Trajectories of particles around lower dimensional rotating and non-rotating black holes

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Certificate of Examination

This is to certify that the dissertation titled **Trajectories of particles around lower dimensional rotating and non-rotating black holes** submitted by **Shailesh Kumar** (Reg. No. MS 13059) for the partial fulfilment of BS-MS dual degree program of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Kinjalk Lochan	Dr. H.K. Jassal	Dr. Sanjib Dey
(Supervisor)		

Dated:

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Kinjalk Lochan at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Shailesh Kumar

(Candidate)

Dated:

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are correct to the best of my knowledge.

Dr. Kinjalk Lochan

(Thesis Supervisor)

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Abstract

According to Einstein's Theory of General Relativity, gravity bends spacetime and objects follow this curved geometry. If a particle or object comes in the gravitation field of given mass source, it gets attracted because mass source bends spacetime and object follows the structure of spacetime. This is all happening due to the potential field generated by the mass source. So curvature can be thought of as the force field.

Black holes are objects whose potential field is so high that classically nothing can escape. Therefore it is essential to get the region for which we are accessible to get all information about the particle and below which we do not have access to the physics of the particle. In simple words, a horizon is a region which separates two regions by a null boundary.

In this thesis, first, we studied possible trajectories of a particle around a given mass source in Newtonian theory, and then we find the correction to this theory. We apply this general theory of relativity to Schwarzschild and BTZ black hole and find the trajectories of massive and massless particles. All in all, the aim is to look at potential field produced by black holes. We also go through the general mathematical structure of identifying horizon and describe the physics of lightcones in different coordinate systems to make sure that the physics is well defined on the horizon. At last, we focused on the null trajectories around black hole geometry in the presence and absence of cosmological constant. We obtain the location of photon sphere, the position of photons at which it exhibits circular motion.

Chapter 1

Orbits in Newtonian theory

1.1 Introduction

According to Newton's theory, 'Time' is an absolute physical parameter which means it does not change from one frame to another frame of reference but it was not the case. Though the foundations of classical physics were developed by Sir Isaac Newton $(1642 - 1727)$, it was Einstein who proved that time is not absolute instead it is relative. When Albert Einstein formulated special theory of relativity (1905), he knew that this theory is applicable for objects moving with constant velocity. He was willing to generalise his approach to accelerated objects. So he worked on this significant problem for the next ten years, and he gave a theory in 1915 named as the General theory of relativity which says that gravity is not a force. It is the bending of space-time structure produced by the mass source.

So my thesis work includes the study of trajectories of a particle in Newtonian and General relativistic theory. How null trajectories behave in presence and absence of cosmological constant.

1.2 Planetary Orbits in Newtonian theory

According to Newtonian theory, gravity is a force. So given a mass source, we can look for possible trajectories of a particle which is moving under the gravitational field of the mass source[1].

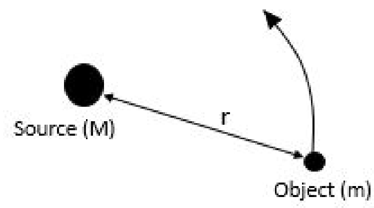


Figure 1.1: motion of an object around a given source in Newtonian theory

If the source has mass M and object is of unit mass, from the principle of energy conservation

$$E = \frac{1}{2}v^2 - \frac{GM}{r}$$

Where G is the Newtonian gravitational constant $G = 6.67 \times 10^{11}\ m^3 kg^{-1} s^{-2}$. Object's velocity v includes radial (v_r) and angular (v_ϕ) velocity given by-

$$v_r = \frac{dr}{dt} \qquad \& \qquad v_\phi = r\frac{d\phi}{dt}$$

Now if we take a vector $\vec{r}$ joining source and object and take a dot product with angular momentum ($\vec{L}$), it turns out to be zero. Which simply means motion of the object lies in a plane i.e. $\vec{r} \cdot \vec{L} = 0$. So from the principle of angular momentum conservation

$$L = r^2 \frac{d\phi}{dt}$$

Combining these with conservation principles, we get the following expression,

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = E - \frac{L^2}{2r^2} + \frac{GM}{r} = E - V_{eff}$$

Above expression tells us that kinetic energy equals to total energy minus centrifugal potential. Now we can define our effective potential

$$V_{eff} = \frac{L^2}{2r^2} - \frac{GM}{r} \tag{1.1}$$

The ultimate aim is to find effective potential. Because it contains all information about the possible trajectories of the particle around the given source and this is what we say possible orbits. Now using this effective potential, if we look for circular orbits,

$$\frac{dV_{eff}}{dr} = 0 \qquad \Rightarrow \qquad r_0 = \frac{L^2}{GM}$$

At this value of $r = r_0$, the double derivative test becomes positive i.e.

$$\frac{d^2V_{eff}}{dr^2}\,|_{r=r_0} = \frac{G^4M^4}{L^6} > 0$$

So potential at this value r_0 is going to be minimum.

$$V_{min} = -\frac{G^2M^2}{2L^2} \tag{1.2}$$

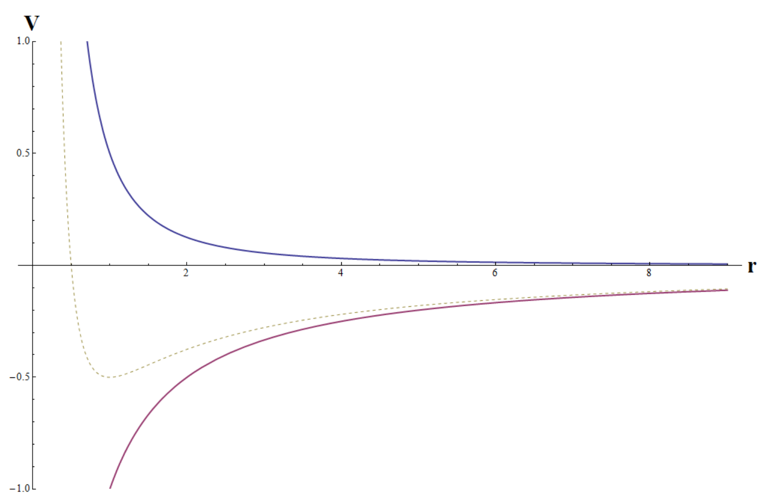


Figure 1.2: Potential curve for an unit mass moving around a source

The figure 1.2 is the plot of effective potential. The blue curve is for $\frac{1}{r^2}$ part of the effective potential, the pink curve is for $-\frac{1}{r}$ part of the effective potential, and the dotted curve is representing the whole effective potential field. By making constant total energy E lines parallel to r axis, we can look for different possible orbits for this kind of potential curve. At the minimum of this potential curve, we expect to get circular orbits as $r = constant$, and as we move above from this minimum value, we hope to get different orbits. We will see this mathematically in further calculations and discussions.

Now we shall try to obtain an equation which will tell about all possible orbits for the object. That equation is known as *Orbit equation.*

$$\frac{d\phi}{dr} = \frac{d\phi}{dt}\frac{dt}{dr} \tag{1.3}$$

We know $\frac{d\phi}{dt}$ and $\frac{dt}{dr}$, given by,

$$\frac{d\phi}{dt} = \frac{L}{r^2} \qquad \& \qquad \frac{dr}{dt} = \sqrt{2(E - V_{eff})}$$

putting these two expressions in equ(2), we get the following simple first order differential equation in r and ϕ variables

$$\frac{d\phi}{dr} = \frac{L}{r^2}\frac{1}{\sqrt{2(E - V_{eff})}}$$

$$\frac{d\phi}{dr} = \frac{L}{r^2}\frac{1}{\sqrt{2(E - \frac{L^2}{2r^2} + \frac{GM}{r})}}$$

assuming $u = 1/r$ and solving the differential equation, we get

$$r = \frac{1}{u} = \frac{r_0}{1 - \cos\phi} \tag{1.4}$$

This equation is known as *Orbit Equation.* Where e is eccentricity and r_0 is a constant.

$$e = \sqrt{1 + \frac{2Er_0^2}{L^2}} = \sqrt{1 + \frac{2EL^2}{G^2M^2}} \qquad and \qquad r_0 = \frac{L^2}{GM}$$

Since the motion lies in a plane, therefore $r\cos\phi = x$ and $r^2 = x^2 + y^2$, and finally we get an expression which explains all possibles orbits in Newtonian theory.

$$(1 - e^2)x^2 + y^2 - 2er_0x - r_0^2 = 0 \tag{1.5}$$

If we look at points p or q in the given fig1.3, at these points *change in velocity* is zero. So, for Circular orbits, acceleration and change in velocity

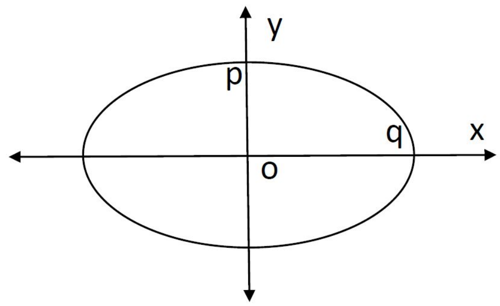


Figure 1.3: Elliptical orbit equation plot

both are zero which means $r = constant$. But for elliptical orbits change in velocity at p and q points is zero but acceleration is non zero which makes the object to change r.

Looking at the energy conservation equation,

$$\frac{1}{2}\dot{r}^2 = E - \frac{L^2}{2r^2} + \frac{GM}{r}$$

at point p, $\quad \dot{r} = 0 \quad \Longrightarrow \quad E - \frac{L^2}{2r^2} + \frac{GM}{r} = 0$

Solving for r, we get two roots of r here,

$$r = \frac{-GM \pm \sqrt{G^2M^2 + 2EL^2}}{2E}$$

At these two values of r, $f = -\frac{\partial V_{eff}}{\partial r} \neq 0$ which means acceleration $\ddot{r}$ is non zero. But for Circular orbits $\frac{dV_{eff}}{dr} = 0$ gives,

$$r = \frac{L^2}{GM}$$

For this value of r, $f = -\frac{\partial V_{eff}}{\partial r} = 0$ which means acceleration $\ddot{r} = 0$.

This is why for elliptical case r is changing ($\ddot{r} \neq 0$) and for circular case r is not changing ($\ddot{r} = 0$).

Here are some plots (x vs y) of orbits for different values of eccentricity e. These plots describe orbits in the $x - y$ plane which mathematically can be seen from the orbit equation (1.4).

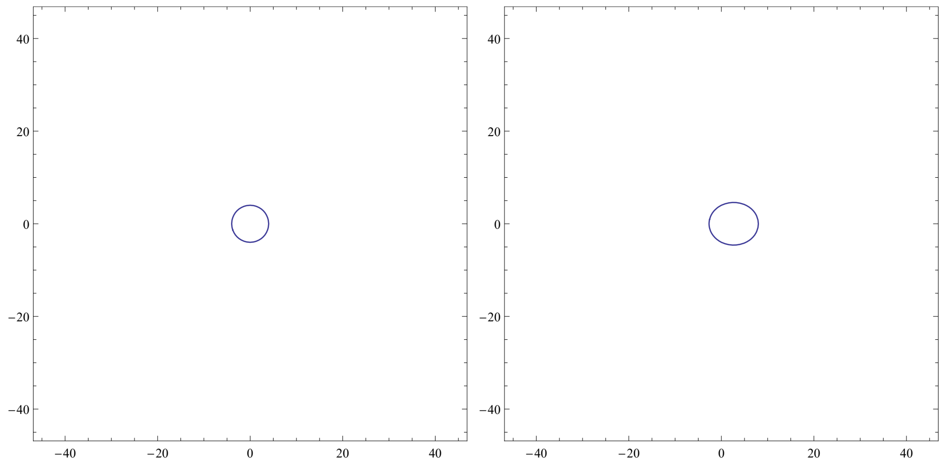


Figure 1.4: Sr.No.1: x-y plot for circular orbit; $e = 0$
Sr.No.2: x-y plot for Elliptical orbits; $e = 0.5$

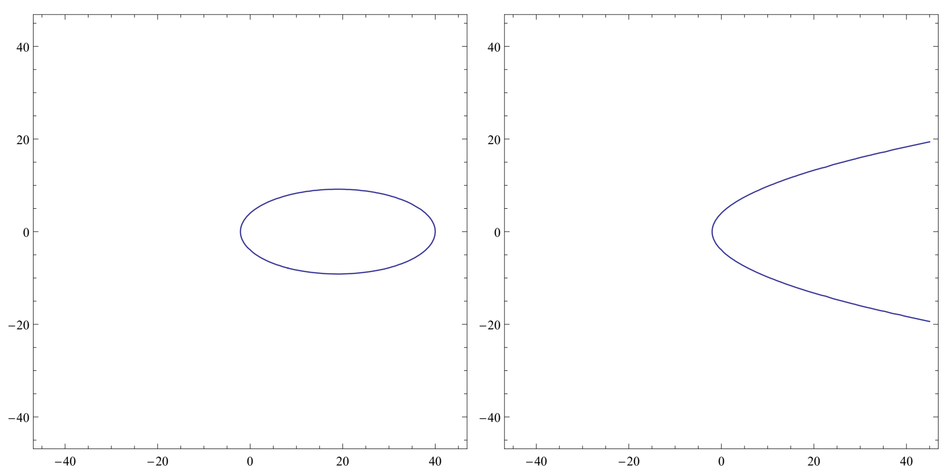


Figure 1.5: Sr.No.3 : x-y plot for Elliptical orbits; $e = 0.9$
Sr.No.4 : x-y plot for Parabolic orbits $e = 1$

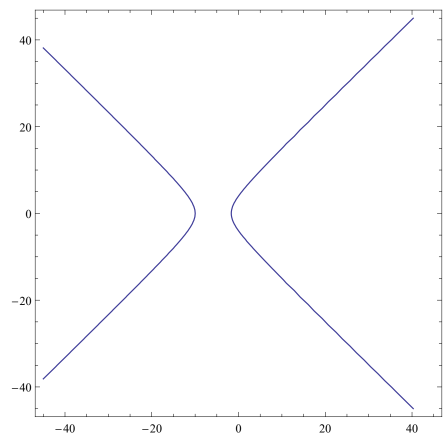


Figure 1.6: Sr.No.5 : x-y plot for Hyperbolic orbits

Sr. No.	**e**	$\mathbf{r}_0$	**Orbit Equation**	v **Type**
1.	0	4	$x^2 + y^2 = 16$	Circle
2.	0.5	4	$\frac{3}{4}x^2 - 4x + y^2 = 16$	Ellipse
3.	0.9	4	$\frac{19}{100}x^2 - \frac{36}{5}x + y^2 = 16$	Ellipse
4.	1.0	4	$-8x + y^2 = 16$	Parabola
5.	1.4	4	$-\frac{69}{100}x^2 - \frac{56}{5}x + y^2 = 16$	Hyperbola

Table 1.1 : Possible orbits for different values of e

If we look at Sr.No.5 above, Hyperbola always comes in pair. So, which of the trajectories is possible, will be decided by initial conditions. Initial conditions can be fixed by E and L for an unit mass body because eccentricity e depends on E and L.

All these five plots named as Sr.No. (Serial Number), can be seen from the Table 1.1. The orbit equation and its type are also mentioned in the table. Table 1.2 gives a general description of the same for different ranges of constant total energy E and eccentricity e. Orbit is bounded or unbounded depends whether the object comes back to its initial position or not. If the object comes back to its initial position, the orbit is Bounded. If it does not then orbit is unbounded.

Sr. no.	**Energy (E)**	**Eccentricity (e)**	**Orbit**	**Bound/UnB**
1.	$E > 0$	$e > 1$	Hyperbola	Unbound
2.	$E = 0$	$e = 1$	Parabola	Unbound
3.	$V_{min} < E < 0$	$0 < e < 1$	Ellipse	Bound
4.	$V_{min} = E$	$e = 0$	Circle	Bound

Table 1.2 : Possible orbits for different ranges of total energy E

Combining all plots in one and considering all given values of eccentricity e keeping r_0 same presented in the table (1.1). We get the following x vs y plot for different orbit equations.

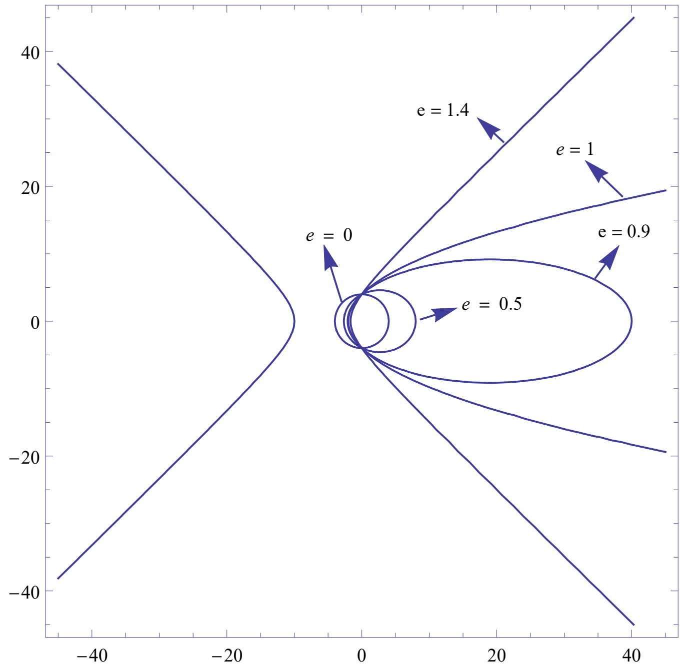


Figure 1.7: orbits with different values of e

Chapter 2

Orbits in General Relativity Theory

2.1 Introduction

We have seen in the previous discussion in Newtonian theory that Circular, Elliptical, Parabolic and Hyperbolic all these orbits are possible and if we look at the potential, as r becomes smaller and smaller potential becomes positive infinity which means the object is repelled by the source or in other words it escapes from the source. This happens because centrifugal barrier term $\frac{L^2}{2r^2}$ becomes larger than attractive potential term -$\frac{GM}{r}$ for small values of r.

Now in general relativistic theory, it is not true. An object can be captured by the source for small values of r. So, we shall see a correction term to the Newtonian theory which makes the potential negative infinity for small values of r. As we know that Newtonian method does not provide us with the time evolution description of the potential field $\phi(x)$ whereas general relativistic theory does provide.

$$\bigtriangledown^2\phi(x) = 4\pi G\rho(x) \tag{2.1}$$

Given a source $\rho(x)$ (mass density) and mass M, we can solve this equation. This equation is known as *Newton's field equation*[2]. If we consider a spherically symmetric source, the gravitational potential is given by

$$\phi(x) = -\frac{GM}{x} \tag{2.2}$$

This is the non-relativistic solution of Newton's field equation, and most importantly this field equation does not involve time component.

In General relativity, the notion of the potential field lies in the metric tensor and the second derivative of the metric tensor is curvature tensor. This abstract statement is known as *Einstein Field Equation*. Which mathematically can be written as

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.3}$$

Where $G_{\mu\nu}$ is *Einstein tensor* and $T_{\mu\nu}$ is *Energy-Momentum tensor*. Here we'll consider the same spherically symmetric source and introduce the time coordinate because time is not absolute. So we'll try to see the time evolution of the potential field.

2.2 Weak field limit in General relativity

Under weak field limit, a general relativistic field becomes the Newtonian potential field. Consider an object moving in a weak and static gravitational field. Weak field means small perturbation to the flat space, i.e. $h_{\mu\nu} << 1$. The Newtonian limit is achieved by keeping in mind three main points-

1. Particle moves slowly w.r.t. light
2. Field is static, i.e. $\frac{\partial g_{\mu\nu}}{\partial t} = 0$
3. Gravitational field is weak

Now considering the geodesic equation

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{2.4}$$

Since particles are moving slowly relative to the speed of light c. Therefore,

$$\frac{dx^i}{d\tau} << \frac{d(ct)}{d\tau}$$

Taking this approximation and staticity condition, the geodesic equation becomes

$$\frac{d^2x^\lambda}{d\tau^2} + \frac{1}{2}\Big(\frac{dt}{d\tau}\Big)^2 \bigtriangledown h_{00} = 0$$

For slowly moving objects $\frac{dx^0}{d\tau} \simeq 1$ because $v << c$. The above equation becomes,

$$\frac{d^2\vec{x}}{d\tau^2} = -\frac{1}{2} \bigtriangledown h_{00}$$

comparing this equation with Newton's equation of motion,

$$\frac{d^2\vec{x}}{dt^2} = \vec{F} = -\frac{\partial \phi}{\partial x^i} = -\vec{\bigtriangledown}\phi$$

We get perturbation to the flat metric,

$$h_{00} = \frac{2\phi}{c^2}$$

If the signature of the flat metric is $\eta_{\mu\nu} = (1, -1, -1, -1)$, the components of the metric tensor under weak field limit become,

$$g_{00} = \eta_{00} + h_{00} = (1 + \frac{2\phi}{c^2})$$

For spherically symmetric source, the potential field is given by $\phi(r) = -\frac{GM}{r}$. Therefore,

$$g_{00} = \left(1 - \frac{2GM}{r}\right) \qquad and \qquad g_{rr} = \left(1 - \frac{2GM}{r}\right)^{-1} \tag{2.5}$$

2.3 Schwarzschild Black hole

Before taking a direct jump to the orbits, let us try to see the notion of distance or metric in this theory. Since Einstein field equation is highly non-linear so we can look for symmetries here. We shall consider *Schwarzschild black hole* which is a spherically symmetric static black hole. Metric tells us about the local geometry of the surface. Therefore the Schwarzschild metric tells us the geometry of space-time outside a massive source[3]. Consider spherical coordinate system (t, r, θ, ϕ), the general spherically symmetric metric is given by,

$$ds^2 = A(r,t)dt^2 - C(r,t)dr^2 - B(r,t)dtdr - D(r,t)r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.6}$$

Since we are considering static metric, the components of metric tensor will be independent of time. Therefore the metric can be written as following-

$$ds^2 = A(r)dt^2 - C(r)dr^2 - B(r)dtdr - D(r)r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

We can define a new time coordinate,

$$ct^{'} \longrightarrow ct^{'} \equiv ct + f(r) \tag{2.7}$$

This will eliminate $dtdr$ term which is non-diagonal term. The differential form of the above definition of new time is

$$dt = dt^{'} - \frac{df}{dr}dr$$

The metric ds^2 in this definition can be written as follows,

$$ds^2 = Adt^{'2} + \Big(A\Big(\frac{df}{dr}\Big)^2 - B\frac{df}{dr} - C\Big)dr^2 - dt^{'}dr\Big(2A\frac{df}{dr} + B\Big) - Dr^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Now we can choose $f(r)$ which eliminates $dt^{'}dr$ term. For this to be eliminated the following condition must be met

$$\frac{df}{dr} = -\frac{B}{2A}$$

Now I'll write $Dr^2 \to r^{'2}$ and define coefficient of dr^2 as $B^{'}(r)$.

$$ds^2 = Adt^{'2} - B^{'}(r)dr^2 - r^{'2}(d\theta^2 + \sin^2\theta d\phi^2)$$

just for writing simplicity, omitting prime from dt and r, the metric becomes

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.8}$$

Elimination of cross term of $dtdr$ ensures the time-reversal symmetry i.e. if we make transformation $t \to -t$, metric remains invariant. There can not be any cross terms of angular parts because in that case, we'll lose spherical symmetry property. Because in crossing angular terms, if we transform $\phi \to -\phi$ or $\theta \to -\theta$, it will flip the sign of the metric. And a metric is a scalar quantity which should be invariant under any kind of transformation. So keeping in mind the property of spherical symmetry, we do not consider any cross-terms in angular parts. This given metric equ(2.8) is general spherically symmetric and static. Components of the metric tensor are given by,

$$g_{tt} = A \qquad g_{rr} = -B \qquad g_{\theta\theta} = -r^2 \qquad g_{\phi\phi} = -r^2 \sin^2\theta$$

$$g^{tt} = \tfrac{1}{A} \qquad g^{rr} = -\tfrac{1}{B} \qquad g^{\theta\theta} = -\tfrac{1}{r^2} \qquad g^{\phi\phi} = -\tfrac{1}{r^2 \sin^2\theta}$$

There are 9 non-zero Christoffel symbols exist, given below

(i) $\Gamma^t_{rt} = \frac{A'}{2A}$ **(ii)** $\Gamma^r_{tt} = \frac{A'}{2B}$ **(iii)** $\Gamma^r_{rr} = \frac{B'}{2B}$

(iv) $\Gamma^r_{\theta\theta} = -\frac{r}{B}$ **(v)** $\Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{B}$ **(vi)** $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$

(vii) $\Gamma^\phi_{r\phi} = \frac{1}{r}$ **(viii)** $\Gamma^\phi_{\phi\theta} = \cot\theta$ **(ix)** $\Gamma^\theta_{r\theta} = \frac{1}{r}$

Now $T_{\mu\nu}$ is zero outside the spherically symmetric source and and non-zero inside the source. Therefore Einstein field equation outside a spherically symmetric source becomes

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

Where $R_{\mu\nu}$ is *Ricci Tensor* and R is *Ricci Scalar*. If we multiply by $g^{\mu\nu}$ on both sides, we get,

$$R = 0$$

putting back Ricci scalar (R) into the equation, we get Ricci tensor,

$$R_{\mu\nu} = 0$$

We will try to find the solution of this equation.

$$R_{tt} = \frac{A''}{2B} - \frac{A'^2}{4AB} - \frac{A'B'}{4B^2} + \frac{A'}{rB}$$

$$R_{rr} = -\frac{A''}{2A} + \frac{A'^2}{4A^2} + \frac{A'B'}{4AB} + \frac{B'}{rB}$$

Where prime over A and B, is derivative with respect r.

Now considering two of the Einstein's field equations,

$$\frac{R_{tt}}{g_{tt}} - \frac{1}{2} R = 0 \qquad \& \qquad \frac{R_{rr}}{g_{rr}} - \frac{1}{2} R = 0$$

Subtracting these two equations, we get

$$BA' + AB' = 0$$

$$AB = constant = C_1$$

Considering another Einstein's field equation,

$$R_{\theta\theta} = -\frac{rA'}{2AB} + \frac{rB'}{2B^2} + 1 - \frac{1}{B}$$

using the result $\quad B = \frac{C_1}{A} \quad \implies \quad \frac{B'}{B} = -\frac{A'}{A},$

$$R_{\theta\theta} = 1 - \frac{1}{B} + \frac{r}{2B}\Big[-\frac{A'}{A} + \frac{B'}{B}\Big]$$
$$R_{\theta\theta} = 1 - \frac{1}{B} - \frac{rA'}{C_1}$$

using the fact that $R_{\mu\nu} = 0$,

$$0 = 1 - \frac{1}{B} - \frac{rA'}{AB}$$
$$rA = C_1 r - C_2$$
$$A = C_1 - \frac{C_2}{r}$$

We can rescale time coordinate in equ(2) $t \to \frac{t}{\sqrt{C_1}}$ to set $C_1 = 1$ and under the weak field approximation limit, looking at the equ(2), we get constant C_2. Therefore,

$$A(r) = \Big(1 - \frac{2GM}{r}\Big) \qquad and \qquad B(r) = \Big(1 - \frac{2GM}{r}\Big)^{-1}$$

Finally, the expression for the spherically symmetric static metric is given by,

$$ds^2 = \Big(1 - \frac{2GM}{r}\Big)dt^2 - \Big(1 - \frac{2GM}{r}\Big)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (2.9)$$

There are two singularities here. One is $r = 2GM$ and another one is $r = 0$. $r = 2GM$ is known as *Schwarzschild radius* (r_{sch}) and it is a coordinate singularity also known as event horizon. Whereas $r = 0$ is true singularity because if we calculate Kretschmann scalar, it turns out that it is inversely proportional to r^6. At $r = 2GM$ this scalar is going to be finite but at $r = 0$ it blows up.

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = \frac{48G^2M^2}{r^6}$$

Kretschmann Scalar :

In general relativity, Kretschmann scalar is the best quantity which tells about the singularity whether it is coordinate singularity or true singularity. Here we'll try to obtain this scalar. There are 9 non-zero Christoffel symbols.

(i) $\Gamma^t_{rt} = \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1}$ **(ii)** $\Gamma^r_{tt} = \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)$

(iii) $\Gamma^r_{rr} = -\frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1}$ **(iv)** $\Gamma^r_{\theta\theta} = -r\left(1 - \frac{2GM}{r}\right)$

(v) $\Gamma^r_{\phi\phi} = -r\left(1 - \frac{2GM}{r}\right)\sin^2\theta$ **(vi)** $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$

(vii) $\Gamma^\phi_{r\phi} = \frac{1}{r}$ **(viii)** $\Gamma^\phi_{\phi\theta} = \cot\theta$

(ix) $\Gamma^\theta_{r\theta} = \frac{1}{r}$

There are 6 non-zero independent Riemann tensors.

(i) $R_{trtr} = -\frac{2GM}{r^3}$ **(ii)** $R_{t\theta t\theta} = -\frac{GM}{r}\left(1 - \frac{2GM}{r}\right)$

(iii) $R_{t\phi t\phi} = -\frac{GM}{r}\left(1 - \frac{2GM}{r}\right)\sin^2\theta$ **(iv)** $R_{r\phi r\phi} = \frac{GM}{r}\left(1 - \frac{2GM}{r}\right)\sin^2\theta$

(v) $R_{\theta\phi\theta\phi} = -2GMr\sin^2\theta$ **(vi)** $R_{r\theta r\theta} = -\frac{GM}{r}\left(1 - \frac{2GM}{r}\right)^{-1}$

Now we need to find Riemann tensors with upper indices. Contravariant Riemann tensors can be written as follows,

$$R^{trtr} = g^{\alpha r} g^{\beta r} g^{\gamma r} R^t_{\alpha\beta\gamma} = g^{rr} g^{tt} g^{rr} R^t_{rtr}$$

Similarly, it can be done for other components. All components of the Kretschmann scalar is given by

(i) $R^{t\theta t\theta} R_{t\theta t\theta} = (g^{\theta\theta})^2 (R^t_{\theta t\theta})^2$ **(ii)** $R^{t\phi t\phi} R_{t\phi t\phi} = (g^{\phi\phi})^2 (R^t_{\phi t\phi})^2$

(iii) $R^{r\phi r\phi} R_{r\phi r\phi} = (g^{\phi\phi})^2 (R^r_{\phi r\phi})^2$ **(iv)** $R^{\theta\phi\theta\phi} R_{\theta\phi\theta\phi} = (g^{\phi\phi})^2 (R^\theta_{\phi\theta\phi})^2$

(v) $R^{r\theta r\theta}R_{r\theta r\theta} = (g^{\theta\theta})^2(R^r_{\theta r\theta})^2$ **(vi)** $R^{trtr}R_{trtr} = (g^{rr})^2(R^r_{rtr})^2$

The general form of the Kretschmann scalar for the Schwarzschild black hole is,

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 4R^{t\theta t\theta}R_{t\theta t\theta} + 4R^{t\phi t\phi}R_{t\phi t\phi} + 4R^{r\phi r\phi}R_{r\phi r\phi} + 4R^{\theta\phi\theta\phi}R_{\theta\phi\theta\phi} + 4R^{r\theta r\theta}R_{r\theta r\theta} + 4R^{trtr}R_{trtr}$$

The final expression of the Kretschmann scalar is,

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{48G^2M^2}{r^6} \tag{2.10}$$

Now we can see that if $r = 2GM$, the scalar is going to have some finite value and if $r = 0$, scalar becomes undefined. Therefore $r = 0$ is the actual singularity of the schwarzschild black hole.

Motion lies in a plane :

As we have seen in the Newtonian theory that $\vec{r} \cdot \vec{L} = 0$, i.e. the motion of the object lies in a plane. Similarly here, we'll obtain the geodesic equation, and from there we'll try to understand the motion of the object. The geodesics of the purpose is given by,

(i) $$0 = \frac{d^2t}{d\lambda^2} + 2\frac{GM}{r}\Big(r - 2GM\Big)^{-1}\frac{dt}{d\lambda}\frac{dr}{d\lambda}$$

(ii) $$0 = \frac{d^2\theta}{d\lambda^2} - \sin\theta\cos\theta\Big(\frac{d\phi}{d\lambda}\Big)^2 + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda}$$

(iii) $$0 = \frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} + 2\cot\theta\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda}$$

(iv) $$0 = \frac{d^2r}{d\lambda^2} + \frac{GM}{r^2}\Big(1 - \frac{2GM}{r}\Big)\Big(\frac{dt}{d\lambda}\Big)^2 - r\Big(1 - \frac{2GM}{r}\Big)\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} - \sin^2\theta(r - 2GM)\Big(\frac{d\phi}{d\lambda}\Big)^2 - \frac{GM}{r^2}\Big(1 - \frac{2GM}{r}\Big)^{-1}\Big(\frac{dr}{d\lambda}\Big)^2$$

Considering $\theta = \frac{\pi}{2}$ i.e. equatorial plane then from geodesic equation(ii)

$$\frac{d\theta}{d\lambda} = 0 \quad \Rightarrow \quad \frac{d^2\theta}{d\lambda^2} = 0$$

Here λ is a free parameter. In real space, if you give a value to λ, I can fix the point in the space. So for any value of λ say $\lambda \rightarrow \lambda^{'}$, θ remains $\frac{\pi}{2}$ i.e. $\frac{d^2\theta}{d\lambda'^2} = 0$ Considering transformation in λ, $\lambda^{'} \rightarrow \lambda + \epsilon\lambda$. Where ϵ is very very small. We know

$$\frac{d}{d\lambda}\Big(\frac{d\theta}{d\lambda}\Big) = 0 \quad \Rightarrow \quad \frac{d}{d\lambda}\Big(\frac{d\theta}{d\lambda'}\frac{d\lambda^{'}}{d\lambda}\Big) = 0$$

We know derivative of λ w.r.t. $\lambda^{'}$ which is $\frac{d\lambda^{'}}{d\lambda} = (1+\epsilon)$ and using the commutativity property of derivative, we need to find $\frac{d\theta}{d\lambda}$, which is given by $\frac{d\theta}{d\lambda} = (1+\epsilon)\frac{d\theta}{d\lambda'}$.

$$\begin{aligned} \therefore \qquad \frac{d}{d\lambda'}\Big(\frac{d\theta}{d\lambda}\Big) =&0 \quad \Rightarrow \quad (1+\epsilon)\frac{d}{d\lambda'}\Big(\frac{d\theta}{d\lambda'}\Big) = 0 \\ \frac{d^2\theta}{d\lambda'^2} =&0 \end{aligned}$$

When we solve both differential equations $\frac{d^2\theta}{d\lambda^2} = 0$ & $\frac{d^2\theta}{d\lambda'^2} = 0$ and using the initial condition i.e. $\theta(\lambda = 0) = \frac{\pi}{2}$, we get θ for any value of λ which turns out to be $\theta(\lambda^{'}) = \frac{\pi}{2}$.

2.3.1 Constants of motion

If the object's path is $x^\mu = x^\mu(\tau)$ and k^μ is a direction of symmetry vector. $u^\mu = \frac{dx^\mu}{d\tau}$ is the tangent to the object's path. Physically it is the 4-velocity vector of the object in free fall which follows the geodesic. For example $k^\mu = (1\ 0\ 0\ 0)^T$ along time t-axis. Components of k^μ along the path is $u_\mu k^\mu$. The rate of change of $u_\mu k^\mu$ in the direction of u^ν is [4],

$$\begin{aligned} u^\nu(k^\mu u_\mu)_{;\nu} =& u^\nu k^\mu_{;\nu} u_\mu + u^\nu k^\mu u_{\mu;\nu} \\ =& u^\nu u_{\mu;\nu} k^\mu + u^\nu u^\mu k_{\mu;\nu} \\ =& u^\nu u_{\mu;\nu} k^\mu + \frac{1}{2} u^\nu u^\mu (k_{\mu;\nu} + k_{\nu;\mu}) \end{aligned}$$

Considering the path is a geodesic, 1_{st} term becomes 0 and 2_{nd} term is *Killing equation* which is also equal to zero. Therefore whole expression becomes 0.

$$u^\nu(k^\mu u_\mu)_{;\nu} = 0 \tag{2.11}$$

So, $k^\mu u_\mu$ is a constant along a geodesic. Considering the direction of symmetry, we can find the constant of motion in that direction. We'll try to obtain potential field for Schwarzschild black hole in the further calculation.

Considering Schwarzschild black hole, we can have two directions of symmetries. Symmetry means, a transformation under which the metric does not change. So for this type of black hole one symmetry in time t direction and another in ϕ direction.

Symmetry in t direction : $k^\mu = (1\ 0\ 0\ 0)^T$

$$\begin{aligned} k^\mu u_\mu =& g_{\mu\nu} k^\mu u^\nu \\ =& g_{00} k^0 u^0 \\ =& g_{00} u^0 = E \\ E =& \Big(1 - \frac{2GM}{r}\Big)\frac{dt}{d\lambda} \end{aligned} \tag{2.12}$$

Here E is one of the constants of motion.

Symmetry in ϕ direction : $k^\mu = (0\ 0\ 0\ 1)$

$$\begin{aligned} k^\mu u_\mu =& g_{\mu\nu} k^\mu u^\nu \\ =& g_{33} u^3 = -L \\ L =& r^2 \frac{d\phi}{d\lambda} \end{aligned} \tag{2.13}$$

Here L is another constant of motion. Which is angular momentum per unit mass.
Now considering the inner product of four velocity

$$u^\mu u_\mu = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \epsilon$$

Where $\epsilon = 1$ for massive particles/objects and $\epsilon = 0$ for massless particles/objects. Expanding this equation taking all non-zero components of the metric tensor,

$$\frac{1}{2}\Big(\frac{dr}{d\lambda}\Big)^2 + \Big(\frac{\epsilon}{2} - \epsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}\Big) = \frac{E^2}{2} \tag{2.14}$$

Effective potential is given by,

$$V(r) \equiv \frac{\epsilon}{2} - \epsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \tag{2.15}$$

For a unit mass particle, first two terms are same as Newtonian effective potential. But the third term is an extra term which is the correction to the Newtonian theory. This tells us that for small values of r effective potential becomes negative infinity which means particle can fall into the black hole.

For circular orbits r will be constant. Therefore $\frac{dr}{d\lambda} = 0$. This sets the condition that

$$\frac{dV}{dr} = 0$$

For massive particles, this gives us two possible roots of r.

$$r = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2L^2}}{2GM}$$

Negative root of r corresponds to maximum of $V(r)$ which is unstable because double derivative test turns out to be less than zero whereas positive root of r corresponds to minimum of $V(r)$ which is stable because double derivative test turns out to be positive. If we take discriminant equals zero, it will give us *inner most stable circular orbits* i.e. r_{ISCO}.

$$L^2 - 12G^2M^2 = 0$$

At this value of L^2, both roots of r collapses into one root.

$$r = r_{ISCO} = 6GM$$

This is for a massive particle. For massless particles $\epsilon = 0$ which gives us only one root of r.

$$r = 3GM$$

This $r = 3GM$ is maxima of the potential because the double derivative test is negative at this value of r. Therefore, orbit is unstable. The potential curve can be seen in the fig(2.1). Here solid curves are implying trajectories of a massless particle and dotted curves are implicating the trajectories of

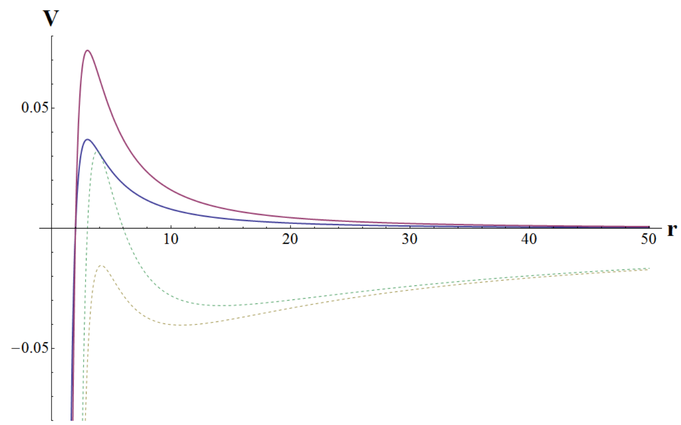


Figure 2.1: $V(r)$ plot for a particle moving around a schwarzschild black hole

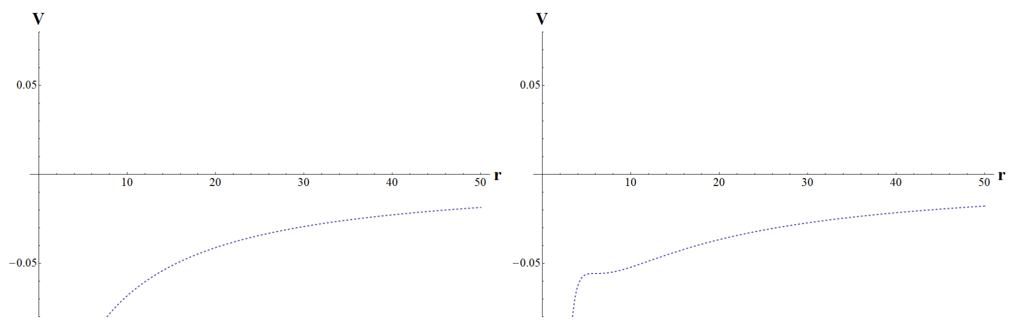


Figure 2.2: Left:$V(r)$ plot of massive particles for $L^2 = 8$ geometrical usits Right:$V(r)$ plot of massive particles for $L^2 = 12$ geometrical units

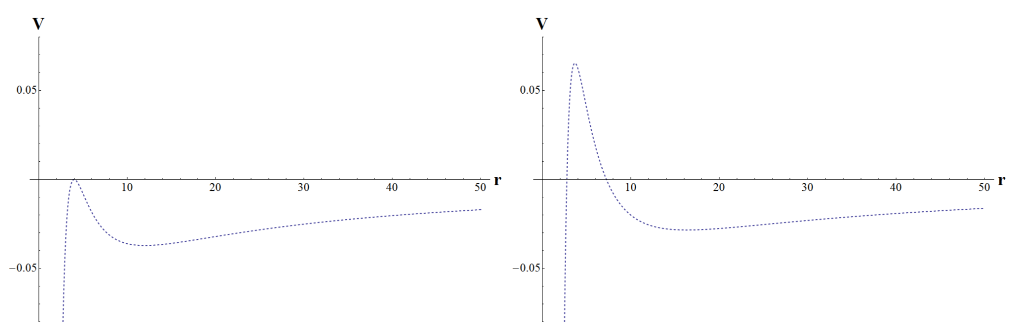


Figure 2.3: Left:$V(r)$ plot of massive particles for $L^2 = 16$ geometrical units Right:$V(r)$ plot of massive particles for $L^2 = 20$ geometrical units

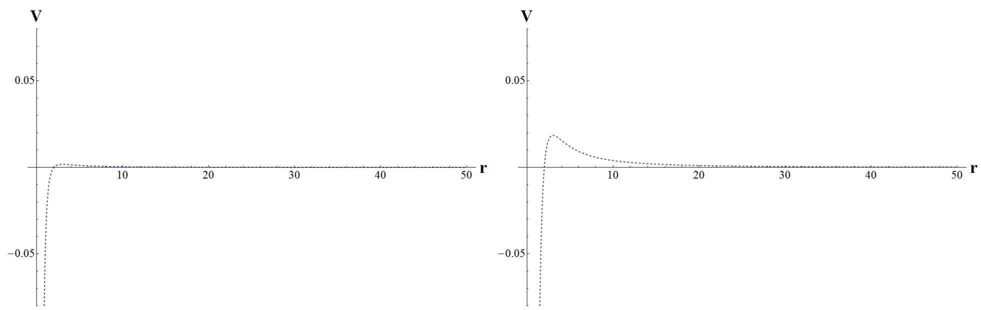


Figure 2.4: Left:$V(r)$ plot of massless particles for $L^2 = 0.1$ geometrical units Right:$V(r)$ plot of massless particles for $L^2 = 1$ geometrical units

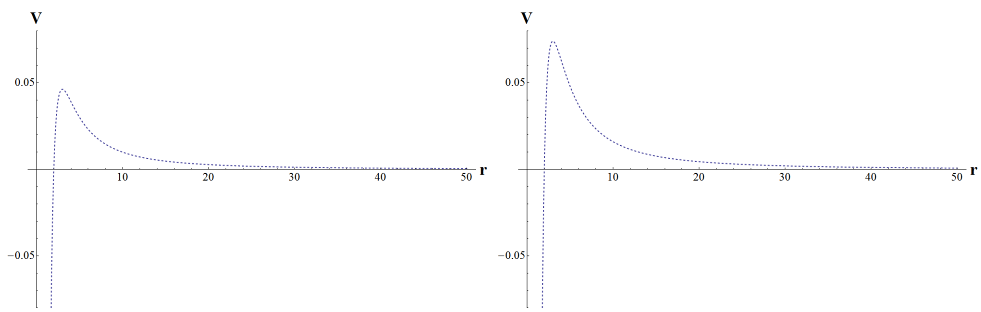


Figure 2.5: Left:$V(r)$ plot of massless particles for $L^2 = 2.5$ geometrical units Right:$V(r)$ plot of massless particles for $L^2 = 4$ geometrical units

massive particles. The solid pink curve is for $L^2 = 4$ geometrical units, and the solid blue curve is for $L^2 = 2$. Similarly, the green dotted curve is for $L^2 = 18$ geometrical units, and the brown dotted curve is for $L^2 = 15$.

I have provided some of the effective potential plots for massive and massless particles for different values of angular momentum L, i.e. fig(2.2), fig(2.3) and fig(2.4), fig(2.5) respectively.

We see that for small values of angular momentum, particle just falls into the black hole. But for intermediate and large values of angular momentum, particle can form elliptical orbits. For small values of r and for a freely falling particle, it can not increase its angular momentum. It will fall into the black hole. Well, to remain stable, particle should increase its angular momentum. But it is difficult to maintain angular momentum at an orbit with minimum radius, so it starts expressing spiral motion fast and it will fall into the black hole at last.

We now have results that Schwarzschild solutions exhibit stable circular orbits for $r > 6GM$ and unstable orbits for $3GM < r < 6GM$. The thing should be noted that these are geodesics. So there is nothing to stop the particle moving below $r = 3GM$. It can fall below $r = 3GM$ and can emerge from this value of r, if it lies in $r > 2GM$ region.

Last four effective potential plots for massless particles are implicating the unstable circular orbits. Below $r = 3GM$ particle is captured by the black hole. Any perturbation is being made to the particle at this location, will start moving inward or outward. If small perturbation is being made inward, the particle will simply fall into the black hole, and if perturbation is being made outward, the particle will simply escape to $r \to \infty$. Same physics happens for massive particles at $r = 6GM$. The double derivative of effective potential at this value of r becomes 0.

If we try to solve $\frac{dr}{d\phi}$, we get a solution which is not ellipse, and we get a precession to this theory for non-circular orbits[3]. Using equ(2.13) & equ(2.14) and solving for $\epsilon = 1$,

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4E^2}{L^2} - \frac{r^4}{L^2} + \frac{2r^3GM}{L^2} - r^2 + 2GMr$$

Let $r = \frac{L^2}{GMq}$,

$$\left(\frac{dq}{d\phi}\right)^2 - \frac{2G^2M^2q^3}{L^2} + q^2 - 2q + \frac{L^2}{G^2M^2} = \frac{E^2L^2}{G^2M^2}$$

Differentiating the above equation w.r.t. ϕ,

$$\frac{d^2q}{d\phi^2} - \frac{3G^2M^2}{L^2}q^2 + q - 1 = 0 \tag{2.16}$$

$\frac{dq}{d\phi}$ can not be zero for non-circular orbits i.e. $\frac{dq}{d\phi} \neq 0$. And if we follow the same procedure in Newtonian theory, we'll get the following expression,

$$\frac{d^2q}{d\phi^2} - 1 + q = 0$$

Under this Newtonian limit, the 2^{nd} term in equ(2.16) becomes zero. And the solution of the limiting equation is given by,

$$q_1 = 1 + e\cos\phi$$

Where e is eccentricity and which contains the information of semi-major and minor axes.

Now in general relativistic case, we add a correction term to this solution.

$$q \approx q_1 + q_2$$

And we obtain the following expression,

$$\frac{d^2 q_2}{d\phi^2} + q_2 - \frac{3G^2M^2}{L^2}(1 + e\cos\phi)^2 \tag{2.17}$$

Where q_2 is the correction term to q_1. When we solve this equation setting large ϕ, we get q_2 which is given by,

$$q_2 \approx 1 + e\cos\left(\phi - \frac{3G^2M^2}{L^2}\phi\right)$$

Now Perihelion advance of the planet, i.e. $\triangle\phi$ can be written as,

$$\triangle\phi = 2\pi\delta = 6\pi\frac{G^2M^2}{L^2} \tag{2.18}$$

Therefore the angular advancement during each orbit is given by $\triangle\phi$. The whole process is known as the precession of orbits.

2.4 Orbits for BTZ black hole

So far, we have seen orbits for Schwarzschild black hole in four-dimensional space-time which is spherically symmetric and static. Now we shall opt a lower dimensional black which is not static, i.e. three-dimensional rotating black hole with negative cosmological constant. The purpose of considering this type of black hole is to see the calculation simplification along with results so that we can do the similar process with other lower and higher dimensional black holes.

Solutions of this type of black holes were produced by *Banadõs, Teitelboim* and *Zanelli* (BTZ). The action for BTZ black hole is given by [5][6],

$$S = \frac{1}{2\pi}\int\sqrt{-g}\Big[R + \frac{2}{l^2}\Big]dtd^2x \tag{2.19}$$

where l is a length scale in order to have horizon which is given by $\frac{1}{l^2} = -\Lambda$. The metric takes the following form using this action,

$$ds^2 = -N^2dt^2 + \frac{1}{N^2}dr^2 + r^2(N^\phi dt + d\phi)^2$$

where $N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}$ and $N^\phi = -\frac{J}{2r^2}$ with limits -$\infty < t < \infty$, $0 < r < \infty$ and $0 \le \phi \le 2\phi$. And M, J are total mass of the black hole and angular momentum of the black hole respectively. The rearranged metric is,

$$ds^2 = -(N^2 - r^2N^{\phi 2})dt^2 + N^{-2}dr^2 + r^2d\phi^2 + r^2dtd\phi + r^2N^\phi d\phi dt \quad (2.20)$$

Now if we look at the horizon, take $N^2(r) = 0$. Which gives us two roots of r. One is the inner event horizon and another one is the outer event horizon.

$$r_\pm = l\Big[\frac{M}{2}\Big(1 \pm \sqrt{1 - \Big(\frac{J}{Ml}\Big)^2}\Big)\Big]^{\frac{1}{2}}$$

To Exist horizon $M > 0$ and $|J| \le Ml$, these two conditions should be satisfied. But for $|J| = Ml$, these two roots of r gets coincided. Now the components of the metric tensor are given by the following matrix,

$$g_{\mu\nu}= \begin{bmatrix} -\Big(-M + \frac{r^2}{l^2}\Big) & 0 & -\frac{J}{2} \\ 0 & \Big(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\Big)^{-1} & 0 \\ -\frac{J}{2} & 0 & r^2 \end{bmatrix}$$

Since this is a non-diagonal matrix, so we would need to find adjoint of the matrix. The final expression of the inverse of $g_{\mu\nu}$ i.e. $g^{\mu\nu}$ is,

$$g^{\mu\nu}= \begin{bmatrix} \frac{r^2}{\Big(-\frac{J^2}{4r^2}+Mr^2-\frac{r^4}{l^2}\Big)} & 0 & \frac{J}{2\Big(-\frac{J^2}{4}+Mr^2-\frac{r^4}{l^2}\Big)} \\ 0 & \Big(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\Big) & 0 \\ \frac{J}{2\Big(-\frac{J^2}{4}+Mr^2-\frac{r^4}{l^2}\Big)} & 0 & \frac{\Big(M-\frac{r^2}{l^2}\Big)}{\Big(-\frac{J^2}{4}+Mr^2-\frac{r^4}{l^2}\Big)} \end{bmatrix}$$

Now we'll look for the possible symmetries in the metric. One of the directions of symmetries is time t and another one is ϕ if we consider the simultaneous transformation of t and ϕ. Otherwise, the individual transformation will flip the sign in the diagonal terms of the metric tensor.

Symmetry in t direction : $\quad k^\mu = (1 \ 0 \ 0 \ )^T$

$$\begin{aligned} E =& - g_{\mu\nu} k^\mu u^\nu = -g_{tt} k^t u^t - g_{t\phi} k^t u^\phi \\ =& \Big(- M + \frac{r^2}{l^2}\Big) \frac{dt}{d\lambda} + \frac{J}{2} \frac{d\phi}{d\lambda} \end{aligned} \tag{2.21}$$

where E is one of the constants of motion. Let another constant of motion is L which is mainly angular momentum of the particle.

Symmetry in ϕ direction : $\quad k^\mu = (0 \ 0 \ 1)^T$

$$\begin{aligned} L =& g_{\mu\nu} k^\mu u^\nu = g_{\phi\phi} k^\phi u^\phi + g_{\phi t} k^\phi u^t \\ =& r^2 \frac{d\phi}{d\lambda} - \frac{J}{2} \frac{dt}{d\lambda} \end{aligned} \tag{2.22}$$

Considering the inner product of four-velocity,

$$u^\mu u_\mu = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = -\epsilon$$

Where $\epsilon = 1$ for massive particles/objects and $\epsilon = 0$ for massless particles/objects. Expanding this equation taking all non-zero components of the metric tensor,

$$-\Big(- M + \frac{r^2}{l^2}\Big) \Big(\frac{dt}{d\lambda}\Big)^2 - \frac{J}{2} \frac{dt}{d\lambda} \frac{d\phi}{d\lambda} - \frac{J}{2} \frac{d\phi}{d\lambda} \frac{dt}{d\lambda} + r^2 \Big(\frac{d\phi}{d\lambda}\Big)^2 + \frac{1}{N^2} \Big(\frac{dr}{d\lambda}\Big)^2 = -\epsilon$$

Except for radial velocity, eliminating all differential terms, the expression becomes,

$$r^2 \Big(\frac{dr}{d\lambda}\Big)^2 = - \epsilon \Big(- M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\Big) r^2 + E^2 r^2 - \frac{L^2 r^2}{l^2} + M L^2 - JEL \tag{2.23}$$

$$\frac{r^2}{l^2 M^2} \Big(\frac{dr}{d\lambda}\Big)^2 = - \epsilon \Big(- \frac{r^2}{l^2 M} + \frac{r^4}{l^4 M^2} + \frac{J^2}{4 M^2 l^2}\Big) + \frac{E^2 r^2}{l^2 M^2} - \frac{L^2 r^2}{l^4 M^2} + \frac{L^2}{l^2 M} - \frac{JEL}{l^2 M^2}$$

defining our variables again,

$$\frac{r^4}{l^4M^2} \equiv \tilde{r}^4 \Rightarrow \tilde{r} = \frac{r}{l\sqrt{M}} \quad and \quad \frac{L^2}{l^2M} \equiv \tilde{L}^2 \Rightarrow \tilde{L} = \frac{L}{l\sqrt{M}}$$

$$\frac{JEL}{l^2M^2} = \frac{J}{lM}\frac{E}{\sqrt{M}}\frac{L}{l\sqrt{M}} = \tilde{J}\tilde{E}\tilde{L} \Rightarrow \tilde{J} = \frac{J}{lM}$$

$$\frac{E^2r^2}{l^2M^2} = \frac{E^2}{M}\Big(\frac{r}{l\sqrt{M}}\Big)^2 = \tilde{E}^2\tilde{r}^2 \Rightarrow \tilde{E} = \frac{E}{\sqrt{M}}$$

$$\tilde{\lambda} \equiv \frac{\lambda}{l} \quad and \quad \tilde{\phi} \equiv \phi\sqrt{M} \quad and \quad \tilde{t} \equiv \frac{\sqrt{M}}{l}t$$

The final expression, after this transformation, becomes,

$$\tilde{r}^2\Big(\frac{d\tilde{r}}{d\tilde{\lambda}}\Big)^2 = \tilde{E}^2\tilde{r}^2 - \tilde{J}\tilde{E}\tilde{L} - \epsilon\Big(\tilde{r}^4 - \tilde{r}^2 + \frac{\tilde{J}^2}{4}\Big) - \tilde{L}^2\tilde{r}^2 + \tilde{L}^2$$

for writing convenience, I'll remove tilde from the above expression.

$$r^2\Big(\frac{dr}{d\lambda}\Big)^2 = E^2r^2 - JEL - \epsilon\Big(r^4 - r^2 + \frac{J^2}{4}\Big) - L^2r^2 + L^2 \tag{2.24}$$

using equ(17), rearranging the terms,

$$\frac{1}{2}\Big(\frac{dr}{d\lambda}\Big)^2 = \frac{1}{2}(\epsilon M + E^2) - \frac{1}{2}\Big(\frac{\epsilon r^2}{l^2} + \frac{\epsilon J^2}{4r^2} + \frac{L^2}{l^2} - \frac{ML^2}{r^2} + \frac{JEL}{r^2}\Big) \tag{2.25}$$

Now we can define our effective potential and total energy as

$$V(r) \equiv \frac{1}{2}\Big(\frac{\epsilon r^2}{l^2} + \frac{\epsilon J^2}{4r^2} + \frac{L^2}{l^2} - \frac{ML^2}{r^2} + \frac{JEL}{r^2}\Big) \tag{2.26}$$

$$\bar{E} \equiv \frac{1}{2}(\epsilon M + E^2)$$

Effective potential for massive and massless particles is given by the following expressions,

$$V(r)_{massive} = \frac{1}{2}\Big(\frac{r^2}{l^2} + \frac{J^2}{4r^2} + \frac{L^2}{l^2} - \frac{ML^2}{r^2} + \frac{JEL}{r^2}\Big) \tag{2.27}$$

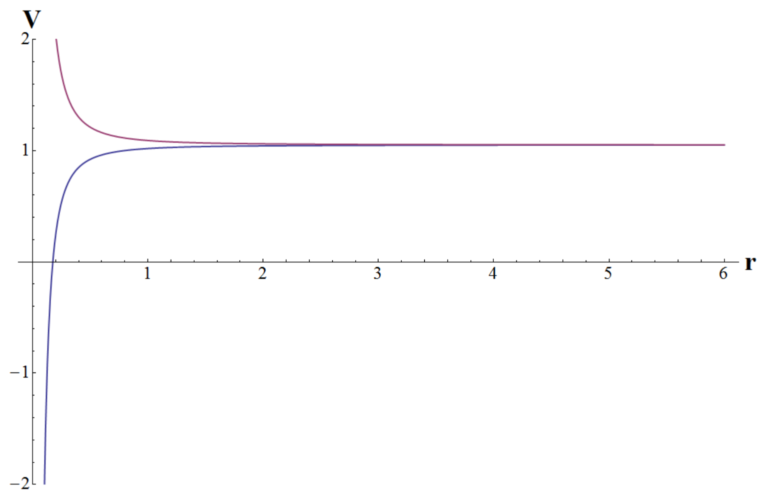


Figure 2.6: Effective potential plot for a massless particle; Blue: L=0.145 ; l=0.1 ; M=10 ; J=1.005 ; E=1 & Pink: L=0.145 ; l=0.1 ; M=10 ; J=2 ; E=1

$$V(r)_{massless} = V(r) = \frac{1}{2}\Big(\frac{L^2}{l^2} - \frac{ML^2}{r^2} + \frac{JEL}{r^2}\Big) \tag{2.28}$$

Here I have shown effective potential plots for massless and massive particles in fig(2.6) and fig(2.7) respectively which tells us about the possible trajectories of the particle. Here in plots, I have varied different parameters in both cases. Varying parameters can be seen in the footnote of each figure. We expect to get to different types of trajectories in each case which will depend on the physical parameters, we are considering.

The potential field can take two different signs, i.e. effective potential can approach towards positive infinity or negative infinity. I have varied parameters in such a way that I can get two different curves. Here in fig(2.6) and fig(2.7), I have varied J and M values to get two different curves. The potential-plot fig(2.6) considers two different values of J.

We may also consider other values of physical parameters which can bring two different signs in the potential of a massless particle. The similar thing we can see for the potential of a massive particle, i.e. two different values of M fig(2.7). It depends on physical parameters which can take different values. This is what I have shown in both potential graphs fig(2.6) and fig(2.7) for massless and massive cases.

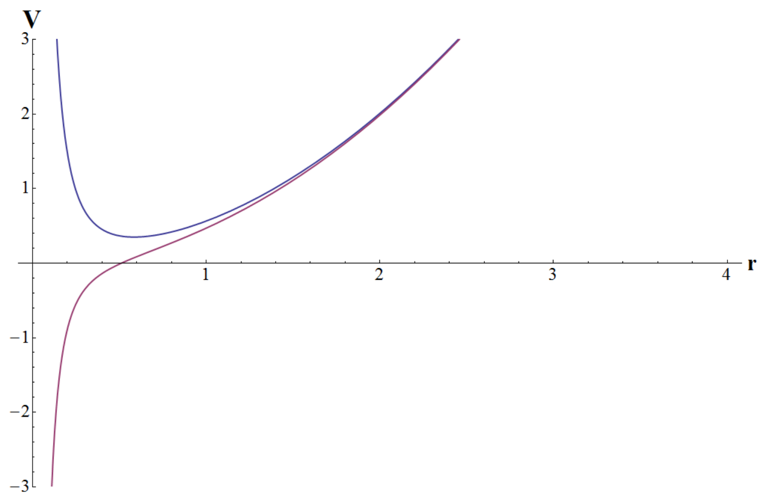


Figure 2.7: Effective potential plot for a massive particle; Blue: L=0.145 ; l=1.004 ; M=1 ; J=0.505 ; E=1 & Pink: L=0.145 ; l=1.004 ; M=10 ; J=0.505 ; E=1

Now we can also consider one signature of potential field and vary one of the parameters. So fig(2.8), fig(2.9), fig(2.10) and fig(2.11) are for different values of l and L in both massive and massless cases.

For massless case, if we consider, $\quad ML^2 = JEL$

$$V(r) = \frac{L^2}{2l^2}$$

the potential becomes constant. This is why we are getting a straight line in part of the potential of a massless particle fig(2.6). Constant potential physically means force $F = -\frac{\partial V}{\partial r} = 0$, i.e. acceleration is zero which means there is no change in velocity. So particle is moving with a constant velocity without getting attracted or repelled by the source. As other physical parameters are being changed then potential takes positive or negative infinity for small values of r.

$\frac{ML^2}{r^2}$ term is the important term because fixing M and increasing L values will flip the sign to the potential. In fig(2.9), if we set $L > 0.43$, potential will tend to negative infinity. Similar explanation can be made for different

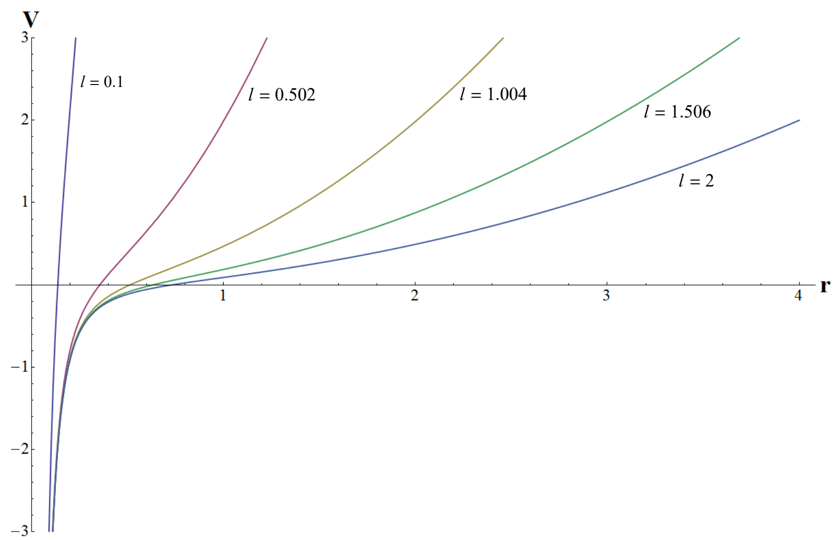


Figure 2.8: l varying effective potential plot for a massive particle for fixed values of L=0.145 ; M=10 ; J=0.505 ; E=1

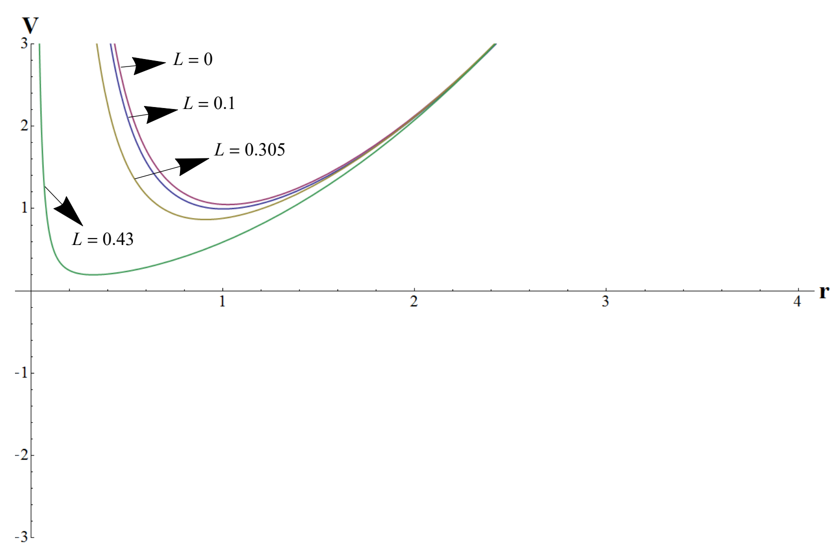


Figure 2.9: L varying effective potential plot for a massive particle for fixed values of l=1.004 ; M=10 ; J=2 ; E=1

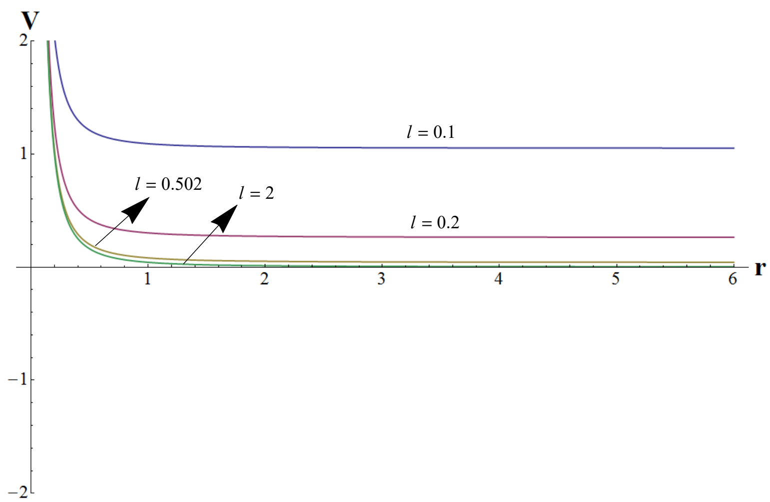


Figure 2.10: l varying effective potential plot for a massless particle for fixed values of L=0.145 ; M=10 ; J=2 ; E=1

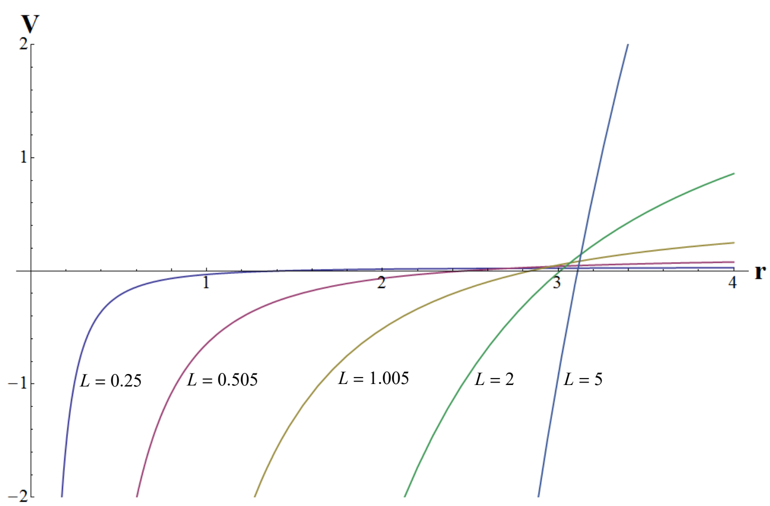


Figure 2.11: L varying effective potential plot for a massless particle for fixed values of l=1.004 ; M=10 ; J=2 ; E=1

values of l. But the whole point is that first, third & fourth and first & second terms are significantly changing terms in the effective potentials of massive and massless cases respectively.

Therefore this is all about trajectories of a particle around a BTZ black hole. Different cases have been considered here to get the physical meaning of the potential field. This also tells us that for small values of r, a particle can fall into the black hole contrast to the Newtonian theory.

Chapter 3

Null rays in black hole geometry in presence of cosmological constant

When we talk about black hole physics, it is essential to figure out the physics at the horizon. Horizon is a boundary which separates timelike, spacelike and null hypersurfaces. Here we shall try to obtain the mathematical structure to find the horizon of a given metric and then what is happening to the trajectories of particles considering different hypersurfaces. We shall see the physical interpretations of a particle's motion using light cone structure in the presence and absence of cosmological constant.

3.1 Mathematical Structure of Horizon

We consider a smooth function F(x) of given spacetime coordinates. The family of hypersurfaces is given by $\Sigma \equiv F(x) - constant = 0$. The normal vector to this hypersurface is[7],

$$n^{\mu} = g^{\mu\nu}\partial_{\nu}F(x)$$

and covariant form of the normal vector is,

$$n_{\mu} = \partial_{\mu}F(x)$$

Normalization of the normal vector will give us the essential information i.e.

$$n_{\mu}n^{\mu} = g^{\mu\nu}\partial_{\mu}F(x)\partial_{\nu}F(x) \tag{3.1}$$

Now considering r constant surfaces, we can look for different timelike, spacelike and null hypersurfaces. We shall see this description for Schwarzschild and BTZ black holes in further discussions.

3.1.1 Horizon of Schwarzschild Black hole

The metric of the static spherically symmetric black hole in sign (-,+,+,+) is given by,

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (3.2)$$

We have already seen in chapter 2 that $r \to 0$ is essential singularity and $r = 2GM$ is just a coordinate singularity. Now we shall see, why $r = 2GM$ is called horizon of Schwarzschild black hole.

Considering a hypersurface $\Sigma \equiv r - const. = 0$, the norm of the normal vector is given by,

$$n^{\mu}n_{\mu} = g^{\mu\nu}\partial_{\mu}\Sigma\partial_{\nu}\Sigma$$

Since we are dealing with $r = cosnt$ surface, Therefore

$$n^{\mu}n_{\mu} = g^{rr}\partial_{r}\Sigma\partial_{r}\Sigma$$

So norm of the normal vector will be just the contravariant form of the metric tensor, i.e. g^{rr}.

$$n^{\mu}n_{\mu} = (1 - \frac{2GM}{r}) \quad (3.3)$$

Now we can set different inequalities for different values of r. Therefore $r = const.$ surface will be timelike if $r > 2GM$, spacelike if $r < 2GM$ and null if $r = 2GM$. For $r = 2GM$ surface, inner product of normal vector becomes zero which means normal vector, for the given hypesurface, is a null vector and the surface is null surface. So we get different hypersurfaces about $r = 2GM$. In other words, $r = 2GM$ separates hypersurfaces. This is why we call $r = 2GM$ as the horizon of the black hole.

Now we can understand these different hypersurfaces by looking at the physics of lightcones about $r = 2GM$.

Light cones in Schwarzschild black hole

We face a problem in schwarzschild black hole at $r = 2GM$ because we see that $t - r$ plot diverges at $r = 2GM$. Which means light cones gets closed at infinity so we can not make precise physical statements about the light cones in this coordinate system. Setting $ds^2 = 0$ in the metric keeping θ and ϕ constant. We get following expression which is basically the angle of light cone[3].

$$\frac{dt}{dr} = \pm\frac{1}{1 - \frac{2GM}{r}} \quad \Rightarrow \quad t = \pm\Big[r + 2GMln \mid \frac{r - 2GM}{2GM} \mid \Big] + c \tag{3.4}$$

$+ve$ sign is for the outgoing trajectory of a particle and $-ve$ is for the ingoing trajectory of a particle. And we define constant c as u for outgoing trajectories and v for ingoing trajectories. So we have following expression in short notation,

$$t = r^* + u \quad (Outgoing) \qquad and \quad t = -r^* + v \quad (Ingoing)$$

where r^* is defined as, $\quad r^* = r + 2GMln \mid \frac{r-2GM}{2GM} \mid$

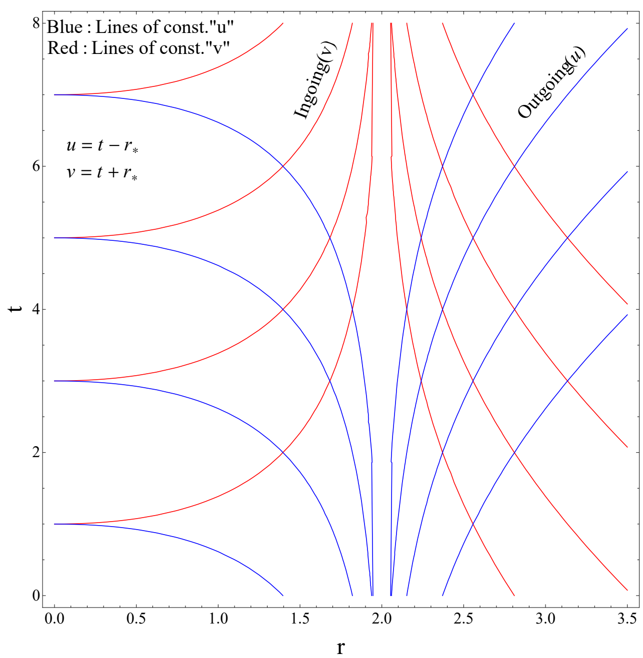


Figure 3.1: t-r plot of schwarzschild black hole

Since the trajectory of a particle must lie inside lightcones, i.e. all real particles move on timelike trajectories. So lightcones will always open up towards $t \to \infty$. In the equ(3.4), for $r = 2GM$, t becomes $\pm\infty$ for v and u respectively. We can see in the fig(3.1) that there could be something wrong with the coordinate system we are using. So we shall investigate another coordinate system in which physics of the lightcones become smooth.

Since physics of lightcones is a coordinate-dependent notion. So the trick is to use those constants u and v as coordinates which are known as Eddington-Finkelstein coordinate system, now we see the tilting over of light cones. So, all in all, we had bad coordinate system (t, r, θ, ϕ) in which we did not have a clear picture of lightcones at horizon whereas it becomes clear in Eddington-Finkelstein coordinate system.

In (v, r) coordinate system our metric becomes,

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dv^2 + 2dvdr + r^2 d\Omega^2 \tag{3.5}$$

Solving for null trajectories setting θ and ϕ constant, we get two differential equations, i.e.

$$\frac{dv}{dr} = 0 \quad and \quad \frac{dv}{dr} = \frac{2}{\Big(1 - \frac{2GM}{r}\Big)} \tag{3.6}$$

and solutions of these two simple differential equations are,

$$v = v_0 \quad and \quad v = v_0 + 2r + 4GMln \mid \frac{r - 2GM}{2GM} \mid \tag{3.7}$$

In similar ways, we have metric for (u, r) coordinates,

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)du^2 - 2dudr + r^2 d\Omega^2 \tag{3.8}$$

Again of null trajectories, setting θ and ϕ constant, we get two differential equations, i.e.

$$\frac{du}{dr} = 0 \quad and \quad \frac{du}{dr} = -\frac{2}{\Big(1 - \frac{2GM}{r}\Big)} \tag{3.9}$$

and solutions of these two differential equations are,

$$u = u_0 \quad and \quad u = u_0 - \Big[2r + 4GMln \mid \frac{r - 2GM}{2GM} \mid \Big] \tag{3.10}$$

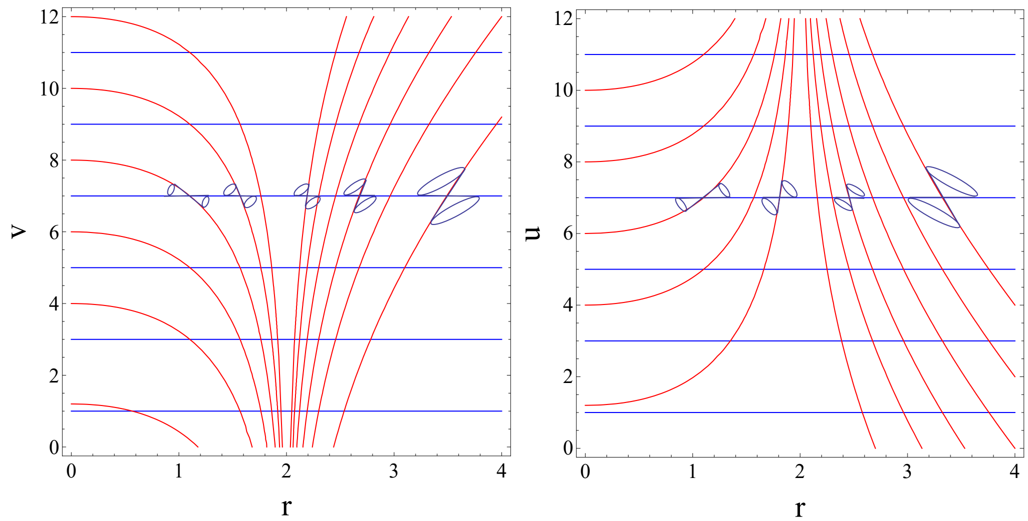


Figure 3.2: (v,r)& (u,r) plot of schwarzschild black hole

Where v_0 and u_0 are constants. Now we can see plots of (v, r) and (u, r) in the Fig(3.2).
In (v, r) plot, for region $r > 2GM$, particle can move towards future-directed timelike path. At $r = 2GM$ particle is having two choices either it will cross the barrier and reaches towards $r \to 0$ point. Or it can move on the horizon. Massless particle will move on null surface i.e. $r = 2GM$. The key point is that once particle crosses the horizon, it will never come back to $r > 2GM$ region. Because all future-directed paths will be ended up at $r = 0$

In (u, r) plot, for region $r > 2GM$, particles can move through past directed path. So we have separated our space-time in two directions, i.e. future one and past one. And light cones are smooth at the horizon. They do tilt over as r starts decreasing. Whereas light cones were closed in (t, r) coordinate system. So now we have seen the coordinate dependent notion of lightcones. So this is why Eddington-Finkelstein coordinate system is good to describe the physics of a particle about the horizon.

3.1.2 Horizon of BTZ black hole

We have already gone through the introduction of BTZ black hole in section 2.4. Here we shall try to identify the horizon and what is the behaviour of lightcones at the horizon. The first thing is to notice that Kretschmann scalar for BTZ black hole turns out to be

$$K = \frac{12}{l^4} \tag{3.11}$$

So It is independent of r which means we do not have any true singularity. In fact $r_\pm$ are coordinate singularities.

The metric of BTZ black hole can be taken from section 2.4 and consider a constant surface $\Sigma \equiv r - const. = 0$,

$$ds^2 = -(N^2 - r^2 N^{\phi 2})dt^2 + N^{-2}dr^2 + r^2 d\phi^2 + r^2 dt d\phi + r^2 N^\phi d\phi dt \tag{3.12}$$

where, $N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}$ and $N^\phi = -\frac{J}{2r^2}$

Now the inner product of normal vector is given by,

$$n^\mu n_\mu = g^{rr} = \frac{(r+r_+)(r-r_+)(r+r_-)(r-r_-)}{r^2 l^2}$$

Now we can set different inequalities for different values of r. Therefore, $r = const.$ surface is timelike if $r > r_+$ and $r < r_-$, spacelike if $r_- < r < r_+$. So r_+ and r_- are the region of separating timelike, spacelike and null hypersurfaces. Therefore, r_+ is outer event horizon and r_- outer event horizon.

Light cones for BTZ black hole

As we already discussed lightcone behaviour for Schwarzschild black hole. Taking limit $J \to 0$, the calculation gets simplified and now the horizon at $r = l\sqrt{M}$, but the description remains same as we have seen for Schwarzschild black hole.
If we take limit $J \to 0$, the metric equ(3.15) with constant ϕ reduces to,

$$ds^2 = -\Big(-M + \frac{r^2}{l^2}\Big)dt^2 + \Big(-M + \frac{r^2}{l^2}\Big)^{-1} dr^2 \tag{3.13}$$

So, to avoid the divergence at the horizon in (t, r, ϕ) coordinate system. We introduce Eddington-Finkelstein coordinate system so that metric can become regular and lightcones do not close up.

$$t = -r^* + v \quad (Ingoing) \qquad t = r^* + u \quad (Outgoing)$$

Where r^* is given by, $\quad r^* = -\frac{l}{2\sqrt{M}} ln\left|\frac{l\sqrt{M}+r}{l\sqrt{M}-r}\right|$

The metric in (v, r) coordinate is given by,

$$ds^2 = -\Big(-M + \frac{r^2}{l^2}\Big)dv^2 + 2dvdr \tag{3.14}$$

Now we can look for radial null trajectories by setting $ds^2 = 0$ and $\phi = const$. We get two differential equations,

$$\frac{dv}{dr} = 0 \qquad and \qquad \frac{dv}{dr} = \frac{2}{-M + \frac{r^2}{l^2}} \tag{3.15}$$

and solutions of these two differential equations are,

$$v = v_0 \qquad and \qquad v = v_0 - \frac{1}{l\sqrt{M}} ln\Big|\frac{l\sqrt{M}+r}{l\sqrt{M}-r}\Big| \tag{3.16}$$

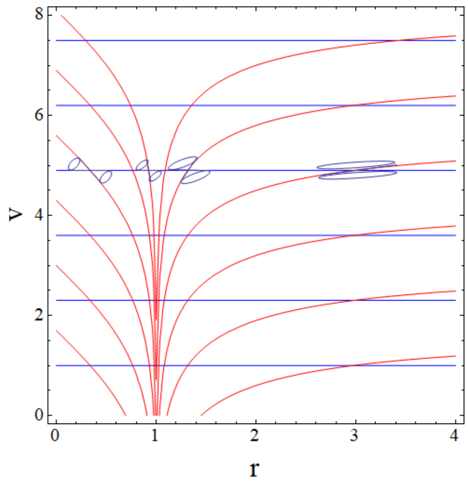
On the similar note, as we have seen in Schwarzschild black hole, we expect to get future directed light cones. Since physical particle always moves on timelike trajectories, so particle will remain inside the light cone, and when it reaches the horizon, it has two options to move. If the particle is massless that will be the null trajectory of the particle and if it is massive then it will hit at $r = 0$ boundary. So the critical point is that once the particle reaches the horizon $l\sqrt{M}$, it will never come back. Metric also become regular and lightcones do not close up at horizon. We can see the behaviour of lightcones in the fig(3.3).

Similarly, we can see the behaviour of lightcones in (u, r) coordinate system. The metric in this coordinates is given by,

$$ds^2 = -\Big(-M + \frac{r^2}{l^2}\Big)du^2 - 2dudr \tag{3.17}$$

Again for radial null trajectories setting ds^2 with $\phi = const$. We get two differential equations.

$$\frac{du}{dr} = 0 \qquad and \qquad \frac{du}{dr} = \frac{2}{\Big(M - \frac{r^2}{l^2}\Big)} \tag{3.18}$$



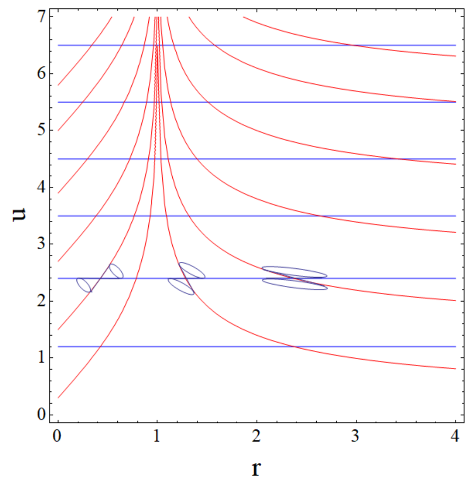


Figure 3.3: behaviour of lightcones in (v,r) coordinates

Figure 3.4: behaviour of lightcones in (u,r) coordinates

And solutions of these two differential equations are,

$$u = u_0 \qquad and \qquad u = u_0 + \frac{1}{l\sqrt{M}} ln\Big|\frac{l\sqrt{M}+r}{l\sqrt{M}-r}\Big| \tag{3.19}$$

Now we can see the behaviour of lightcones in this coordinate system in fig(3.4). We see that metric is regular on the horizon, and lightcones do not close up instead they tilt over as expected. The trajectories, in this case, will be past directed, and once particle crossed the horizon barrier, it will hit $r = 0$ point.

Therefore, Eddington-Finkelstein coordinates are good at describing the behaviour of lightcones specifically at the horizon. This gives us an idea that the behaviour of lightcone is a coordinate-dependent notion, i.e. in different coordinate system metric becomes regular and lightcones tilt over instead of getting closed at the horizon.

3.2 Photon sphere

In this section, we shall be looking for the position r of the photon, i.e. at which value of r, photon starts doing circular orbits. That is known as *photon sphere*. Due to the strong potential field of the black hole, photons make circular orbits. Here we'll be considering the potential field of black holes in presence and absence of cosmological constants.

3.2.1 Photon sphere for Schwarzschild black hole

From equation(2.14) in Chapter 2, we have effective potential for Schwarzschild black hole

$$V(r) \equiv \frac{\epsilon}{2} - \epsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \tag{3.20}$$

For photons ϵ will be zero. The effective potential field reduces to

$$V(r)_{photons} = \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \tag{3.21}$$

A derivative of this potential with respect to r, will give me the location of photon sphere. Which turns out to be at,

$$r = 3GM \tag{3.22}$$

The double derivative test of effective potential is less than zero at $r = 3GM$ which means this is unstable circular orbit location of photons. If a small perturbation is being made inward or outward to photons, it will fall into the black hole or can escape to infinity respectively.

3.2.2 Photon sphere for BTZ black hole

The expression of effective potential for BTZ black hole is given by equation (2.26),

$$V(r) = \frac{1}{2}\Big(\frac{\epsilon r^2}{l^2} + \frac{\epsilon J^2}{4r^2} + \frac{L^2}{l^2} - \frac{ML^2}{r^2} + \frac{JEL}{r^2}\Big) \tag{3.23}$$

For photons ϵ will be zero. The effective potential field reduces to,

$$V(r)_{photons} = \frac{1}{2}\Big(\frac{L^2}{l^2} - \frac{ML^2}{r^2} + \frac{JEL}{r^2}\Big) \tag{3.24}$$

Now we can find the location of photon sphere by taking derivative of effective potential with respect to r,

$$\frac{ML^2 - JEL}{r^3} = 0 \tag{3.25}$$

Which means photon sphere is located at $r \to \infty$. In other words, if we say that there is no photon sphere, is also correct. This statement can be verified by plotting the potential curve for massless particles. Which we already have seen in fig(2.6). There is no such point, i.e. r which gives us circular orbits. Potential is constant at $ML^2 = JEL$ which is $V = \frac{L^2}{2l^2}$, at this equality, each r is photon sphere. If this equality does not hold photon sphere will be at infinity, i.e. no photon sphere.

If we consider $l \to \infty$ limit in equ(3.31), the result will not get affected, and the location of photon sphere is $r \to \infty$.

3.2.3 Potential field and photon sphere of de sitter schwarzschild black hole

The metric of de sitter schwarzschild black hole in presence of cosmological constant is given by,

$$ds^2 = -\Big(1 - \frac{2GM}{r} - \frac{r^2}{l^2}\Big)dt^2 + \Big(1 - \frac{2GM}{r} - \frac{r^2}{l^2}\Big)^{-1} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.26}$$

Considering $\theta = constant$ we can have two directions of symmetry i.e. one in time (t) another one in ϕ. Therefore,

$$E = -g^{\mu\nu}k^{\mu}u^{\nu} = \Big(1 - \frac{2GM}{r} - \frac{r^2}{l^2}\Big)\frac{dt}{d\lambda} \tag{3.27}$$

This E is one of the constants of motion and another one in ϕ direction which is L,

$$L = g_{\mu\nu}k^{\mu}u^{\nu} = r^2\frac{d\phi}{d\lambda} \tag{3.28}$$

Where $k^t = (1\ 0\ 0\ 0)^T$ and $k^{\phi} = (0\ 0\ 0\ 1)^T$. u^{μ} and u^{ν} is four velocity. Now we can use inner product of four velocity i.e. $g_{\mu\nu}u^{\mu}u^{\nu} = -\epsilon$,

$$g_{tt}u^t u^t + g_{rr}u^r u^r + g_{\phi\phi}u^{\phi}u^{\phi} = -\epsilon \tag{3.29}$$

Where ϵ is one for massive particles and zero for massless particles. Plugging all expressions in terms of E and L, we get the following expression,

$$\frac{1}{2}\Big(\frac{dr}{d\lambda}\Big)^2 + \frac{\epsilon}{2}\Big(1 - \frac{2GM}{r} - \frac{r^2}{l^2}\Big) + \frac{L^2}{2r^2}\Big(1 - \frac{2GM}{r} - \frac{r^2}{l^2}\Big) \tag{3.30}$$

This is nothing but conservation of energy. Now we can define our total energy as $\tilde{E} = \frac{E^2}{2}$ and the effective potential is,

$$V(r) = \frac{\epsilon}{2}\Big(1 - \frac{2GM}{r} - \frac{r^2}{l^2}\Big) + \frac{L^2}{2r^2}\Big(1 - \frac{2GM}{r} - \frac{r^2}{l^2}\Big) \tag{3.31}$$

If we take limit $l \to \infty$, we recover Schwarzschild black hole effective potential. For photons, the effective potential reduces to,

$$V_{photon} = \frac{L^2}{2}\Big(\frac{1}{r^2} - \frac{2GM}{r^3} - \frac{1}{l^2}\Big) \tag{3.32}$$

taking a derivative of $V(r)$ with respect to r, to find the position of photon sphere,

$$r = 3GM \tag{3.33}$$

Here if we take limit $l \to \infty$ in the V_{photon} expression, the location of photon sphere does not get affected. It remains at the same position i.e. $r = 3GM$. This is the unstable circular orbit of photons because second derivative test at $r = 3GM$ is negative. So any small perturbation made to the particle, will cause the particle to fall or escape from the black hole which depends on in which direction the perturbation is being made.

So we find that there is no effect of cosmological constant, i.e. length scale l, on photon sphere for mentioned black holes. For both cases, BTZ black hole and de sitter schwarzschild black hole, it disappears in the derivative of potential while finding r. If we take limit $l \to \infty$, it does not make any effect on the location of the photon sphere.

Chapter 4

Summary

4.1 Conclusion and Discussion

Chapter 1 introduces the particle's trajectory around a given mass source. We see that particle, approaches to a given mass source, can escape to infinity. Though we get all different kinds of orbits, e.g. elliptical, circular, parabolic and hyperbolic. The key point is, in Newtonian theory particle always escapes to infinity. We find in chapter 2 that there was a correction term to the Newtonian theory i.e. $-\frac{1}{r^3}$ which makes particle to fall into the source. Here our mass source is Schwarzschild black hole and BTZ black hole. Though we do get circular, elliptical and parabolic orbits but for small values of r, particle falls into the black hole. We realise that the Newtonian theory assumes time as an absolute parameter whereas in relativity space and time are treated on equal footing. Considering time as a relative parameter, Einstein found the relativistic generalisation of the Newtonian theory of gravity.

In Chapter 3, we have a general mathematical structure for identifying horizon of a given metric. We face a problem that in the specific coordinate system, it takes an infinite amount of time for a particle to cross the horizon. We find Eddington-Finkelstein coordinates which make the metric regular at horizon for both Schwarzschild and BTZ black holes. So lightcones become coordinate dependent notion, and we get future and past-directed lightcones but once a particle crosses the horizon, it will never come back.

Above results and discussion include physics of massive and massless particles. Further, we understand the effect of cosmological constant on null trajectories in black hole geometry. We determine the location of circular photon orbits for Schwarzschild, BTZ black holes and Schwarzschild black hole in the presence of cosmological constant. We get photon sphere at $r \to \infty$ and if we see potential plot of massless particle, it becomes clear that there is no photon sphere for BTZ black hole. However, we do get photon sphere location at $r = 3GM$ for Schwarzschild black hole and de sitter Schwarzschild black hole. These orbit locations are unstable, so any small perturbation made to the particle will compel the particle to fall into the black hole or to escape depending upon the direction of perturbation. So cosmological constant is not producing any effect on photon sphere as the effect of l is getting disappeared in the derivative of effective potential in both cases BTZ and de sitter schwarzschild black hole with cosmological cosntant.

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