Kakeya Sets in Harmonic Analysis

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of BS-MS dual degree in Science*

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Certificate of Examination

This is to certify that the dissertation titled **"Kakeya sets in Harmonic Analysis** submitted by **Aswin G S** (Reg. No. MS13017) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: 18 April, 2018

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. Shobha Madan at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Aswin G S
(Candidate)

Dated: April 2018

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Shobha Madan
(Supervisor)

Acknowledgment

The thesis work carried out in the past one year has been a great experience in my student life. I put on record, my feeling of appreciation to everyone who, directly or indirectly, lent their assistance in this venture.

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Abstract

Kakeya sets (or Besicovitch sets) were first introduced as a solution to a geometrical problem. But, as it turns out, they have applications in solving many seemingly unrelated problems in v various areas of mathematics. This dissertation aims at studying the appearance of Kakeya sets in Harmonic analysis.

We begin with a brief introduction to the Kakeya Needle Problem, which asks for the smallest area of a set in which a unit line segment can be continuously turned around. Besicovitch's solution that such sets can have arbitrarily small area, is explained.

The first application of Kakeya sets in Harmonic analysis was seen in disproving the multiplier problem of the ball, and as a result invalidating the spherical convergence of multiple Fourier series. When the more regularized Bochner-Riesz means are considered, it is proven to be L^p- bounded, at least in large dimensions.

The second part of the thesis begin by investigating the Kakeya conjecture, and its known result in the two dimensional case. A result on the hausdroff dimension of line segments and its extended lines is also briefly explained.

The Kakeya conjectue in the finite field case is easily solved by polynomial method, as explained in Chapter 4.

The last part of the thesis contains a recent study on closed sets with Kakeya property. It is proven that there are no non trivial closed sets with Kakeya property, other than those which can be covered by a null set of parallel lines or concentric circles.

Contents

Chapter 1

Introduction

In 1917, S.Kakeya posed the *Kakeya needle problem:* What is the smallest area required for a set within which one can rotate a unit line segment(a needle) by 360°. Such sets are now called the Kakeya sets. Clearly, a disk is an example. It can be easily seen that a Deltoid also qualifies to be a Kakeya set with a much lesser area $\frac{\pi}{8}$ and for a long time, people believed that Deltoid is in fact the solution for Kakeya problem. Around at the same time, A.S. Beiscovitch, a Russian mathematician was trying to give a counter example to the following problem in analysis:
Given a function of two variables, Riemann-integrable on a plane domain, does there always exist a pair of mutually perpendicular directions such that the repeated simple integration along the two directions exists and gives the value of the integral over the domain?
The problem reduces to that of existence of a plane measure zero which is the union of segments of all directions each of length greater than 1. He constructed such a set (its called the Besicovitch set) and published the result in a Russian journal in 1920. As we can see, the Kakeya problem and Besicovitch's counter example are closely related. Both involves line segments in all directions, while Kakeya problem has an extra requirement of continuous movement of these line segments within arbitrarily small area. Due to the civil wars and blockade in Russia, this problem didn't reach Besicovitch. Few years later, after being aware about the Kakeya problem, He published the solution in 1928. The movement part was handled using 'Pal joins', suggested by a Hungarian mathematician J. Pal, and thus settling the Kakeya needle problem.

Though Kakeya problem is interesting on its own, it has gradually been realized that this type of problem is connected to many other, seemingly unrelated problems in Harmonic analysis, number theory, and arithmetic combinatorics. They

have been long used to construct various counter examples in analysis, starting of course the Riemann integration problem, as we have seen above. Kakeya type constructions was first introduced into Harmonic analysis by Charles Fefferman in his famous result on Ball multiplier problem. We briefly discuss the problem here. One primary question in Fourier analysis is to study about the p norm convergence of Fourier series. Let ψ be a measurable function with compact support on $\mathbb{R}^n$. Define $S_R f(x) = \int \psi(\xi/R)\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$ for $f \in \mathcal{S}(\mathbb{R}^n)$. Do $S_R f$ converges to f in L^p as $R \longrightarrow \infty$?
Fefferman proved that if ψ is the characteristic function of the ball, the partial integrals will not converge to the function in p- norm for $p \neq 2$. In his proof, we can observe a beautiful interplay between multidimensional Fourier analysis and Euclidean geometry.

Since Besicovitch set can have lebesgue measure zero, our usual measure theoretical tools are not enough to analyse these sets, instead we use the Hausdroff measure theory. It is conjectured that any Besicovitch set in $\mathbb{R}^n$ should have the Hausdroff dimension n, called the Kakeya conjecture. It is considered to be one of the hard problems in geometric measure theory. Davies in 1971, gave the proved the conjecture in $n = 2$ case. Works by J.Bourgain, T.Wolff, T.Tao, I.Laba, etc have shown progresses in the estimates on the Hausdroff dimension, But so far, there has been no major breakthrough event in tackling this problem. Kakeya conjecture is inter-connected with two major unsolved problems in Harmonic Analysis; Restriction conjecture and Bochner- Riesz conjecture. Wolff first asked the equivalent of Kakeya problem in finite fields. The motivation was to avoid the technical difficulties one might face in the real case. He conjectured that for a Kakeya set $K \subset \mathbb{F}^n$, the size of K should be at least q^n multiplied by a constant (C_n), where $|\mathbb{F}| = q$. Dvir proved the result in 2009, using polynomial method.

Chapter 2

The Kakeya Needle Problem

The Classical Kakeya problem was to find the minimum area required for a planar set K in which a unit needle can be continuously turned around to come back to its original position. A.S.Besicovitch in 1928 proved that there is no minimum area, i.e., K can have area as small as we please. His construction used Pal joins, a tool for translating a line segment from one position to another parallel position consuming arbitrary small area, and thus made such sets to be multiply connected and to have large diameter. A result by F.Cunningham gave the final statement in this category, by coming up with a bounded simply connected set as a solution to Kakeya Needle Problem.

In this chapter, we study the Kakeya Needle Problem and its solution given by Besicovitch. We will be considering a modified construction due to Perron and Schoenberg, which is much simpler to understand comparing to Besicovitch's original one.

2.1 Kakeya Needle Problem

The problem was first posed by the Japanese mathematician S.Kakeya, in 1917. He stated the problem as follows:

Kakeya Needle Problem : In the class of figures in which a segment of length 1 can be turned around through 360°, remaining always within the figure, which one has the smallest area ?

We have the following result due to Besicovitch[1].

Theorem 2.1. *Given $\epsilon > 0$, there exists a set E in $\mathbb{R}^2$, within which a unit segment can be turned around through $360°$, and its plane measure $\mu(E) < \epsilon$.*

An elementary proof of the theorem contains two key observations.

- For any given $\epsilon > 0$, There exists a movement by which a line segment can be translated to a parallel position in the plane, and the area covered by the movement is less than ϵ. These movements are called *Pál Joints.*
- There exist a set in a plane which contains lines segments in all direction, but has plane measure zero.

Proof. At first, we take a square of side 2 and divide it into four congruent right triangles by joining the center to the vertices. The hypotenuse of each triangle is divided into a large number n of equal parts. Joining each point of division to the center of the square, we have $4n$ "elementary" triangles, each of height 1.

The directions of the various segments which join the vertex of each elementary triangle to every point of its base have a range of $360°$. The same will remain true if we give arbitrary parallel translations to the elementary triangles. As we shall show, parallel translations can be given to these elementary triangles which achieve such a degree of overlapping that the total area covered by the triangles in their new position is as small as we please.

Now if we place an end-point of the unit segment successively at the vertices $O_1, O_2, \ldots$ of the first elementary triangle, the second one, and so on, in their position after translations and in each case rotate it in the positive directions from one side of the triangle to the other, the segment would turn through $360°$. But this movement would not be continuous, for in moving from one triangle to the next one of the segment would not remain within the area of the figure. We eliminate this difficulty by means of Pal's joins, as follows:

Let DEF and GHI (as shown in Figure 2.1) be a pair of consecutive elementary triangles after a parallel translation, and ϵ an arbitrarily small positive number. The sides DF and GH are parallel. Take a point K on HI so that $HK/HI < \epsilon/8$.

Suppose that the lines DF and GK meet in the point L and the triangle LMN is congruent to GHK. We have (denoting area by the sign | |)

$$| LMN |=| GHK |< \frac{\epsilon}{8} | GHI | .$$

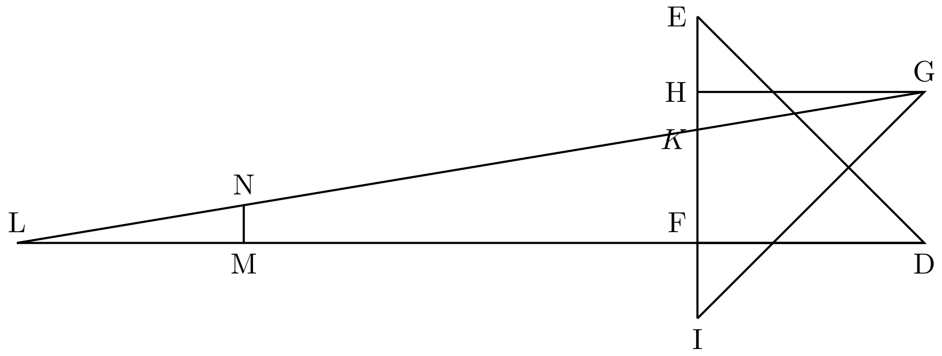

FIGURE 2.1: Pal Joins

The figure consisting of the lines GL, DL and of the triangle LMN will be called the join. We see that the area of the join is less than $\epsilon/8$ times the area of an elementary triangle.Now, if we Connect every pair of consecutive elementary triangles using these joins, we shall get totally $4n$ joins of total area less than $< \epsilon/2$. The join added to the triangles DEF, GHI permits the unit segment to come from the triangle DEF to GHI remaining always on the area of the triangles or of the join. For, from the position of the segment on the side DF we let the segment slide down along the line DL until its lower end-point reaches L, then rotate about L until it reaches the side LN and then slide up until its top end reaches G, that is, gets in the second triangle. Thus the problem is reduced to finding parallel translations of elementary triangles such that the area covered by them be small.

We consider the coordinate plane and an integer $p \geqq 2$. We construct the isosceles right triangle $\Delta = OAB$ of our original square with its hypotenuse of length 2 on the x-axis. The base AB of Δ is divided into $n = 2^{p-2}$ equal parts and n elementary triangles with vertex O are constructed.

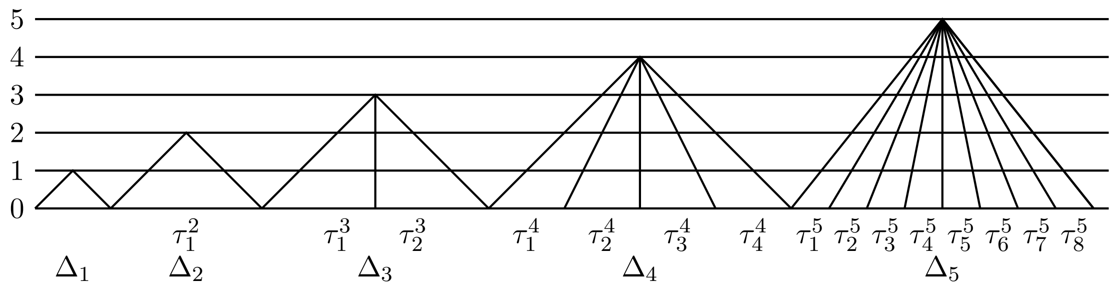

FIGURE 2.2

We next draw the lines $y = k/p$, for $k = 1, 2, \ldots, p$ (Figure 2.2) and call the line $y = k/p$ the line of level k, or simply level k. We then construct p isosceles right triangles $\Delta_1, \Delta_2, \ldots \Delta_p$ each with hypotenuse on the x-axis, and the opposite vertices on the $1, 2, \ldots, p$ levels respectively. Note that $\Delta_p = \delta$. For each $k, k = 3, \ldots, p$, the base of Δ_k is divided into 2^{k-2} equal parts and the elementary triangles are constructed on the subintervals of each base. Notice that Δ_{k+1} is divided into twice as many elementary triangles as Δ_k; Δ_2 is not divided into elementary triangles, Δ_3 is divided into two, Δ_4 into four, Δ_5 into eight, and so on (see Figure 2.2).

We shall now assign labels to the elementary triangles in each Δ_k. These will be labeled from left to right as $\tau_1^k, \tau_2^k, \tau_3^k, \ldots, \tau_j^k, \ldots, \tau_{2^{k-2}}^k$. The superscript k shows that τ_j^k is part of Δ_k and the subscript j says that τ_j^k is the j-th elementary triangle in Δ_k counting from left to right. Δ_2 is not divided into elementary triangles. We shall say it coincides with the elementary triangle τ_1^2; Δ_3 has τ_1^3 and τ_2^3 as elementary triangles; Δ_4 has τ_1^4, τ_2^4, τ_3^4 and τ_4^4; and so on.

Note a simple relationship between the elementary triangles of Δ_k and of Δ_{k+l}.

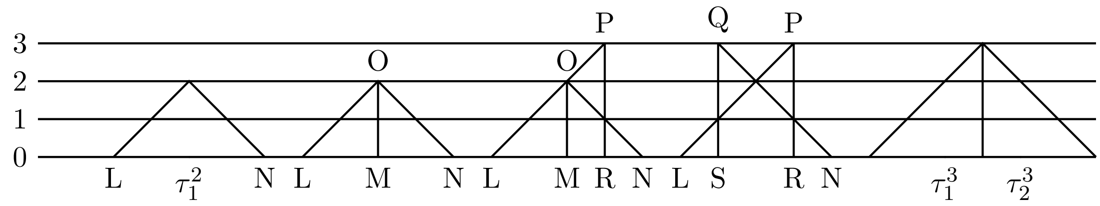

FIGURE 2.3

Let us start with $\tau_1^2 = LON$ (Figure 2.3). Bisect it by the median OM into two triangles OLM and ONM and expand them to similar triangles PLR and QNS to the level 3. We shall call this operation the bisection and expansion. The result of this operation is a pair of triangles congruent to the pair τ_1^3 and τ_2^3 of Δ_3. Similarly is defined the operation of bisection and expansion of the triangles τ_j^k for any $k > 2$: τ_j^k bisected into two triangles by the median from its vertex, and each of the triangles is expanded to the next level. The operation transforms τ_j^k into parallel translates of τ_{2j-1}^{k+1} and τ_{2j}^{k+1}, and applied to the set of all triangles τ_j^k, or to any set of their parallel translates, transforms the set into a set of parallel translates of elementary triangles of Δ_{k+1}. Figure 2.4 represents a particular case of $k = 3$.

The part of Δ_k (or of any elementary triangle τ_j^k of Δ_k) which lies between levels $k-1$ and k will be called the "top end" of Δ_k (or of τ_j^k). Notice that the top end of Δ_k is congruent to Δ_1 and that the sum of the areas of the top ends of all elementary triangles of Δ_k is equal to the area of the top of Δ_k, that is to $\mid \Delta_1 \mid$.

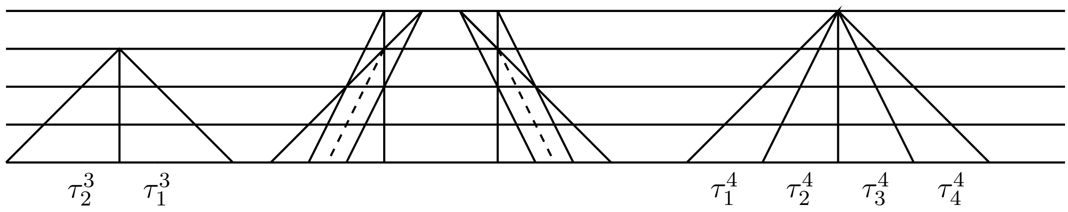

FIGURE 2.4

Now let us look at the change in area when bisection and expansion are applied to a triangle. Consider an elementary triangle $\tau_j^k = LMN$ with vertex N at level k (see Figure 2.5). Let NP be the median of LMN, bisecting it into the two subtriangles LPN and MPN. If we expand LPN upwards and to the right to the level $k+1$, we get a similar triangle LRQ. If we expand MPN upward to the left to level $k+1$, we get a similar triangle MTS.

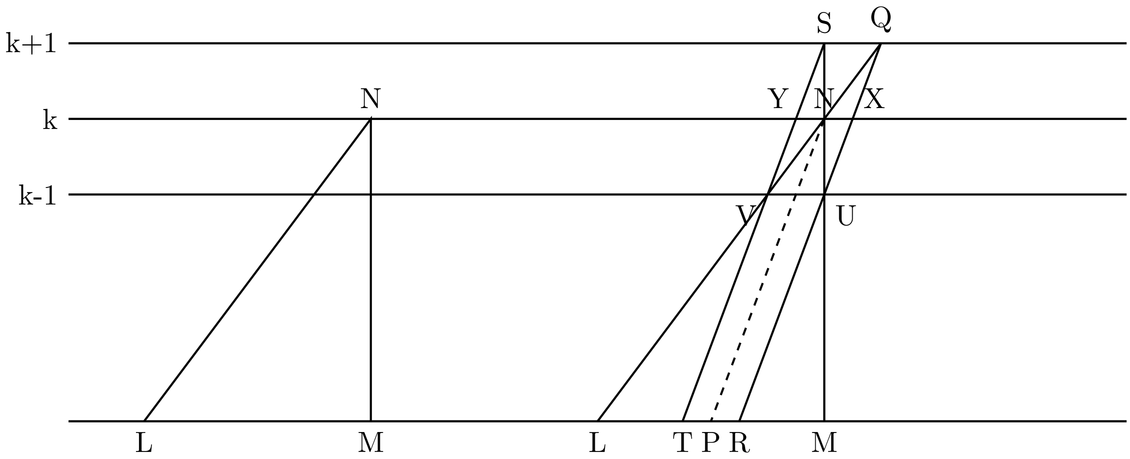

FIGURE 2.5

The two triangles LRQ and MTS together cover the triangle LMN and the two "end pieces" SNV and QNU. We have

$$| SNV |=| QNU |=| NUV |$$

so that the two overlapping triangles cover an area equal to the area of the original triangle LMN plus twice the area of the top end of LMN.

Observe that for constructing the end pieces to LMN, one merely has to know the top end of the triangle LMN. The rule is this: produce the sides LN and MN to the level $k+1$ and join the end-points to the points of the sides on the level $k-1$.

If we start with a complete set of elementary triangles of Δ_k or of their parallel translates and apply bisection and expansion to each of these triangles, we shall arrive at a set of parallel translates of all elementary triangles of Δ_{k+1} and introduce an increase in area at most equal to twice the sum of the areas of the top ends of these triangles. (The "at most" is necessary here because of possible overlapping.) The sum of the areas of the top ends of a complete set of elementary triangles is equal to the area of Δ_1, so that the total increase in area is at most $2 \mid \Delta_1 \mid$.

This conclusion leads immediately to the complete solution of our problem. We start with the triangle $\Delta_2 = \tau_1^2$ and apply to it the bisection and expansion which will transform it into parallel translates of the elementary triangles of Δ_3. Applying the same operation to each of the new triangles we shall get a set of parallel translates of elementary triangles of Δ_4, and so on.

After $p - 2$ such operations we shall arrive at a set of parallel translates of elementary triangles of $\Delta_p = \Delta$. As the increment of the area at each operation is $\leqq 2 \mid \Delta_1 \mid$, the area of the final figure will be

$$\leqq \mid \Delta_2 \mid +2(p-2) \mid \Delta_1 \mid = \frac{4}{p^2} + 2(p-2) \mid \Delta_1 \mid = \frac{4}{p^2} + 2(p-2)\frac{1}{p^2} = \frac{2}{p}.$$

Taking $p > 16/\epsilon$ we shall get the area $< \epsilon/8$. With similar translations for the other $3n$ elementary triangles we shall get the total area covered by the translates $< \epsilon/2$. Adding $4n - 1$ joins of total area $< \epsilon/2$ we shall get a figure of area $< \epsilon$ on which the unit segment can turn round through 360°, which represents a solution of the problem.

The following figure shows the geometric appearance of sets we obtain by the above mentioned construction.

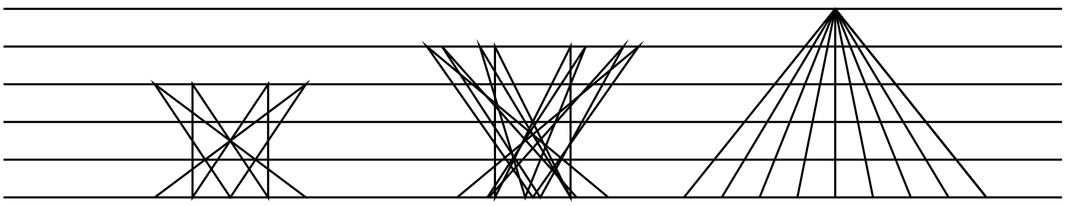

FIGURE 2.6

□

It can be easily proved that the as ϵ goes to 0, these sets converges to a set of measure zero, containing a line segment in all direction(Besicovitch set).
It is clear that due Pal joins, the Kakeya sets constructed are highly multiply connected, and complex in appearance. It was Besicovitch who first posed the problem to construct a simply connected Kakeya set, which was solved by F. Cunningham [3].
We state his result without the proof.

Theorem 2.2. *Given $\epsilon > 0$, there exist a simply connected Kakeya set of area less than ϵ contained in a disc of radius 1.*

Chapter 3

Multiple Fourier Series and Ball Multiplier

The goal of this chapter is to understand the sense in which the Fourier series of a function converges to itself. During the course of time, we will also see how Kakeya sets enter into the frame and plays an important part in achieving the goal. Let us consider a function f on the n-dimensional torus $\mathbb{T}^n$, and its Fourier series. The question is whether the equality shown below involving an infinite sum on one side, is a true statement in the L^p space.

$$f(x) \stackrel{?}{=} \sum_{m \in \mathbb{Z}^n} \hat{f}(m) e^{2\pi i x \cdot m}$$

for $n = 1$, we can prove that the partial sums $S_N(f) = \sum_{-N}^{N} \hat{f}(m) e^{2\pi i x \cdot \xi}$ converges to f as $N \to \infty$ in L^p, due to fact that Hilbert transform is bounded in L^p for $1 < p < \infty$. If $n > 1$, we address the problem by introducing a localizing factor $\phi(\xi)$, that is zero outside a compact set and then to study the convergence of $S_R(f) = \sum_{m \in \mathbb{Z}^n} \phi(m/R) \hat{f}(m) e^{2\pi i x \cdot m}$ as $R \to \infty$.
There is an equivalent problem in $\mathbb{R}^n$. Again, we consider ϕ be a measurable function with compact support on $\mathbb{R}^n$. Define $S_R f(x) = \int \phi(\xi/R) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$ for $f \in \mathcal{S}(\mathbb{R}^n)$. The question we can ask here is whether $S_R f$ converges to f in L^p as $R \longrightarrow \infty$?

3.1 Introduction to Theory of Multipliers

Definition 3.1. Let m be a measurable function in $\mathbb{R}^n$. Consider the map $\mathcal{M}_m$ defined by

$$\mathcal{M}_m f = \widecheck{(m \cdot \hat{f})},$$

where $f \in \mathcal{S}(\mathbb{R}^n)$.

If $\|\mathcal{M}_m(f)\|_p \leqslant C\|f\|_p$ for any Schwartz class function, then the operator can be extended to L^p space. In such a case, We call $\mathcal{M}_m$ to be a Multiplier operator on $L^p(\mathbb{R}^n)$ and m, to be a bounded multiplier function. The collection of all bounded multiplier functions on $L^p(\mathbb{R}^n)$ is denoted by $\mu_p(\mathbb{R}^n)$. It can be verified that it is a normed space with $\|m\|_{\mu_p(\mathbb{R}^n)} = \|\mathcal{M}_m\|_{op}$.

Similar definition exists for Multiplier operators on $L^P(\mathbb{T}^n)$. The difference is that here, we begin with a function on $\mathbb{Z}^n$. i.e., a sequence of numbers $\{m(k) : k \in \mathbb{Z}^n\}$ and the operator is defined by

$$\mathcal{M}_m f = \sum_{k \in \mathbb{Z}^n} m(k)\hat{f}(k)e^{i(k\cdot(.))}$$

The collection of all multiplier sequences forms a normed space $\mu_p(\mathbb{Z}^n)$ with norm as in the previous case.

Examples: For $n = 1$, the multiplier operator corresponding to the function $m(\xi) = -isgn(\xi)$ is called the Hilbert transform, denoted as $\mathcal{H}$. By the theory of singular integrals, it can be proven that $\|\mathcal{H}f\|_p \leqslant C\|f\|_p$ for $1 < p < \infty$. It is easy to see that the multiplier operator corresponding to positive half line $\mathcal{M}_{\chi_{[0,\infty)}} = (I + i\mathcal{H})/2$, where I denotes the identity operator. Therefore, $\mathcal{M}_{\chi_{[0,\infty)}}$ is a L^p multiplier operator for $1 < p < \infty$.

Properties of bounded multiplier functions: The following properties can be proved by carrying out simple computations. Let $m \in \mu_p$, $1 \leqslant p \leqslant \infty$, $x \in \mathbb{R}^n$ and $h > 0$ we have,

$$\begin{aligned}
\|\tau^x(m)\|_{\mu_p} &= \|m\|_{\mu_p} \\
\|D^h(m)\|_{\mu_p} &= \|m\|_{\mu_p} \\
\|e^{2\pi i(.)x} m\|_{\mu_p} &= \|m\|_{\mu_p} \\
\|\widetilde{m}\|_{\mu_p} &= \|m\|_{\mu_p}
\end{aligned}$$

where $\tau^x(m)(y) = m(y-x)$ is the translation operator, $D^h(m)(y) = (1/h^n)m(y/h)$ is the dilation operator and $\widetilde{m}(x) = m(-x)$ is the conjugation operator.

We are mainly interested in two types of multiplier functions - The characteristic function of a ball and a square.

- Consider $m(k) = \chi_{B_R} = \begin{cases} 1, \mid k \mid \leqslant R \\ 0, \text{ else.} \end{cases}$ The corresponding $\mathcal{M}_m$ is denoted by $\widetilde{D}(n, R)$.
- Consider $m(k_1, \dots k_n) = \begin{cases} 1, \mid k_j \mid = R \text{ for some } j \\ 0, \text{ else.} \end{cases}$ $\mathcal{M}_m$ is denoted by $D(n, R)$

Now, we have an interesting result which shows that restriction of a bounded multiplier function is again a bounded multiplier operator.

Theorem 3.2. *Suppose that $m(\xi, \eta) \in \mu_p(\mathbb{R}^{n+m})$, where $1 < p < \infty$. Then for almost every $\xi \in \mathbb{R}$ the function $\eta \to m(\xi, \eta)$ is in $\mu_p(\mathbb{R}^m)$, with*

$$\|m(\xi,)\|_{\mu_p(\mathbb{R}^m)} \leqslant \|m\|_{\mu_p(\mathbb{R}^{n+m})} \tag{3.1}$$

Proof. Since m lies in $L^\infty(\mathbb{R}^{n+m})$, it follows by Fubini's theorem that almost all $\xi \in \mathbb{R}^n$, the function $m(\eta, .) \subset L^\infty(\mathbb{R}^m)$,with

$$\|m(\xi, .)\|_{L^\infty(\mathbb{R}^n)} \leqslant \|m\|_{L^\infty(\mathbb{R}^{n+m})}.$$

Fix f_1, g_1 in $\mathcal{S}(\mathbb{R}^n)$ and f_2, g_2 in $\mathcal{S}(\mathbb{R}^m)$.For all ξ for which (3.1) holds, define

$$M(\xi) = \int_{\mathbb{R}^m} (m(, .)\widehat{f_2})^\vee(y) g_2(y) dy = \int_{\mathbb{R}^m} m(, \eta)\widehat{f}_2(\eta)\check{g}_2(\eta) d\eta$$

and observe that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} (M(.)\widehat{f_1})^\vee(x) g_1(x) dx \right| &= \left| \int_{\mathbb{R}^n} M(\xi)\widehat{f}_1(\xi)\check{g}_1(\xi) d\xi \right| \\ &= \left| \int\int_{\mathbb{R}^{n+m}} m(\xi, \eta)\widehat{f_1 f_2}(\xi, \eta)\widetilde{g_1 g_2}(\xi, \eta) d\xi d\eta \right| \\ &= \left| \int\int_{\mathbb{R}^{n+m}} (m\widehat{f_1 . f_2})^\vee(x, y)(g_1 g_2)(x, y) dx dy \right| \\ &\leqslant \|m\|_{\mu_p(\mathbb{R}^{n+m})} \|f_1\|_p \|f_2\|_p \|g_1\|_{p'} \|g_2\|_{p'}. \end{aligned}$$

Now, because of the identity,

$$\|(M(.)\widehat{f_1})\check{}\|_p = \underset{\text{sup}}{\|g_1\|_{p'}} \leqslant 1 \left| \int_{\mathbb{R}^n} (M(.)\widehat{f_1})\check{}(x) g_1(x) dx \right|$$

we can conclude that $M(\xi) \in \mu_p(\mathbb{R}^n)$, and we also have

$$\|M\|_{\mu_p(\mathbb{R}^n)} \leqslant \|m\|_{\mu_p(\mathbb{R}^{n+m})} \|f_2\|_p \|g_2\|_{p'}.$$

Since $\|M\|_{L^\infty} \leqslant \|M\|_{\mu_p(\mathbb{R}^n)}$ for almost all $\xi \in \mathbb{R}^n$, we obtain

$$\left| \int_{\mathbb{R}^m} (m(\xi, .)\widehat{f_2})\check{}(y) g_2(y) dy \right| \leqslant \|m\|_{\mu_p(\mathbb{R}^{n+gm})} \|f_2\|_p \|g_2\|_p'$$

which gives the required conclusion, by taking the supremum over all g_2 in $L^{p'}$ with norm at most 1. □

3.1.1 Transference theorems

Now, we have the transference theorems using which we can see a correspondence between $\mu_p(\mathbb{Z}^n)$ and $\mu_p(\mathbb{R}^n)$. We begin with the following definition.

Definition 3.3. Consider a bounded function b on $\mathbb{R}^n$. It is said to be regulated at a point t_0 if

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^n} \int_{|t| \leqslant \epsilon} (b(t_0 - t) - b(t_0)) dt = 0. \tag{3.2}$$

b is said to be a regulated function, if it is regulated at all points in $\mathbb{R}^n$.

Indeed, b is regulated at any of its Lebesgue point. Thus, a continuous function is a regulated function. But, this is much weaker idea than of Lebesgue points. For example observe that the function $b(t) = \text{sgn}(t - t_0)$ is regulated at t_0, but it is not a Lebesgue point. We state the following theorems without any proof. Interested reader can refer [17].

Theorem 3.4. *Suppose b is a regulated function at every point $m \in \mathbb{Z}^n$ and lies in $\mu_p(\mathbb{R}^n)$ for some $1 \leqslant p < \infty$. Then, the sequence $\{b(m) : m \in \mathbb{Z}^n\}$ is in $\mu_p(\mathbb{Z}^n)$ and moreover,*

$$\|\{b(m)\}\|_{\mu_p(\mathbb{Z}^n)} \leqslant \|b\|_{\mu_p(\mathbb{R}^n)} \tag{3.3}$$

Also, for all $R > 0$, the sequence $\{b(\frac{m}{R}) : m \in \mathbb{Z}^n\}$ are in $\mu_p(\mathbb{Z}^n)$ and we have

$$\sup_{R>0} \left\| \left\{ b\left(\frac{m}{R}\right) \right\} \right\|_{\mu_p(\mathbb{Z}^n)} \leqslant \|b\|_{\mu_p(\mathbb{R}^n)} \tag{3.4}$$

The second conclusion of the theorem is a consequence of the first, since for any given $R > 0$, the function $b(\xi/R)$ is regulate don $\mathbb{Z}^n$ and has the same norm as b.

Now, we have a converse to the previous theorem. If we have a bounded function b on $\mathbb{R}^n$, and even if $\{b(m)\}_{m\in\mathbb{Z}^n}$ is in $\mu_p(\mathbb{Z}^n)$, it will be a foolish idea to comment on its $\mu_p(\mathbb{R}^n)$ norm, because the whole integer lattice is a set of measure zero. So, we require a more stronger condition as we shall notice below.

Theorem 3.5. *Suppose $b(\psi)$ is a bounded Riemann integrable function over $\mathbb{R}^n$ and that the sequences $\{b(\frac{m}{R})\}$ are in $\mu_p(\mathbb{Z}^n)$ uniformly in $R > 0$, for some $1 \leqslant p < \infty$. Then b is in $\mu_p(\mathbb{R}^n)$ and we have,*

$$\|b\|_{\mu_p(\mathbb{R}^n)} \leqslant \sup_{R>0} \left\| \left\{ b\left(\frac{m}{R}\right) \right\} \right\|_{\mu_p(\mathbb{Z}^n)} \tag{3.5}$$

As a result, we have an important corollary which gives the correlation between the p-convergence of partial Fourier series and L_p boundedness of corresponding multiplier operators.

Corollary 3.6. *Let $1 \leqslant p < \infty, f \in L^P(\mathbb{T}^n)$ and $\alpha \geqslant 0$. Then,*

(a) $\|D(n,R) * f - f\|_{L^P(\mathbb{T}^n)} \to 0$ *as* $R \to \infty \iff \chi_{[-1,1]^n} \in \mu_p(\mathbb{R}^n)$

(b) $\|\widetilde{D}(n,R) * f - f\|_{L^P(\mathbb{T}^n)} \to 0$ *as* $R \to \infty \iff \chi_{B(0,1)} \in \mu_p(\mathbb{R}^n)$

Proof. The convergence is satisfied when f is a trigonometric polynomial, which forms a dense subset for $L^P(\mathbb{T}^n)$. Hence, We can extend the result to the whole space if they are uniformly bounded. The converse of this statements is true by Uniform boundedness principle. So we have,

$$\text{L.H.S. in (a)} \iff \sup_{R>0} \|D(n,R) * f\|_{L^P(\mathbb{T}^n)} \leqslant C_p \|f\|_{L^P(\mathbb{T}^n)}$$

$$\text{L.H.S. in (b)} \iff \sup_{R>0} \|\widetilde{D}(n,R) * f\|_{L^P(\mathbb{T}^n)} \leqslant C_p \|f\|_{L^P(\mathbb{T}^n)}$$

Now we define

$$\widetilde{\chi}_{[-1,1]^n}(x_1, x_2, \ldots, x_n) = \begin{cases} 1, \text{ if } \mid x_j \mid < 1 \ \forall j \\ \frac{1}{2}, \text{ if } \mid x_j \mid = 1 \text{ for some but all } j \\ \frac{1}{2^n}, \text{ if } \mid x_j \mid = 1 \ \forall j \leqslant n \\ 0, \text{ if } \mid x_j \mid > 1 \text{for some } j \end{cases}$$

and

$$\widetilde{\chi}_{B(0,1)}(x) = \begin{cases} 1, \text{ if } \mid x \mid< 1 \\ \frac{1}{2}, \text{ if } \mid x \mid= 1 \\ 0, \text{ if } \mid x \mid> 1 \end{cases}$$

Now both $\widetilde{\chi}_{B(0,1)}$ and $\widetilde{\chi}_{[-1,1]^n}$ are regulated and Riemann integrable. Theorem 3.4 and Theorem 3.5 imply that the uniform boundedness of the operators $D(n, R)$ are $\widetilde{D}(n, R)$ equivalent to the statements that the functions $\widetilde{\chi}_{[-1,1]^n}$ and $\widetilde{\chi}_{B(0,1)}$ are in $\mu_p(\mathbb{R}^n)$ respectively. Since $\widetilde{\chi}_{[-1,1]^n} = \chi_{[-1,1]^n}$ a.e. and $\widetilde{\chi}_{B(0,1)} = \chi_{B(0,1)}$ a.e., the result follows. □

We note here that by much simpler arguments, it can be shown that if $S_R f(x) = \int_{\mathbb{R}^n} \psi(\xi/R)\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$, we have

$$\|S_R f - f\|_{L^P(\mathbb{R}^n)} \to 0 \text{ as } R \to \infty \iff \psi \in \mu_p(\mathbb{R}^n).$$

3.2 Cubic multipliers and Hilbert transform

After the translating the problem of convergence of Multiple Fourier series to a problem of Fourier multipliers, we now focus on the latter. In this section, we consider the case of $\chi_{[-1,1]^n}$ and we prove that this function is in fact a bounded multiplier function, using Hilbert transform. We can actually prove a more general theorem, where the multipliers are characteristic functions of polygons in $\mathbb{R}^n$. As for now, we will be proving this result only for the case $n = 2$.

Theorem 3.7. *Let $\mathbb{P}$ be any closed polygon(convex polyhedron) lying in $\mathbb{R}^2$ (resp., $\mathbb{R}^n$) and having a non empty interior. Then the Fourier multiplier $\mathcal{M}_{\chi_P}$ is bounded on $L^p, 1 < p < \infty$. therefore polygonal summations are valid in L^p for $1 < p < \infty$.*

The outline of the proof is as follows. Using Hilbert transform, We first show that the Half plane multiplier operators are L^p bounded. Now, A polygonal multiplier operator is obtained by finite composition of Half plane multiplier operators and hence the result follows. [16]

Theorem 3.8. *Let P be a point of $\mathbb{R}^2$, $\mathbf{v} \in \mathbb{R}^2$ be a unit vector, and set*

$$E_{\mathbf{v}} = \{x \in \mathbb{R}^2 \colon (x - P) \cdot \mathbf{v} \geqslant 0\}.$$

Then the operator

$$f \longmapsto (\chi_{E_{\mathbf{v}}} \cdot \widehat{f})^{\vee}$$

is bounded on $L^P,\ 1 < p < \infty$.

Proof. As we have stated earlier, the 1-dimensional Hilbert transform

$$H\phi = (-i \text{ sgn } \xi \cdot \widehat{\phi})^{\vee}$$

is bounded on $L^p(\mathbb{R}),\ 1 < p < \infty$. It is often useful to consider, instead of the Hilbert transform H, the operator $M = \frac{1}{2}(I + iH)$ because it has the very simple Fourier multiplier $m = \chi_{[0,\infty)}$.

We now express the multiplier for a half-space as amalgam of multipliers for the half-line, using the Fubini's theorem.
After composition with a rotation and a translation, we may assume that $P = 0$ and that the vector $\mathbf{v}$ is the vector $(0, 1)$. Let us drop the subscript and denote the corresponding half-space by E.
Fix $1 < p < \infty$. Since the Schwartz functions are dense in $L^p(\mathbb{R}^2)$, it suffices for us to perform the estimates for a a Schwartz function f. For almost every $x_1 \in \mathbb{R}$, the function $f_{x_1}(x_2) \equiv f(x_1, x_2)$ is certainly in $L^p(\mathbb{R}^1)$ and, by Fubini's theorem,

$$\int_{x_i \in \mathbb{R}} \|f_{x_1}\|^p_{L^p(\mathbb{R})} dx_1 = \|f\|^p_{L^p(\mathbb{R}^2)}.$$

Now we have

$$(\chi_E \widehat{f})^{\vee}(x_1, x_2) = (2\pi)^{-2} \int_0^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(t_1, t_2) \times e^{i\xi_1 t_1} e^{i\xi_2 t_2} dt_1 dt_2 e^{-i\xi_1 x_1} e^{-i\xi_2 x_2} d\xi_1 d\xi_2.$$

The two inside integrals give rise to a Schwartz function, so all integrals converge absolutely. By Fubini's theorem, the last line equals

$$\frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\frac{1}{2\pi} \int_0^{\infty} \left[\int_{\mathbb{R}} f(t_1, t_2) e^{-2\pi i \xi_2 t_2} dt_2 \right] e^{i2\pi\xi_2 x_2} d\xi_2 \right] \times e^{-i2\pi\xi_1 t_1} dt_1 e^{i2\pi\xi_1 x_1} d\xi_1$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} M(f_{t_1})(x_2) e^{-i2\pi t_1 \xi_1} dt_1 e^{i2\pi\xi_1 x_1} d\xi_1 \tag{3.6}$$

Since, for almost every t_1, $f_{t_1} \in L^p(\mathbb{R})$, we see that Mf_{t_1} makes sense for each t_1. Also

$$\int_{t_1 \in \mathbb{R}} \|Mf_{t_1}(\cdot)\|^p_{L^p(\mathbb{R})} dt_1 \leqslant C \int_{t_1 \in \mathbb{R}} \|f_{t_1}(\cdot)\|^p_{L^p(\mathbb{R})} dt_1 \\ = C\|f\|^p_{L^p(\mathbb{R}^2)}. \tag{3.7}$$

Hence we see that $Mf_{t_1}(x_2)$ is in $L^p(\mathbb{R}^2)$ as a function of the variables (t_1, x_2). In particular, for almost every x_2, the function

$$t_1 \mapsto F_{x_2}(t_1) \equiv M(f_{t_1})(x_2)$$

lies in $L^p(\mathbb{R})$. By equation 3.7 we get

$$\int_{x_2 \in \mathbb{R}} \|F_{x_2}(\cdot)\|^p_{L^p(\mathbb{R})} dx_2 = \int_{t_1 \in \mathbb{R}} \|Mf_{t_1}(\cdot)\|^p_{L^p(\mathbb{R})} dt_1 \leqslant C \cdot \|f\|_{L^p(\mathbb{R}^2)}.$$

In summary, we may rewrite the right-hand side of 3.6 as

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} \frac{1}{2\pi} \int_{\mathbb{R}} F_{x_2}(t_1) e^{it_1\xi_1} dt_1 e^{-i\xi_1 x_1} e^{-\epsilon|\xi_1|^2} d\xi_1$$

by Gauss-Weierstrass summation. But this equals

$$\lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} \check{F}_{x_2}(-\xi_1) e^{-i\xi_1 x_1} e^{-\epsilon|\xi_1|^2} d\xi_1 = F_{x_2}(x_1),$$

for almost every x_1. We have already noted that the latter function has $L^p(\mathbb{R}^2)$ norm dominated by $C\|f\|_{L^p(\mathbb{R}^2)}$. Hence the proof. □

Now consider,

$$E_1 \equiv \{(x, y) \in \mathbb{R}^2 \colon (-1, 0) \cdot [(x, y) - (1, 0)] \geqslant 0\},$$

$$E_2 \equiv \{(x, y) \in \mathbb{R}^2 \colon (1, 0) \cdot [(x, y) - (-1, 0)] \geqslant 0\},$$

$$E_3 \equiv \{(x, y) \in \mathbb{R}^2 \colon (0, -1) \cdot [(x, y) - (0, 1)] \geqslant 0\},$$

$$E_4 \equiv \{(x, y) \in \mathbb{R}^2 \colon (0, 1) \cdot [(x, y) - (0, -1)] \geqslant 0\}.$$

Then E_1, E_2, E_3, E_4 are four half-planes whose common intersection is the unit sphere $Q = \{(x, y) \colon \ | x | \leqslant 1, | y | \leqslant 1\}$ in $\mathbb{R}^2$. Let T_j be the multiplier operator associated to χ_{E_j}, that is,

$$T_j \colon f \longmapsto (\chi_{E_j} \cdot \widehat{f})^{\vee}.$$

Then $T_1 \circ T_2 \circ T_3 \circ T_4$ is the multiplier operator associated to the closed unit square.

We know from the above theorem that each T_j is bounded on L^p. $1 < p < \infty$. As a result, $T_1 \circ T_2 \circ T_3 \circ T_4$ is certainly bounded on L^p for the same range of p. Therefore the multiplier operator associated to the unit square is bounded on L^p, $1 < p < \infty \ldots$ As a result, square summability is valid for double Fourier series, $1 < p < \infty$. Indeed, by the exact same arguments, we obtain the proof for Theorem 3.7.

3.3 Multiplier Problem for Ball

In this section, we investigate the L^p boundedness of Ball multiplier operator, $\mathcal{M}_{\chi_B}$, defined by the property, $\widehat{\mathcal{M}_{\chi_B} f} = \chi_{B(0,1)} \hat{f}$. Unlike characteristic function for the cube (or any general polygon), we **cannot** obtain a ball by composing finitely many half-planes, and therefore it is not possible to tackle the multiplier problem for ball by the same way we have treated the case of polygons in the previous section. In fact, Ball multipliers are not in $\mu_p(\mathbb{R}^n)$ for any n, if $p \neq 2$. C.Fefferman proved this surprising result by a clever exploitation of Kakeya set construction. The theorem is stated below [4]:

Theorem 3.9. *The characteristic function of the unit ball in $\mathbb{R}^n$ is not an L^P multiplier when $1 < p \neq 2 < \infty$*

As a consequence, spherical summation is not valid for multiple Fourier series. We first prove some auxiliary results that will be needed for proving the theorem. The method is to first assume that $\mathcal{M}_{\chi_B}$ is an L^p bounded operator, and arrive at a contradiction. We begin by producing a modified construction of Kakeya sets (due to F. Cunningham) which will be used in the proof to give a counter example.

3.3.1 Sprouting Method

We begin with a triangle ABC, with base $b = AB$ and height h_0. M be the mid-point of AB. We choose a number $h_1 > h_0$, and extend the sides AC & BC till it reaches the height h_1, and denote the new end points as F and E respectively (See 3.1).

We obtain two triangles ΔAMF and ΔBME, called sprouts of ΔABC. Union of the two sprouts is denoted by Spr(ABC). The process of constructing more number of triangles from a given triangle by this method is called Sprouting method (similar to Divide and expansion method mentioned in chapter 1). Consider a sequence of numbers $h_0, h_1, h_2, \ldots h_k$ such that $h_{j-1} < h_j \ \forall \ 0 \leqslant j \leqslant k$.

We begin with a triangle ABC, with base $b = AB$ and height h_0. M be the midpoint of AB. We choose a number $h_1 > h_0$, and extend the sides AC & BC till it reaches the height h_1, and denote the new end points as F and E respectively (See 3.1).

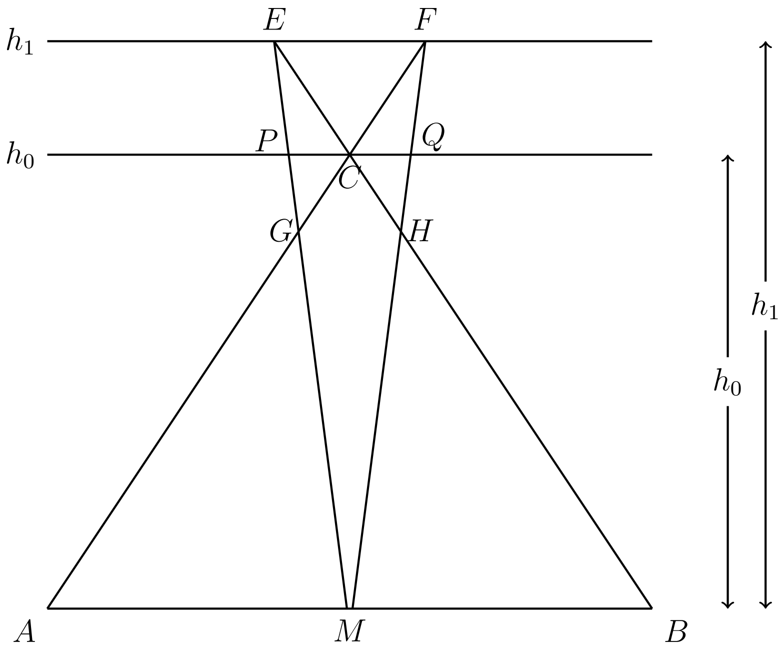

FIGURE 3.1: Sprouting method

We obtain two triangles ΔAMF and ΔBME, called sprouts of ΔABC. Union of the two sprouts is denoted by Spr(ABC). The process of constructing more number of triangles from a given triangle by this method is called Sprouting method (similar to Divide and expansion method mentioned in chapter 1). Consider a sequence of numbers $h_0, h_1, h_2, \ldots h_k$ such that $h_{j-1} < h_j \ \forall \ 0 \leqslant j \leqslant k$.

We can apply the sprouting procedure repeatedly on each triangle at each level, starting from ΔABC at zeroth level. At jth level, We obtain 2^j triangles, each with base length $b_j = \frac{b}{2^j}$ and height h_j, as the sprouts of 2^{j-1} triangles with height h_{j-1}.

Lets fix $b = b_0 = \epsilon$ and $h_0 = \epsilon$. Define the heights $h_j = (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j+1})\epsilon$. Let $E(\epsilon, k)$ be the union of 2^k triangles we obtain at the kth level. We wants to evaluate the total area of $E(\epsilon, k)$. At first, Let's have a detailed look at the first level (Figure 3.2).
We call the difference $Spr(ABC) \backslash ABC$ the arms of the sprouted figure, denoted by $Arm(ABC)$.
It is easy to see that $\mid CP \mid = \mid CQ \mid = \frac{b}{2} . \frac{h_1 - h_0}{h_1}$. Also, If we denote the perpendicular

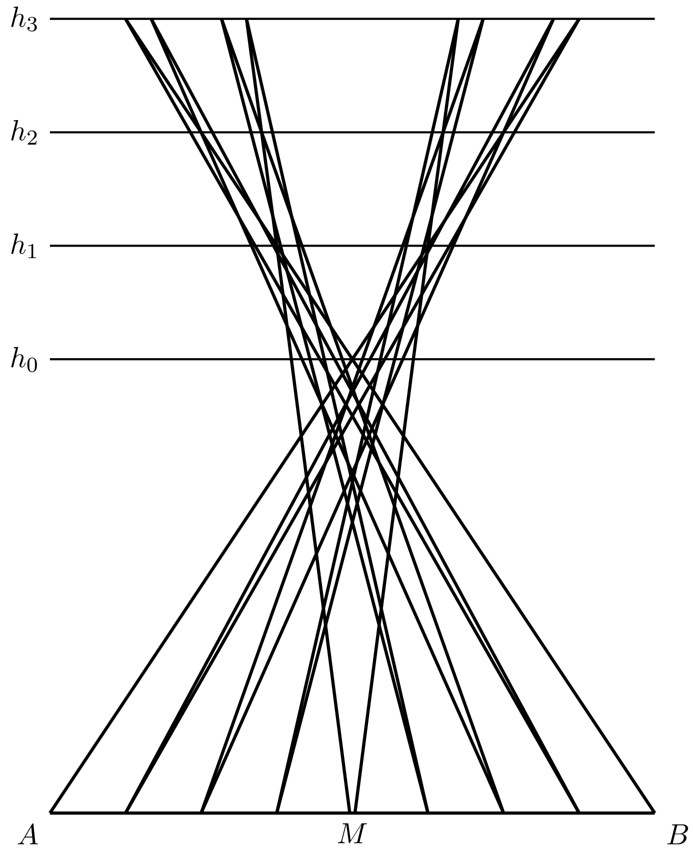

FIGURE 3.2: At the third level of sprouting procedure

height of G from AB by h', we then have $h' = h_0/(1 + \frac{h_1 - h_0}{h_1})$. This expression can be derived from the fact that $\Delta CGP \approx \Delta AGM$. We can easily show that the height of H is also h'.
Now, $\text{Area}(Arm(ABC)) = 2(\mid \Delta CPE \mid + \mid \Delta CGP \mid) = b(h_1 - h_0)^2/(2h_1 - h_0)$.

Hence, at j-th level, the increment in area due to sprouting is $2^{j-1}.\frac{b_{j-1}(h_j - h_{j-1})^2}{2h_j - h_{j-1}}$. Summing over all these areas and adding the area of the original triangle, we obtain the estimate

$$
\begin{aligned}
\mid E(\epsilon, k) \mid &= \frac{1}{2}.\epsilon^2 + \sum_{j=1}^{k} 2^{j-1}.\frac{b_{j-1}(h_j - h_{j-1})^2}{2h_j - h_{j-1}} \\
&\leqslant \frac{1}{2}.\epsilon^2 + \sum_{j=1}^{k} 2^{j-1}.\frac{\epsilon}{2^{j-1}} \frac{\epsilon^2}{(j+1)\epsilon} \\
&\leqslant \frac{1}{2}.\epsilon^2 + \sum_{j=2}^{k} \frac{\epsilon^2}{j^2} \\
&\leqslant \frac{3}{2}\epsilon^2.
\end{aligned}
$$

We do not wish to elaborate on how the Kakeya problem is solved using the the sprouting procedure (An interested reader can look upon Cunningham's paper [reference]). But it can be observed that for a fixed ϵ, the set $E(\epsilon, k)$ contains line segments with more and more directions for larger values of k, but the area always remains lesser than $2\epsilon^2$.

Now, consider a rectangle R in $\mathbb{R}^2$, R' is the union of two copies of R adjacent to R along the two shortest sides as shown in the following diagram. We will now show that a suitable half plane multiplier operator translates R along its longer axis.

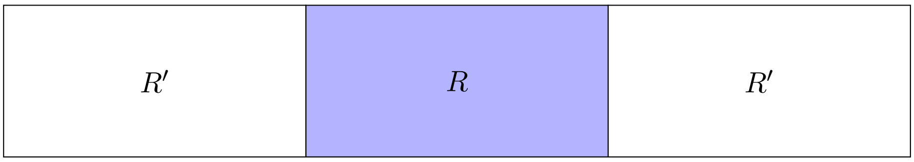

FIGURE 3.3: A rectangle R and its adjacent rectangles R'

Proposition 3.10. *Let R be a rectangle whose center is P in $\mathbb{R}^2$ and let v be a unit vector parallel to its longer side. Consider the half plane,*

$$\mathcal{H} = \{x \in \mathbb{R}^2 : (x - P) \cdot v \geqslant 0\}$$

Then we have $\mid \mathcal{M}_{\mathcal{H}}(\chi_R) \mid \geqslant \frac{1}{10}\chi_{R'}$. where $\mathcal{M}_{\mathcal{H}}$ is the corresponding multiplier operator.

Proof. Since multiplier operators are translation invariant, We can take P to be the origin. Also, applying a rotation, We can assume that $R = [-a, a] \times [-b, b]$ where $0 \leqslant a \leqslant b < \infty$ and $v = (0, 1)$. Now,

$$\begin{aligned} \mathcal{M}_{\mathcal{H}}(\chi_R)(x_1, x_2) &= \chi_{[-a,a]}(x_1)(\widehat{\chi}_{[-b,b]}\chi_{[0,\infty)})^{\vee}(x_2) \\ &= \chi_{[-a,a]}(x_1)\frac{I + iH}{2}(\chi_{[-b,b]})(x_2) \end{aligned}$$

$$\begin{aligned} \mid \mathcal{M}_{\mathcal{H}}(\chi_R)(x_1, x_2) \mid &\geqslant \frac{1}{2}\chi_{[-a,a]}(x_1) \mid H(\chi_{[-b,b]})(x_2) \mid \\ &= \frac{1}{2\pi}\chi_{[-a,a]}(x_1)\left| \log\left|\frac{x_2 + b}{x_2 - b}\right|\right|. \end{aligned}$$

But for $(x_1, x_2) \in R'$,

$$\left| \frac{x_2 + b}{x_2 - b} \right| > 2$$

Hence,

$$| \mathcal{M}_{\mathcal{H}}(\chi_R)(x_1, x_2) | \geqslant \frac{\log 2}{\pi} \geqslant \frac{1}{10}.$$

□

Now, we will see a construction closely related to Kakeya sets, which in turn will lead to the counter example we need in the multiplier problem of ball.

Lemma 3.11. *Let $\delta > 0$ be a given number. Then there exists a measurable subset $E \subset \mathbb{R}^2$ and a finite collection of rectangles R_j such that*

(a) The R_j's are pairwise disjoint.

(b) We have $1/2 \leqslant | E | \leqslant 3/2$.

(c) We have $| E | \leqslant \delta \sum_j | R_j |$.

(d) $| R_j' \cap E | \geqslant \frac{1}{12} | R_j | \ \ \forall j$.

Proof. We begin with a triangle ΔABC as in the sprouting procedure, with $A = (0, 0), B = (1, 0)$, both height and base length 1. For a given $\delta > 0$, choose a k so large such that $k + 2 > e^{1/\delta}$, and choose $E = E(1, k)$. By the previous calculations, (b) is immediately satisfied.

Note that each dyadic interval $[j2^{-k},)(j + 1)2^{-k}]$ is the base of exactly one sprouted triangle, say, $A_jB_jC_j$, where $j \in \{0, 1, \ldots, 2^k - 1\}$. We set $A_j = (j2^{-k}, 0)$, $B_j = ((j + 1)2^{-k}, 0)$. We define R_j inside the angle $\angle A_jC_jB_j$ as shown in figure 3.5, with the length of its longest side $3 \log(k + 2)$. By carefully examining the sprouting construction, we can deduce that the region inside the angles $\angle A_jC_jB_j$ and under the triangle $A_jC_jB_j$ are pairwise disjoint, satisfying (a) in the lemma. By, symmetry we can assume that $| A_jC_j | \geqslant | B_jC_j |$. From the construction of $E(1, k)$, it can be easily seen that the longest possible value that $| A_jC_j |$ can achieve is $\sqrt{5}h_k/2$. We now have that,

$$\frac{\sqrt{5}}{2}h_k < \frac{3}{2}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k+1}\Big) < \frac{3}{2}(1 + \log(k + 1)) < 3 \log(k + 2),$$

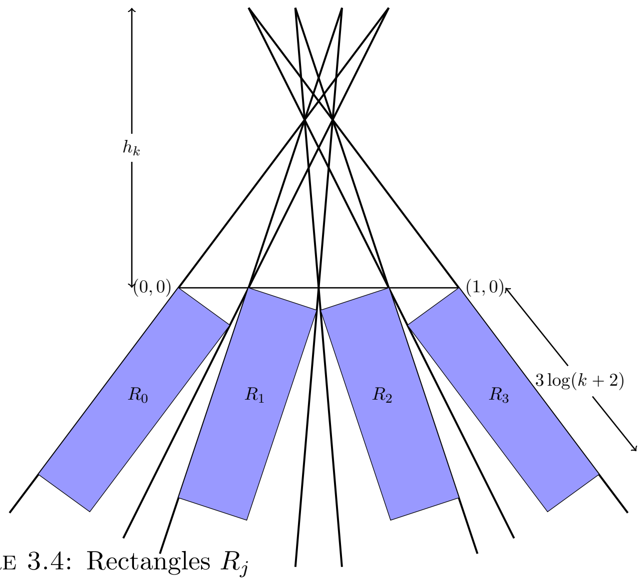

FIGURE 3.4: Rectangles R_j

which implies that R'_j contains the triangle $A_jB_jC_j$. We also have that

$$h_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k+1} > \log(k+2).$$

As a result, we have the following inequality,

$$| R'_J \cap E | \geqslant \text{Area}(A_jB_jC_j) = \frac{1}{2}2^{-k}h_k > 2^{-k-1}\log(k+2). \tag{3.8}$$

Now, by applying the law of sines to the triangle $A_jB_jD_j$ gives

$$| A_jD_j |= 2^{-k}\frac{\sin(\angle A_jB_jD_j)}{\sin(\angle A_jD_jB_j)} \leqslant \frac{2^{-k}}{\cos(\angle A_jC_jB_j)}. \tag{3.9}$$

But the law of cosines applied to the triangle $A_jB_jC_j$ with the estimates $h_k \leqslant| B_jC_j |\leqslant| A_jC_j |\leqslant \sqrt{5}h_k/2$ give that

$$\cos(\angle A_jC_jB_j) \geqslant \frac{h_k^2 + h_k^2 - (2^{-k})^2}{2\frac{5}{4}h_k^2} \geqslant \frac{4}{5} - \frac{2}{5}.\frac{1}{4} \geqslant \frac{1}{2}. \tag{3.10}$$

Combining both the results, we obtain

$$| A_jD_j |\leqslant 2^{1-k} = 2 \mid A_jB_j \mid . \tag{3.11}$$

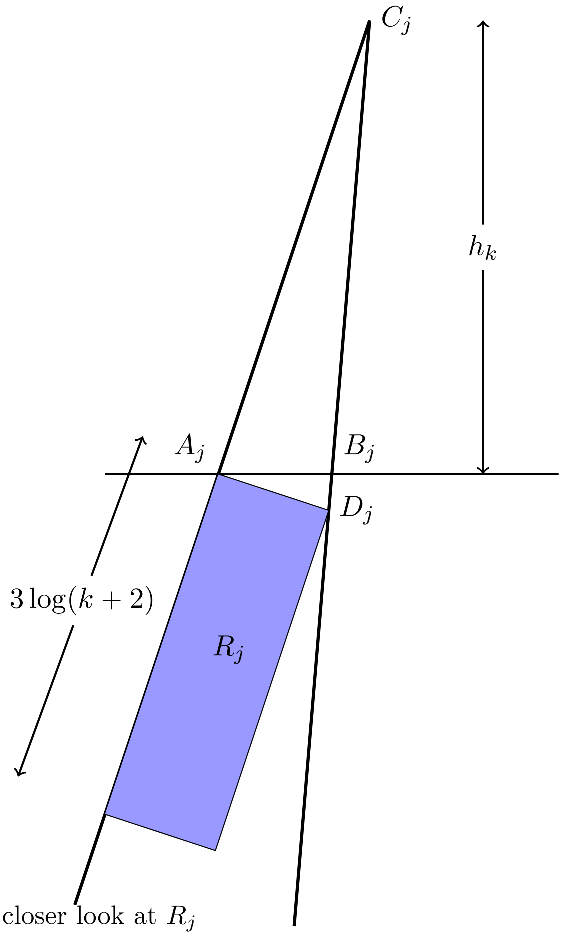

FIGURE 3.5: A closer look at R_j

Using this result and (3.8), we prove that

$$| R_j \cap E | \geqslant 2^{-k-1} \log(k+2) = \frac{1}{12} 2^{-k+1} 3 \log(k+2) \geqslant \frac{1}{12} | R_j | . \tag{3.12}$$

and hence conclusion (d) is proved.
Now, Let us prove the conclusion (c) in the lemma. Applying the law of sines again on triangle $A_jB_jD_j$ gives

$$| A_jD_j | \geqslant 2^{-k} \sin(\angle A_jB_jD_j) \geqslant 2^{-k-1} \angle A_jB_jD_j \geqslant 2^{-k-1} \angle B_jA_jC_j.$$

But the smallest possible value of the angle $\angle B_jA_jC_j$ is attained when $j = 0$, in which case $\angle B_0A_0C_0 = \tan^{-1}(2) > 1$. This gives that

$$| A_jD_j | \geqslant 2^{-k-1}.$$

It follows that each R_j has area at least $2^{-k-1}3\log(k+2)$. Therefore,

$$\left|\bigcup_{j=0}^{2^k-1} R_j\right| = \sum_{j=0}^{2^k-1} \mid R_j \mid \geqslant 2^k 2^{-k-1} 3log(k+2) \geqslant \mid E \mid \log(k+2) \geqslant \frac{\mid E \mid}{\delta},$$

Since $\mid E \mid \leqslant 3/2$ and $k+2 > e^{1/\delta}$. □

An important remark to state here is that the estimates we have in (b) and (d) are independent of δ or number of R_j's. We will need this observation while giving the final proof for the multiplier problem of ball.

Now, We have a lemma regarding the vector-valued inequalities of half plane multipliers. The lemma indeed plays a very important role in proving that ball multiplier is not a bounded L^p-operator.

Lemma 3.12. *(Meyer's lemma) Let $v_1, v_2, \ldots$ be a set of unit vectors in $\mathbb{R}^2$. For each v_j, consider the half plane*

$$\mathcal{H}_j = \{x \in \mathbb{R}^2 : x \cdot v_j \geqslant 0\}$$

and the corresponding multiplier operator $\mathcal{M}_{\mathcal{H}_j}$. If we assume that $\chi_{B(0,1)} \in \mu_p(\mathbb{R}^2)$, then we have the inequality,

$$\Big\| \sum_j (\mid \mathcal{M}_{\mathcal{H}_j}(f_j) \mid^2)^{\frac{1}{2}} \Big\|_p \leqslant C_p \Big\| \sum_j (\mid f_j \mid^2)^{\frac{1}{2}} \Big\|_p, \tag{3.13}$$

where $C_p = \|T\|_{op}$, $T = \mathcal{M}_{\chi_B}$ being the Disk multiplier operator, for all bounded and compactly supported functions f_j.

Proof. We first choose f_j's to be Schwartz functions. We define disks $D_{j,R} = \{x \in \mathbb{R}^2 : \mid x - Rv_j \mid < R\}$. Consider the multiplier operator $T_{j,R}(f) = (\hat{f}\chi_{D_{j,R}})^{\vee}$ associated with each disk. Observe that $\chi_{D(j,R)} \to \chi_{\mathcal{H}_j}$ pointwise as $R \to \infty$.
By passing the limit inside the convergent integral, for $f \in \mathcal{S}(\mathbb{R}^2)$ and every $x \in \mathbb{R}^2$ we have

$$\lim_{R\to\infty} T_{j,R}(f)(x) = \mathcal{M}_{\mathcal{H}_j}(f)(x)$$

Fatou's lemma now yields

$$\Big\| \sum_j (\mid \mathcal{M}_{\mathcal{H}_j}(f_j) \mid^2)^{\frac{1}{2}} \Big\|_p \leqslant \lim_{R\to\infty} \inf \Big\| \sum_j (\mid T_{j,R}(f_j) \mid^2)^{\frac{1}{2}} \Big\|_p. \tag{3.14}$$

Now we note that

$$T_{j,R}(f)(x) = e^{2\pi i R v_j x} T_R(e^{-2\pi i R v_j (.)} f)(x) \tag{3.15}$$

where T_R is the R-dilate of T; $T_R(f) = (\chi_{B(0,R)}\hat{f})^\vee$.Consider the dilation operator $D^R(f)(x) = \frac{1}{R} f(\frac{x}{R})$. Then, We have

$$T_R(f) = D^{1/R}(T(D^R f) \tag{3.16}$$

Setting $g_j = e^{-2\pi i R v_j (.) f_j}$ we can deduce that

$$\Big\| \sum_j (| \mathcal{M}_{\mathcal{H}_j}(f_j) |^2)^{\frac{1}{2}} \Big\|_p \leqslant \lim_{R\to\infty} \inf \Big\| \sum_j (| T_R(f_j) |^2)^{\frac{1}{2}} \Big\|_p. \tag{3.17}$$

Note here that the Dilation of an operator does not change its operator norm. Now, using the vector valued inequality for bounded operator,

$$\Big\| \sum_j (| \mathcal{M}_{\mathcal{H}_j}(f_j) |^2)^{\frac{1}{2}} \Big\|_p \leqslant \lim_{R\to\infty} \inf \Big\| T_R \Big\|_{op} \Big\| \sum_j (| g_j |^2)^{\frac{1}{2}} \Big\|_p = C_p \Big\| \sum_j (| f_j |^2)^{\frac{1}{2}} \Big\|_p \tag{3.18}$$

Now since Schwartz functions are dense in the space of all bounded and compactly supported functions, we can extend the inequality to the required space. □

Having all the auxiliary results, we can now proceed to prove our main theorem. We first consider the Ball multiplier operator T, and assume that it is bounded. Then by Meyer's lemma, we will get the vector inequality involving countably many half plane multiplier operators. Now, If we choose f_j's to be the characteristic functions of R_j's (Rectangles that we have constructed in the previous lemmas), we have seen earlier that Half plane multiplies shift the rectangles, towards the Kakeya set. Hence, the support of its image will be small, and using holders inequality it can be shown that its L^p norm blows up, which gives us the required counter example.

Proof of Theorem 2.7 : By Theorem 3.2, it is enough to look at the case when n = 2. By duality, it suffices to prove the result when $p > 2$. To begin with, we assume that $\|T\|_p \leqslant C_p \|f\|_p$.

Suppose that $\delta > O$ is given. Let E and R_j be as in Lemma 3.11. We let $f_j = \chi_{R_j}$. Let v_j be the unit vector parallel to the long side of R_j and $\mathcal{H}_j$ be the corresponding half-planes. Now using Proposition 3.10 and Lemma 3.11 we obtain

$$\begin{aligned}
\int_E \sum_j \mid \mathcal{M}_{\mathcal{H}_j}(f_j)(x) \mid^2 dx &= \sum_j \int_E \mid \mathcal{M}_{\mathcal{H}_j}(f_j)(x) \mid^2 dx && (3.19)\\
&\geqslant \int_E \frac{1}{100}\chi_{R'_j}(x)dx && (3.20)\\
&= \frac{1}{100}\sum_j \mid E \cap R_j \mid && (3.21)\\
&\geqslant \frac{1}{1200}\sum_j \mid R_j \mid && (3.22)
\end{aligned}$$

Now, using Holder's inequality with exponents $p/2$ and $(p/2)' = p/(p-2)$ gives

$$\begin{aligned}
\int_E \sum_j \mid \mathcal{M}_{\mathcal{H}_j}(f_j)(x) \mid^2 dx &\leqslant \mid E \mid^{\frac{p-2}{p}} \left\| \sum_j (\mid \mathcal{M}_{\mathcal{H}_j}(f_j) \mid^2)^{\frac{1}{2}} \right\|_p^2 && (3.23)\\
&\leqslant C_p^2 \mid E \mid^{\frac{p-2}{p}} \left\| \sum_j (\mid f_j \mid^2)^{\frac{1}{2}} \right\|_p^2 && (3.24)\\
&= C_p^2 \mid E \mid^{\frac{p-2}{p}} \left(\sum_j \mid R_j \mid^2 \right)^{\frac{2}{p}} && (3.25)\\
&\leqslant C_p^2 \delta^{\frac{p-2}{p}} \sum_j \mid R_j \mid && (3.26)
\end{aligned}$$

where we have used Lemma 3.12, disjointness of R_j's and (c) part of Lemma 3.11. Combining (3.22) and (3.26), we obtain the inequality

$$\sum_j \mid R_j \mid \leqslant 1200 C_p \delta^{\frac{p-2}{p}} \sum_j \mid R_j \mid$$

Which gives us the contradiction as δ was arbitrarily chosen. □

3.4 Bochner-Riesz Operators

Once we've shown that ball multiplier is not an L^p multiplier, one question which can be posed is how bad does its operator norm blow up. To start answering this question, we can consider a much smoother multiplier function like $(1- \mid \xi \mid^2)_+^\lambda$ (where + sign indicates that the function takes the value zero outside the unit disk) and consider the multiplier operator corresponding to it.

Definition 3.13. For a function $f \in \mathcal{S}(\mathbb{R}^n)$ we define its Bochner-Riesz multiplier of complex order λ with Re $\lambda > 0$ to be the operator

$$B^{\lambda}(f)(x) = \int_{\mathbb{R}^n} (1- \mid \xi \mid^2)_+^{\lambda} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \tag{3.27}$$

We are investigating whether Bochner-Riesz operator is an L^P multiplier in the case of $n = 2$.

Proposition 3.14. *For all* $1 \leqslant p \leqslant \infty$ *and* $Re\lambda > \frac{n-1}{2}$, B^{λ} *is a bounded operator on* $L^p(\mathbb{R}^n)$.

Proof. It can be seen that B^{λ} is a convolution operator with kernel

$$K_{\lambda}(x) = \frac{\Gamma(\lambda+1)}{\pi^{\lambda}} \frac{J_{\frac{n}{2}+\lambda}(2\pi \mid x \mid)}{\mid x \mid^{\frac{n}{2}+\lambda}} \tag{3.28}$$

Where Γ, J are Gamma function and Bessel function respectively[refer Appendix A]. Now using the estimates for J, We can obtain for $\mid x \mid \leqslant 1$,

$$\mid K_{\lambda}(x) \mid \leqslant \frac{\Gamma(\text{Re}\lambda + 1)}{\pi^{\text{Re}\lambda}} C_0 e^{\pi^2 |\text{Im}\lambda|^2} \tag{3.29}$$

Where C_O is a constant that depends only on $n/2+$ Re λ. For $\mid x \mid \geqslant 1$, we have

$$\mid K_{\lambda}(x) \mid \leqslant C_0 \frac{e^{\pi |Im\lambda| + \pi^2 |Im\lambda|^2}}{\pi^{Re\lambda}(2\pi \mid x \mid)^{\frac{1}{2}}} \frac{\Gamma(\text{Re}\lambda + 1)}{\mid x \mid^{\frac{n}{2}+\text{Re}\lambda}} \tag{3.30}$$

$$= \frac{C(n,\lambda)}{\mid x \mid^{\frac{n+1}{2}+\text{Re}\lambda}} \tag{3.31}$$

Hence for Re$\lambda > \frac{n-1}{2}$, K_{λ} is a smooth integrable function, and hence B^{λ} is a bounded operator on L^p for $1 \leqslant p \leqslant \infty$. □

Proposition 3.15. *When* $\lambda > 0$ *and* $p \leqslant \frac{2n}{n+1+2\lambda}$ *or* $p \geqslant \frac{2n}{n-1-2\lambda}$, *the operators* B^{λ} *are not bounded on* $L^p(\mathbb{R}^n)$

Proof. Let h be a Schwartz function whose Fourier transform is equal to 1 on the ball $B(0,2)$ and vanishes off the ball $B(0,3)$. Then,

$$B^{\lambda}(h)(x) = \int_{|\xi| \leqslant 1} (1- \mid \xi \mid^2)^{\lambda} e^{2\pi i \xi \cdot x} dx = K_{\lambda}(x),$$

and it suffices to show that K_{λ} is not in $L_p(\mathbb{R}^n)$ for the claimed range of ps. Now, we have,

$$\sqrt{2}/2 \leqslant \cos(2\pi \mid x \mid -\frac{\pi(\pi+1)}{4} - \frac{\pi\lambda}{2}) \leqslant 1 \tag{3.32}$$

for all x lying in the annuli and

$$A_k = \left\{x \in \mathbb{R}^n : k + \frac{n+2\lambda}{8} \leqslant \mid x \mid \leqslant k + \frac{n+2\lambda}{8} + \frac{1}{4}\right\}, \qquad k \in \mathbb{Z}^+.$$

According to the asymptotics for Bessel function, K_λ is a smooth function equal to

$$\frac{\Gamma(\lambda+1)}{\pi^{\lambda+1}} \frac{\cos(2\pi \mid x \mid -\frac{\pi(\pi+1)}{4} - \frac{\pi\lambda}{2})}{\mid x \mid^{\frac{n+1}{2}+\lambda}} + O(\mid x \mid^{-\frac{n+3}{2}-\lambda}) \tag{3.33}$$

For $\mid x \mid \geqslant 1$. When $x \in A_k$, the argument of the cosine in the above equation lies in $[2\pi k, 2\pi k + \frac{\pi}{4}]$.
Consider the range of ps that satisfy

$$\frac{2n}{n+1+2\lambda} \geqslant p > \frac{2n}{n+3+2\lambda}. \tag{3.34}$$

If we can show that B^λ is unbounded in this range, it will also have to be unbounded in the bigger range $\frac{2n}{n+1+2} \geqslant p$. This follows by interpolation between the values $p = \frac{2n}{n+3+2\lambda} - \delta$ and $p = \frac{2n}{n+1+2\lambda} + \delta, \delta > 0$, for λ fixed.
Using 3.32 and 3.34, we then obtain that

$$mid K_\lambda \mid_p^p \geqslant C' \sum_{k=n+2\lambda}^{\infty} \int_{A_k} \mid x \mid^{-p\frac{n+1}{2}-p\lambda} dx - C'' - C''' \int_{|x|\geqslant 1} \mid x \mid^{-p\frac{n+3}{2}-p\lambda} dx, \tag{3.35}$$

where C'' is the integral of K_λ in the unit ball. It can be easily proved that the integral outside the unit ball converges, but the series diverges.

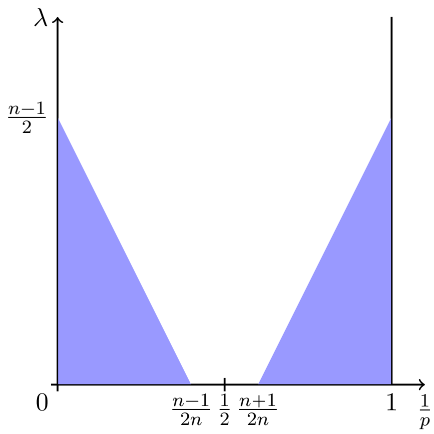

FIGURE 3.6: B^λ is unbounded when $(\frac{1}{p}, \lambda)$ lies in the shaded area.

The conclusion for $p \geqslant \frac{2n}{n-1-2\lambda}$ follows by duality. $\square$

We have proven that B^λ is unbounded for any pair $(\frac{1}{p}, \lambda)$, shown as a shaded region in the Figure 3.6.
It is conjectured that the B^λ is L^p bounded for the unshaded area. As it turns out, Kakeya sets again comes into the frame in the form of Kakeya conjecture,which will be elaborated in the next chapter, and has a close relation with the Bochner-Riesz conjecture.

Bochner- Riesz Conjecture: For $\lambda > 0$, B^λ is a bounded L^p-operator, for all $1 \leqslant p \leqslant \infty$ such that $\mid \frac{1}{p} - \frac{1}{2} \mid < \frac{2\lambda+1}{2n}$ holds.

- For $n = 2$, it is a known result called Carleson-Sjolin Theorem.
- Bochner-Resz conjecture $\implies$ Kakeya conjecture.

Chapter 4

Hausdorff dimension of Besicovitch sets

As we have seen in the previous chapter, There are sets (We call them Besicovitch sets) which contain line segments in all directions but with lebesgue measure zero. It implies that Lebesgue measure theory is just not enough for studying Besicovitch sets, and hence the need for Hausdorff measures, which is commonly used to study sets in fractal geometry. The primary question we are interested is whether Besicovitch sets can have full dimension in $\mathbb{R}^2$ since it contains line segments in all direction, or is it lesser than that, since it can have lebesgue measure zero.

There is a generalization to this problem in the n-dimensional case. A *Besicovitch set* in $\mathbb{R}^n$ is defined to be a set containing a unit line segment in all directions. Lets call it E. It is conjectured that E has dimension n. Davies presented the proof for the case $n = 2$. When $n \geqslant 3$, Kakeya conjecture sill is one of the most infamous unsolved problems at the intersection of geometric measure theory, incidence combinatorics and real-variable harmonic analysis.

In this chapter, we begin by introducing Hausdorff measure and dimension, and move on to present solution for Kakeya conjecture for the case $n = 2$. We will also discuss an interesting result by T. Keleti on comparing the Hasudroff dimension of a set of line segments and its extended line set.

4.1 Introduction

4.1.1 Hausdorff Measure

Definition 4.1. For $E \subset \mathbb{R}^n$, and two given numbers s and δ, Consider any open cover $\{U_i\}$ with each diameter, $| U_i | \leqslant \delta$. Let $\mathcal{C}$ be the collection of all such δ covers. Define,

$$\mathcal{H}^s_\delta(E) = \inf\Big\{ \sum_{i=1}^{\infty} | (U_i) |^s : U_i \in \mathcal{C} \Big\}. \tag{4.1}$$

Following facts can be easily checked.

- $E \subset F \Rightarrow \mathcal{H}^s_\delta(E) \leqslant \mathcal{H}^s_\delta(F)$
- $\delta_1 \geqslant \delta_2 \Rightarrow \mathcal{H}^s_{\delta_1}(E) \leqslant \mathcal{H}^s_{\delta_2}(E)$

Last fact tells us that as we decrease δ towards 0, $\mathcal{H}^s_\delta(E)$ either goes to infinity, or monotonically increases to a non-negative number. We define Hausdorff measure to be the limit.

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E) \tag{4.2}$$

Hausdorff measure is a metric outer measure. Indeed If we consider two sets E and F with $d(E, F) = \inf \{d(x, y) : x \in E, y \in F\} > 0$, we can see that any open δ- covering for $E \bigcup F$ can be written as disjoint union of two open coverings for E and F, provided δ is less than $\frac{1}{2}d(E, F)$. Hence we achieve the equality

$$\mathcal{H}^s\big(E \bigcup F\big) = \mathcal{H}^s(E) + \mathcal{H}^s(F).$$

Now by a Theorem in measure theory [ref.], the σ-algebra of all Hausdorff measurable sets contains Borel sets.

Properties of Hausdorff Measure:

(a) Let $F \subset \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then, $\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$.

(b) Let $F \subset \mathbb{R}^n$, $f : F \longrightarrow \mathbb{R}^m$ be a mapping such that

$$| f(x) - f(y) | \leqslant C | x - y |^\alpha, \forall x, y \in F \text{ (Holder condition)}$$

Then for every s,

$$\mathcal{H}^{\frac{s}{\alpha}}(f(F)) \leqslant C^{\frac{s}{\alpha}} \mathcal{H}^s(F). \tag{4.3}$$

Proof. Let U_i be a $\delta-$ cover for $f(F)$.Since f is an open map, $V_i = f(U_i)$ forms an open cover for $f(F)$. It is easy to observe that $\mid V_i \mid \leqslant C \mid U_i \mid^{\alpha}$. We obtain

$$\mathcal{H}^{\frac{s}{\alpha}}_{C\delta^{\alpha}}(f(F)) \leqslant \sum \mid V_i \mid^{\frac{s}{\alpha}} \leqslant \sum C^{\frac{s}{\alpha}} \mid U_i \mid^{s}$$

As $\delta \to 0$,we have $C\delta^{\alpha} \to 0$ and hence we get the required result. □

Some additional comments:

- For a Lipchitz mapping ($\alpha = 1$), $\mathcal{H}^s(f(F)) \leqslant C^s\mathcal{H}^s(F)$
- If f is an isometry, $\mathcal{H}^s(f(F)) = \mathcal{H}^s(F)$. In particular, Hausdorff measure is translation invariant.

(c) Let $F \subset \mathbb{R}^n$. If $s_1 \leqslant s_2$, then $\mathcal{H}^{s_1}(F) \geqslant \mathcal{H}^{s_2}(F)$. This is immediate after an easy observation that $\mathcal{H}^{s_1}_{\delta}(F) \geqslant \mathcal{H}^{s_2}_{\delta}(F)$ for $\delta < 1$.

(d) For $t \geqslant s$,

$$\sum \mid U_i \mid^{t} \leqslant \delta^{t-s} \sum \mid U_i \mid^{s} \Rightarrow \mathcal{H}^t_{\delta}(F) \leqslant \delta^{t-s}\mathcal{H}^s_{\delta}(F)$$

Hence, $\mathcal{H}^s(F) < \infty \Rightarrow \mathcal{H}^t(F) = 0$.

(e) for $t > n$, we have $\mathcal{H}^t(\mathbb{R}^n) = 0$.

Proof. It is enough to show that $\mathcal{H}^t(B) = 0$ for an n-dimensional unit cube B, since $\mathbb{R}^n$ can be written as countable union of translates of B. Let $\{U_i\}$ be a δ-cover for B. Now,

$$\mathcal{H}^t_{\delta}(B) \leqslant \sum \mid U_i \mid^{n} . \mid U_i \mid^{t-n} \leqslant \left(\sum \mid U_i \mid^{n}\right).\delta^{t-n} \tag{4.4}$$

Since by the definition of lebesgue measure, $\sum \mid U_i \mid^{n} \to$ Vol(B) as $\delta \to 0$. So there exists δ_0 , $C > 0$ such that $\sum \mid U_i \mid^{n} \leqslant C$ for every $\delta \leqslant \delta_0$. Now it is clear from (4.4) that $\mathcal{H}^t_{\delta}(B) = 0$. □

Informally speaking, $\mathcal{H}^1$ is the linear measure of a set, $\mathcal{H}^2$ measures the area, $\mathcal{H}^3$ measures volume and so on.

Proposition 4.2. *For any set $E \subset \mathbb{R}^n$, there exists a G_δ-set G containing E such that* $\dim_H(G) = \dim_H(E)$.

Proof. For each n, consider an open cover $\{U^n_i\}$ for E such that $\sum_i \mid U^n_i \mid \leqslant \dim_H(E) + 1/n$. Now define $G = \bigcap_n \bigcup_i U^n_i$. Clearly, G is a G_δ-set, contains E

and contained in $\bigcup_i U_i^n$ for each n. So we have,

$$\dim_H(E) \leqslant \dim_H(G) \leqslant \dim_H(E) + 1/n \text{ for all } n.$$

The result follows immediately. □

4.1.2 Hausdorff dimension

Definition 4.3. Let $F \subset \mathbb{R}^n$. Define Hausdorff dimension to be

$$\dim_H(F) = \inf \{s \geqslant 0 : \mathcal{H}^s(F) = 0\} \tag{4.5}$$

$$= \sup \{s \geqslant 0 : \mathcal{H}^s(F) = \infty\} \tag{4.6}$$

The existence of Hausdorff dimension is ensured by property (e) and (f) listed above.

Properties of Hausdorff dimension:

- Monotonicity : If $E \subset F$, then $\dim_H(E) \leqslant \dim_H(F)$.
- Countable Stability : For a countable collection of sets $\{F_i\}$,

  $$\dim_H\left(\bigcup_i F_i\right) = \sup\{\dim_H(F_i)\} \tag{4.7}$$

  Proof. By monotonicity, $\dim_H(F_i) \leqslant \dim_H\left(\bigcup F_i\right)$ for any i, So we have LHS $\geqslant$ RHS. Conversely, if $s > \sup_i\{\dim_H(F_i)\}$, then $\mathcal{H}^s(F_i) = 0$ for all i and hence $\mathcal{H}^s\left(\bigcup F_i\right) = 0$. So, $\dim_H\left(\bigcup F_i\right) \leqslant s$, which leads to LHS $\leqslant$ RHS. □

- Any open set F in $\mathbb{R}^n$ contains an open ball of n-dimensional volume. Hence, $\dim_H(F) = n$.
- Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ satisfies holder condition with parameter α, then

  $$\dim_H(f(F)) \leqslant \left(\frac{1}{\alpha}\right) \dim_H(F)$$

  Proof. choose $s > \dim_H(F)$. Then we have $\mathcal{H}^s(F) = 0$. Using (4.4), $\mathcal{H}^{\frac{s}{\alpha}}(f(F)) = 0 \implies \dim_H(f(F)) \leqslant s$. Since we have chosen s arbitrarily, the result follows. □

- A set $F \subset \mathbb{R}^n$ with $\dim_H(F) < 1$ is totally disconnected.

  Proof. Choose $x, y \in F$. Consider $f : \mathbb{R}^n \Rightarrow [0, \infty)$ defined by $f(z) = \mid z - x \mid$. Clearly, $\mid f(z) - f(w) \mid \leqslant \mid z - w \mid$ and by the above result,

  $$\dim_H(f(F)) \leqslant \dim_H(F) < 1.$$

  Thus $f(F) \subset \mathbb{R}$ is of $\mathcal{H}^1$ measure (or length) 0. Hence it has a dense compliment. Choosing r with $r \notin f(F)$ & $0 < r < f(y)$, we have,

  $$F = \{z \in F : \mid z - x \mid < r\} \cup \{z \in F : \mid z - x \mid > r\},$$

  Where x and y belongs to distinct open sets. Hence, singletons are the only non-trivial connected sets in F. □

4.2 Tools to calculate Hausdorff dimension

There are two important methods used to get bounds for the dimension of a set. One is slicing the set with parallel lines and studying the dimension of the intersection, whereas the second method is to analyse the dimension of its image under an orthogonal projection on a line. More details can be found in [14] and [15].

4.2.1 Slicing theorems

Theorem 4.4. *Let F be a Borel subset of* $\mathbb{R}^2$*. Let* L_x *be the vertical line with abscissa* x*. If* $1 \leqslant s \leqslant 2$*, then,*

$$\int_{-\infty}^{\infty} H^{s-1}(F \cap L_x)dx \leqslant \mathcal{H}^s(F) \tag{4.8}$$

Proof. For $\epsilon > 0$, Let $\{U_i\}$ be a δ -cover of F such that

$$\sum \mid U_i \mid^s \leqslant \mathcal{H}^s_\delta + \epsilon$$

Now each U_i is contained in a square S_i, whose sides are parallel to the co-ordinate axis with a magnitude of $\mid U_i \mid$. Let, χ_i be the indicator function of S_i. Now,

$\{S_i \cap L_x\}$ forms a δ-cover for $F \cap L_x$.

$$\begin{aligned}
\mathcal{H}_\delta^{s-1}(F \cap L_x) &\leqslant \sum_i | S_i \cap L_x |^{s-1} \\
&\leqslant \sum_i | U_i |^{s-2} | S_i \cap L_x | \\
&= \sum_i | U_i |^{s-2} \int \chi_i(x,y)dy \\
\int \mathcal{H}_\delta^{s-1}(F \cap L_x)dx &\leqslant \sum | U_i |^{s-2} \int\int \chi_i(x,y)dydx \\
&= \sum | U_i |^s \leqslant \mathcal{H}_\delta^s(F) + \epsilon
\end{aligned}$$

Since ϵ was chosen arbitrarily, the result follows. □

Corollary 4.5. *Let F be a Borel subset of $\mathbb{R}^2$ with $\dim_H(F) \geqslant 1$. Then $\forall\ x \in \mathbb{R}$ a.e.,*

$$\dim_H(F \cap L_x) \leqslant \dim_H(F) - 1.$$

Proof. Choose $s > \dim_H(F)$ so that $\mathcal{H}^s(F) = 0$. By Theorem 4.4, $\mathcal{H}^{s-1}(F \cap L_x) = 0$ for almost all $x \in \mathbb{R}$. Hence, $s - 1 \geqslant \dim_H(F \cap L_x)\ \forall\ x \in \mathbb{R}$ a.e. □

4.2.2 Projection Theorems

Let L_θ be the line in $\mathbb{R}^2$ that passes through origin at an angle θ consider the Projection $proj_\theta : \mathbb{R}^2 \longrightarrow L_\theta$. Clearly, $proj_\theta$ is a lipschitz mapping, and its image lies in a line. Hence,

$$\dim_H(proj_\theta(F)) \leqslant \min\{\dim_H(F), 1\}\ \forall\ \theta \in [0, \pi] \tag{4.9}$$

In fact the opposite inequality in (1.5) is true for almost all θ. We state the following theorem without a proof. Interested reader can refer [14] for the proof.

Theorem 4.6. *Let $F \subset \mathbb{R}^2$ be a Borel set. Then the following conditions hold.*

1. *If $\dim_H(F) \leqslant 1$, then $\dim_H(proj_\theta(F)) = \dim_H(F)\ \forall\ \theta \in [0, \pi]$ a.e.*

2. *If $\dim_H(F) > 1$, then $proj_\theta(F)$ has positive length and has dimension 1 for almost all $\theta \in [0, \pi]$*

4.3 Dimension of Besicovitch sets

Having enough material, we can now proceed to the main theorem in this chapter. The key idea is to associate a line to a unique point in $\mathbb{R}^2$, and observe a duality behavior between the slices of line set and projections of point set. We utilize the duality to get a lower bound on dimension of Besicovitch set.

Duality arguments had been used since Besicovitch [6] and Davies [5], using polar reciprocation as an association between lines and points; fix a circle C in $\mathbb{R}^2$, and find the pole corresponding to each line (polar) with respect to C. Besicovitch used the technique to construct a very simple example of a set of measure zero containing a line in all directions. The following proof is a modified version of the original one by Davies [5]. By a plane set, we mean a subset of $\mathbb{R}^2$.

Theorem 4.7. *Any plane set E containing a line segment in every direction must have a Hausdorff dimension 2.*

Duality argument: For a given point (a, b) in the plane, define $L(a, b) = \{a+bx : x \in \mathbb{R}\}$. L is a bijection from $\mathbb{R}^2$ to the collection of all lines in $\mathbb{R}^2$. For any plane set F, its line set is defined to be $L(F) = \bigcup_{(a,b)\in F} L(a, b)$. Let $c = \tan(\theta)$. Observe that, $L(F) \cap L_c = \{(c,(a, b) \cdot (1, c)) : (a, b) \in F\}$ is just a scaled copy of $proj_\theta(F)$ by a factor of $\sqrt{(1 + c^2)}$. It is easy to see that

$$\dim_H(L(F) \cap L_c) = \dim_H(proj_\theta F) \tag{4.10}$$

$$\mathcal{L}(L(F) \cap L_c) = 0 \iff \mathcal{L}(proj_\theta F) = 0 \tag{4.11}$$

Proof of Theorem 3.7 : Let E be a set containing line segments in all direction. We can find two parallel lines L_x and L_y such that the set of directions of segments contained in E that intersects both is of linear measure greater than zero. By translating and scaling we may assume that $L_x = L_0$ and $L_y = L_1$. If we prove the result for the E_1, subset of E containing line segments that intersects both L_0 and L_1, clearly it follows for E. Hence, without loss of generality, let us assume that $E = E_1$.

Consider the collection of lines we get by extending each line segment of E, we denote it by E'. We may assume E' to be a Borel set for now. Proof of this claim will be given in Lemma 4.10. Take $F = \{(a, b) : L(a, b) \subset E'\}$. Since L is a continuous map, $F = L^{-1}(E')$ is Borel. Since the set of all directions of lines in

E' has positive linear measure, $proj_0 F$ contains an interval in X-axis. Hence,

$$\dim_H F \geqslant \dim_H(proj_0 F) = 1 \tag{4.12}$$

We observe that $L(F) \cap L_c = E \cap L_c \ \forall \ c \in [0,1]$. Combining results from Corollary 4.5, Theorem 4.6 and duality argument, we obtain

$$\begin{aligned}\dim_H E &\geqslant \dim_H(L(F) \cap L_c) + 1 = \dim_H(proj_\theta F) + 1 \\ &= \min\{2, 1 + \dim_H(F)\}\end{aligned}$$

Though the results from Corollary 4.5, Theorem 4.6 are true in almost all cases, we can find a θ and $c = \tan\theta$ where they are satisfied. Combining the last inequality with (4.12) implies that $\dim_H E \geqslant 2$, which completes the proof. □

4.3.1 Kakeya conjecture

Statement of Kakeya Conjecture: If E is a set in $\mathbb{R}^n$ containing unit line segments in all directions, then $\dim_H(E) = n$.

There is a corresponding conjecture for the Minkowski dimension. Except for the cases $n = 1$ and 2, no one could come up with a complete solution for the conjecture till date, although rapid progress has been made during the last few decades. We look at the known progress in Kakeya conjecture. Let's denote the Hausdorff [resp. Minkoswki] dimension of Besicovitch set in $\mathbb{R}^n$ to be $d(n)$ [resp. $d_M(n)$].

- Davies(1971) solved the case $n = 2$.
- Bourgain(1991) proved that $d(n) \geqslant \frac{(n+1)}{2}$ using "Bush" argument; In the same paper he shows that infact $d(n) \geqslant \frac{(n+1)}{2} + \epsilon_n$ for some $\epsilon_n > 0$.
- Wolff's agrument (1995) gives $d(n) \geqslant \frac{(n+2)}{2}$.
- Bourgain (1998) proved that $d(n) \geqslant 13n/25 + 12/25$.
- Tao,Laba and Katz(1999) have shown that $d_M(3) \geqslant 5/2 + 10^{-10}$.
- Katz,Tao (2001) have shown that $d(n) \geqslant (2 - \sqrt{2})(n-4) + 3$ for $n > 4$.

4.4 Lines and line segments

Here, we study the question if extending the line segments to full lines can increase the Hausdroff dimension. We pose the following problem :
Line Segment Extension Conjecture: If A is the union of line segments in $\mathbb{R}^n$, and B is the union of the corresponding full lines, then the Hausdroff dimensions of A and B agree.

Though Line Segment Extension Conjecture in its own is not an interesting result, it can have very strong consequences. We state the following result by T.Keleti [10] without the proof.

Theorem 4.8.
- *Line Segment Extension Conjecture for n would imply that every Besicovitch set in $\mathbb{R}^n$ has Hausdroff dimension at least $n-1$.*
- *If the Line Segment Extension Conjecture holds for every $n \geqslant 2$, then every Besicovitch set in $\mathbb{R}^n$ has packing and upper Minkowski dimension n.*

Now, we present some evidence for the conjecture. We have the following theorem proving the conjecture for $n = 2$:

Theorem 4.9. *Let S be a collection of line segments in $\mathbb{R}^2$, and $\mathcal{L}(S)$ be the collection of lines we get by extending S. Then $\dim_H(\cup S) = \dim(\cup \mathcal{L}(S))$.*

Here, $\cup\mathcal{S}$ (or $\cup\mathcal{L}(\mathcal{S})$) denotes the union of all lines sements (or lines) in $\mathcal{S}$ (or $\mathcal{L}(\mathcal{S})$) as a subset of $\mathbb{R}^2$.
Now we have a lemma which can save us from many measurability assumptions while doing the calculation.

Lemma 4.10. *For any collection $\mathcal{S}$ of closed line segments in $\mathbb{R}^n$ there exists a collection $\mathcal{S}' \supset \mathcal{S}$ of closed line segments with $\dim_H(\cup\mathcal{S}') = \dim_H(\cup\mathcal{S})$ such that $\mathcal{L}(\mathcal{S})$ is Borel.*

Proof. We can suppose that for some fixed $\delta > 0$ and bounded open set $B \subset \mathbb{R}^n$ each $s \in \mathcal{S}$ is contained in B and has length at least δ since we can write $\mathcal{S}$ as as countable union $\mathcal{S} = \cup_j \mathcal{S}^j$ of such sub collections and if $\mathcal{S}^{j'}$ is good for $\mathcal{S}^j$, then $\cup_j \mathcal{S}^{j'}$ is good for $\mathcal{S}$.
Let $A = \cup\mathcal{S}$. Then $A \subset B$. We can find a G_δ set $A' \supset A$ with $A' \subset B$ and $\dim_H(A') = \dim_H(A)$. We write $A' = \cap_{k=1}^{\infty} G_k$, G_k's are open, $B \supset G_1$ and $G_k \supset G_{k+1}$. Let $\mathcal{S}_k$ be the collection of those closed line segments inside G_k that

have length at least δ. Let $\mathcal{S}' = \cap_{k=1}^{\infty}\mathcal{S}_k$.
Then $\mathcal{S}' \supset \mathcal{S}$, since each $\mathcal{S}_k$ contains $\mathcal{S}$. Since for any k, $\cup\mathcal{S}' \subset \cup\mathcal{S}_k \subset G_k$ by construction, we have $\cup\mathcal{S}' \subset \cap_k G_k = A'$, so $\dim_H \cup\mathcal{S} \leqslant \dim_H \cup\mathcal{S}' \leqslant \dim_H A' = \dim_H \cup\mathcal{S}$, hence $\dim_H \cup\mathcal{S} = \dim_H \cup\mathcal{S}'$.
We claim that $\mathcal{L}(\mathcal{S}') = \bigcap_{k=1}^{\infty}\mathcal{L}(\mathcal{S}_k)$. It is easy to see that $\mathcal{L}(\mathcal{S}') \subseteq \bigcap_{k=1}^{\infty}\mathcal{L}(\mathcal{S}_k)$. Conversely, if $l \in \bigcap_{k=1}^{\infty}\mathcal{L}(\mathcal{S}_k)$, then for each k, the set $l \cap G_k$ contains a closed line segment of length at least δ. Since, B is bounded and $B \supset G_1 \supset G_2 \supset ...$, this implies that there exists a closed line segment$s \subset l$ of length at least δ that is contained in every G_k, so $s \in \mathcal{S}'$ and $l \in \mathcal{L}(\mathcal{S}')$.
Since G_k is open, $\mathcal{L}(\mathcal{S}_k)$ is also open, so $\mathcal{L}(\mathcal{S}') = \cap_k\mathcal{L}(\mathcal{S}_k)$ is Borel. □

Proof of Theorem 3.9 : S can be written as countable union of sub-collections S_i with the property that, each line segment $s \in S_i$ intersect two fixed segments e_i and f_i, which are opposite sides of a rectangle. Hence if we prove the result on each sub-collection, it follows for S.
So, we make the assumptions that S has this property, $\mathcal{L}(S)$ is Borel, and $\cup\mathcal{L}(S)$ is analytic. These assumptions are made in order to smoothly apply some results.
Now, Choose a number u such that $u < \dim_H(\cup\mathcal{L}(S))$. We will now show that $u \leqslant \dim_H(\cup S)$. Which will imply that, $\dim_H(\cup\mathcal{L}(S)) \leqslant \dim_H(\cup S)$ and the result follows.
For $u = 1$, It is clear, since $\cup S$ is not a totally disconnected set.
For $1 < u \leqslant 2$, we uses the following theorem.

Marstrand's slicing theorem [13]: If $u > 1$ and an analytic subset A of the plane has positive u-dimensional Hausdorff measure, then in almost every direction, positively many lines meet A in a set of Hausdorff dimension atleast $u - 1$.
In other words, when the assumptions are satisfied, for almost every unit vector w, there exists $T \subset \mathbb{R}$ of positive lebesgue measure such that $\forall\ t \in T$, we have

$$\dim_H((\cup\mathcal{L}(S)) \cap l_{w,t}) \geqslant u - 1$$

where $l_{w,t} = \{a \in \mathbb{R}^2 : a \cdot w = t\}$.
Choose distinct parallel lines l_0, l_1 such that both separates e, f and they are orthogonal to a unit vector w with the above property. Then every line segment of S intersect both l_0 and l_1.
Without loss of generality, we assume that $w = (1,0), l_0 = v_0, l_1 = v_1$ where v_t is the vertical line with x-coordinate t. Thus, $\dim_H(\mathcal{L}(S) \cap v_t) \geqslant u - 1\ \forall\ t \in T$.

Since T has a positive Lebesgue measure, $dim_H((\cup\mathcal{L}(S)) \cap v_t) \geqslant u-1 \ \forall \ t \in \mathbb{R} \ a.e.$
Since $(\cup\mathcal{L}(S)) \cap v_t = (\cup S) \cap v_t$ for every $t \in [0,1]$, we obtain,
$\dim_H((\cup S) \cap v_t) \geqslant u - 1$ and hence $\dim_H(\cup S) \geqslant u$.

□

Chapter 5

Kakeya Problem in finite field

5.1 Introduction

Let $\mathbb{F}_q$ be a finite field with $\mid \mathbb{F} \mid = q$. A Kakeya set $K \subset \mathbb{F}z - q^n$ is a set containing a line in every direction. Formally,

$$\forall\ x \in \mathbb{F}^n,\ \exists y \in \mathbb{F}^n \text{ such that } \{y + tx : t \in \mathbb{F}\} \subseteq K.$$

T.Wolff first posed the equivalent of Kakeya problem in finite field case, in his survey. The motivation was to try and understand better the more complicated questions regarding Kakeya sets in $\mathbb{R}^n$. It was asked by Wolff that

Question: Can we find a estimate of the form $\mid K \mid \geqslant C_n q^n$.?

The first estimate on this problem given by Wolff was of the form $\mid K \mid \geqslant C_n q^{\frac{n+2}{2}}$, using arguments such as counting the incident points of K with lines with different directions. Bounds were improved later on using similar arguments in incidence geometry, but nobody could come up with a complete solution until 2005, when Z.Dvir produced a lower bound of $C_n q^{n-1}$. With a simple observation that product of Kakeya sets also forms a Kakeya set, we could instantly improve the lower bound to $C_{n,\epsilon} q^{n-\epsilon}$. Dvir's strategy was to show that there are no non-zero low degree polynomials which vanish on K, and thus K should have some minimum number of elements in it. Following the initial publication of Dvir's result, T. Tao and N. Alon observed that his arguments can be modified to obtain the required lower bound of $C_n q^n$, and thus solves the Kakeya problem in finite field situation completely.

In this chapter, we reproduce Dvir's proof to achieve the bound of $C_n q^{n-1}$ and its modification for $C_n q^n$.

5.2 Nikodym sets and the first bound $[C_n q^{n-1}]$.

Throughout this chapter we follow the notation $\mathbb{F}$ for an arbitrary field and $\mathbb{F}_q$ for a finite field with cardinality q.
We begin with introducing Nikodym sets, which are closely related to Kakeya sets.

Definition 5.1 (Nikodym set). A set $M \subset \mathbb{F}^n$ is called a Nikodym set if

$$\forall y \notin M, \ \exists\, x \ \text{ such that } \{y + tx : t \in \mathbb{F}_q^*\} \subseteq M.$$

Given a Kakeya set K, it is easy to construct a Nikodym set. We can define $M := \{tx : t \in \mathbb{F}, x \in K\}$. For each $x \in \mathbb{F}^n$, there exists $y \in \mathbb{F}^n$ such that $\{y + tx \ : t \in \mathbb{F}\} \subset K$. Hence, $\{sy + stx : s, t \in \mathbb{F}\} \subset M$. Now, putting t = 1/s implies $\{sy + x : s \in \mathbb{F}_q^*\} \in M$.

5.2.1 Polynomial Method

The polynomial method is used to impose an algebraic structure on a geometric problem. It gives us a correlation between the size of the zero set of a polynomial to its degree. It relies on two simple lemmas, which we have shown below.The first lemma gives an upper bound on the number of solutions for a multi-variable polynomial.

Lemma 5.2 (Schwartz-Zippel Lemma). *Let $d \geqslant 0$. If $p \in \mathbb{F}_q[x_1, x_2, ...x_n]$ is a non-trivial polynomial of degree at most d, then $| \{x \in \mathbb{F}^n \ : p(x) = 0\} | \leqslant dq^{n-1}$.*

Proof. We prove the lemma by induction.
Let $E(f)$ be the solution set of $f \in \mathbb{F}_q[x_1, ..x_n]$. case $k = 1$ follows from fundamental theorem of algebra.
Now, assuming the theorem for $k = n - 1$, we consider a polynomial $f(x)$ with the mentioned properties. Let, $f(x_1, x_2, ...x_n) = \sum_{i=1}^{r} g_i(x_2, ..., x_n)x_1^i$, where g_r is a polynomial in $n-1$ variables, with degree at most $d-r$. By induction hypothesis, $| E(g_r) | \leqslant (d - r)q^{n-2}$.

Now, for each of the $(n-1)$-tuples $(x_2, ..., x_n) \in E(g_r)$, we possible can have all q choices for x_1 as solutions for f which is now considered as a polynomial in one variable.
If $(x_2, ..., x_n) \notin E(g_r)$ (for which there are at most q^{n-1} choices), then f is a polynomial in x_1 with degree r, and hence has at most r solutions. Counting the total number of solutions for f, we obtain,

$$| E(f(x_1, ..., x_n)) | \leqslant (d-r)q^{n-1} + r.q^{n-1} = dq^{n-1}.$$

Hence proved. □

Lemma 5.3. *Consider a field $\mathbb{F}$. Let $E \subset \mathbb{F}^n$ be a set such that $| E | < \binom{n+d}{d}$ for some $d \geqslant 0$. Then there exist a non-zero polynomial $f \in \mathbb{F}[x_1, ..., x_n]$ with degree $\leqslant d$ such that f vanishes on E.*

Proof. Let $V \in \mathbb{F}[x_1, ..., x_n]$ be the set of all polynomials of degree at most d. By a combinatorial argument, it can be shown that $\dim(V) = \binom{n+d}{d}$. Let $W = \mathbb{F}^{|E|}$ be the collection of all functions from E to $\mathbb{F}$. The space clearly has a dimension $| E |$ over $\mathbb{F}$. Consider the restriction map $\Phi : V \to W$ defined by $\Phi(f) = f_{|E}$. By the rank-nullity theorem, the linear map Φ has a non-trivial kernel. Hence, the result is proved. □

5.2.2 The bound $[\approx q^{n-1}]$

The following result by Dvir [11] first introduced the Polynomial method in solving finite field Kakeya problem.

Theorem 5.4. *Let $K \subset \mathbb{F}_q^n$ be a Kakeya set. Then $| K | \geqslant C_n q^{n-1}$, where C_n depends only on n.*

Corollary 5.5. *For every integer n and every $\epsilon > 0$ there exists a constant $C_{n,\epsilon}$, depending only on n and ϵ such that any Kakeya set $K \subset F_n$ satisfies $| K | \geqslant C_{n,\epsilon} q^{n-\epsilon}$.*

Proof. We observe that for $t > 0$, $K^t \subseteq \mathbb{F}^{nt}$ is also a Kakeya set. Hence by Theorem 5.4,

$$| K |^t = | K^t | \geqslant C_{n,t} q^{nt-1} \implies | K | \geqslant C_{n,t} q^{n-1/t}.$$

Hence proved. □

Proof of Theorem 4.4 : We begin with any Nikodym set $M \subset \mathbb{F}_q^n$, and prove that $\mid M \mid \geqslant C_n q^n$. As we have seen before, given a Kakeya set, we can define a Nikodym set M with $\mid M \mid = q \mid K \mid$. Hence we achieve the required lower bound.
Claim: $\mid M \mid \geqslant \binom{n+q-2}{n}$.
We prove the claim by contradiction. assume that it is not the case. then by Lemma 5.3, there exist a non-zero polynomial $f \in \mathbb{F}[x_1, ..., x_n]$ with degree $\leqslant q-2$ such that $f(x) = 0$ for all $x \in M$.
If $x \notin M$, there exists $y \in \mathbb{F}^n$ such that $l_{x,y} = \{x + ty : t \in \mathbb{F}_q^*\} \subseteq M$. Now, $f(x+ty)$ is a polynomial in t with degree less than $q-1$ but vanishes on $q-1$ points. Hence it has to be a zero polynomial on $l_{x,y}$ which implies that $f(x) = 0$. Since x is arbitrary, it follows that f vanishes on the whole space $\mathbb{F}_q^n$, which contradicts the Schwartz-Zippel lemma.
Once the claim is proved, we have the inequality $\mid M \mid \geqslant (1/n!)(q-1)^n \geqslant (\frac{1}{2^n n!})q^n$ for $q \geqslant 2$. Hence proved. □

5.3 Improving the bound to $\approx q^n$

T.Tao and N.Alon suggested a slight improvement on Dvir's proof to achieve an optimal bound.

Theorem 5.6. *Let $K \subset \mathbb{F}_{\shortparallel}{}^n$ be a Kakeya set. Then $\mid K \mid \geqslant (1/n!)q^n$.*

Proof. We claim that $\mid K \mid \geqslant \binom{n+q-1}{n}$. If this is not true, again by Lemma 5.3, there exist a nonzero polynomial $P \in \mathbb{F}_q^n[x_1, ..., x_n]$ of degree d at most $q-1$ such that $P(x) = 0$ for all $x \in K$.
Write $P = \sum_{i=0}^d P_i$ with $P_d \neq 0$, where P_i is the i-th homogeneous component of P. Let, $v \in \mathbb{F}_q^n \backslash 0$ be an arbitrary direction. There exists x such that $\{x + tv : t \in \mathbb{F}_q\} \subseteq K$. For such a fixed x and v, $P(x + tv)$ is a polynomial of degree at most $q-1$, but vanishes for sll $t \in \mathbb{F}$, and hence is a zero polynomial. In particular, $P_d(v)$, the coefficient of t^d is zero. Also by definition, $P_d(0) = 0$. Since P_d vanishes on whole of $\mathbb{F}_q^n$, it is identically zero by Schwartz-Zippel lemma. Thus we reach a contradiction.

$$\mid K \mid \geqslant \binom{n+q-1}{n} \geqslant (1/n!)q^n.$$

□

Remark: The modification is in fact an argument using Projective spaces. The idea is that a low-degree polynomial which vanishes on a line must also vanish on the point at infinity where the line touches the hyperplane at infinity(for the polynomial P we've considered, P_d is its restriction on the hyperplane at infinity). Thus a polynomial which vanishes on a Kakeya set vanishes at the entire hyperplane at infinity. This means that P_d is identically zero, and hence the contradiction.

In the finite field setting we also might care about the constant C_n (this does not appear in the real case since we are taking a limit). There is a better bound of $\mid K \mid \geqslant (\frac{1}{2^n})q^n$ on Kakeya sets, which uses a more sophisticated polynomial argument with zeros of high multiplicities.

Chapter 6

Closed sets with Kakeya property

As we have seen earlier, a line segment can be continuously moved around in a set of arbitrarily small area. A similar question was posed by Cunningham by replacing line segments with circular arcs and it is proved that movement can be done within an area as small as we please, proved we begin with cirular arcs of angle short enough. It leads us to the question of finding out all such planar sets which can be moved from one position to another within a set of arbitrarily small area.

We begin by precisely formulating these notions.

6.1 Definitions

A *rigid motion*, α is function on $\mathbb{C}$ defined by $\alpha(x) = ux + c$, where $u, c \in \mathbb{C}$ and $u \in S^1$. It is an isometry of the plane that preserves orientation.
A *continuous movement*, $M : \mathbb{C} \times [0,1] \longrightarrow \mathbb{C}$ is a continuous map such that $M_t = M(.,t)$ is a rigid motion for every $t \in [0,1]$ and $M_0 = Id$, the identity map.
If M is a continuous movement, the set of points touched by a moving set A is

$$W_M(A) = \{M_t(x) : t \in [0,1],\ x \in A\}.$$

Definition 6.1. A set A is said to have *Kakeya property* or in short property (K) if there exist a rigid motion $\alpha \neq Id$ such that for every $\epsilon > 0$ there exists a continuous movement M such that $M_1 = \alpha$ and $\mu(W_M(A)) < \epsilon$.

Definition 6.2. A set A is said to have *Strong Kakeya property* or in short property (K^s) if for any rigid motion α and for every $\epsilon > 0$ there exists a continuous movement M such that $M_1 = \alpha$ and $\mu(W_M(A)) < \epsilon$.

A planar set A is said to be *trivial (K)-set*, if A can be covered by a null set which is either the union of parallel lines or the union of concentric circles. A nontrivial connected component of a set is connected component having at least two points

6.2 Main results

. The following theorems characterizes all possible closed sets with Kakeya property. [12]

Theorem 6.3. *Let $A \subset \mathbb{C}$ be a closed set having property (K). Then, the union of the nontrivial connected components of A is a trivial (K) set. If A is nonempty, closed, connected and has property (K), then A is a line segment, a half line, a line, a circular arc, a circle or a singleton.*

Theorem 6.4. *If $A \subset \mathbb{C}$ is a nonempty closed and connected set having property (K^s), then a is a line segment, a circular arc or a singleton.*

We make a few observations before proving the theorems.

- A circle has property (K) but when we move a circle C to a circle disjoint from C, the moving circle has to touch all points inside C, hence cannot have the property K^s.
- Any line can be translated along itself without consuming area, and hence has property (K). But it is impossible to rotate a line by any amount of angle in finite area. In the case of line segments, we have seen that they have property (K^s).
- Every set of Hausdroff dimension less than 1 has property (K^s). For such a set $A \subset \mathbb{R}^2$, and a translation map $T_c(x) = x + c$, $c \in \mathbb{R}^2$, we define the movement by $M_t(x) = x + tc, t \in [0, 1]$. We claim that the 2- dimensional lebesgue measure $\mu(W_M(A)) = 0$. Let l be a line orthogonal to c. It is clear that the orthogonal projections of both $W_M(A)$ and A on l are same, say B, and thus $\dim_H(B) < 1$. Now, If $\mu(W_M(A)) > 0$, it contains a ball of positive measure, and the projection contains an interval, which gives us the

contradiction. In similar fashion, we can prove that even when the movement is a rotation, zero area consumed.

6.2.1 Some auxiliary results and the proof of theorems

In this section we formulate some auxiliary results needed for the proof of Theorem 6.3 and Theorem 6.4.
As we have used earlier, translation by a vector c will be denoted by T_c. The rotation about a point c by an angle Φ is denoted by $R_{c,\Phi}$. i.e. $R_{c,\Phi}(x) = e^{i\Phi}(x-c)+c \, \forall \, x \in \mathbb{C}$. If α is a rigid motion and $\alpha^2 \neq Id$, then we define the *elementary movement* E^{α} as follows. If $\alpha = T_c$, then put $E^{\alpha}(t) = T_{tc}$ for every $t \in [0,1]$ If $\alpha = R_{a,\Phi}$ and $\alpha^2 \neq Id$, then we can assume $\mid \Phi \mid < \pi$ and define $E^{\alpha}(t) = R_{a,t\Phi}$ for every $t \in [0,1]$.

Now denote L, the space of all functions $x \mapsto ux + v$, where $u, v \in \mathbb{C}$. Clearly it is a linear space under pointwise addition and constant multiplication. Now consider the operator norm $\|f\| = \sup\{\mid f(x) \mid : x \in \mathbb{C}, \mid x \mid \leqslant 1\}$. It is easy to see that $\|f\| = \mid u \mid + \mid v \mid$ for $f \in L$.

Now consider two rigid motions, f_1 and f_2 with $f_i(x) = u_i x + v_i$, $u_i, v_i \in \mathbb{C}, \mid u_i \mid = 1$. Their inverses are given by $f_1^{-1}(x) = u_1^{-1}x - u_1^{-1}v_1$ and $f_2^{-1}(x) = u_2^{-1}x - u_2^{-1}v_2$. Now, for $\mid x \mid \leqslant 1$, we have

$$\begin{aligned}
\mid f_1^{-1}(x) - f_2^{-1}(x) \mid &= \mid (u_1^{-1} - u_2^{-1})x + u_2^{-1}v_2 - u_1^{-1}v_1 \mid \\
&\leqslant \mid u_1^{-1} - u_2^{-1} \mid + \mid u_2^{-1}v_2 - u_1^{-1}v_2 + u_1^{-1}v_2 - u_1^{-1}v_1 \mid \\
&\leqslant \mid u_1 - u_2 \mid + \mid v_2 \mid\mid u_1 - u_2 \mid + \mid v_2 - v_1 \mid \\
&\leqslant (1 + \mid v_2 \mid)(\mid u_1 - u_2 \mid + \mid v_1 - v_2 \mid
\end{aligned}$$

Hence we have

$$\|f_1^{-1} - f_2^{-1}\| \leqslant (1 + \mid v_2 \mid)\|f_1 - f_2\|. \tag{6.1}$$

Indeed, if a set has property (K), there exist a rigid motion α to which we can reach by continuous movements but within arbitrarily small area. The next lemma states that we can choose these movements so close to the elementary movements corresponding to α, so that they can be approximated by translations or rotations.

Lemma 6.5. *If $A \subset \mathbb{R}^2$ has property (K), then there exists a rigid motion α such that $\alpha^2 \neq Id$ and the following conditions are satisfied. For every $\epsilon > 0$, there is* *a continuous movement M such that $M_1 = \alpha, \mu(W_M(A)) < \epsilon$, and $\|M_t - E_t^\alpha\| < \epsilon$ for every $t \in [0, 1]$*

By continuum, we mean a compact connected set. A connected set $A \subset \mathbb{R}^2$ is said to be irreducible between a and b, If these two points cannot be joined by any closed, connected, proper subset of A. In other words, A should be minimal among all closed connected sets containing a and b. Informally, an irreducible set looks like a curve between two points.

Lemma 6.6. *If $A \subset \mathbb{R}^2$ be a continuum which is irreducible between two distinct points a and b, and suppose that $\mathbb{R}^2 \setminus A$ is connected. Let D be an open disc not containing the points a and b. Then every neighbourhood of every point of $A \cap D$ intersects at least two of the connected components of $D \setminus A$.*

We skip the proof for Lemma 6.5 and Lemma 6.6 now. We will look at them in the next section.

Lemma 6.7. *Let $A \subset D \subset \mathbb{R}^2$ be arbitrary and $G \subset D\backslash A$. Suppose that M is a continuous movement, $t \in [0, 1]$, and $M_s^{-1}(x) \in D$ for every $s \in [0, t]$ and $x \in G$. If G and $M_t^{-1}(G)$ are subsets of distinct connected components of $D \setminus A$, then $G \subset W_M(A)$.*

Proof. Choose $u \in G$. We consider the continuous map $\gamma : [0, t] \to D$ defined by $\gamma(s) = M_s^{-1}(u)$. Observe that $\gamma(0) = u$ and $\gamma(1) = M_t^{-1}(u) \in M_t^{-1}(G)$ are lying n distinct connected components of $D \setminus A$. Since $\gamma([0, 1]) \subset D$, it follows that there exists an $s \in [0, t]$ such that $\gamma(s) = a \in A$. Hence u $= M_s(a) \in W_M(A)$. Since u was chosen arbitrarily, the result follows. ☐

Proof of Theorem 5.3 and 5.4 : First of all, We note that once Theorem 6.3 is proved, we just need to look at the possible cases in Theorem 6.4. By the first two observations we made, it is clear that full circles, lines or half lines can have property (K^s), and hence the result follows.

Let $A \subset \mathbb{R}^2$ be a closed set having property (K). By Lemma 6.5, there exist a rigid motion α with properties mentioned in the lemma. Let A' be the union of all non-trivial connected components of A. It is clear that $\mu(A) = 0$. We shall prove that if α is translation by a vector v, then A' can be covered by lines parallel to v, and if α is a rotation around a point c, A' can be covered by concentric circles with centre c. Since A' has measure zero, it cannot meet parallel lines in positive length. More formally, for $w \in S^1$ orthogonal to v and $T = \{l : l \text{ is parallel to } v,\ \lambda(l \cap A') > 0\}$, we have $\lambda\{d(w,l) : l \in T\} = 0$, where λ is the 1-dimensional lebesgue measure. Similarly, it cannot meet positively many concentric circles in positive length. Therefore A' is a trivial (K)-set.

We shall only prove the statement when α is a rotation; the latter case can be solved by similar arguments.

Lets assume that $\alpha = R_{0,\phi}$. Take A_1 to be a connected component of A'. We have to show that A_1 can be covered by a circle. Assume this is not true. Then, $\Gamma = \{\mid x \mid : x \in A_1\}$ is a nondegenerate interval. Choose $r_1, r_2 \in \Gamma$ such that $0 < r_1 < r_2 < 1$.(The last inequality can be achieved by looking at a similar copy of A_1 inside the unit ball centered at 0).

Take U to be the annulus $\{x : r_1 < \mid x \mid < r_2\} \subset B(0,1)$. The set A_1 contains a minimal connected closed subset C that intersects the circles $\mid x \mid = r_1$ and $\mid x \mid = r_2$. We may assume that $C \subset \bar{U}$. Indeed, connected components and quasi-components coincide in the compact set $C \cap \bar{U}$. Using this fact, one can prove that there is a connected component C_1 of $C \cap \bar{U}$ which intersects the circles $\mid x \mid = r_1$ and $\mid x \mid = r_2$. So by minimality, $C_1 = C$.

Let $C*$ denote the set of all points $p \in C \cap U$ with the following property: if $B(p,r) \subset U$, then every neighborhood of p intersects at least two connected components of $B(p,r) \setminus C$. We will prove that if $p \in C*$, then C contains an arc of the circle $\{x : \mid x \mid = \mid p \mid\}$. First choose p, r that satisfies the aforementioned condition. Put $D = B(p,r)$. There exists a t_0 such that $E_t^\alpha(p) \in D\ \forall t \in [0,t_0]$. We prove that the arc $I = \{E_t^\alpha(p) : t \in [0,t_0]\}$ is in C. Suppose it is not true. Then there exists a $t_1 < t_0$ such that $q = E_{t_1}^\alpha(p) \notin\ D \setminus C$. Since $D \setminus C$ is open, there exist a ball $B(q,\delta) \notin D \setminus C$. Now, $B(p,\delta)$ intersects $D \setminus C$ in at least two connected components. Therefore, we can choose an open disc G whose closure is contained in $B(p,\delta) \setminus C$ and G and $B(q,\delta)$ belongs to different connected components.

Now by Lemma 6.5, for a given $\epsilon > 0$, we choose a movement M so that $\|M_t - E_t^\alpha\| < \epsilon$ for every $t \in [0,t_1]$. Now, by (6.1), we obtain

$$\mid M_s^{-1}(x) - E_s^{\alpha^{-1}}(x) \mid \leqslant \|M_s^{-1} - E_s^{\alpha^{-1}}\| < \epsilon \quad \forall x \in G \subset B(0,1) \tag{6.2}$$

Hence, if ϵ is small enough, we can have $M_s^{-1}(G) \subset D\ \ \forall s \in [0,t_1]$. Since $E_{t_1}^{\alpha^{-1}}$ is a rigid motion, $\bar{E}_{t_1}^{\alpha^{-1}}(G) \subset B(q,\delta)$.Therefore if ϵ is small enough, we have $M_{t_1}^{-1}(G) \subset B(q,\delta)$. Using, Lemma 6.7, we obtain that $G \subset W_M(C)$. But, $\mu(W_M(C)) < \epsilon$ and therefore cannot be bounded below by $\mu(G)$, which gives us the contradiction, and hence $I \subset C$.

It is clear that if $a, b \in C$ and $| a |= r_1$ & $| b |= r_2$, then C is irreducible between a and b. If $\mathbb{R}^2 \backslash C$ is connected, then by Lemma 6.6, $C* = C \cap U$. But then, by what we have just proved, C intersects positively many circular arcs in positive length, which is impossible since C is connected and $\mu(C) = 0$.

Therefore, $\mathbb{R}^2 \backslash C$ is not connected. Since $C \subset \bar{B}(0, r_2)$, There exist a bounded connected component of $\mathbb{R}^2 \backslash C$ inside $B(0, r_2)$. We denote it by V. We show that, C contains a full circle of centre 0. This is clear if $V = B(0, r_1)$, since the boundary circle is contained inside C. Hence we assume that's not the case. Then, $V \cap U \neq \varnothing$. We prove that $\bar{V}$ is an annulus around 0. if not, there exists $x_1 \in \bar{V}$ and $x_2 \notin \bar{V}$ with $| x_1 |=| x_2 |$. Which implies that for small enough ϵ, $B(x_2, \epsilon) \cap V = \varnothing$. Since C being irreducible is nowhere dense, it is clear that we can choose points $y_1 \in B(x_1, \epsilon)$ and $y_2 \in B(x_2, \epsilon)$ such that $| y_1 |=| y_2 |$, and they belong to different connected components of $\mathbb{R}^2 \setminus C$. This easily implies that there is an $\eta > 0$ such that for every $| y_1 | - \eta < r <| y_1 |$, there exist a point $p \in \partial\bar{V}$ with $| p |= r$. If $p \in U \cap \partial\bar{V}$, then it can be proved that $p \in C*$. Consider an open disc $D \subset U$ around p. Now, D has to intersect at least two connected components of $U \setminus C$ since $p \in \partial\bar{V}$. This is true for any disc $B(p, \delta) \subset D$ as well, and thus intersects at least two different connected components of $D \setminus C$, proving that $p \in C*$. This implies that for every $| y_1 | - \eta < r <| y_1 |$, C contains a sub arc of the circle $| x |= r$. As we have seen above, this gives us a contradiction.

Therefore, $\bar{V}$ must be an annulus, and hence its boundary circle is contained in C. We have proved that $A_1 \cap \{x : r_1 <| x |< r_2\}$ contains a full circle of centre 0 for every $r_1, r_2 \in \Gamma$ with $r_1 < r_2 < 1$. Thus A_1 contains a dense subset of $\{x :| x |\in \Gamma\} \cap B(0, 1)$. Since A_1 is closed, it contains the whole set, which is clearly impossible. Thus we reach a contradiction, which tell us that Γ cannot be a non degenerate interval, and hence Theorem 6.3 is proved.

□

Appendix A

Bessel function

A.1 An Interesting Identity

Let Re $\mu > -\frac{1}{2}$, Re $\nu > -1$, and $t > 0$. Then the following identity is valid:

$$\int_0^1 J_\mu(ts)s^{\mu+1}(1-s^2)^\nu ds = \frac{\Gamma(\nu+1)2^\nu}{t^{\nu+1}}J_{\mu+\nu+1}(t).$$

To prove this identity we use the following formula which says that, for Re $\mu > \frac{1}{2}$,

$$J_\mu(t) = \frac{1}{\Gamma(\frac{1}{2})}\Big(\frac{t}{2}\Big)^\mu \sum_{j=0}^\infty (-1)^j \frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+\mu+1)}\frac{t^{2j}}{(2j)!}.$$

Hence we have

$$\begin{aligned}
\int_0^1 J_\mu(ts)s^{\mu+1}(1-s^2)^\nu ds &= \frac{\left(\frac{t}{2}\right)^\mu}{\Gamma\left(\frac{1}{2}\right)}\int_0^1 \sum_{j=0}^\infty \frac{(-1)^j\Gamma(j+\frac{1}{2})t^{2j}}{\Gamma(j+\mu+1)(2j)!}s^{2j+\mu+\mu}(1-s^2)^\nu s ds \\
&= \frac{1}{2}\frac{\left(\frac{t}{2}\right)^\mu}{\Gamma\left(\frac{1}{2}\right)}\sum_{j=0}^\infty \frac{(-1)^j\Gamma(j+\frac{1}{2})t^{2j}}{\Gamma(j+\mu+1)(2j)!}\int_0^1 u^{j+\mu}(1-u)^\nu du \\
&= \frac{1}{2}\frac{\left(\frac{t}{2}\right)^\mu}{\Gamma\left(\frac{1}{2}\right)}\sum_{j=0}^\infty \frac{(-1)^j\Gamma(j+\frac{1}{2})t^{2j}}{\Gamma(j+\mu+1)(2j)!}\frac{\Gamma(\mu+j+1)\Gamma(\nu+1)}{\Gamma(\mu+\nu+j+2)} \\
&= \frac{2^\nu\Gamma(\nu+1)}{t^{\nu+1}}\frac{\left(\frac{t}{2}\right)^{\mu+\nu+1}}{\Gamma\left(\frac{1}{2}\right)}\sum_{j=0}^\infty \frac{(-1)^j\Gamma(j+\frac{1}{2})t^{2j}}{\Gamma(j+\mu+\nu+2)(2j)!} \\
&= \frac{\Gamma(\nu+1)2^\nu}{t^{\nu+1}}J_{\mu+\nu+1}(t).
\end{aligned}$$

A.2 The Fourier Transform of Surface Measure on $\mathbf{S}^{n-1}$

Let $d\sigma$ denote surface measure on $\mathbf{S}^{n-1}$ for $n \geqslant 2$. Then the following is true:

$$\widehat{d\sigma}(\xi) = \int_{\mathbf{S}^{n-1}} e^{-2\pi i \xi.\theta} d\theta = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi|\xi|).$$

To see this, we use the result:
Let K be a function on the line. For $n \geqslant 2$ and when $x \in \mathbb{R}^n \backslash \{0\}$, we have

$$\int_{\mathbf{S}^{n-1}} K(x.\theta) d\theta = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_{-1}^{+1} K(s|x|)(\sqrt{1-s^2})^{n-3}.$$

Therefore we can write

$$\begin{aligned}
\widehat{d\sigma}(\xi) &= \int_{\mathbf{S}^{n-1}} e^{-2\pi i \xi.\theta} d\theta \\
&= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_{-1}^{+1} e^{-2\pi i |\xi| s} (1-s^2)^{\frac{n-2}{2}} \frac{ds}{\sqrt{1-s^2}} \\
&= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \frac{\Gamma(\frac{n-2}{2}+\frac{1}{2})\Gamma(\frac{1}{2})}{(\pi|\xi|)^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi|\xi|) \\
&= \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi|\xi|).
\end{aligned}$$

A.3 The Fourier Transform of a Radial Function on $\mathbb{R}^n$

Let $f(x) = f_0(|x|)$ be a radial function defined on $\mathbb{R}^n$, where f_0 is defined on $[0, \infty)$. Then the Fourier transform of f is given by the formula

$$\widehat{f}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty f_0(r) J_{\frac{n}{2}-1}(2\pi r|\xi|) r^{\frac{n}{2}} dr.$$

To obtain this formula, use polar coordinates to write

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi.x}dx \\
&= \int_0^\infty \int_{\mathbf{S}^{n-1}} f_0(r)e^{-2\pi i\xi.r\theta}d\theta r^{n-1}dr \\
&= \int_0^\infty f_0(r)\frac{2\pi}{(r|\xi|)^{\frac{n-2}{2}}}J_{\frac{n-2}{2}}(2\pi r|\xi|)r^{n-1}dr \\
&= \frac{2\pi}{|\xi|^{\frac{n-2}{2}}}\int_0^\infty f_0(r)J_{\frac{n}{2}-1}(2\pi r|\xi|)r^{\frac{n}{2}}dr.
\end{aligned}$$

As an application we take $f(x) = \chi_{B(0,1)}$, where $B(0,1)$ is the unit ball in $\mathbb{R}^n$. We obtain

$$(\chi_{B(0,1)})^\wedge(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}}\int_0^1 J_{\frac{n}{2}-1}(2\pi|\xi|r)r^{\frac{n}{2}}dr = \frac{J_{\frac{n}{2}}(2\pi|\xi|)}{|\xi|^{\frac{n}{2}}},$$

in view of the result in Appendix A.1. More generally, for Re $\lambda > -1$, let

$$m_\lambda(\xi) = \begin{cases} (1-|\xi|^2)^\lambda & \text{for } |\xi| \leqslant 1, \\ 0 & \text{for } |\xi| > 1. \end{cases}$$

Then

$$m_\lambda{}^\vee(x) = \frac{2\pi}{|x|^{\frac{n-2}{2}}}\int_0^1 J_{\frac{n}{2}-1}(2\pi|x|r)r^{\frac{n}{2}}(1-r^2)^\lambda dr = \frac{\Gamma(\lambda+1)}{\pi^\lambda}\frac{J_{\frac{n}{2}+\lambda}(2\pi|x|)}{|x|^{\frac{n}{2}+\lambda}},$$

using again the identity in Appendix A.1.

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