) If $ab > 1/4$ then $f = 0$.

(iii) If $ab < 1/4$ then there are infinitely many functions satisfying the given conditions.

Proof We will first show that actual values of a and b do not matter, we are only interested in the value of the product as whole. Let us assume that the function and its Fourier transform satisfy the given conditions.Now we define a function $f_1(x) = f(kx)$ for some $k \neq 0$. Then

$$\hat{f}_1(\xi) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} f(kx) e^{-ix\xi} dx = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} k^{-1} f(x) e^{-ik^{-1}x\xi} dx = k^{-1} \hat{f}(k^{-1}\xi)$$

From the given conditions, we get for f_1

$$|f_1(x)| \leq Ce^{-ak^2x^2} = Ce^{-mx^2}$$

and

$$|\hat{f}_1(\xi)| = |k^{-1}\hat{f}(k^{-1}\xi)| = k^{-1}|\hat{f}(k^{-1}\xi)| \leq k^{-1}De^{-bk^{-2}\xi^2} = D'e^{-n\xi^2}$$

where $m = ak^2$ and $n = bk^{-2}$. Now we can see that the product

$$mn = ak^2bk^{-2} = ab.$$

Thus exact values are not important, we are mainly concerned with the product.

(i) Let us start with $ab = 1/4$. Since exact values of a, b are not important, for simplicity we can consider $a = \frac{1}{4}\pi$ and $b = \pi$.
We know that if f is a even function then so is $\hat{f}$, then for a even function f we can write its Fourier transform series as $\hat{f}(y) = \sum_{n\in\mathbb{N}} c_n z^{2n}, where\ y \in \mathbb{C}$.
Let us define a function $h(y) = \hat{f}(\sqrt{y}) = \sum_{n\in\mathbb{N}} c_n z^n, where\ y \in \mathbb{C}$, then for $\hat{f}$ we have

$$\begin{aligned}
|\hat{f}(y)| &= \frac{1}{\sqrt{2}} |\int_{\mathbb{R}} f(x) e^{-ixy} dx| \\
&\leq \frac{1}{\sqrt{2}} \int_{\mathbb{R}} |f(x)||e^{-ixy}| dx \\
&= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} |f(x)| e^{Im(y)x} dx \\
&\leq \frac{C}{\sqrt{2}} \int_{\mathbb{R}} e^{-x^2/4\pi} e^{Im(y)x} dx \\
&= \frac{C}{\sqrt{2}} \int_{\mathbb{R}} g(x) e^{-i(iIm(y))x} dx \ (\ where\ g(x) = e^{-x^2/4\pi}) \\
&= C\hat{g}(iIm(y)) \\
&= C'e^{-Im^2(y)}
\end{aligned}$$

Since Fourier transform of a Gaussian is Gaussian function.

For $y = Re^{it}$ we get

$$|h(y)| = |\hat{f}(\sqrt{y})| \leq C'e^{-Im^2(\sqrt{y})}$$

and since $\sqrt{Re^{it}} = \sqrt{Rcos(t) + iRsin(t)} = a + ib$

$\Rightarrow Rcos(t) = a^2 - b^2 and Rsin(t) = 2ab$ we get $b = \pm\sqrt{R}sin(t/2)$.

So we get

$$|h(y)| \leq C'e^{-\pi Rsin^2(t/2)}.$$

If $y \in \mathbb{R}^+$ then we have $y = R$ and then

$$|h(y)| = |\hat{f}(\sqrt{y})| \leq De^{-\pi R}(from\ given\ conditions\ of\ decay).$$

Let $M = max(C', D)$ be such that it satisfies both the above inequalities. Now, we define a plane $D_\delta = \{Re^{it}|0 \leq t \leq \delta, R \geq 0\}\ where\ 0 \leq \delta \leq \pi$ and a function

$$g_\delta(y) = exp(\frac{i\pi Rye^{-i\delta/2}}{sin(\delta/2)}).$$

Then for $y = Re^{it}$, we get

$$|g_\delta(Re^{it})| = exp(\frac{-\pi Rsin(t - \delta/2)}{sin(\delta/2)})$$

For $t = 0$ we get $|g_\delta(R)| = e^{\pi R}$ and for $t = \pi$ we get $|g_\delta(Re^{i\pi})| = e^{-\pi R}$,then from above we have

$$|g_\delta(R)h(R)| \leq M\ and\ |g_\delta(Re^{it})h(Re^{it})| \leq M$$

The function $g_\delta h$ is limited on the boundary of D_δ and as function is analytic, by Phragmén-Lindelőf, it is bounded on whole of D_δ. Also

$$\frac{sin(t - \delta/2)}{sin\delta/2} \to -cost\ as\ \delta \to \pi$$

and $g_\delta h \leq M$ gives that

$$|h(y)| \leq Me^{-\pi rcost}\ for\ 0 \leq t \leq \pi.$$

For $-\pi \leq t \leq 0$ we can get similar result. For all $z = re^{it} \in \mathbb{C}$ we get

$$|e^{\pi z}h(z)| = |e^{\pi R(cost+isint)}h(z)| = |e^{\pi Rcosth(z)}| \leq M.$$

By Liouville's theorem, we get that $e^{\pi z}h(z)$ is constant for all $z \in \mathbb{C}$.Thus, we get that $\hat{f}(y) = Ke^{-\pi y^2}$ and by Fourier inversion formula we get that $f(x) = K'e^{-x^2/4\pi}$.

Also, if f odd then so is its $\hat{f}$, and it can be written as power series $\hat{f}(y) = \sum_{n\in\mathbb{N}} c_n y^{2n+1}$ then $y^{-1}\hat{f}$ will be analytic and even. Then treating it as even function we get $\hat{f}(y) = yKe^{-\pi y^2}$ and which can be bounded for all $y \in \mathbb{C}$ only if $K = 0$.

Now any function can be split into $f = f_{even} + f_{odd}$ such that

$$f_{even}(x) = \frac{f(x) + f(-x)}{2} \quad and \quad f_{odd}(x) = \frac{f(-x) - f(-x)}{2}$$

Since f satisfies the given conditions, so does f_{even} and f_{odd}. Therefore, for $ab = 1/4$ we get $f, \hat{f}$ are Gaussian functions.

(ii) For $ab > 1/4$, let us assume that $a > 1/4\pi$ and $b > \pi$. Then from the given conditions we will have

$$|f(x)| \leq Ce^{-ax^2} \leq Ce^{-x^2/4\pi} \; as \; a > 1/4\pi$$

$$|\hat{f}(\xi)| \leq De^{-b\xi^2} \leq De^{-\xi^2/4\pi} \; as \; b > \pi$$

then from (i) we get that $f = C'e^{-x^2/4\pi}$ for some C', but that would mean that $C'e^{-x^2/4\pi} \leq Ce^{-ax^2}$ for all $x \in \mathbb{R}$ which is only possible if $C' = 0$ as $a > 1/4\pi$.

(iii) For $ab < 1/4$ we can assume that $a = b < 1/2$. Since Hermite functions are polynomials multiplied with $e^{-x^2/2}$ and any polynomial function is bounded by e^{kx^2} for some $k > 0$, so there exists $K >$) such that

$$|\hat{H}_n(x)| = |H_n(x)| \leq Ke^{-(k+1/2)x^2}$$

by choosing k such that $k + 1/2 > a$ we get that $e^{-(k+1/2)x^2} \leq e^{-ax^2}$.

Therefore we get infinite family of functions satisfying given conditions of decay.

2.3.2 Beurling's Theorem

Let us suppose f and $\hat{f}$ satisfy

$$|f(x)| \leq Ce^{-ax^2} \; \forall \, x \in \mathbb{R}$$

and

$$|\hat{f}(\xi)| \leq De^{-b\xi^2} \; \forall \, \xi \in \mathbb{R}$$

for some $a, b, C, D \in \mathbb{R}$ then

$$\int_{\mathbb{R}} f(x)e^{-a'x^2}dx \leq C \int_{\mathbb{R}} e^{-(a+a')x^2}dx < \infty.$$

Similarly for $\int_{\mathbb{R}} \hat{f}(\xi)e^{-b'\xi^2}d\xi$.

Now, let us compute

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)\hat{f}(\xi)e^{-a'x^2}e^{-b'\xi^2}dxd\xi$$

i.e.

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)\hat{f}(\xi)e^{-(a'x^2+b'\xi^2)}dxd\xi$$

As we know that $x^2 + \xi^2 > x\xi \Rightarrow e^{-(x^2+\xi^2)} < e^{-x\xi}$

Therefore computing the integral with $e^{-|x||\xi|}$,will give us a stronger result and the result will also be true with $e^{-(x^2+\xi^2)}$.

However, a much stronger result is Beurling-Hőrmander Theorem:

Theorem 2.15. ***Beurling-Hőrmander Theorem***

If $f \in L^1(\mathbb{R})$ satisfies

$$\int\int_{\mathbb{R}\times\mathbb{R}} |f(x)||\hat{f}(\xi)|e^{2\pi|x||\xi|}dxd\xi < \infty$$

then $f = 0$.

Due to time constraint, the proof of the theorem could not be completed fully. However, one can consult appendix section of the paper by Aline Bonami, Bruno Demange, Philippe Jaming, on Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms, to see the proof of the theorem.

Bibliography

[1] Shobha Madan, *Poisson Summation Formula(Notes)*.

[2] Stein and Shakarchi, *Fourier Analysis: An Introduction*.

[3] Kehe Zhu, *Operator Theory In Function Spaces*. Marcel Dekker Inc. , 1990

[4] Belá Bollobás, *Linear Analysis: An Introductory Course(Second Edition).*Cambridge University Press,1999

[5] Shobha Madan, *Vandermonde(Notes)*

[6] Stein and Weiss, *Introduction To Fourier Analysis On Euclidean Spaces.* Princeton University Press,1971

[7] Loukas Grafakos, *Classical Fourier Analysis(Third Edition).* Springer,2014

[8] Jens Gerlach Christensen, *Uncertainty Principles.*

[9] Javier Duoandikoetxea, *Fourier Analysis.*2003

[10] Terence Tao, *An Uncertainty Principle for Cyclic Groups Of Prime Order.*Mathematical Research Letters 12, 121127

[11] Aline Bonami, Bruno Demange, Philippe Jaming ; *Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms.*Revista Matemática Iberoamericana, European Mathematical Society, 2003,19, pp.23-55.