Finite Coxeter Groups

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Certificate of Examination

This is to certify that the dissertation titled **"Finite Coxeter Groups"** submitted by **Pragnya Das** (Reg. No. MS13014) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 20, 2018

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Pranab Sardar at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Pragnya Das
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Dated: April 20, 2018

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Pranab Sardar
(Supervisor)

Acknowledgment

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Abstract

Finite Coxeter groups, named after H.S.M Coxeter, is an abstract group generated by finite set of reflections with some defining properties. In this dissertation we define Finite Coxeter groups and give some of its properties, which is discussed in chapter 2 and 3. In chapter 4 we give a presentation of Coxeter groups. In chapter 5 and 6 we define Coxeter graphs and classification and construction of finite Coxeter Groups is discussed based on that.

Contents

Chapter 1

Finite Reflection Groups

1.1 Introduction

In this chapter the prerequisites needed throughout this report is mentioned. Moreover,we discuss the finite subgroups of orthogonal transformations in dimension 2 and 3 as motivation for the latter chapters.

1.2 Definitions

Definition 1.1. *For a real vector space, say V, of dimension n over field* $\mathbb{R}$ *an inner product*$<-,->:V \times \ V \to \mathbb{R}$ *satisfies the following four properties.For any, u,v and w* $\in$ *V and c* $\in \mathbb{R}$*,*

1. $< u+v, w >=< u, w > + < v, w >$
2. $< cv, w >= c < v, w >$
3. $< v, w >=< w, v >$
4. $< v, v >\geq$ *0 and equality holds if and only if v=0*

Example:V=$\mathbb{R}^n$,v=$\{x_1, x_2, ..., x_n\}$,w=$\{y_1, y_2, ..., y_n\}$,then the standard inner product is

$$(v, w) = \sum_{i=1}^{n} x_i y_i$$

Remark. *Given an n-dimensional vector space V over* $\mathbb{R}$ *with the inner product* $<-,->$ *there is a linear isomorphic transformation T:V*$\rightarrow \mathbb{R}^n$ *such that* $<v,w>=<Tv,Tw>$. *Hence for the most part of the thesis we shall work with* $\mathbb{R}^n$ *with the standard inner product.*

Definition 1.2. *For an n dimensional vector space V over* $\mathbb{R}$

$$O(V) = \{T \in Aut_{\mathbb{R}}(V) :< Tv, Tw >=< v, w >, v, w \in V\}$$

Example: O(n,$\mathbb{R}$)={ A$\in GL(n,\mathbb{R})$ | AA'=A'A=I,Where A' is transpose matrix of A and I is the identity matrix.}

Lemma 1.1. *Determinant of orthogonal matrices can be only 1 or –1.*

Definition 1.3. *For n=2,3 ,Rotation subgroup in O(2,*$\mathbb{R}$*);*

$$H := \{R \in O(2,\mathbb{R}) \mid det(R) = 1\}$$

Non-identity elements in H ,R$\in$ *H, are called rotations.*

Remark. *The rotation subgroup is an index 2 subgroup of O(2,*$\mathbb{R}$*) or O(3,*$\mathbb{R}$*) and hence it is normal in*$O(\mathbb{R}^n)$ *fro n=2,3.*

Definition 1.4. *Reflection in terms of linear transformation is represented by:*

$$S_r x = x - 2\frac{(x,r)}{(r,r)} r \forall x \in V$$

Where 0$\neq$ *r*$\in V$*,P=*$r^{\perp}$ *and P is a hyperplane of* $\mathbb{R}^n$*,for n=2,3.*

The vector $\pm\frac{r}{\|r\|}$ *above is generally called the roots of* S_r*.*

Remark. *S*$\in$ *O(n,R)*$, forn = 2, 3.Sodet(S) = -1, if Sisareflectionin R^2$ *or* $\mathbb{R}^3$*.*

Definition 1.5. *Points in a unit sphere of in* $\mathbb{R}^3$ *those are fixed by non-zero rotations are called poles.i.e.,If* $\{x\in\mathbb{R}^3 \mid \|v\|=1\}$ *be a unit sphere of* $\mathbb{R}^3$ *and 1*$\neq$*T be a rotation in* $\mathbb{R}^3$*,If Tx=x,then x is a pole and these are precisely the points where axis of rotation T intersects the unit sphere.*

1.3 Finite subgroups in O(2,$\mathbb{R}$)

Rotation R in O(2,$\mathbb{R}$) is represented by the matrix $\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$ and reflection S in O(2,$\mathbb{R}$) is represented by the matrix $\begin{pmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{pmatrix}$ wrt to standard basis in $\mathbb{R}^2$.

Theorem 1.1. *A finite subgroup of O(2,$\mathbb{R}$) is either a cyclic group denoted by H or a dihedral group denoted by H_2^n.*

Proof. We have any orthogonal transformation can be either be a rotation or a reflection which follows from lemma 1.1.

Let H := {R$\in G|\ det(R) = 1$},subset of rotational transformations.

We will prove that H is cyclic.

Let T$\in$H be rotation with minimum angle θ. And T' be any other rotation in H with angle ϕ. Then we have for some m$\in$Z:

$$m\theta \leq \phi \leq (m+1)\theta$$

$$\Rightarrow 0 \leq \phi - m\theta \leq \theta$$

$$\Rightarrow \phi - m\theta = 0$$

$$\Rightarrow \phi = m\theta$$

This proves that T'=T^m

So H is cyclic as H=$< T >$

If H=G then there is nothing to prove.But if H $\neq G$ then there exists S$\in$ Gn H

And ST^k is a reflection as det(ST^k)=-1.

Let S' be an arbitrary reflection then we have det(SS')=1

So S'$\in$ SH.

So G=H$\cup$ SH and [G:H]=2

G=$< R, S >$={$1, R, ..., R^{n-1}, S, SR, ..., SR^{n-1}$} which is isomorphic to D$_{2n}$,dihedral group of order 2n.

□

1.4 Finite subgroups in O(3,$\mathbb{R}$)

The following results can be derived for 3-dimensional cases.

1. Let R $\in$ O($\mathbb{R}^3$) be a rotation,then R is a rotation about a fixed axis, i.e., R has eigen vector v having eigen value 1 s.t. $R|_P$, (P=$v^{\perp}$) is a 2-dimesional rotation.

2. S is a reflection in O($\mathbb{R}^3$) is a transformation s.t.

$$Sx=x \ \forall x \in P$$

$$Sx=-x \ \forall x \in P^{\perp}$$

Where P is plane passing through origin.

3. Let S $\in$ O($\mathbb{R}^3$) s.t. det S =-1 then geometrical effect of S is the same as reflection through a plane P, followed by a rotation about the line through the origin orthogonal to P.

The above result gives us an interpretation of rotation and reflection in 3-dimensional case.

1.5 Extending the orthogonal transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$

Finite subgroups G$\leq$ O(3,$\mathbb{R}$) can be obtained by the following ways:

1. We first extend rotations in O(2,$\mathbb{R}$) to get rotational subgroup in O(3,$\mathbb{R}$). Let W be a the y-z plane in $\mathbb{R}^3$. Let R be rotation in O(2,$\mathbb{R}$); set Rx=x for all x$\in W^{\perp}$ then

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix} isamatrixrepresentingrotationinVwrttobasis\{e_1, e_2, e_3$$

} in V where $e_1 \in W^{\perp}$ and $\{e_2, e_3\}$ in W.

Remark. *We can extend each rotation in rotation subgroup H$\leq$O(2,$\mathbb{R}$) to get rotation in O(3,$\mathbb{R}$) that forms rotation subgroup in O(3,$\mathbb{R}$).*

2. We can also extend reflectionS in O(2,$\mathbb{R}$) to get rotational subgroup in O(3,$\mathbb{R}$). Let W be y-z plane in $\mathbb{R}^3$. Let S be a reflection in O(2,$\mathbb{R}$); set Sx=-x for all

x$\in W^{\perp}$,then

$$R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & sin\theta & -cos\theta \end{bmatrix} isamatrixrepresentingrotationinO(3,$$

R)$wrttobasis\{e_1, e_2, e_3\}$ in $\mathbb{R}$ where $x_1 \in W^{\perp}$ and $\{e_2, e_3\}$ in W.

Remark. *we can extend each rotation and reflection in dihedral subgroup H_2^m $\leq$O(2,$\mathbb{R}$) to get rotations in O(3,$\mathbb{R}$) that forms rotation subgroup denoted by H_3^m in O(3,$\mathbb{R}$) and referred to as dihedral group O(3,$\mathbb{R}$).*

1.6 Symmetry Group of Regular convex polyhedra

We can look at the finite subgroups G$\leq$ O(3,$\mathbb{R}$) as the symmetric groups of regular convex polyhedra.

Theorem 1.2. *There can be only 5 possible regular convex polyhedra;Tetrahedron, cube,octahedron,icosahedron and dodecahedron*

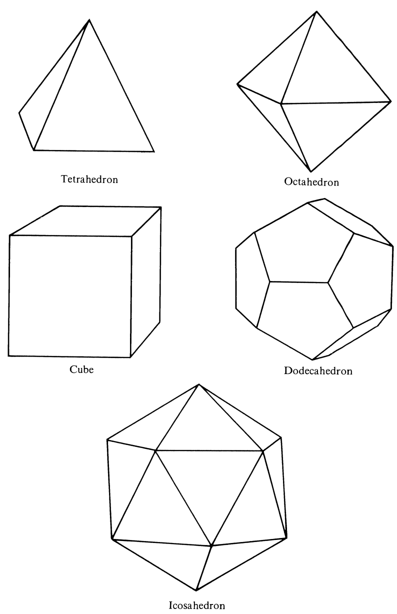


Figure 1.1. The 5 polyhedra discussed

Remark. *Since cube and octahedron are geometrically same and so are icosahedron and dodecahedron we discussing only 3 cases are enough.*

Symmetric groups of these 3 Platonic solids are tabulated below:

Rotation group for tetrahedron($\mathcal{T}$)			
Angle of rotation	Axes of rotation	no.of axes of rotation	order of rotation
$\frac{2\pi}{3}$ and $\frac{4\pi}{3}$	rotation along axes joining vertices with center of opposite faces	4	3
π	axes joining the mid point of opposite edges	3	2

Table 1.1: Table for rotation subgroup of tetrahedron

Rotation group for cube($\mathcal{W}$)			
Angle of rotation	Axes of rotation	no.of axes of rotation	order of rotation
$\frac{\pi}{2}, \pi, \frac{3\pi}{2}$	axes joining the center of opposite faces	3	4
$\frac{2\pi}{3}$ and $\frac{4\pi}{3}$	axes joining extreme opposite vertices	4	3
π	axes joining the mid point diagonally of opposite edges	6	2

Table 1.2: Table for rotation subgroup of cube

Order of $\mathcal{T}$, $\mid \mathcal{T} \mid$=12

Order of $\mathcal{W}$, $\mid \mathcal{W} \mid$=24

Order of $\mathcal{I}$, $\mid \mathcal{I} \mid$=60

1.7 Finite reflection groups in $\mathbb{R}^3$

Lemma 1.2. *If V=$\mathbb{R}^3$ and the finite subgroup $G \leq O(3,\mathbb{R})$, then G is a permutation group on its set of poles.*

Rotation group for IIcosahedron($\mathcal{I}$)			
Angle of rotation	Axes of rotation	no.of axes of rotation	order of rotation
$\frac{2\pi}{3}$ and $\frac{4\pi}{3}$	axes joinign the center of opposite faces	10	3
$\frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$	axes joining extreme opposite vertices	6	5
π	axes joining the mid point diagonally of opposite edges	15	2

Table 1.3: Table for rotation subgroup of Icosahedron

Let C_3^n denote the cyclic rotation subgroup in $O(3, \mathbb{R})$ and H_3^n denote the dihedral subgroup in O(3,R)$.thisgivesusthefollowingresult:$

Table 1.4: Order of rotation subgroups in $\mathbb{R}^3$

G	$\mid G \mid$	orbits	order of set of poles	order of stabilizers
C_3^n	n	2	2	n,n
H_3^n	2n	3	2n+2	2,2,n
$\mathcal{T}$	12	3	14	2,3,3
$\mathcal{W}$	24	3	26	2,3,4
$\mathcal{I}$	60	3	62	2,3,5

Remark. *The above table give us the complete list of finite rotational subgroups.*

Let G≤O(3,R) and -1∉G and H ≤ G be a rotation subgroup,then define :

$$G^* = H \cup -(1)H$$

And if -1∈G and H≤G be a rotation subgroup which has a subgroup K such that [H:K]=2, then define:

$$H]K := K \cup \{-T : T \in H \backslash K\}$$

Finally we have If G$\leq O(3,\mathbb{R})$ and G is finite then G is one of the following:

(a) $\mathrm{C}_3^n, n \geq 1; H_3^n, n \geq 2;\ \mathcal{T};\ \mathcal{W}\ ;\ \mathcal{I}$

(b) $(\mathrm{C}_3^n)^*, n \geq 1; (H_3^n)^*, n \geq 2;\ (\mathcal{T})^*;\ (\mathcal{W})^*;(\mathcal{I})^*$

(c) $\mathrm{C}_3^{2n}]C_3^n, n \geq 1; H_3^{2n}]H_3^n, n \geq 2; H_3^n]C_3^n, n \geq 2;\ \mathcal{W}]\mathcal{T}$

Chapter 2

Introduction To Coxeter Groups

2.1 Introduction

Coxeter group, named after H.S.M Coxeter, is an abstract group generated by finite set of reflections with some defining properties. In this chapter,Coxeter group will be defined with its root system followed by some properties of Coxeter groups. Here we take V=$\mathbb{R}^n$

2.2 Definition

Definition 2.1. *Let* $\boldsymbol{r}$ *be a non zero vector and P be the hyper plane perpendicular to r passing throough the origin i.e., P=*$r^{\perp}$*. Define*

$$S_r x := x\text{-}2\frac{(x,r)}{(r,r)}r$$

as the reflection on the hyperplane P, then generally $\pm\frac{r}{\|r\|}$ *are called roots along reflection* S_r*.*

Remarks

- $S_{\lambda r}$=S_r.$\forall\lambda \in \mathbb{R}\backslash\{0\} {S_r}^2$=1
- $S_r \in$ O($\mathbb{R}$)

Definition 2.2. *Let $G \leq O(V)$*

$V_T := \{x \in V | \ Tx{=}x\}$ *where* $T\in G$ $V_o{:=}\cap\{V_T \ | T \in G\}$

If $V_o(G) = 0$ *then G is called an effective group.*

Remarks

- V_0 is a vector subspace of V
- $T|_{V o} = 1$(identity) on V_o for all T $\in$ G.

Proposition 2.1. *$T(V_o) = V_o$ and $T({V_o}^{\perp}) {=} V_o^{\perp}$ $\forall$ $T \in G hence if V = V_o \bigoplus {V_o}^{\perp}$ then $T{=}1{\oplus}T'$ for any $T \in G$ and $T'{=}T|_{V_o^{\perp}} \in O(V_o)^{\perp}$*

Proposition 2.2. *If $G'{=}\{T' : \ T'{=}T|_{V_o^{\perp}}, \ T\in G\} \leq O(V_o)^{\perp}$ then there exists an isomorphism from G to G' and $V_o(G'){=}0$ that is G' is effective.*

Proof. let

$$\Phi : G \longrightarrow G'$$

$$\text{s.t. } \Phi(T) = T|_{V_o^{\perp}}$$

It is clear that Φ is a group homomorphism and $| G |{=}|G'|$.

we just need to show isomorphism

Let T$\in Ker(\Phi)$

$\Rightarrow \Phi(T) = 1$

$\Rightarrow T'{=}1$

$\Rightarrow T|_{V_o^{\perp}}{=}1$

$\Rightarrow Tx = x \ \ \forall$ x $\in {V_o}^{\perp}$

$\Rightarrow T = 1$

If not then x $\in V_o$,that is x=0 as x $\in {V_o}^{\perp} \cap V_o$,which is a contradiction.

so Φ is an isomorphism.

$V_0(G')$= { x $\in V|T'x{=}x$ }

$\Rightarrow x \in {V_o}^{\perp} \cap V_o = \{0\}$

$V_0(G^{'}) = 0$

So $G^{'}$ is effective.This completes the proof.

□

Proposition 2.3. *If r is a root of a reflection $S_r \in G \leq O(V)$. And if $T \in G$, then Tr is a root of the reflection S_{Tr}=TS_rT^{-1}∈G.*

Proof. Let P=$r^{\perp}$ and $P^{'}$=TP. Then $P^{'}$ is a hyperplane as P is a hyperplane.

We have Tr=x,

Let $P^{'}$=$x^{\perp}$.

Let y=Tz∈ $P^{'}$

⇒ TS_rT^{-1}y = TS_rz = Tz = y

⇒ TS_rT^{-1}x = TS_rr = -Tr = -x

So it is clear that TS_rT^{-1}=S_x ∈ G and Tr is a root of the reflection S_{Tr}.

□

Proposition 2.4. *Let $G \leq O(V)$ is generated by reflections along roots r_1,....,r_k. Then G is effective iff {r_1,......,r_k} contains a basis for V.*

Proof. Let W ={$r_i^{\perp}$ | 1≤i≤k}

T$|_W$=1_W ∀ T ∈ G

⇒ W⊆ V_o(G)

If x ∈ V_o(G)

⇒ $Tx = x$∀ T ∈ G

⇒ x∈ $r_i^{\perp}$ ∀ i

⇒ x∈ W.

⇒ W =V_o(G)

G is effective iff V_o(G)=0 iff W=0 iff $W^{\perp}$ =V

$W^{\perp}$= $\sum_{i=1}^{k}$$r_i^{\perp\perp}$

We know that,{r_1,......,r_k} spans $r_i^{\perp\perp}$

⇒ {r_1,......,r_k} spans $W^{\perp}$

G is effective iff {r_1,......,r_k} spans V that is iff it contains a basis for V. □

Definition 2.3. ***Root system of G:*** *Let $A = \{r \in V \mid r$ is a root of S_r, where $S_r \in$ generating set of reflection of $G\}$ and $B = \{ Tr \; \forall \; r \in A$ and $\forall \; T \in G \}$ $\dot{T}$hen Δ:=A$\cup$ B is called the root system of group G.*

Proposition 2.5. *Let $G \leq O(V)$ be generated by a finite set of reflections and that G is effective.If Δ is finite then G is finite.*

Remark. *Even if G is not effective it follows from proposition 2.1 that if Δ is finite then G is finite.*

Definition 2.4. ***Coxeter group G:***

A finite effective subgroup $G \leq O(V)$ that is generated by a set of reflections is called Coxeter group.

Definition 2.5. ***t-base,*** Π***:***

Let $t \in V$ s.t $(t,r) \neq 0$, $\forall \; r \in \Delta$

Let $\Delta_t^+ := \{r \in \Delta \mid (t,r) > 0\}$ and $\Delta_t^- := \{r \in \Delta \mid (t,r) < 0\}$.

Let $\Pi \subseteq \Delta_t^+$ be a mininal subset such that r=$\sum_{r \in \Pi} \lambda_i r_i$ for any $r \in \Delta_t^+$ and $\lambda_i \geq 0$. Then Π is called a t-base for G.

Note: From now onwards G will denote a Coxeter group with root system

Δ and t-base Π.

Remarks

- If r $\in \Delta$ then -r $\in \Delta$ since if r $\in \Delta_t^+$ then -r $\in \Delta_t^-$ and vice versa.
- $|\Delta_t^+| = |\Delta_t^-|$
- If v $\in$ V is such that v=$\sum_{i=1}^{k} \lambda_i r_i$ where $r_i \in \Pi$ and $\lambda_i \geq 0$,then we say that v is t-positive.
- If v $\in$ V is such that v=$\sum_{i=1}^{k} \lambda_i r_i$ where $r_i \in \Pi$ and $\lambda_i \leq 0$,then we say that v is t-negetive.

Proposition 2.6. *If r_i,$r_j \in \Pi$ where $i \neq j$, and λ_i and λ_j are positive real numbers, then x=$\lambda_i x_i$-$\lambda_j x_j$ is neither positive nor negative.*

Proposition 2.7. *Suppose $\{v_1, v_2, ..., v_m\} \subseteq V$ be such that $(v_i,v)>0$ where $1 \leq \imath \leq m$ for some $v \in V$. If $(v_i,v_j) \leq 0$ whenever $i \neq j$, then $\{v_1, v_2, ..., v_m\}$ is a linearly independent set.*

2.3 t-base Π is a basis for V

Theorem 2.1. *If Π is a t-base for Δ then Π is a basis for V.*

For the proof of the theorem we need the following lemma and some of the remarks mentioned above.

Lemma 2.1.1. *If r_i,$r_j \in \Pi$ where $i \neq j$,and if S_i is the reflection along r_i*

then $S_i r_j \in \Delta_t^+$ and $(r_i,r_j) \leq 0$

Proof. $S_i r_j \in \Delta$ by proposition 2.2. so S_ir$_j$ is either +ve or -ve, by remark already mentioned. Also we have,

$$S_i r_j{=}r_j\text{-}2\frac{(r_i,r_j)}{(r_i,r_j)}\mathrm{r}_j$$

Coefficient of r_j is positive so $S_i r_j \in \Delta_t^+$ and also $(r_i,r_j) \leq 0$ as both coefficients has to be positive.

□

Proof. (of theorem 2.1)Since G is effective so V is spanned by Δ and Δ is linear combintation of elements in Π so V is spanned by Π and by the last remark and lemma 2.1.1 Π is linearly independent.So,Π

is a basis for V.

□

Corollary 2.1.1. ***Uniqueness of*** Π

t-base Π is unique for Δ.

Proof. Let Π_1 and Π_2 be two t-bases. Since both are basis for V,so let Π_1 be an ordered basis and A be the change of basis matrix from Π_1 to Π_2. Since each element in Π_2 is a non negative linear combination of elements in Π_1,so entries in A are non-negative. Let B be the change of matrix from Π_2 to Π_1. AB=1,entries in A are non zero entries and $B = A^{-1}$ so entries in B are also non negative as each element in Π_1 is a non-negative linear combination of elements in Π_2.Let $a_1, ...a_n$ and $b_1, ..., b_n$ be row of A and columns of B respectively. Since AB=1 so a_1^t $\perp$b$_i$ $2 \leq$ i $\leq$ n.B is non singular $\Rightarrow$b$_2, ..., b_n$ are linearly independent.

$\Rightarrow$ There exists at most one j for which j^{th} entry is zero for $b_2, ..., b_n$

$\Rightarrow$a$_1$ has atmost one nonzero entry.

Similarly each a_i has atmost one non zero entry.

A is non singular $\Rightarrow$ A has exactly one non-zero entry in each row and

each column.

$\Rightarrow$ Each root in Π_1 is a positive multiple of roots in Π_2.

But,only the root is a positive multiple of itself so A is a permutation matrix.And hence $\Pi_1 = \Pi_2$. □

Definition 2.6. ***Simple roots:*** *The roots $r_1, ..., r_n$ in the base Π are called fundamental roots or simple roots.*

Definition 2.7. ***Fundamental reflections:*** *The reflections $S_1, .., S_n$ along roots $r_1, ..., r_n$ are called fundamental reflections of G.*

Let $G_t := < S_i$:$1 \leq i \leq n >$ be subgroup of G.

Remarks

- Let S_i be reflection along $r_i \in \Pi$=$\{r_1, ..., r_n\}$ $.If$ $r$$\in$$\Delta^+$ but r $\neq$r$_i$ then S_ir $\in$$\Delta^+$.
- If v $\in$ V, there exisits a transformation T $\in$ G_t such that (Tv,r_i) $\geq$ 0, $\forall$ $r_i \in \Pi$

2.3.1 G=G_t

Theorem 2.2. *The fundamental reflections $S_1, ..., S_n$ generate G. i.e., G=G_t.*

For the proof we need the following:

Lemma 2.2.1. *:If $r \in \Delta^+$,Tr $\in \Pi$ for some $T \in G_t$.*

Proof. If r $\in \Pi$ choose T=1. If r $\notin \Pi$, then (r,r_i) $\geq$ 0,else Π will not be a basis for V,as $\Pi \cup \{$r$\}$ will be linearly independent. Set a_1=S_{i_1}r $\in \Delta^+$ and (a_1, t) <(r,t). If a_1 $\in \Pi$ then T=S_{i_1}. If not we continue the process which terminates for some $a_k \in \Pi$ as Δ_t^+ is finite. So now we can choose T=$S_{i_1,.....,}$S$_{i_k}$ $\in G_t$.And the claim follows. □

Proof. **of theorem 2.2**:

G = <S$_r$ | $r \in \Delta$ > and $wehave$S$_r$=S_{-r}. So it is enough to prove if r $\in \Delta^+$,then S_r $\in$G$_t$ as G_t is already a subgroup of G. If r $\in \Delta^+$ then by the claim 2.2.1 Tr $\in \Pi$ for some T in G_t. Suppose Tr=r_i,then S_r=TS_iT^{-1} $\in$G$_t$.

Hence G=G$_t$.

□

2.4 Some Properties of a Finite Coxeter group

Property(1) If T$\in$G and TΠ=Π then T=1.

Property(2) If T$\in$G and T(Δ_t^+)=(Δ_t^+) then TΠ = Π.

Property(3) If T $\in$G and T(Δ_t^+)=(Δ_t^+) then T=1(this property follows from property 1 and property 2)

Property(4) If $r_i, r_j \in$ Π,then there exists an integer $P_{ij} > 1$ such that $\frac{r_i r_j}{\|r_i\|\|r_j\|} = -cos\frac{\pi}{P_{ij}}$.

2.5 Example of a Coxeter Group(Dihedral group,H_2^n)

$H_2^n = < S, T \mid T$ is represented by the matrix $\begin{pmatrix} cos(\frac{2\pi}{n}) & sin(\frac{2\pi}{n}) \\ sin(\frac{2\pi}{n}) & -cos(\frac{2\pi}{n}) \end{pmatrix}$, where $1 \le k \le n$ and S is represented by the matrix: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ >

By the result,Sr=-r,if r is a root for reflection S and by some simple calculations we get:

$\Delta(H_2^n) = \{(cos\frac{k\pi}{n},sin\frac{k\pi}{n})$ where $0 \le k \le$ n-1 }.Now since $\Delta(H_2^n)$ contains a basis for V=$\mathbb{R}^2$. So H_2^n is effective and hence

is a Coxeter group.Now we will see what are the possible t-bases for $\Delta(H_2^n)$.This follows from the following result,that can be proved by detailed

computation.

Lemma 2.5.1. *Suppose $r_i \in \Delta_t^+(H_2^n)$ for some $t \in V$, and r_i= $(cos\frac{k\pi}{n}, sin\frac{k\pi}{n})$ where k=0,1,2,.....,2n-1.If $r_i \in \Pi_t$ and $r_j \in \Pi_t$ then r_j=$(cos\frac{k\pm(n-1)\pi}{n}, sin\frac{k\pm(n-1)\pi}{n})$* This gives us the t-base for dihedral group H_2^n. We will now look at dihedral group for n=4 explicitly.

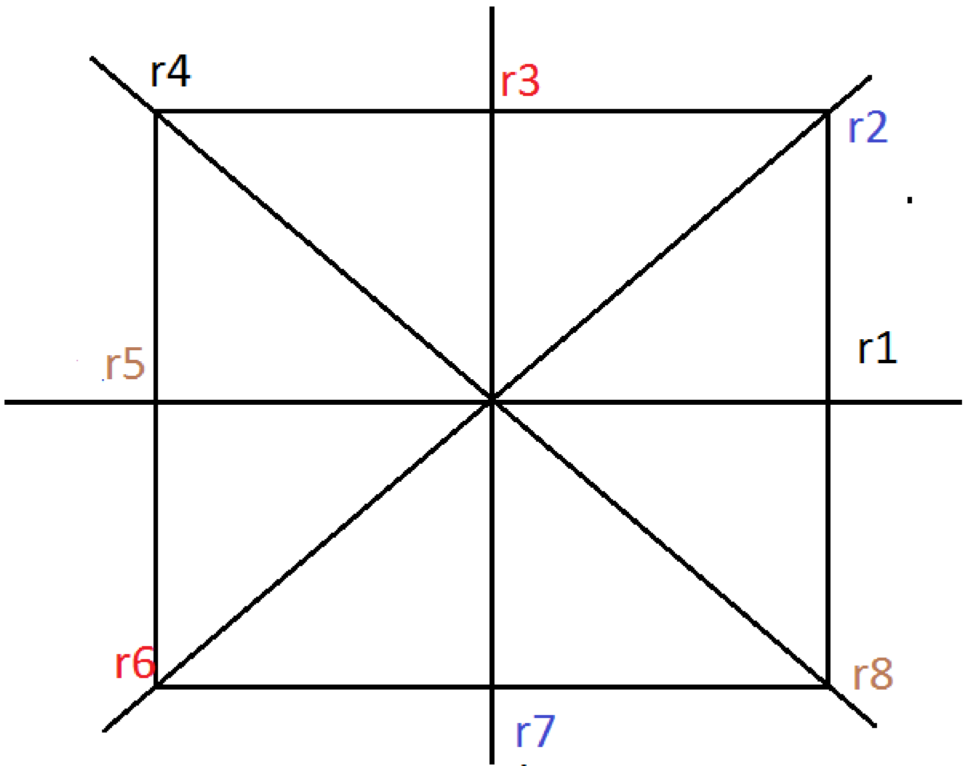
The case n=4:, $\mathbf{H_2^4}$: Here the root system by our above

mentioned formula is:

$\Delta(H_2^4)$={(1,0), $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, (0,1), $(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, (-1,0), $(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$, (0.-1), $(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$}

And the possible(depending on t) t-bases are:

- {(1,0),$(-\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}})$}
- {$(\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}})$,(-1,0)}
- {(0,1),$(\pm\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$}
- {$(\pm\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$,(0,-1)}



Example of Coxeter group H_2^4 ,with its roots and possible t-base colored in same color

Chapter 3

Fundamental Regions

3.1 Introduction

In this chapter first we will give a description of **fundamental region** of a group and then find ways for obtaining the **fundamental region** for any finite subgroup of orthogonal transformations. Then we specialize to Coxeter groups. We assume V=$\mathbb{R}^n$.

3.2 Definitions

Definition 3.1. *Relatively Open Set: Suppose* $Y \subseteq X \subseteq V$. *Then Y is said to be* ***relatively open*** *wrt to another set X if* $Y = X \cap U$ *for some open set U in V.*

Definition 3.2. *A subset F of V is said to be a fundamental region of G in V iff the following conditions are satisfied:*

(a) *F is open in V.*

(b) $F \cap TF = \phi$ *if* $1 \neq T \in G$

(c) $V = \cup\{\overline{(TF)} \mid T \in G\}$

Definition 3.3. *Suppose* $X \subseteq V$ *is a linear subspace s.t.* $T(X) = X\ \forall\ T \in G$.

F is a fundamental region for G in X iff the following conditions are satisfied:

(a) *F is relatively open in X.*

(b) *F$\cap$TF=ϕ if 1$\neq$T$\in$G*

(c) *X=$\cup\{\overline{(TF)}\cap X|$ T$\in$G.$\}$*

Some useful notation and convention: Here $x_o \in$ V is such that Tx$_o \neq x_o$,for any 1$\neq$T $\in$G. .Let $G = \{1, T_1, ..., T_{N-1}\}$ and $_ix_o = x_i$. So orb(x_o)=$\{x_1, ..., x_{N-1}\}$.

Let $[x_o x_i] := \{x_o + \lambda(\mathrm{x}_i - x_o) \mid 0\leq \lambda \leq 1\}$,

L_i:= {x$\in$V | d(x,x$_o$) $< d(x, x_i)$,1$\leq i \leq N-1\}$, L_i is an open half space. F := $\cap\{$L$_i|1\leq i \leq N-1\}$.

Remark: Further it will be proved that F as defined above is a fundamental region for G in V. For that we will need some results which are mentioned in the next section.

3.3 Lemmas to be used

Lemma 3.1. *Suppose dim(V) $\geq$ 2 and $x_1, x_2 \in$ V are linearly independent. For each $\lambda \in \mathbb{R}$ and define V_λ=$(x_1 + \lambda x_2)^\perp$. If $\lambda \neq \mu$, then $V_\lambda$and V_μ are distinct (n-1) dimensional subspace of V.*

Proof. V_λ and V_μ are subspaces of V.Let W := $< x_1 + \lambda x_2 >$, dim(W)=1 Let

$$T_{(x_1+\lambda x_2)} : \mathrm{V}\rightarrow \mathrm{F\ s.t.}$$
$$T_{(x_1+\lambda x_2)}(\mathrm{y})=(\mathrm{y}, x_1 + \lambda x_2)$$

Ker $T_{(x_1+\lambda x_2)}$ = V$_\lambda$

dim(Im($T_{(x_1+\lambda x_2)}$))=1

$\Rightarrow$dim(Ker($T_{(x_1+\lambda x_2)}$))=n-1 ,where n is the dimension of V.

Let y $\in V_\lambda \cap$V$_\mu$

$\Rightarrow (y$,x$_1 + \lambda x_2$)=(y,$x_1 + \mu x_2$)

$\Rightarrow (y$,x$_2(\lambda - \mu)$)=0

$\Rightarrow (\lambda - \mu)$(y,x$_2$)=0

$\Rightarrow y = 0$ as $\lambda\neq\mu$ and x$_2 \neq 0$.

So V_λ $\cap$V$_\mu$=$\{0\}$. ∎

Corollary 3.1.1. *If dim V=n $\geq$ 2, then there are infinitely many subspaces og dim(n-1).*

Lemma 3.2. *If dim(V) $\geq$1, then V is not the union of any finite number of proper subspaces.*

Proof. Let dim V=1, then the only proper subgroup of V is $\{0\}$ and V$\neq\bigcup_{i=1}^{n}\{0\}$.

The proof is by induction on dim V for all vector spaces of dimension $\leq$ n-1.Let dim(V)=n.

Suppose the lemma holds. Suppose that V=$\bigcup_{i=1}^{n} V_i$ where V_i is a proper subspace of V.

Let W be a subspace of V s.t. dim W=n-1.

W=W$\cap$V=W$\cap\bigcup_{i=1}^{n} V_i$=$\bigcup_{i=1}^{n}(W\cap V_i)$

$\Rightarrow W = W\cap V_i$ for some i

$\Rightarrow W = V_i$ for some i.

$\Rightarrow$ Every n-1 subspace of V $\in \{V_1, V_2, ..., V_n\}$

$\Rightarrow$ There are finitely many n-1 subspaces of V.Which is a contradiction by Corollary 3.3.1.

So V $\neq \bigcup_{i=1}^{n} V_i$.

□

3.4 Fundamental Region for a group G in V

In this section it is proved that one fundamental region of a finite group G $\leq$O(V) in V is F as defined above. Then some of the basic properties of a fundamental region is given.

Theorem 3.1. *Let $G \leq O(V)$ be a finite subgroup.The set F=$\cap\{Li$:1$\leq i \leq N-1\}$ is a fundamental region for G in V.*

Proof. (a) **F is open** Since each L_i is open,so $\cap L_i$ is open for $1 \leq i \leq N-1$ and hence F is open.

(b) **F$\cap T_i$F=0 $\forall$ 1$\neq T_i \in$ G.**

T$_i$F=$T_i(\cap L_j)$

$\Rightarrow$T$_iF = \{T_ix \in V \mid d(T_ix, T_ix_o) < d(T_ix, T_ix_j, 1 \leq j \leq N-1\}$

$\Rightarrow$T$_iF = \{T_ix \in V \mid d(T_ix, T_ix_o) < d(T_ix, T_iT_jx_o, 1 \leq i \leq N-1\}$

$\Rightarrow$T$_iF = \{y \in V \mid d(y, x_i) < d(y, T_kx_o), k \neq i, 0 \leq k \leq N-1\}$.

Since $\{T_iT_j : 1 \leq j \leq N-1\}$=G$\setminus\{T_i\}$. Thus T$_i$F={ x : d(x,x$_i$) $<$ $d(x, x_j), i \neq j\}$

Let 0$\neq$ x $\in$ F$\cap T_i$F,

$\Rightarrow d(x, x_o) <$ d(x,x$_i$) and d(x,x$_i$) $<$ d(x,x$_o$) since i$\neq$ 0. Which is a contradiction.

So F$\cap T_i$F=0

(c) **V=$\cup$ $\{\overline{T_iF} \mid 0 \leq i \leq N-1\}$**

we have V $\supseteq \cup \{\overline{T_iF} \mid 0 \leq i \leq N-1\}$

Now,let x $\in$ V and choose i such that d(x,x$_i$)is minimal i.e., d(x,x$_i$) $\leq d(x, x_j)$ $\forall$ j

$\Rightarrow x\in \overline{T_iF}$

$\Rightarrow x\in\cup \{\overline{T_iF} \mid 0 \leq i \leq N-1\}$

So V=$\cup$ $\{\overline{T_iF} \mid 0\leq i \leq N-1\}$.

Hence F is a fundamental region for G in V.

□

Remark. *F is convex since each L_i is convex and it is also connected.*

Corollary 3.1.1. *If F is a fundamental region for a group $G \leq O(V)$ in V and $T \in G$ then TF is also a fundamental region for G in V.*

Proof. (a) T is continuous and F is open so TF is open.

(b) if x $\in TF \cap T_iTF$ proof for this is analogous to that in the proof for theorem 3.1 replacing x_o with Tx_o.

(c) since T$_i$TF=T$_j$F for some j where T$_j$ $\neq$T so V=$\cup$ $\{\overline{T_iF} \mid 0 \leq i \leq N-1\}$ implies V=$\cup$ $\{\overline{T_iTF} \mid 0 \leq i \leq N-1\}$

So TF is also a fundamental region.

□

Corollary 3.1.2. *If F is a fundamental region for G in V and $X \subseteq V$ is an invariant subspace under G than $F_X = F \cap X$ is a fundamental region for G in X.*

Proof.

F∩X is relatively open as F is open in X.

F∩T_iF=0 $T_i \in$ G as F is a fundamental region for G and X is invariant under G.

X=V∩X=∪ $\{\overline{T_iF} \mid 0 \leq i \leq N-1\}$∩X= $\{\overline{T_iF} \cap X \mid 0 \leq i \leq N-1\}$

□

3.5 Fundamental Region For a Coxeter Group G ≤O(V) in V

Suppose G is a Coxeter group with t-base Π=$\{r_1, ...r_n\}, say.$

Let F := $\{$v$\in V \mid$ (v,r_i) $> 0, r_i \in \Pi\} = \cap_{i=1}^{n}\{v\in V(v, r_i) > 0\}$

In this section we will prove that F is a fundamental region for G.

Theorem 3.2. *F is a fundamental region for G.*

Proof. (a) F is open because of the way it is defined.

(b) Let x ∈ F∩TF, and 1≠ T=R^{-1} ∈ G. So we have Rx∈ F as x∈ TF.Hence $(x, r) > 0$ for all r ∈ Δ_t^+ So,Δ_t^+=Δ_x^+.

⇒ Π_t=Π_x. Similarly we have Π_t=Π_{Rx}

⇒ Π_t=Π_{Rx}=RΠ_x=RΠ_t

⇒ $R = T = 1$

⇒ F∩TF=ϕ

(c) If y∈ V then there exists a T∈ G such that (Ty,r_i) ≥ 0 for all r_i in Π.

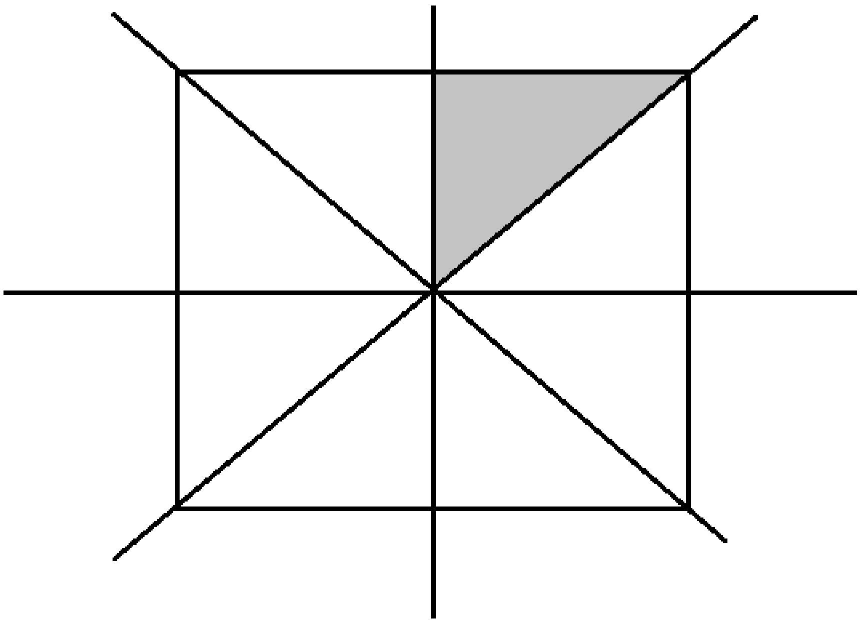
So Ty∈ $\overline{F}$.

⇒ $y \in \overline{T^{-1}(F)}$

⇒ $V \subseteq \cup$ $\{\overline{R_iF} \mid$ R∈G $\}$

$\Rightarrow V = \cup \{\overline{R_i F} \mid \text{R}\in\text{G} \}$

□



The shaded region is a fundamental region of H_2^4

Chapter 4

Presentation of Coxeter Group

4.1 Introduction

Our main aim in this chapter is to show that Coxeter group G has a presentation $< S_i, ..., S_n | (S_iS_j)^{P_{ij}} = 1 >$ *where* P_{ij} is the order of S_iS_j and S_i's are the fundamental reflections of Coxeter group G.

4.2 Definitions

Definition 4.1. *If* $T = S_{i_1} ..... S_{i_n}$ *where* S_{ij}*'s are fundamental reflections in G such that there is no other word in* $S_i's$ *representing T having less than k fundamental reflections as factor than* ***k is called the length of T*** *and write l(T)=k.*

Remark. *l(1)=0.*

Definition 4.2. $n(T) := | T(\Delta_t^+) \cap \Delta_t^- |$

In other words n(T) is the number of positive roots sent to negative roots by T.

Definition 4.3. *Suppose* $i \neq j$*. Then* $(S_iS_j.....)_m$ *is the product of* S_i *and* S_j *appearing alternately m times starting with* S_i*.*

$(.........S_iS_j)_m$ *is the product of* S_i *and* S_j *appearing alternately m times ending with* S_i*.*

$(.....S_iS_j......)_m$ *is the product of* S_i *and* S_j *appearing alternately m times.*

Definition 4.4. *Partial words: Let* $W = S_{i_1.....}S_{i_k}$ *and* $W_j = S_{i_1.....}S_{i_j}$ *where* $1 \leq j \leq k$ *then* W_j *is called a partial word of W.*

Theorem 4.1(Coxeter). *Every relation $W=S_{i_1.....}S_{i_k}=1$ in a Coxeter group G is a consequence of the relations of the form $(S_iS_j)^{P_{ij}}=1$.*

Proof. The following lemmas will be used in the proof of this theorem.

Lemma 4.1.1. *If $T\in$ G then $l(TS_i)=$* $\begin{cases} l(T)-1 & ifTr_i\in\Delta_t^- \\ l(T)+1 & ifTr_i\in\Delta_t^+ \end{cases}$

If S_i is a fundamental reflection in G.

Proof of this lemma follows from the fact that n(T)=L(T) and n(TS_i)=n(T)-1 if $Tr_i \in \Delta_t^-$ and n(TS_i)=n(T)+1 if $Tr_i \in \Delta_t^+$.

Lemma 4.1.2. *If S_i and S_j are fundamental reflection in G and $1\leq m \leq P_{ij}$ then $(S_iS_j....)_{m-1}r_i \in \Delta_t^+$.*

Lemma 4.1.3. *Let $T\in G$ and i and j are fixed and $l(TS_i)=l(TS_j)=l(T)-1$ then $l(T(....S_iS_j....)_m)=l(T)-m$ if $0\leq m \leq P_{ij}$*

Proof: Let u be the maximal length of partial words of W and $p=P_{ij}$.

We can write $W=W_1S_iS_jW_2$ such that $l(W_1S_i)$=u and all partial words of W_1 are of length less than u. Denote $W‘=W_1(S_jS_i......)_{2p-2}W_2$. We have $(S_iS_j)^p=1$ in G

$\Rightarrow S_iS_j=(S_jS_i......)_{2p-2}$ in G

$\Rightarrow W_1S_iS_jW_2=W_1(S_jS_i......)_{2p-2}W_2$. in G

$\Rightarrow W=W‘$ in G.

So W and W‘ are equal as elements in G.Except for W_1S_i all partial words of W coincides with partial words of W‘.

Set $W_1S_i=T$. Replacing W_1S_i with $T(S_iS_j....)_m$ where $2\leq m \leq 2p-2$ we see that $T(S_iS_j....)_m$ coincides with $W_1S_j, W_1S_jS_i, ..., W_1(S_jS_i......)_{2p-2}$ which are the partial words of W'as elements of G. In this step we used the identity $S_i{}^2=1$.

$l(T(S_iS_j....)_m) <$ u by lemma 4.1.3.

So we have by the above step replaced W with W' where the latter has partial words of length less than or equal to u and one partial word of length less than that of W having length u.

Repeating the above steps after a certain number of steps we obtain the empty word, that is we conclude that applying the identity $(S_iS_j)^p=1$ and $(S_i)^2=1$ we get W=1 as elements in G.

□

Corollary 4.1.1. *G has a presentation* $< S_i, ..., S_n \mid (S_iS_j)^{P_{ij}}$ *=1>*

Let G be a finite group having presentation

$< T_i, ..., T_n \mid (T_iT_j)^{P_{ij}}=1, 1 \leq i,j \leq n > where P_{ii}=1$ for all i, $P_{ij} = P_{ji} \geq 2$ if $i \neq j$. Let $S=\{T_1, ...T_n\}$.

Next we will prove that G is infact a Coxeter group.

Definition 4.5. *If* $S=S_1 \sqcup S_2$ *where* S_1 *and* S_2 *are non empty and* $P_{ij}=2$ *if* $T_i \in S_1$ *and* $T_j \in S_2$ *or vice-versa then G is called decomposable otherwise it is called indecomposable.*

Remark. *If G is decomposable then clearly it is the direct product of two subgroups with the same type of presentation, therefore we will assume that G is indecomposable.*

Let H := $< S_1, ..., S_n >$ be the group of non-singular transformations of $\mathbb{R}^n$ where

$$S_je_i = e_i + 2cos\frac{\pi}{P_{ij}}e_j$$

and $\{e_1, ...e_n\}$ is the standard basis of $\mathbb{R}^n$. Denote $A=[\alpha_{ij}]$ be the n $\times$ n matrix where $\alpha_{ij} = -cos\frac{\pi}{P_{ij}}$.

Let the coloumn of A be $\{a_1, ...a_n\}$ and $(P_i) = (a_i)^{\perp} \subseteq \mathbb{R}^n$.

Remark. $S_ix = x$ *if* $x \in P_i$ *and* $S_ie_i = -e_i$.

Theorem 4.2. $H \leq O(V)$ *is a Coxeter subgroup and G is isomorphic to H.*

Proof. We will need the following lemmas for the proof of this theorem.

Lemma 4.2.1. *There is a homomorphism* ϕ *from G to H,i.e.* $\phi(T_i)=S_i$ $1 \leq i \leq n$.

Remark. *From the above lemma we have H is finite.*

Lemma 4.2.2. *If H is as above(or finite group of invertible transformations on* $\mathbb{R}^n$*). Then there is an inner product* $(-,-)_C$ *such that* $(x,y)_C = (Tx, Ty)_C = (x, y)_C, \forall T \in G\}$ *such that* $H \leq O(\mathbb{R}^n, (-,-)_C)$.

Lemma 4.2.3. *If W is a subspace of* $\mathbb{R}^n$ *such that W is invariant under T for all T in H then* $W=\mathbb{R}^n$ *or W=0.*

Lemma 4.2.4(*Schur's lemma*). *Suppose the only H-invariant subspaces in $\mathbb{R}^n$ are $\mathbb{R}^n$ and 0. If S is a non-zero linear transformation on $\mathbb{R}^n$ such that ST=TS for all T $\in$ H then S is non-singular.*

Lemma 4.2.5. *If S is a linear transformation and λ is an eigen value of S, then S=λI*

Let B:=$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the bilinear transformation such that B(x,y)=(Ax,y), where A=$[\alpha_{ij}]$.

Remark. *B is a symmetric form as A is a symmetric matrix and B is invariant under transformations in H. Also $S_i x = x - B(x, e_i) e_i$*

Lemma 4.2.6. *Let $T_i \in H$ be represented by matrix M_i wrt standard basis of $\mathbb{R}^n$ and $P=\sum_{i=1}^{n} M_i^t M_i$. If T$\in$H is represented by a matrix M wrt standard basis of $\mathbb{R}^n$ then $M(P^{-1}A)=(P^{-1}A)M$.*

Remark. *From the above lemma we see that B is a positive scalar multiple of inner product C and hence B is also an inner product on $\mathbb{R}^n$*

Using these lemmas and remarks we will prove that H is infact a Coxeter group. B is invariant under transformations in H so **H$\leq$O($\mathbb{R}^n$).** Since S_ix=x-B(x,e_i)e_i and B(e_i, e_i)=1 and B(x,e_i)=0 if x$\in$ $(e_i)^\perp$ so S_i is a orthogonal reflection of $\mathbb{R}^n$ with root $r_i = e_i$.Since $\{e_1, ...e_n\}$ is a basis for $\mathbb{R}^n$ so H is effective. So H is a **Coxeter group**. H has a postive definite graph as B(r_i, r_j) = α_{ij}. So the matrix of $\{r_1,...,r_n\}$ is A which is postive definite. As G is indecomposable so the Coxeter graph of H is connected. Also we have $\{S_1,....S_n\}$ are fundamental reflections of H so H has the same representation as that of G and hence H is isomorphic to G.

□

Chapter 5

Clasification of Coxeter group

5.1 Introduction

In this chapter we will introduce Coxeter graph and establish a relationship between Coxeter graph and Coxeter group. This will lead to classification of Coxeter groups.

5.2 Definitions

Definition 5.1. ***Marked graph:*** *A Marked graph is finite set of points called nodes such that any two distinct nodes may or may not be joined by a line called Branch and if there is a branch joining the i^{th} and j^{th} nodes then it is marked with a real number $P_{ij} > 2$.*

Definition 5.2. ***Coxeter graph:*** *If for a marked graph every mark P_{ij} is an integer then it is a Coxeter graph.*

Remark. *If P_{ij}=3 then the labelling is not done on the branch.*

Definition 5.3. ***Quadratic forms of a marked graph:*** *Let G be a marked graph with n nodes we associate a quadratic form Q_G with G such that*

$$Q_G(\lambda_1, ..., \lambda_n) = \sum_{ij} \alpha_{ij}\lambda_i\lambda_j$$

where $\alpha_{ij} = -cos\frac{\pi}{P_{ij}}$, if there is a branch joining i^{th} and j^{th} nodes.Otherwise α_{ij}=2 and α_{ii}=1.

We denote A=[α_{ij}]

Remark. *A is a symmetric matrix.*

Definition 5.4. ***Positive definite Marked graph:*** *If A is positive definite for a marked graph G then G is called positive definite.*

Remark. *Given a marked graph G by det(G) we denote the determinant of the matrix associated to that graphas discussed above.*

Definition 5.5. ***Marked graph G for a set of vectors:*** *If $\{x_1,....,x_n\}$ is a finite set of mutually obtuse vectors then we define a marked graph G with n nodes $x_i's$ and if i $\neq$ j then the i^{th} and j^{th} nodes are joined by a branch iff $(x_i, x_j) \neq 0$ and it is labelled P_{ij}, where $\frac{(x_i,x_j)}{\|x_i\|\|x_j\|} = -cos\frac{\pi}{P_{ij}}$.*

Remark. *For a Coxeter group G we denote the graph associated to it by $\mathbb{G}$.*

Definition 5.6. ***Reducible and Irrreducible graphs:*** *If the t-base Π is not a union of two non empty orthogonal subsets then $\mathbb{G}$ is irreducible. Else it is reducible.*

Remark. *If G is Coxeter group then the marked graph corresponding to G is a Coxeter graph.*

Definition 5.7. ***Connected nodes:*** *Two distinct nodes a and b are connected iff there are nodes $a_1,....,a_n$ in G such that a=a_1 joined by a branch with a_2 ,a_2 is joined by a branch with $a_3,....,a_{n-1}$ is joined by a branch with a_n=b.*

Remark. *If all the nodes in a graph are connected then the graph is called connected graph. So a Coxeter graph of a Coxeter group is connected iff it is irreducible.*

Definition 5.8. ***Subgraph*** *A marked graph $\mathbb{H}$ is called a subgraph of a graph G if $\mathbb{H}$ can be obtained either by deleting some of the nodes of G or by decreasing the marks on some branches or by both.*

Theorem 5.1. *If G_1 and $G_2 \leq$ O(V) are Coxeter groups having the same Coxeter graphs then they are geometrically the same.i.e., G_1=$T^{-1}G_2T$ for some $T \in$ O(V)*

Proof. If Π_1=(x_1,...,x_n) and Π_2=(y_1,....y_n) be the t-bases for G_1 and G_2 respectively then $(x_i, x_j) = (y_i, y_j)$ for all i and j since G_1 and G_2 have the same Coxeter graph.

Let T be a linear transformation defined as:

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

such that $Tx_i = y_i$ then since $T(x_i, x_j) = (Tx_i, Tx_j) = (Ty_i, Ty_j) = (x_i, x_j)$ so $T \in O(V)$.

As $x_i = y_i$ so $S'_i = TS_iT^{-1}$ for all $S_i \in G_1$ and S'_i in G_2.

As S_i generates G_1 and S'_i generates G_2 so $G_2 = TG_1T^{-1} so G_1 = T^{-1}G_2T$ as required.

□

Remark. *The Coxeter Graph of a Coxeter group is positive definite.*

5.3 Positive definite graphs

The following are some of the marked graphs.

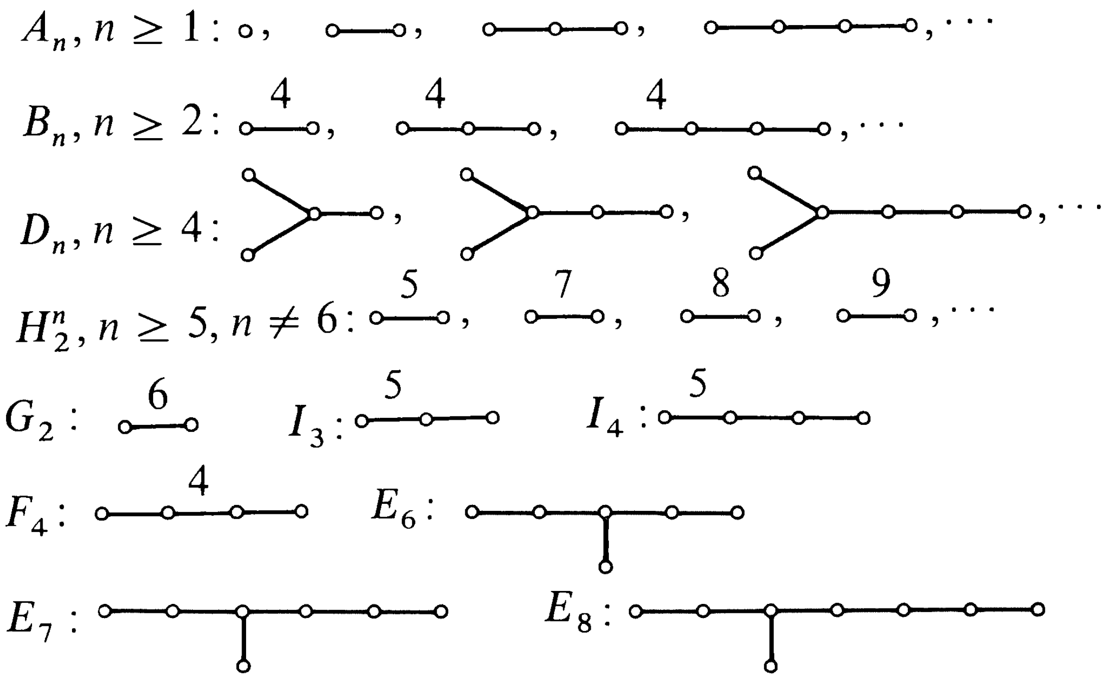


Figure 5.1

In this section we will prove that these graphs are positive definite.

Remark. *The k^{th} principal minor of matrix of marked graph A_n is the det of matrix of marked graph A_k. So if we prove that det(A_k) is positive for all k then A_n will be positive definite. This is true form all other marked graphs for figure 4.1.*

Theorem 5.2. *The marked graphs $A_n, B_n, D_n, H_2^n, G_2, I_3, I_4, F_4, E_6, E_7, E_8$ are all postive definite.*

Proof.

Lemma 5.2.1. *Let G be a marked graph and a_1 be one of its nodes connected to only one other node say a_2 by a branch marked P_{12}. Denote subgraphs $G\backslash\{a_1\}$ by G_1 and $G\backslash\{a_1, a_2\}$ by G_2. Then det G=det(G_1) $- \cos\frac{\pi}{P_{12}}$ det(G_2)*

By a simple calculation we have det(A_1)=1 and det(A_2) $= \frac{3}{4}$ and by applying induction on A_n we have det(A_n) $= \frac{n+1}{2^n}$

Now using the above lemma we get the following:

(a) $det(A_n) = det(A_{n-1}) - \frac{1}{4}det(A_{n-2}) = \frac{n+1}{2^n} > 0$

(b) $det(B_n) = det(A_{n-1}) - \frac{1}{2}det(A_{n-2}) = \frac{1}{2^{n-1}} > 0$

(c) $det(D_n) = det(A_{n-1}) - \frac{1}{4}det(A_{n-3}) = \frac{1}{2^{n-2}} > 0$

(d) $det(I_3) = det(A_2) - \alpha^2 det(A_1) = \frac{3-\sqrt{5}}{8} > 0$

(e) $det(I_4) = det(A_3) - \alpha^2 det(A_2) = \frac{7-3\sqrt{5}}{8} > 0$

(f) $det(F_4) = det(B_3) - \frac{1}{4}det(A_2) = \frac{1}{16} > 0$

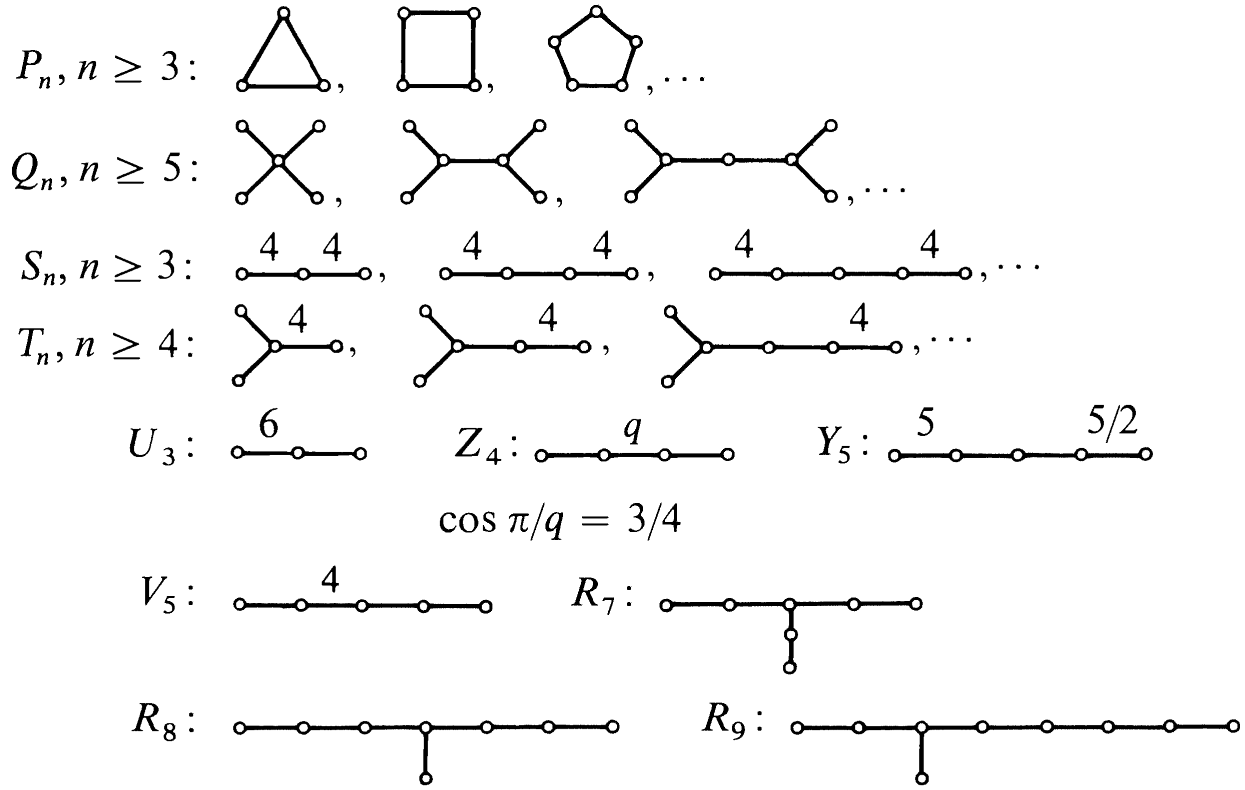
(g) $det(E_n) = det(D_{n-1}) - \frac{1}{4}det(A_{n-2}) = \frac{9-n}{2^n} > 0$

Since determinant of all the minors of the matric associated to the graphs is positive so the above graphs are positive definite.

Since H_2^n and G_2 are graphs of Coxeter graph of dihedral groups H_2^n so they are positive definite by remark already mentioned in chapter 2.

□

5.4 Marked graphs with determinant zero



THE MARKED GRAPHS WITH DETERMINANT ZERO

Figure 5.2

Definition 5.9. ***Cycle in a marked Graph*** *is a subgraph of form P_n.*

Definition 5.10. ***Branch point*** *in marked graph G is a node having 3 or more branches emanating from it.*

Theorem 5.3. *Marked graph $P_n, Q_n, S_n, T_n, U_3, V_5, Z_4, Y_5, R_7, R_8, R_9$ have determinant zero.*

Proof. Using lemma 5.2.2 we have the following:

(a) $det(Q_n) = det(D_{n-1}) - \frac{1}{4}det(D_{n-3}) = 0$

(b) $det(S_n) = det(B_{n-1}) - \frac{1}{2}det(B_{n-2}) = 0$

(c) $det(T_n) = det(B_{n-1}) - \frac{1}{4}det(B_{n-3}) = 0$

(d) $det(U_3) = det(A_2) - \frac{3}{4}det(A_1) = 0$

(e) $det(Y_5) = det(I_4) - \beta^2 det(I_3) = 0$

(f) $det(V_5) = det(B_4) - \frac{1}{4}det(A_3) = 0$

(g) $det(R_7) = det(E_6) - \frac{1}{4}det(A_5) = 0$

(h) $det(R_8) = det(E_7) - \frac{1}{4}det(D_6) = 0$

(i) $det(R_9) = det(E_8) - \frac{1}{4}det(E_7) = 0$

Z_4 by finding the matrix and den its determinant and for P_n the rows of the matrix for P_n add upto zero so they are linearly dependent and hence $\det(P_n) = 0$.

□

Theorem 5.4. *If G is a connected positive definite Coxeter graph then G is one of the following graphs $A_n, B_n, D_n, H_2^n, G_2, I_3, I_4, F_4, E_6, E_7, E_8$.*

Proof. We first note

Lemma 5.4.1. *A non nonempty subgraph $\mathbb{H}$ of a positive definite graph G is also positive definite.*

Case 1: Subgraph of the form $\mathbf{P}_n$. If a marked graph has a cycle as a subgraph then this positive definite graph will have a subgraph that is not positive definite leading to a contradiction to lemma 5.4.1. **A cycle cannot be a subgraph for G**.

Case 2: Branch points of G.G can have only one branch point with 3 branches emanating from it. Else Q_n would be a subgraph of G contradicting lemma 5.4.1.

Case 3: If G has $H_2^n n \geq$7 as a subgraph. Then G= H_2^n.otherwise U_3 would be a subgraph contradicting lemma 5.4.1 and same reason hold for if G_2 is a subgraph of G then G=G_2

Now the only possible marks on G could be 3,4 or 5. These cases are dealt with in the next steps.

Case 4: If B_2 is a subgraph. It can occur as subgraph in G only once else S_n would be a subgraph. Also G cannot have a branch point else T_n would be a subgraph. If H_2^5 is also a subgraph thenG=H_2^5 or G=I_3 or G=I_4. These are the only possibilities for this particular case with H_2^5 as subgraph else Z_4 and Y_5 would be its subgraphs.

Case 5: If B_2 is a subgraph and H_2^5 is not a subgraph. Then G=B_n or F_4 else V_5 would be a subgraph of G.

We are done with branches marked with 4 and 5 so the only case remaing is if all the branches of G are unmarked or P_{ij}=3.

case 6: If the branches of G are all unmarked. If G has no branch point the G=A_n. If G has a branch point then G=D_n or E_6 or E_7 or E_8 no other possiblities are there as except for these graphs anyother case will lead to subgraph as R_7, R_8 and R_9 on G.

□

5.5 Crystallographic Groups

Definition 5.11. *A lattice in vector space V(dim v=n) is a discrete set of points obtained by taking all integer linear combination of n-linearly independent vectors in V,i.e.,If X=$\{x_1, ...x_n\}$ is a set of linearly independent vectors in V, then lattice L in V is defined as:*

$L := \{v \in V \mid v = \sum_{i=1}^{n} \lambda_i x_i$, *for all* $\lambda_i \in \mathbb{Z}$ *and* $x_i \in X\}$

Definition 5.12. *A subgroup G$\leq$ O(V) is said to satisfy the crystallographic condition iff there is a lattice L invariant under G.*

Remark. *If a Coxeter group G is crystallographic then the only possible values of P_{ij} are 1,2,3,4 or 6.*

From this remark it follows that the only Possible irreducible crystallographic Coxeter groups are $\mathbf{A}_n$,$\mathbf{B}_n$,$\mathbf{D}_n$,$\mathbf{G}_2$,$\mathbf{E}_6$,$\mathbf{E}_7$,$\mathbf{E}_8$ and $\mathbf{E}_9$.

Theorem 5.5. *A Group with graph A_n,B_n,D_n,G_2,E_6,E_7,E_8 and E_9 satisfies crystallographic conditions.*

Proof. Let L$\{ \sum_{i=1}^{i=n} \lambda_i r_i : \lambda_i \in \mathbb{Z}$ and $r_i \in \Pi\}$

Assign relative lengths to the roots of the groups as follows:

P_{ij}	relation between lengths of r_i and r_j
3	$\|r_i\| = \|r_j\|$
4	$\|r_i\| = \sqrt{2}\|r_j\|$ or $\|r_j\| = \sqrt{2}\|r_i\|$
6	$\|r_i\| = \sqrt{3}\|r_j\|$ or $\|r_j\| = \sqrt{3}\|r_i\|$

Case(1) If $P_{ij} = 1$ then $S_i r_j = -r_j$

Case(2) If $P_{ij} = 2$ then $S_i r_j = r_j$

Case(3) If $P_{ij} = 3$ then $S_i r_j = r_i + r_j$

Case(4) If $P_{ij} = 4$ then $S_i r_j = r_i + r_j$ or $2r_i + r_j$ depending on the relative lengths of r_i and r_j

Case(5) If $P_{ij} = 6$ then $S_i r_j = r_i + r_j$ or $3r_i + r_j$ depending on the relative lengths of r_i and r_j
In all the above cases $S_i r_j \in$L for all So we have S_iL=L,hence TL=L as required since S_i generated G.

□

Chapter 6

Construction Of Coxeter Group

6.1 Introduction

In this chapter we Construct Coxeter group and for that we will show that the graphs listed in previous chapter are actually the graphs of Coxeter groups.

6.2 Construction of Coxeter group with graph $\mathbf{A}_n$

Let the symmetric group S_{n+1} be viewed as a group of linear transformation on $\mathbb{R}^{n+1}$ such that any $T \in S_{n+1}$ permutes the basis vectors $e_1, \ldots, e_{n+1}$.

then we have, $S_{n+1} = < S_1, ...S_n >$ where $S_i = (e_i e_{i+1})$.

Proposition 6.1. *$S_{n+1} \leq O(\mathbb{R}^{n+1})$*

Proof. $S_i(e_{i+1} - e_i) = -(e_{i+1} - e_i)$

$S(e_i + e_{+1}) = e_i + e_{+1}$

$S_i(e_j) = e_j$, if $j \neq i$, $j \neq i+1$

$(e_{i+1} - e_i)^{\perp}$ is spanned by $\{e_j \mid j \neq i, j \neq i+1\} \cup \{e_i + e_{+1}\}$

So $S_i's$ are reflection along root $r_i = e_{i+1} - e_i$ and $S_{n+1} = < S_1, ...S_n >$. Hence $S_{n+1} \leq O(\mathbb{R}^{n+1})$.

□

Remark. *The root system of S_{n+1} $=\{e_i - e_j : i \neq j, 1 \leq i, j \leq n+1\}$. Reason being the conjugates of any transposition is the set of all transpositions and hence the set of conjugate reflections is the set of all transpositions.*

Let V be the subspace of of $\mathbb{R}^{n+1}$ spanned by $\{r_1, ..., r_n\}$ and A_n be the group of transformations in S_{n+1} restricted to V.

Then we have $A_n = < S_1, ..., S_n >$ and $A_n \leq O(V)$.

Remark. *A_n is effective since $\{r_1, r_2, ...r_n\}$ forms a basis for V.*

So from this remark we have A_n is a Coxeter group. Also $\{r_1, ..., r_n\}$ is a t-base for A as any root in A_n is a positive linear combination of $\{r_1, ..., r_n\}$.

Remark. *The Coxeter graph of A_n is A_n as $\frac{(r_i, r_j)}{\|r_i\|\|r_j\|} = \frac{-1}{2}$, $1 \leq i, j \leq n+1$.*

6.3 Construction of Coxeter group B_n with graph B_n

Let $K_n :=$ $<S_{e_1}, ..., S_{e_n}>$ where $S_{e_i}(\lambda_1, ..., \lambda_i, ..., \lambda_n) = (\lambda_1, ..., -\lambda_i, ..., \lambda_n)$ is linear transformation on $\mathbb{R}^n$.

And as before S_n is a group of linear transformation but here $S_n = < S_2, ..., S_n >$where S_i is a reflection with root $r_i = e_i - e_{i-1}$.

Let $J \subseteq \{e_1, ..., e_n\}$ define f_J: $\mathbb{R}^n \to \mathbb{R}^n$ such that

$$f_J(e_i) = \begin{cases} -e_i & if e_i \in J \\ e_i & if e_i \notin J \end{cases}$$

Remarks:

1. K_n is abelian and $|K_n| = 2^n$.
2. K_n is normalised by S_n.
3. $K_n \cap S_n = 1$.

Let $B_n := < K_n \cup S_n >$ and $f_i = f_{\{ei\}} = S_{ei}$. Then we have $B_n = < f_1, ..f_n, S_2, ...S_n >$. But $f_i = T_i f_1 T_i^{-1}$ where T_i is a reflection in S_n whose root is $e_i - e_1$.

So we have B_n = $< f_1, S_2, ...S_n >$ where f_1 is a reflection with root $r_1 = e_1$.

Remark. *For B_n we can conclude the following*

1. *root system for B is $\{\pm e_i : 1 \leq i \leq n\} \cup \{e_i \pm e_j : i \neq j, 1 \leq i, j \leq n\}$*
2. *$\{r_1, ...r_n\}$ is a t-base for B_n as any root in B_n is a linear combination of $\{r_1, ...r_n\}$.*
3. *Since $\{r_1, ...r_n\}$ forms a basis for $\mathbb{R}^n$ so B_n is effective and hence a Coxeter group.*
4. *The Coxeter graph for B_n is B_n as $\frac{(r_1, r_2)}{\|r_1\|\|r_2\|} = \frac{-\sqrt{2}}{2}$. and $\frac{(r_i, r_j)}{\|r_i\|\|r_j\|} = \frac{-1}{2}$, $2 \leq i, j \leq n+1$.*

6.4 Construction of a Coxeter group D_n with Coxeter graph D_n

Let $L_n \leq K_n$ such that $L_n = \{f_J : \mid J \mid \text{ is even}\}$. L_n is a subgroup of K_n. Let $D_n = < L_n \cup S_n >$. L_n is generated by elements like $S_{e_i}S_{e_j}$ $i \neq j$.

For $i \neq j, S_{e_i+e_j} \in D_n$ and $S_{e_i-e_j}S_{e_i}+_{e_j} = S_{e_i}S_{e_j}$.

Let $T \in S_n$ be such that $Te_1 = e_i$ and $Te_2 = e_j$.Then we have $TS_{e_1+e_2}T^{-1} = S_{e_i+e_j}$

$\Rightarrow D_n = < S_1, S_2, ..., S_n >$ with roots $r_1 = e_1 + e_2$ and $r_i = e_i - e_{i-1}$ for $2 \leq i \leq n$

Remark. *The following can be concluded about D_n,*

1. *Root system for D_n is $\{e_i \pm e_j : i \neq j, 1 \leq i, j \leq n\}$.*
2. *$\{r_1, ...r_n\}$ is a t-base for D_n as any root in D_n is a linear combination of $\{r_1, ...r_n\}$.*
3. *Since $\{r_1, ...r_n\}$ forms a basis for $\mathbb{R}^n$ so D_n is effective and hence a Coxeter group.*
4. *The Coxeter graph for D_n is D_n.*

6.5 Construction of Coxeter groups with Coxeter Graph $G_2, I_3, I_4, F_4, E_6, E_7, E_8$

In this section the method we use to find a Coxeter group is by extending the base of Coxeter groups A_n or B_n or D_n and obtaining base and hence Coxeter groups having Coxeter graph $G_2, I_3, I_4, F_4, E_6, E_7, E_8$. We will give one example to find the Coxeter group with graph F_4 rest follows similarly hence the detailed proof is omitted and just a summary is given for the rest.

6.5.1 Construction of Coxeter group with Coxeter Graph F_4

We will extend the base of the group B_3 to get the required Coxeter group.

Let the base for Coxeter group with Coxeter Graph F_4 be $\{r_1, r_2, r_3, r_4\}$. Where $\{r_2, r_3, r_4\}$ is the base for B_3. Let $r_1 = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ then using the crystallographic condition and the required values of (r_i, r_j) for all i and j we have $r_1 = (\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2})$. This Coxeter group has Coxeter graph F_4.

Similarly we can do for the rest of the groups.

For G_2 we extend the base of the group A_1.

For I_3 we verify the conditions for the base of symmetric group of icosahedron.

For I_4 we extend the base of the group I_3.
For E_6, E_7 and E_8 we extend the base of the group A_5, A_6 and A_7.
We get the following result:

Graph	Base
A_n	$r_i = e_{i+1} - e_i, 1 \leq i \leq n.$
B_n	$r_1 = e_1, r_i = e_i - e_{i-1}, 2 \leq i \leq n.$
D_n	$r_1 = e_1 + e_2, r_i = e_i - e_{i-1}, 2 \leq i \leq n.$
H_2^n	$r_1 = (1, 0), r_2 = (-\cos \pi/n, \sin \pi/n).$
G_2	$r_1 = e_2 - e_1, r_2 = e_1 - 2e_2 + e_3.$
I_3	$r_1 = \beta(2\alpha + 1, 1, -2\alpha), r_2 = \beta(-2\alpha - 1, 1, 2\alpha),$ $r_3 = \beta(2\alpha, -2\alpha - 1, 1).$
I_4	$r_1 = \beta(2\alpha + 1, 1, -2\alpha, 0), r_2 = \beta(-2\alpha - 1, 1, 2\alpha, 0),$ $r_3 = \beta(2\alpha, -2\alpha - 1, 1, 0), r_4 = \beta(-2\alpha, 0, -2\alpha - 1, 1).$
F_4	$r_1 = -(1/2)\Sigma_1^4 e_i, r_2 = e_1, r_3 = e_2 - e_1, r_4 = e_3 - e_2.$
E_6	$r_1 = (1/2)(\Sigma_1^3 e_i - \Sigma_4^8 e_i), r_i = e_i - e_{i-1}, 2 \leq i \leq 6.$
E_7	$r_1 = (1/2)(\Sigma_1^3 e_i - \Sigma_4^8 e_i), r_i = e_i - e_{i-1}, 2 \leq i \leq 7.$
E_8	$r_1 = (1/2)(\Sigma_1^3 e_i - \Sigma_4^8 e_i), r_i = e_i - e_{i-1}, 2 \leq i \leq 8.$

figure 6.1

6.6 Algorithm to find the Root system of any of the above mentioned groups

Here we will give the basic steps of the algorithm and one example to illustrate the algorithm. Rest will be tabulated.

1. Let $\Gamma_o = \{r_1, ..., r_n\}$ be basis of V. This is actually our t-base for the group G, whose root system we want to find. For $1 \leq i \leq n$ find $S_i r_j$ such that $(r_i, r_j) < 0$. Denote the this set along with Γ_o as Γ_1.

2. For all $r \in \Gamma_1 \backslash \Gamma_o$ for which $(r, r_i) < 0$ find $S_i r$ and denote this set along with Γ_1 by Γ_2.

3. We repeat the above step and get the following set of vectors

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq ....$$

4. The process ends after a finite number of steps for all the groups that we need to find a root system and also we found that every vector obtained by this method is a linear combination of the vectors in the base. Let Γ_k be the set after which this process procedure is terminated then we denote $\Gamma_k \cup (-\Gamma_k)$ by Γ^*.

5. Verify that $S_i\Gamma^* = \Gamma^*, 1 \leq i \leq n$.So $T\Gamma^* = \Gamma^*$ for all $T \in G$. Hence Γ^* is the root system for G.

6.6.1 Example to demonstrate the algorithm

Let the group whose roots we need to find be G_3.

step(1) $\Gamma_o = \{r_1 = e_2 - e_1, r_2 = e_1 - 2e_2 + e_3\}$. We have$(r_1, r_2) = -3 < 0$ So we find $S_1 r_2$ and $S_2 r_1$ whcih are $3r_1 + r_2$ and $r_1 + r_2$ respectively. So $\Gamma_1 = \{r_1, r_2, r_1 + r_2, 3r_1 + r_2\}$

step(2) $(r_1, r_1 + r_2) = -1 < 0$ and $(r_1, 3r_1 + r_2) = 3 > 0$. $(r_2, r_1 + r_2) = 3 > 0$ and $(r_2, 3r_1 + r_2) = -3 < 0$. So we find $S_1(r_1 + r_2) = 2r_1 + r_2$ and $S_2(3r_1 + r_2) = 3r_1 + 2r_2$. Hence $\Gamma_2 = \{r_1, r_2, r_1 + r_2, 3r_1 + r_2, 2r_1 + r_2, 3r_1 + 2r_2\}$

step(3) After this we get $(r_1, 3r_1 + 2r_2) = 0$ and $(r_1, 2r_1 + r_2) = 3 > 0$. $(r_2, 2r_1 + r_2) = 0$ and $(r_2, 3r_1 + 2r_2) = 3 > 0$. So our $\Gamma^* = \Gamma_2 \cup -\Gamma_2$.

step(4) For all r_i and $r_j \in \Gamma_2$ we have if $(r_i, r_j) < 0$ then $S_i r_j \in \Gamma_2$. If $(r_i, r_j) = 0$ then $S_i r = r \in \Gamma_2$. If $(r_i, r_j) > 0$ then we find that $S_i r_j = S_i^2 r_k = r_k \in \Gamma_2$ for some $r_k \in \Gamma_2$. So we get $S_i\ \Gamma_2 = \Gamma_2$ So we have $S_i\Gamma^* = \Gamma^*$. Hence it follows that the root system of $G_2 = \Gamma^*$.

6.7 The root system and its order for various groups

Base	Group	$\lvert\Delta\rvert$	Root system Δ
A_n	$\mathscr{A}_n$	$n^2 + n$	$\pm(e_i - e_j), 1 \le j < i \le n + 1.$
B_n	$\mathscr{B}_n$	$2n^2$	$\pm e_i, 1 \le i \le n; \pm e_i \pm e_j, 1 \le j < i \le n.$
D_n	$\mathscr{D}_n$	$2n(n-1)$	$\pm e_i \pm e_j, 1 \le j < i \le n.$
H_2^n	$\mathscr{H}_2^n$	$2n$	$(\cos j\pi/n, \sin j\pi/n), 0 \le j \le 2n - 1.$
G_2	$\mathscr{G}_2$	12	$\pm(e_i - e_j), 1 \le j < i \le 3; \pm(1, -2, 1), \pm(-2, 1, 1), \pm(1, 1, -2).$
I_3	$\mathscr{I}_3$	30	$\pm e_i, 1 \le i \le 3; \beta(\pm(2\alpha + 1), \pm 1, \pm 2\alpha)$, and all even permutations of coordinates.
I_4	$\mathscr{I}_4$	120	$\pm e_i, 1 \le i \le 4; (1/2)(\pm 1, \pm 1, \pm 1, \pm 1); \beta(\pm 2\alpha, 0, \pm(2\alpha + 1), \pm 1)$, and all even permutations of coordinates.
F_4	$\mathscr{F}_4$	48	$\pm e_i, 1 \le i \le 4; \pm e_i \pm e_j, 1 \le j < i \le 4; (1/2)\Sigma_1^4 \varepsilon_i e_i, \varepsilon_i = \pm 1.$
E_8	$\mathscr{E}_8$	240	$\pm e_i \pm e_j, 1 \le j < i \le 8; (1/2)\Sigma_1^8 \varepsilon_i e_i, \varepsilon_i = \pm 1, \Pi_1^8 \varepsilon_i = -1.$
E_7	$\mathscr{E}_7$	126	Those roots of $\mathscr{E}_8$ orthogonal to $u = (1/2)(1, 1, 1, 1, 1, 1, 1, -1).$
E_6	$\mathscr{E}_6$	72	Those roots of $\mathscr{E}_7$ orthogonal to $r_8 = e_8 - e_7.$

figure 6.2

6.8 Order of the Coxeter groups

For finding the order of these groups we need the following 2 lemmas.

Lemma 6.2. *If $H \leq G$ that fixes the dual basis element s_i then,* $H =< S_1, ...S_{i-1}, S_{i+1}, ...S_n >$.

Lemma 6.3. *Let G be irreducible and its Coxeter graph does not have any mark over its branches that is P_{ij}=3 for all i and j. Then G is transitive as a permutation group on its root system.*

6.8.1 Order of Group A_n

A_n is isomorphic to symmetric group S_{n+1} so $\mid A_n \mid = \mid S_{n+1} \mid$=(n+1)!

6.8.2 Order of Group B_n

$B_n = K_n \rtimes S_n$ So we have $\mid B_n \mid = \mid K_n \mid \mid S_n \mid = 2^n.n!$

6.8.3 Order of Group D_n

$\mid D_n \mid = \frac{|B_n|}{2} = 2^{n-1}.n!$

6.8.4 Order of group H_2^n

As these are the dihedral groups so $\mid H_2^n \mid$=2n.

For the rest of the groups we will follow the following method.

1. We know that for a group G and any element g$\in G$, $\mid G \mid$ = [G:stab(g)] [stab(g)] = [orb(g)] [stab(g)], where orb(g)=orbit of g and stab(g)=stabilizer of g. We will use this result to find the order of the groups.

2. The element of the group which we will choose to find the stabilizer and orbit of is a root r$\in\Delta$ such that r is orthogonal to all but one elements in Π.

3. To find the orbit of the group we follow a similar procedure as that of finding the root system of group. But here in place of (r,r_i) $<$ 0 we find (r,r_i) $\neq$ 0.And compute $S_i r$, which gives us orb(r).

4. To find the stabilizer, if r is not orthogonal to r_i only,then r is a scalar multiple of dual basis vector s_i .So stabilizer of r fixes s_i and Lemma 6.2 gives us stab(r).

For example,Let us Consider group E_6. We have root $r=e_7+e_8$ is orthogonal to all the roots in the base except r_1.$Sostab(r)=$ ¡$S_2, S_3, ..S_6 >$ which is A_5. So $| stab(r) |=| A_5 |=6!$

$| E_6 |=6!\ 72$

As $| orbE_6 |= 72$

Similarly we can do for the rest of the groups.The results are tabulated below.

Group G	root(r)	$r_i s.t(r, r_i) \neq 0$	stab(r)	$\| stab(r) \|$	$\| orb(r) \|$	$\| G \|$
F_4	$e_4 - e_3$	r_4	B_3	48	24	$2^7 3^2$
I_3	$\beta(1, 2\alpha, 2\alpha + 1)$	r_2	$A_1 \times A_1$	4	30	120
I_4	e_4	r_4	i_3	120	120	14400

Group G	root(r)	$r_i s.t(r, r_i) \neq 0$	stab(r)	$\| stab(r) \|$	$\| orb(r) \|$	$\| G \|$
E_6	$e_7 + e_8$	r_1	A_5	6!	72	$2^7.3^4.5$
E_7	$e_1 + e_8$	r_2	D_6	$2^5.6!$	126	$2^{10}.3^4.5.7$
E_8	$(1/2)(\sum_{i=1}^{7} e_i - e_8)$	r_8	E_7	$2^5.6!.126$	240	$2^{14}.3^5.5^2.7$

Next we just tabulate the results we already discussed before.

Group G	$\| G \|$
A_n	(n+1)!
B_n	$2^n.n!$
D_n	$2^{n-1}.n!$
H_2^n	2n
G_2	12

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