# Construction of a Class of Quantum Dynamical Semigroups Associated With Formal Lindbladians via Hudson-Parthasarathy Flows

**A thesis submitted for the partial fulfillment**

**of the degree of**

**Doctor of Philosophy**


*Supervised by:*

**Dr. Lingaraj Sahu**

*Submitted by:*

**Preetinder Singh**

**Department of Mathematical Sciences**

**Indian Institute of Science Education and Research Mohali**

**Mohali-140306**



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*Dedicated*

*to*

*my parents*

# Declaration

The work presented in this thesis has been carried out by me under the guidance of Dr. Lingaraj Sahu at Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Date:
Place:

Preetinder Singh

In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Lingaraj Sahu
(Supervisor)

**ABSTRACT**

The efforts made in the dissertation are to understand strongly continuous quantum dynamical semigroups(QDS) by yielding examples of Lindbladians which could generate Markov semigroups. Such semigroups come into picture when one studies the dynamics of open quantum systems. The QDS, which are non-commutative analogue of the expectation semigroup of Markov processes in the classical case, are the semigroups of completely positive maps on $C^*$-algebras or von Neumann algebras satisfying continuity conditions. The uniformly continuous QDS are completely characterized on hyperfinite von Neumann algebras by Lindblad and on $C^*$-algebras by Christensen, Evans by a bounded generator known as Lindbladian.

However, for the case of a strongly continuous QDS, structure of the generator is not well understood. Davies, Kato, Chebotarev, Fagnola showed that under certain assumptions, unbounded generators have a similar Lindblad form. Conversely, in various attempts, given a Lindblad like unbounded operators, the QDS were generated but these QDS need not be Markov(Conservative).

Here, we study a class of Lindbladians expressed as bilinear forms on a GNS space of a UHF algebra. Using quantum stochastic dilations it was proved that the Hudson-Parthasarathy (HP) type quantum differential equation associated with Lindblad form exhibits unique unitary solution. The QDS thus constructed by taking the vacuum expectation semigroup of the homomorphic co-cycle is conservative, therefore is the unique $C_0$-contraction semigroup associated with the given form.

Next, for a class of Lindbladians on UHF algebra, existence of associated Evans-Hudson flows was proved. The expectation semigroup associated with the given Lindbladian is Markov. The arguments used here to solve stochastic differential equations associated with the Lindbladian reveal that the local structure of the UHF algebra is immensely helpful.

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**Preetinder Singh**

# Contents

# Chapter 1

# Introduction

In this thesis the main objects of study are quantum dynamical semigroups. Quantum dynamical semigroups (QDS) are the semigroups of completely positive maps on nice algebras of operators satisfying some continuity conditions. Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the von Neumann algebra of bounded linear operators on $\mathcal{H}$.

**Definition.** A *quantum dynamical semigroup* on a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a semigroup $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ of completely positive maps on $\mathcal{A}$ with the following properties:

$(i)$ $\mathcal{T}_t(I) \leq I$, for all $t \geq 0$.

$(ii)$ $\mathcal{T}_t$ is a ultra-weakly continuous operator i.e. normal for all $t \geq 0$.

$(iii)$ for each $a \in \mathcal{A}$, the map $t \to \mathcal{T}_t(a)$ is continuous with respect to the ultra-weak topology on $\mathcal{A}$.

A QDS is called *Markov* or *Conservative* if $\mathcal{T}_t(I) = I$ for every $t$. QDS appear naturally when one studies the evolution of irreversible open quantum systems describing the time evaluation. The notion of QDS extends the semigroups of probability transition maps for classical Markov processes.

The generator of a semigroup $(\mathcal{T}_t)_{t\geq 0}$ on a Banach space is defined as the limit of operators $\dfrac{\mathcal{T}_t - I}{t}$ as $t$ tends to 0 and write $\{\mathcal{T}_t = e^{t\mathcal{L}} \ : \ t \in \mathbb{R}\}$. For a uniformly continuous (or norm continuous) semigroup on $C^*$ or von Neumann algebras, the generator is bounded, conditionally completely positive map. In [25], Lindblad proved that for hyper-finite von Neumann algebras, which includes the case of $\mathcal{B}(\mathcal{H})$, that a bounded operator $\mathcal{L}$ is a generator of a uniformly continuous QDS if and only if $\mathcal{L}$ can be written as $\mathcal{L}(X) = \phi(X)+G^*X+XG$, where $\phi$ is completely positive. In the same year, Gorini, Kosaakowski and Sudarshan [17] proved the similar result for finite-dimensional Hilbert spaces.

**Theorem.** [25, 17] A bounded map $\mathcal{L}$ on the von Neumann algebra $\mathcal{B}(\mathcal{H})$ is the infinitesimal generator of a uniformly continuous QDS $(\mathcal{T}_t)_{t\geq 0}$ if and only if it can be written as

$$\mathcal{L}(X) = \sum_{n=1}^{\infty} L_n^* X L_n + G^*X + XG, \text{ for all } X \in \mathcal{B}(\mathcal{H}),$$

where $L_n$'s and $G$ are in $\mathcal{B}(\mathcal{H})$ and the series on the right side converges strongly, with $G$ generator of a contraction semigroup in $\mathcal{H}$. The QDS is Markov if and only if

$$Re(G) = -\frac{1}{2}\sum_{n=1}^{\infty} L_n^* L_n.$$

In [8] Christensen and Evans proved that for general $C^*$-algebras, the generator of a uniformly continuous QDS exhibits the similar structure. More precisely, if $\mathcal{A}$ is a $C^*$-algebra acting on a Hilbert space $\mathcal{H}$ and suppose that $\{\mathcal{T}_t = e^{t\mathcal{L}} : t \in \mathbb{R}\}$ is a norm continuous semigroup of completely positive maps of $\mathcal{A}$ into $\mathcal{A}$, then there exists a completely positive map $\theta$ from $\mathcal{A}$ into the ultraweak closure $\bar{\mathcal{A}}$ of $\mathcal{A}$ and an operator $k$ in $\bar{\mathcal{A}}$ such that the generator $\mathcal{L}$ is given by $\mathcal{L}(a) = \theta(a) + k^*a + ak$. However, often the QDS governing the dynamics of physical system are not uniformly continuous, rather strongly continuous.

For the case of a strongly continuous QDS, structure of the generator is not well understood. The problem of constructing QDS with an unbounded generator $\mathcal{L}$, could be handled using Hille-Yosida theorem as it is done by Matsui in [27] for certain class of semigroups on UHF algebras. In general, the infinitesimal generator $\mathcal{L}$ is not given explicitily (with some manifold as a domain), but it is given formally which is called formal Lindbladian with unbounded coefficients. Kato [22] and Davies [11] studied some unbounded operators or forms similar to the Lindblad form on $\mathcal{B}(\mathcal{H})$ and gave a construction of one-parameter semigroups, so-called minimal semigroup. However, these semigroups need not preserve the identity, that is need not be Markov. In [5], Chebotarev listed out some sufficient conditions for a QDS to be conservative. Later on these conditions are simplified in [7, 6] by Chebotarev and Fagnola. Generally such unbounded operator or form referred as Lindbladian. Under certain assumptions, Davies in [12] showed that the unbounded generator have a similar form as for the bounded case, thus extends the Lindblad's result to strongly continuous QDS. Holevo in [19] investigated the structure of covariant QDS. An expository article giving the development of QDS theory is written by Fagnola [16]. In [2], Bahn, Ko and Park discuss conservative QDS generated by noncommutative unbounded elliptic operators. Recently, in [1] authors give a structure theorem for ultra-weakly continuous QDS on $\mathcal{B}(\mathcal{H})$ under the assumption of existence of rank one projection in the domain of generator.

In this thesis, we have considered Hudson-Parthasarathy (HP) and Evans-Hudson (EH) quantum stochastic differential equations associated with unbounded Lindbladians and construct the QDS by taking vacuum expectation of homomorphic cocycles. There are various attempts to study quantum stochastic differential equations with unbounded coefficients, for example see [16, 36] and references therein.

In this introductory chapter, we have given a historical background of the development of the theory of Markov semigroups and their dilations. The main results of the thesis is discussed briefly.

In chapter second, the more basic theory which are the results from Hilbert space theory, von Neumann algebras, $C^*$-algebras, main results as well as characterization of Completely positive maps on von Neumann algebras are included and general semigroup theory on Banach spaces are given to make the thesis self-contained.

In the third chapter of the thesis, then the notion of QDS, characterization of uniform continuous QDS are given. The theory of strongly continuous QDS is presented. As the last section of this chapter, the theory of quantum stochastic calculus developed by Hudson and Parthasarathy [20] is discussed briefly.

In the fourth chapter, the results proved in [34] are explained. Briefly, a class of unbounded Lindblad form are defined on the GNS space of UHF $C^*$-algebra and properties of structure maps are studied. Finally, exploring the local structure of UHF algebra, it is shown that the associated HP equation admits a unitary solution. This implies that the expectation semigroup of the homomorphic co-cycle implemented by this unitary is conservative and therefore the unique (also minimal) $C_0$-contraction semigroup associated with the given form.

# Main Results

Before listing the main results, we shall introduce some notions and give important observations for sake of clarity.

For a separable Hilbert space $\mathcal{H}$, let $\Gamma_{sym}(\mathcal{H})$ denotes the symmetric Fock space over $\mathcal{H}$. For any $u \in \mathcal{H}$, we denote by $e(u)$, the exponential vector in $\Gamma_{sym}(\mathcal{H})$ associated with $u$:

$$e(u) = \bigoplus_{n\geq 0} \frac{1}{\sqrt{n!}} u^{\otimes n}.$$

Given a contraction $T$ on $\mathcal{H}$, the second quantization $\Gamma(T)$ on $\Gamma_{sym}(\mathcal{H})$ is defined by $\Gamma(T)e(u) = e(Tu)$ and extends to a contraction on $\Gamma_{sym}(\mathcal{H})$. Moreover, if $T$ is an isometry (respectively unitary), then so is $\Gamma(T)$.

Let us write $\Gamma_{sym}$ for the symmetric Fock space $\Gamma_{sym}(L^2(\mathbb{R}_+, \mathbf{k}))$, where $\mathbf{k}$ is a Hilbert space with an orthonormal basis $\{e_l : 1 \leq l \leq m\}$.

Let us consider the UHF $C^*$-algebra $\mathcal{A}$, the $C^*$-inductive limit of the infinite tensor product of the matrix algebra $M_N(\mathbb{C})$,

$$\mathcal{A} = \overline{\bigotimes_{j\in\mathbb{Z}^d} M_N(\mathbb{C})}^{c^*}.$$

For $x \in M_N(\mathbb{C})$ and $j \in \mathbb{Z}^d$, $x^{(j)}$ denotes an element of $\mathcal{A}$ with $x$ in the $j^{th}$ component and identity everywhere else. We shall call the elements of the form $\prod_{i\geq 1} x_i^{(j_i)}$ to be simple tensor elements in $\mathcal{A}$. For a simple tensor element $x$ in $\mathcal{A}$, let $x_{(j)}$ be the $j^{th}$ component of $x$. Support '$supp(x)$' of $x$ is defined to be the subset $\{j \in \mathbb{Z}^d; x_{(j)} \neq I\}$. For a general element $x \in \mathcal{A}$ such that $x = \sum_{n=1}^{\infty} c_n x_n$ with simple tensor elements $x_n$ and complex coefficients $c_n$, define $supp(x) = \bigcup_{n\geq 1} supp(x_n)$. For any $\Delta \subset \mathbb{Z}^d$, let $\mathcal{A}_\Delta$ denotes the $*$-sub algebra generated by the elements of $\mathcal{A}$ with support in $\Delta$. For $j = (j_1, j_2, \cdots, j_d) \in \mathbb{Z}^d$, define $|j| = max\{|j_i| \ ; \ 1 \leq i \leq d\}$ and set $\Delta_n = \{j \in \mathbb{Z}^d; |j| \leq n\}$, $\partial\Delta_n = \{j \in \mathbb{Z}^d; |j| = n\}$. We say an element $x \in \mathcal{A}$ is local if $x \in \mathcal{A}_{\Delta_p}$ for some $p \geq 1$. Denote by $\mathcal{A}_{loc}$, the dense $*$-algebra generated by local elements. Consider the unique normalized trace $tr$ on $\mathcal{A}$. The algebra elements in $\mathcal{A}$ are represented as vectors in the Hilbert space $\mathbf{h}_0 = L^2(\mathcal{A}, tr)$, the GNS Hilbert space for $(\mathcal{A}, tr)$, and as a bounded operator on $\mathbf{h}_0$ by left multiplication.

Consider a formal element of the type

$$r := \sum_{n=1}^{\infty} W_n \text{ such that } \sum_{n=1}^{\infty} \|W_n\| = \infty,$$

where each $W_n$ belongs to $\mathcal{A}_{\partial\Delta_n}$. Let us denote formally

$$\sum_{n=1}^{\infty} W_n^* \text{ by } r^*.$$

Now, if we set $\mathcal{C}_r(x) = [r, x] = \sum\limits_{n=1}^{\infty} [W_n, x]$ for $x \in \mathcal{A}_{loc}$, clearly it is well defined since $[W_n, x] = 0$ for all $n > m$ when $x$ is in finite dimensional algebra $\mathcal{A}_{\Delta_m} \subseteq \mathcal{A}_{loc}$. We have observed that the operator $(\mathcal{C}_r, \mathcal{A}_{loc})$ is densely defined, closable operator along with its adjoint. Furthermore, the operator $G := -\frac{1}{2}\mathcal{C}_r^* \bar{\mathcal{C}}_r$ generates a $C_0$-contraction semigroup $\mathcal{S}_t$ in $\mathbf{h}_0$.

Now consider the Lindblad form, $\mathcal{L}(X)$, where $X \in \mathcal{B}(\mathbf{h}_0)$ with the domain $\mathcal{A}_{loc} \times \mathcal{A}_{loc} \subseteq$

$Dom(G) \times Dom(G)$ given by

$$\langle u, \mathcal{L}(X)v\rangle \equiv \langle u, XGv\rangle + \langle Gu, Xv\rangle + \langle \bar{\mathcal{C}}_r u, X\bar{\mathcal{C}}_r v\rangle. \tag{1.1}$$

By definition of $G$, it is clear that $\langle u, \mathcal{L}(I)v\rangle = \langle u, Gv\rangle + \langle Gu, v\rangle + \langle \bar{\mathcal{C}}_r u, \bar{\mathcal{C}}_r v\rangle = 0$. Let $\mathcal{A}_{loc} \otimes \mathcal{E}$ be the linear span of $\{x \otimes e(f) \;:\; x \in \mathcal{A}_{loc},\ f \in L^2(\mathbb{R}_+, \mathbb{C})\}$. Then the set $\mathcal{A}_{loc} \otimes \mathcal{E}$ is a dense subspace of $\mathbf{h}_0 \otimes \Gamma_{sym}$.

**Main Results.**

**1**. *Consider the HP type QSDE in* $\mathcal{A}_{loc} \otimes \mathcal{E}$

$$U_t = I + \int_0^t U_s G ds + \int_0^t U_s \bar{\mathcal{C}}_r a^\dagger(ds) - \int_0^t U_s \mathcal{C}_r^* a(ds), \tag{1.2}$$

*where* $a^\dagger, a$ *are creation and annihilation processes respectively. The QSDE admits a unitary solution* $U_t$. *Moreover, the expectation semigroup* $(\mathcal{T}_t)_{t\geq 0}$ *on* $\mathcal{B}(\boldsymbol{h}_0)$ *of the homomorphic co-cycle* $J_t(X) = U_t^*(X \otimes I)U_t$ *is the unique (minimal) semigroup associated with the formal Lindbladian* $\mathcal{L}$ *in* (1.1) *and is conservative.*

Next,we deal with the structure maps on $\mathcal{A}_{loc}$ in the UHF algebra. In this section, we deal with the structure maps on $\mathcal{A}_{loc}$ in the UHF algebra. For $W_k \in \mathcal{A}_{\partial\Delta_k}$, define the operators:

$$\delta_k(X) = [X, W_k], \qquad \delta_k^\dagger(X) = (\delta_k(X^*))^* = [W_k^*, X],$$

for every $X \in \mathcal{A}_{loc}$. Consider the Lindbladian:

$$\mathcal{L}(X) = \frac{1}{2}\sum_{k=1}^{\infty}\{W_k^*\delta_k(X) + \delta_k^\dagger(X)W_k\}, \text{ for all } X \in \mathcal{A}_{loc}. \tag{1.3}$$

Though each component $W_k^*\delta_k(.) + \delta_k^\dagger(,)W_k$ are bounded maps, $\mathcal{L}$ is unbounded due to presence of infinitely many components (like in [27]). For $n \geq 1$, define a bounded map $\mathcal{L}^{(n)}(X) = \frac{1}{2}\sum_{k=1}^n\{W_k^*\delta_k(X) + \delta_k^\dagger(X)W_k\}$, for all $X \in \mathcal{A}$. Note that for $X \in \mathcal{A}_{\Delta_n}$, $\delta_k(X) =$

$\delta_k^\dagger(X) = 0$ and $\mathcal{L}^{(k)}(X) = \mathcal{L}^{(n)}(X)$ for every $k \geq n$.

**2**. *The associated HP equation to* (1.3) *does not make sense. However, there exist a homomorphic co-cycle* $J_t : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\Gamma_{sym})$ *satisfying the Evans-Hudson equation, for* $X \in \mathcal{A}_{loc}$,

$$J_t(X) = X \otimes I + \int_0^t J_s(\mathcal{L}(X))ds + \sum_{j=1}^{\infty} \int_0^t J_s(\delta_j(X))a_j^\dagger(ds) + \sum_{j=1}^{\infty} \int_0^t J_s(\delta_j^\dagger(X))a_j(ds).$$

*The expectation semigroup* $(\mathcal{T}_t)_{t\geq 0}$ *of the homomorphic co-cycle* $J_t$ *is conservative minimal semigroup associated with the Lindbladian* (1.3).

**Remark.** The main differences between the classes of Lindbladian considered in (4.4) and (1.3) and the one considered in [27] is lack of translation invariance, which considerably affect the physical relevance of the semigroups. Indeed,to show HP or EH dilations of semigroups, the local structure of the algebra is exploited in such a way that importance of approximations by finite dimensional algebras is clearly recognized. Second difference is unlike the existence of only EH dilations of semigroups in [27], both HP and EH dilations are possible for the semigroup generated by the Lindbladian in (4.4), which makes these class of semigroups more interesting.

# Chapter 2

# Preliminaries

In this chapter, we review some of the result and concepts regarding operators on Hilbert spaces, in particular unbounded operators which are essential to understand semigroups and their generators, for detail we refer to [32, 9, 39, 41, 33, 38, 24, 23]. We also discuss the basic notions of operator algebras: $C^*$-algebras and von Neumann algebras, the details can be seen in [13, 14, 3, 10, 21]. Important properties of completely positive maps are discussed, for more details refer to [29, 37, 30]. In the last section, brief introduction to semigroup theory on Banach spaces is given. Most of the material can be found in [15, 31, 41].

## 2.1 Hilbert Space Theory

Let $\mathcal{H}$ be a Hilbert space with inner-product $\langle \cdot, \cdot \rangle$, which is conjugate linear in first and linear in second coordinate and $\| \cdot \|$ be the norm on $\mathcal{H}$. For a linear subspace $M$ of $\mathcal{H}$, define orthogonal complement $M^\perp = \{h \in \mathcal{H};\ \langle g, h \rangle = 0,\ \forall\ g \in M\}$.

**Remark 2.1.1.** *$M$ is dense in $\mathcal{H}$ if and only if $M^\perp = 0$.*

Suppose $\mathcal{H}_1, \mathcal{H}_2, \cdots$ are Hilbert spaces and let $\mathcal{H} = \{(h_n)_{n \geq 1} : h_n \in \mathcal{H}_n, \sum_{n \geq 1} \|h_n\|^2 <$

$\infty\}$. For $h = (h_n)_{n\geq 1}$ and $g = (g_n)_{n\geq 1}$ in $\mathcal{H}$, define

$$\langle h, g\rangle = \sum_{n\geq 1} \langle h_n, g_n\rangle .$$

Then $\langle\cdot,\cdot\rangle$ is an inner-product on $\mathcal{H}$ and $\mathcal{H}$ is called the **direct sum** of $\mathcal{H}_i$'s and is written as $\mathcal{H} = \bigoplus_{i=1}^{\infty} \mathcal{H}_i$. We denote the Banach space of bounded linear operators from a Hilbert space $\mathcal{H}$ to $\mathcal{K}$ by $\mathcal{B}(\mathcal{H},\mathcal{K})$ and $\mathcal{B}(\mathcal{H})$ is a Banach space of bounded linear operators on $\mathcal{H}$.

**Definition 2.1.2.** *An operator $T \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Bh, h\rangle \geq 0$ for all $h \in \mathcal{H}$. We write $T \geq 0$ if $T$ is positive and $S \geq T$, if $S - T \geq 0$.*

**Definition 2.1.3.** *An operator $T \in \mathcal{B}(\mathcal{H})$ is called compact if the closure of the image of the unit ball under $T$ is compact. Equivalently $T$ is compact if and only if for every bounded sequence $\{h_n\}$ in $\mathcal{H}$, $\{Th_n\}$ has convergent subsequence.*

The set of all compact operators form a closed maximal ideal of the ring $\mathcal{B}(\mathcal{H})$. Denote this ideal by $\mathcal{B}_0(\mathcal{H})$.

**Example 2.1.4. (Finite rank operators)** *Suppose the range of $T$ is finite dimensional. Since in a finite dimensional Hilbert space every closed and bounded set is compact, and the image of a unit ball is bounded. We see that $T$ is compact. In particular, for $u, v \in \mathcal{H}$, the rank one operators on $\mathcal{H}$ defined by $|u\rangle\langle v|(w) := \langle v, w\rangle u$ are compact.*

In fact, every compact operator is the uniform limit of finite rank operators. We now give the statement of the **spectral theorem** for compact normal operators.

**Theorem 2.1.5.** *Let $T \in \mathcal{B}_0(\mathcal{H})$ be a compact normal operator, then the set of eigenvalues of $T$ is countable. Suppose $(\lambda_n)_{n\geq 1}$ is a sequence of eigenvalues of $T$ then the eigenspace $M_n$ associated to $\lambda_n$ is a finite dimensional Hilbert space. The sequence $\lambda_n \to 0$ if there are infinitely many eigenvalues. If $P_n$ is the orthonormal projection of $\mathcal{H}$ onto $M_n$ =*

*$Ker((T-\lambda_n)I)$, then $P_nP_m = 0 = P_mP_n$ if $m = n$ and*

$$T = \sum_{n\geq 1} \lambda_n P_n,$$

*where the series converges in the norm topology on $\mathcal{B}(\mathcal{H})$. In addition, if $T$ is self-adjoint, $\lambda_n$'s can be ordered in a decreasing sequence, $|\lambda_1| \geq |\lambda_2| \geq \cdots$ which converges to 0.*

Let $T$ be a compact operator. For a self-adjoint compact operator $T^*T$, let $(\lambda_n)_{n\geq 1}$ be the decreasing sequence of eigenvalues in the above sense. We define the $n^{th}$ **singular value** of $T$ to be the positive square-root of the $n$th eigenvalue of the operator $T^*T$. Denote by $s_n(T)$: the $n$th singular value of $T$.

**Definition 2.1.6.** *A compact operator $T$ is said to be a **trace class operator** if the series $\sum_{n\geq 1} s_n(T)$ is convergent. The set of all trace-class operators is denoted by $\mathcal{B}_1(\mathcal{H})$. For $T \in \mathcal{B}_1(\mathcal{H})$, define the **trace** of $T$ to be $trT = \sum_{n\geq 1} \langle e_n, Te_n\rangle$, where $(e_n)_{n\geq 1}$ is a orthonormal basis for $\mathcal{H}$ and the trace norm $\|\cdot\|_1$ on $\mathcal{B}_1(\mathcal{H})$ by $\|A\|_1 = \sum_{n\geq 1} s_n(T)$. The space $\mathcal{B}_1(\mathcal{H})$ is a Banach space with respect to the trace norm.*

It is easy to see that the series $\sum_{n\geq 1} \langle e_n, Te_n\rangle$ converges and the sum is independent of the choice of basis. There is an interesting relation between the classes $\mathcal{B}_0(\mathcal{H}), \mathcal{B}_1(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ which is shown in the following theorem.

**Theorem 2.1.7.** *For the spaces $\mathcal{B}_0(\mathcal{H}), \mathcal{B}_1(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ the following is true:*

*(i) $\mathcal{B}_1(\mathcal{H}) \cong \mathcal{B}_0(\mathcal{H})^*$. That is, the map $K \mapsto tr(K\ \cdot)$ is an isometric isomorphism of $\mathcal{B}_1(\mathcal{H})$ on $\mathcal{B}_0(\mathcal{H})^*$.*

*(ii) $\mathcal{B}(\mathcal{H}) \cong \mathcal{B}_1(\mathcal{H})^*$. That is, the map $A \mapsto tr(A\ \cdot)$ is an isometric isomorphism of $\mathcal{B}(\mathcal{H})$ on $\mathcal{B}_1(\mathcal{H})^*$.*

### 2.1.1 Unbounded Operators

Most of the operators which we come across while solving problems arising from the physical significance are not bounded. We introduce some basic concepts and results concerning unbounded operators, necessary to understand the semigroup theory. The closed graph theorem states that an operator which is everywhere defined and whose graph is closed must be bounded, suggesting that a nice unbounded operator will only be defined on dense linear subset of the Hilbert space $\mathcal{H}$. An operator(unbounded) $T$ is a linear map with its domain, a linear subspace which is usually dense into $\mathcal{H}$. We denote by $Dom(T)$, the domain of the operator $T$.

**Definition 2.1.8.** *The **graph** of a linear operator $T$ is the set $\Gamma(T) := \{(h, Th) : h \in Dom(T)\}$ and is denoted by $\Gamma(T)$. The dual of the graph $\Gamma$ is given by $\Gamma^*(T) := \{(-Th, h)\ ;\ h \in Dom(T)\}$.*

An operator $T$ is **closed** if $\Gamma(T)$ is a closed subset of $\mathcal{H} \times \mathcal{H}$. Let $T_1$ and $T_2$ be operators on $\mathcal{H}$. If $\Gamma(T_1) \subseteq \Gamma(T_2)$, then $T_2$ is said to be an extension of $T_1$ and we write $T_1 \subseteq T_2$.

**Definition 2.1.9.** *An operator $T$ is **closable** if it has a closed extension. The smallest closed extension which exists, is called the **closure** of $T$, denoted by $\overline{T}$.*

**Definition 2.1.10.** *Let $T$ be a densely defined linear operator on $\mathcal{H}$. For a fixed $h \in Dom(T)$, if the linear map $\Phi_h(g) = \langle h, Tg\rangle$ with domain $Dom(T)$ can be extended to a bounded linear functional given by $\langle f, g\rangle$ on $\mathcal{H}$, then we say $h \in Dom(T^*)$ and $T^*(h) = f$. The operator $T^*$ is called **adjoint** of $T$.*

It is easy to see that

$$\Gamma(T^*) = [\Gamma^*(T)]^\perp,$$

where $S^\perp := \{u \in \mathcal{H} : \langle u, s\rangle = 0\}$. If the domain of $T^*$ is dense, then we can define $T^{**} = (T^*)^*$. There is a simple relationship between the adjoint and closure of an operator $T$.

**Theorem 2.1.11.** *Let $T$ be a densely defined operator on $\mathcal{H}$. Then the following holds:*

*(i)* *$T^*$ is closed.*

*(ii)* *$T$ is closable if and only if $Dom(T^*)$ is dense, in that case $\overline{T} = T^{**}$.*

*(iii)* *If $T$ is closable, $(\overline{T})^* = T^*$.*

**Proposition 2.1.12.** *Let $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $T : \mathcal{H} \to \mathcal{K}$ is densely defined, then*

$$(Range\ T)^{\perp} = Ker\ T^*.$$

*If $T$ is also closed then*

$$(Range\ T^*)^{\perp} = Ker\ T.$$

Now we define the resolvent of an operator. The knowledge of a resolvent helps us to understand the nature of semigroups.

**Definition 2.1.13.** *Let $T$ be a closed operator on $\mathcal{H}$. A complex number $\lambda$ is in the* ***resolvent set****, denoted by $\rho(T)$, if $\lambda I - T$ is a bijection from $Dom(T)$ onto the dense range of $(\lambda I - T)$ with a bounded inverse. For $\lambda \in \rho(T)$, $\mathcal{R}(\lambda, T) = \mathcal{R}_\lambda(T) := (\lambda I - T)^{-1}$ is called the* ***resolvent*** *of $T$ at $\lambda$.*

**Definition 2.1.14.** *A densely defined operator $T$ on $\mathcal{H}$ is called* ***symmetric*** *if $T \subseteq T^*$. Equivalently, $T$ is symmetric if and only if $\langle Th, g\rangle = \langle h, Tg\rangle$ for all $h, g \in Dom(T)$. An operator $T$ is called* ***self-adjoint*** *if $T$ is symmetric and $Dom(T^*) = Dom(T)$.*

The adjoint $T^*$ of a symmetric densely defined $T$ is an extension of $T$, but is not symmetric always. The Symmetry of $T^*$ requires $T^* = T^{**}$. We recall that $T^{**}$ is the closure of $T$ and generally all that can be true is:

$$T \subset T^{**} \subset T^*$$

for densely-defined, symmetric $T$. Since $T^{**}$ is the closure of $T$, it is symmetric. The distinction between closed symmetric operators and self-adjoint operators is significant. For self-adjoint operators, the spectral theorem holds and they generates a one-parameter unitary groups.

**Definition 2.1.15.** *A symmetric operator $T$ is called **essentially self-adjoint** if its closure $\overline{T}$ is self-adjoint. If $T$ is closed, subset $D \subseteq Dom(T)$ is called **core** for $T$ if closure of the restriction $\overline{T \restriction D} = T$.*

In general, symmetric densely defined operators do not possess unique self-adjoint extension. In contrast, an essentially self-adjoint operator has a unique self-adjoint extension. So for a self-adjoint operator $T$, one need not to give exact domain of $T$, but just some core for $T$. The following results show equivalence conditions for an operator to be self-adjoint or essentially self-adjoint.

**Theorem 2.1.16.** *Let $T$ be a symmetric operator on $\mathcal{H}$. Then the following are equivalent:*

*(i)* *$T$ is self-adjoint.*

*(ii)* *$T$ is closed and $Ker(T^* \pm iI) = \{0\}$.*

*(iii)* *$Range(T^* \pm iI) = \mathcal{H}$.*

**Theorem 2.1.17.** *Let $T$ be a symmetric operator on $\mathcal{H}$. Then the following are equivalent:*

*(i)* *$T$ is essentially self-adjoint.*

*(ii)* *$Ker(T^* \pm iI) = \{0\}$.*

*(iii)* *$Range(T^* \pm iI)$ is dense in $\mathcal{H}$.*

Here we state spectral theorem for the self-adjoint operators.

**Theorem 2.1.18. Spectral Theorem** *Let $(T, \mathcal{D}(T))$ be a self-adjoint operator on $\mathcal{H}$, then there exists a right continuous projection valued function $\mathbb{E} : \mathbb{R} \to \mathcal{P}(\mathcal{H})$, where $\mathcal{P}(\mathcal{H})$ is the space of orthogonal projections on $\mathcal{H}$, such that $T$ is*

$$T = \int_{\mathbb{R}} \lambda \; \mathbb{E}(d\lambda). \tag{2.1}$$

The above function $\mathbb{E} : \mathbb{R} \to \mathcal{P}(\mathcal{H})$ satisfies the following:

(*i*) $\lim_{t\to\infty} \mathbb{E}(t) = I$ strongly.

(*ii*) $\lim_{t\to-\infty} \mathbb{E}(t) = 0$ strongly.

(*iii*) $\mathbb{E}(s)\mathbb{E}(t) = \mathbb{E}(s \wedge t),\; s \wedge t = \min\{s, t\}$

and is called **spectral measure** for $T$. The spectral integration in (2.1) is in the sense that

$$\langle u, Tv\rangle = \int t \; \mu_{u,v}(dt),$$

where $\mu_{u,v}$ is the complex measure given by $\mu_{u,v}((-\infty, t]) = \langle u, E(t)v\rangle$ .

**Polar decomposition for closed operators**

There exists a special decomposition for operators on a Hilbert space which is analogous to the decomposition $z = |z|e^{i \arg z}$ for complex numbers. An arbitrary bounded operator $T$ can be written as $T = U|T|$ *uniquely*, where $|T|$ is positive self-adjoint and $U$ is a partial isometry. We discuss the polar decomposition in case of unbounded operators. For the bounded case, polar decomposition is easy to construct since we can set $|T| = \sqrt{T^*T}$ in view of the existence of positive square root. In case of unbounded operators the following theorem helps us to generalize the existence of polar decomposition for unbounded operators.

**Theorem 2.1.19. (von Neumann)** *Let $T$ be a closed, densely defined operator on $\mathcal{H}$. Then the operator $T^*T$ is self-adjoint operator on $\mathcal{H}$ and $Dom(T^*T)$ is a core for $T$.*

$T^*T$ is positive self-adjoint operator on $\mathcal{H}$. As we can define $|T| = \sqrt{T^*T}$ by spectral theorem and the polar decomposition can be constructed the same way as in case of bounded operators.

**Theorem 2.1.20.** *Let $T$ be a closed, densely defined operator on $\mathcal{H}$. Then, there is a positive self-adjoint operator $|T| = \sqrt{T^*T}$, with $Dom(|T|) = Dom(T)$ and a partial isometry $U$ with domain $(Ker\ T)^\perp$ and co-domain $\overline{Range\ T}$, so that $T = U|T|$. $|T|$ and $U$ are uniquely determined by these properties together with the property $Ker(|T|) = Ker(T)$.*

## 2.2 $C^*$ and von Neumann Algebras

### 2.2.1 $C^*$-algebras

Here we give a brief introduction to $C^*$-algebras and von Neumann algebra on which QDS are discussed in the next chapter.

**Definition 2.2.1.** *A complete normed algebra $\mathcal{A}$ is said to be **Banach algebra** if the norm satisfies $\|xy\| \leq \|x\|\|y\|$ for $x, y \in \mathcal{A}$. It is called $C^*$**- algebra** if it has a $*$-structure and $\|x^*x\| = \|x\|^2$ holds.*

**Example 2.2.2.** *Let $X$ be a locally compact Hausdorff space, the space $C_0(X)$ of all complex valued continuous functions on $X$, vanishing at infinity, with supremum norm and with complex conjugation as the $*$-operation forms a commutative $C^*$-algebra under point-wise addition and multiplication.*

The algebra is called **unital** or **non-unital** according to whether it has identity or not. However every $C^*$-algebra can be made unital by adjoining the identity to it. Example 2.2.2 is important in the way that every commutative $C^*$-algebra is essentially of this form. Explicitly, the following result gives the characterization of commutative $C^*$-algebras.

**Theorem 2.2.3. (Gelfand Naimark)** *Every commutative $C^*$-algebra $\mathcal{A}$ is isometrically isomorphic to $C_0(X)$ for some locally compact Hausdorff space $X$. In case $\mathcal{A}$ is unital, $X$ is compact.*

Let $\mathcal{A}$ be a $C^*$-algebra. A linear functional $\psi : \mathcal{A} \to \mathbb{C}$ is said to be positive if $\psi(x^*x) \geq 0$ for all $x \in \mathcal{A}$. It can be seen that element of $\mathcal{A}$ is positive if and only if $\psi(x)$ is positive for all positive functionals $\psi$ on $\mathcal{A}$. A positive linear functional $\psi$ for which $\psi(1) = 1$ is called a **state** on $\mathcal{A}$. It can be shown that positivity implies boundedness. A state $\psi$ is called **tracial** if $\psi(xy) = \psi(yx)$ for all $x, y \in \mathcal{A}$. It is called **faithful** if $\psi(x^*x) = 0$ implies $x = 0$.

**Definition 2.2.4.** *A* ***representation*** *of a $C^*$-algebra is a pair $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism. If $\mathcal{A}$ is unital, it is assumed that $\pi(1) = 1$.*

**Theorem 2.2.5. (Gelfand-Naimark-Segal Construction)** *For a given state $\psi$ on a $C^*$-algebra $\mathcal{A}$, there exists a Hilbert space $\mathcal{H}_\psi$, a representation $\pi_\psi$ of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_\psi)$ and a vector $\xi_\psi \in \mathcal{H}_\psi$ which is cyclic in the sense that the set $\{\pi_\psi(x)\xi_\psi \ ; \ x \in \mathcal{A}\}$ is total in $\mathcal{H}_\psi$, satisfying*

$$\psi(x) = \langle \xi_\psi, \pi_\psi(x)\xi_\psi \rangle \, .$$

This triple $(\mathcal{H}_\psi, \pi_\psi, \xi_\psi)$ is called the GNS triple for $(\mathcal{A}, \psi)$ and $\mathcal{H}_\psi$ is called **GNS Hilbert space** for the pair $(\mathcal{A}, \psi)$ and it is denoted by $L^2(\mathcal{A}, \psi)$.

### 2.2.2 UHF $C^*$-algebra

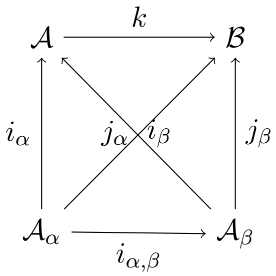
The construction of quantum dynamical semigroups, obtained in this thesis, is carried out on the GNS space of a UHF $C^*$-algebra. In this section we discuss some of the results. Before that let us introduce a special class of $C^*$-algebras, namely approximately finite dimensional $C^*$-algebras (in short AF $C^*$-algebra). The following theorem classifies all the finite dimensional algebras.

**Theorem 2.2.6.** *Every finite dimensional $C^*$-algebra $\mathcal{A}$ is $*$-isomorphic to the direct sum of full matrix algebras, that is:*

$$\mathcal{A} \cong \mathcal{M}_{n_1} \oplus \cdots \oplus \mathcal{M}_{n_k}.$$

*In particular, every non-zero finite dimensional $C^*$-algebra is unital.*

**Definition 2.2.7.** *Let $\{\mathcal{A}_\alpha\}_{\alpha \in I}$ be a directed family of $C^*$-algebras, that is for any $\alpha < \beta$ in the directed set $I$, there is an isometric isomorphism $i_{\alpha,\beta}$ from $\mathcal{A}_\alpha$ into $\mathcal{A}_\beta$ and $i_{\alpha,\beta} = i_{\gamma,\beta}(i_{\alpha,\gamma})$, whenever $\alpha < \gamma < \beta$. Then there exists a universal $C^*$-algebra $\mathcal{A}$, called **Inductive Limit** of the directed family $(\mathcal{A}_\alpha, i_{\alpha,\beta})$ and isometric isomorphism $i_\alpha$ from $\mathcal{A}_\alpha$ into $\mathcal{A}$ such that $i_\alpha = i_\beta(i_{\alpha,\beta})$ and $\mathcal{A} = \bigcup_{\alpha \in I} i_\alpha(A_\alpha)$. The Inductive Limit has universal property that for any $C^*$-algebra $\mathcal{B}$ with isometric isomorphisms $j_\alpha$ from $\mathcal{A}_\alpha$ into $\mathcal{B}$ such that $j_\alpha = j_\beta(i_{\alpha,\beta})$, there exists an isometric isomorphism $k : \mathcal{A} \to \mathcal{B}$ and following diagram*




*commutes.*

**Definition 2.2.8.** *A $C^*$-algebra $\mathcal{A}$ is said to be an **AF $C^*$-algebra** if it is the Inductive Limit of a family of $C^*$-subalgebra $\{\mathcal{A}_n\}_{n \geq 0}$ with isometric embeddings $i_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$ for $n \geq 0$. Here $\mathcal{A}_0 = \mathbb{C}I$ in case of $\mathcal{A}$ is unital and $\mathcal{A} = \overline{\bigcup_{n \geq 0} \mathcal{A}_n}$ with the norm closure. A particular class of AF $C^*$-algebras is called Uniformly hyper-finite $C^*$-algebras or **UHF $C^*$-algebras** if it is an increasing union of unital subalgebras which are isomorphic to full matrix algebras $\{\mathcal{M}_{n_k}(\mathbb{C})\}$ for some sequence of positive integers $\{n_k\}$.*

A unital embedding of $\mathcal{M}_{n_k}(\mathbb{C})$ into $\mathcal{M}_{n_l}(\mathbb{C})$ requires $n_k$ divides $n_l$ $(n_k|n_l)$, thus we get an increasing sequence $n_1|n_2|\cdots$. For a prime number $p$ there exists a unique number $\epsilon_p \in \{1, 2, \cdots \infty\}$, given by $\epsilon_p = sup\{l\ ;\ p^l|n_k\ as\ k \to \infty\}$. Now we define a number $\delta(A)$ associated with the UHF $C^*$-algebra $\mathcal{A}$, known as **supernatural number**, by a formal product:

$$\delta(\mathcal{A}) = \prod_{p:prime} p^{\epsilon_p}.$$

This number gives a complete invariant for the class of UHF $C^*$-algebra by the following result of Glimm:

**Theorem 2.2.9.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two UHF $C^*$-algebras. Then $\mathcal{A}$ is isomorphic to $\mathcal{B}$ if and only if $\delta(\mathcal{A}) = \delta(\mathcal{B})$.*

In particular, we are interested in the class of $N^\infty$ UHF $C^*$-algebras obtained as infinite tensor product of finite dimensional matrix algebra $\mathcal{M}_N(\mathbb{C})$. More explicit, for a fixed pair of positive integers $d$ and $N$, consider the infinite lattice $\mathbb{Z}^d$, then we are interested in

$$\mathcal{A} = \overline{\bigotimes_{j\in\mathbb{Z}^d} \mathcal{M}_N(\mathbb{C})}^{c^*}$$

$\mathcal{A}$ can be interpreted as inductive limit of $N$ by $N$ matrix algebras $\mathcal{M}_N(\mathbb{C})$ using embedding of $\mathcal{M}_N(\mathbb{C})$ to $\mathcal{M}_{N^2}(\mathbb{C})$ as $A \to A \otimes I$. For details we refer [22, 37, 36].

### 2.2.3 Locally Convex Topologies in $\mathcal{B}(\mathcal{H})$

For a Hilbert space $\mathcal{H}$, the Banach space of all bounded linear operators $\mathcal{B}(\mathcal{H})$, is usually equipped with the operator-norm topology. There are many other important topologies with respect to which $\mathcal{B}(\mathcal{H})$ is a locally convex topological vector space such as weak, strong, ultra-weak and ultra-strong topologies. The algebra of operators $\mathcal{B}(\mathcal{H})$ is complete in each of these topologies. Here we give the details of these topologies:

**Norm Topology:** The norm of a bounded operator defines a topology on $\mathcal{B}(\mathcal{H})$ called the norm topology. The function $T \to \|T\|$ is a semi-norm on $\mathcal{B}(\mathcal{H})$ and give rise to the topology of uniform convergence over bounded sets of $\mathcal{H}$.

**Strong (Operator) Topology:** For every $h \in \mathcal{H}$, the function $T \to \|Th\|$ is a semi-norm on $\mathcal{B}(\mathcal{H})$. The collection of all these semi-norms determine the Hausdorff locally convex topology is called the topology of strong point-wise convergence. A base of neighborhoods around origin for this topology is obtained by taking subsets

$$\{T \in \mathcal{B}(\mathcal{H}) \ ; \ \|Th_i\| < \epsilon \ , \ 1 \leq i \leq n\},$$

for each finite sequence $(h_i)_{i=1}^n$ of elements of $\mathcal{H}$ and $\epsilon > 0$. We can also define the strong topology as the coarsest topology on $\mathcal{B}(\mathcal{H})$ such that the maps $T \to Th$ from $\mathcal{B}(\mathcal{H})$ into $\mathcal{H}$ are continuous.

**Weak (Operator) Topology:** For $h, g \in \mathcal{H}$, the collection of the semi-norms $T \to |\langle Th, g\rangle|$ determine the Hausdorff locally convex topology know as weak topology or the topology of weak convergence. In view of polarization identity, we see that the semi-norms $T \to |\langle Th, h\rangle|$ are enough to define weak topology. A base of neighborhoods around origin for this topology is obtained by taking subsets

$$\{T \in \mathcal{B}(\mathcal{H}) \ ; \ |\langle Th_i, g_i\rangle| < \epsilon \ , \ 1 \leq i \leq n\},$$

for each pair of finite sequences $(h_i)_{i=1}^n; (g_i)_{i=1}^n$ of elements of $\mathcal{H}$, $\epsilon > 0$. We can also define the weak topology as the coarsest topology on $\mathcal{B}(\mathcal{H})$ such that the maps $T \to \langle Th, g\rangle$ from $\mathcal{B}(\mathcal{H})$ into $\mathbb{C}$ are continuous.

**Ultra-Strong Topology:** Let $(h_i)_{i=1}^\infty$ be a sequence of elements of $\mathcal{H}$ such that $\sum\limits_{i=1}^{\infty} \|h_i\|^2 < \infty$. Since the series $\sum\limits_{i=1}^{\infty} \|Th_i\|^2$ is convergent, the map $T \to \left(\sum\limits_{i=1}^{\infty} \|Th_i\|^2\right)^{\frac{1}{2}}$, defines a semi-norm on $\mathcal{B}(\mathcal{H})$. The collection of all these semi-norms determine the Hausdorff locally convex topology called ultra-strong topology. A base of neighborhoods around origin for this topology is obtained by taking subsets

$$\left\{ T \in \mathcal{B}(\mathcal{H}) \ ; \ \sum_{i=1}^{\infty} \|Th_i^k\|^2 < \epsilon \ , \ 1 \leq k \leq n \right\},$$

for each $\epsilon > 0$ and for every finite family of sequences $\{(h_i^1)_{i=1}^{\infty}, (h_i^2)_{i=1}^{\infty}, \cdots, (h_i^n)_{i=1}^{\infty}\}$ of elements of $\mathcal{H}$ such that for all $k : 1 \leq k \leq n$,

$$\sum_{i=1}^{\infty} \|h_i^k\|^2 < \infty.$$

This is the topology for which the maps, $T \to (Th_1, Th_2, \cdots)$ from $\mathcal{B}(\mathcal{H})$ into direct sum $\bigoplus H_i : H_i = H$ for all $i$, are continuous.

**Ultra-Weak Topology:** In view of Cauchy-Schwarz inequality and Hölder's inequality we see that, for each pair of sequences $(h_i)_{i=1}^{\infty}; (g_i)_{i=1}^{\infty}$ in $\mathcal{H}$ such that

$$\sum_{i=1}^{\infty} \|h_i\|^2 < \infty, \quad \sum_{i=1}^{\infty} \|g_i\|^2 < \infty,$$

the map $T \to |\sum_{i=1}^{\infty} \langle Th_i, g_i \rangle|$ defines a semi-norm on $\mathcal{B}(\mathcal{H})$. The collection of these semi-norms determine the Hausdorff locally convex topology called ultra-weak topology. A base around origin for this topology is given by taking subsets

$$\left\{ T \in \mathcal{B}(\mathcal{H}) \ ; \ |\sum_{i=1}^{\infty} \left\langle Th_i^k, g_i^k \right\rangle| < \epsilon \ , \ 1 \leq k \leq n \right\},$$

for each $\epsilon > 0$ and for every finite family of pair of sequences $\{((h_i^1)_{i=1}^{\infty}; (g_i^1)_{i=1}^{\infty}), ((h_i^2)_{i=1}^{\infty}; (g_i^2)_{i=1}^{\infty}), \cdots, ((h_i^n)_{i=1}^{\infty}; (g_i^n)_{i=1}^{\infty})\}$ of elements of $\mathcal{H}$ such that for every $k : 1 \leq k \leq n$,

$$\sum_{i=1}^{\infty} \|h_i^k\|^2 < \infty, \quad \sum_{i=1}^{\infty} \|g_i^k\|^2 < \infty.$$

This topology is also coarsest for which the maps $T \to \sum_{i=1}^{\infty} \langle Th_i, g_i \rangle$ from $\mathcal{B}(\mathcal{H})$ into $\mathbb{C}$ are continuous.

The above topologies are compared to give the following diagram, where the symbol $<$ means "finer than":

$$\begin{array}{ccccc} \text{Norm topology} & < & \text{Ultra-strong topology} & < & \text{Strong topology} \\ & & \wedge & & \wedge \\ & & \text{Ultra-weak topology} & < & \text{Weak topology} \end{array}$$

For infinite dimensional Hilbert space, the symbol $<$ can be taken to mean "strictly finer than". The strong(respectively weak) and ultra-strong(respectively ultra-weak) topologies coincide on bounded subsets of $\mathcal{B}(\mathcal{H})$.

### 2.2.4 von Neumann Algebras

For a Hilbert space $\mathcal{H}$, we have discussed many important topologies on $\mathcal{B}(\mathcal{H})$ with respect to which it is a locally convex topological vector space. $\mathcal{B}(\mathcal{H})$ is complete in each of these topologies but a general $C^*$- subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ need not be so. It is known that $\mathcal{A}$ is complete in all of the locally convex topologies except norm topology if and only if it is complete in any one of them and in that case $\mathcal{A}$ is said to be a **von Neumann algebra**. Furthermore, the strong(respectively weak) and ultra-strong(respectively ultra-weak) topologies coincide on norm bounded convex subsets of $\mathcal{A}$.

For a von Neumann algebra $\mathcal{A}$, denote by $\mathcal{A}'$, the commutant of $\mathcal{A}$ which is the set $\{a \in \mathcal{B}(\mathcal{H}) \text{ such that } ax = xa, \ \forall\ x \in \mathcal{A}\}$ and we have $\mathcal{A}'' = (\mathcal{A}')'$. The following result due to von Neumann is of fundamental importance in the study of von Neumann algebras.

**Theorem 2.2.10. (Double commutant theorem)** *Let $\mathcal{A}$ be a non-degenerate $C^*$-algebra in $\mathcal{B}(\mathcal{H})$. Then $\mathcal{A}'' = \overline{\mathcal{A}}^w = \overline{\mathcal{A}}^s$, where $\overline{\mathcal{A}}^w$ and $\overline{\mathcal{A}}^s$ are closure of $\mathcal{A}$ in weak and strong operator topologies of $\mathcal{B}(\mathcal{H})$ respectively.*

In particular, any unital $C^*$-algebra $\mathcal{A}$ is a von Neumann algebra if and only if $\mathcal{A}'' = \mathcal{A}$.

A state $\psi$ on a von Neumann algebra $\mathcal{A}$ is said to be **normal** if $\psi(x_\alpha)$ increases to $\psi(x)$ whenever $x_\alpha$ increases to $x$ for a net $\{x_\alpha\}$ of positive elements in $\mathcal{A}$. We call a linear map $\Psi : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is a von Neumann algebra, to be **normal** if whenever $x_\alpha$ increases to $x$ for a net $\{x_\alpha\}$ of positive elements in $\mathcal{A}$, we have $\Psi(x_\alpha)$ increases to $\Psi(x)$ in $\mathcal{B}$. It can be seen that a positive linear map is normal if and only if it is continuous with respect to the ultra-weak topology. In view of this fact, we shall say that a bounded linear map between two von Neumann algebras is normal if it is continuous with respect to the ultra-weak topologies. Normal states and more generally normal positive linear maps like normal $*$- homomorphisms play a important role in the study of von Neumann algebras. The following result describes the structure of a normal state.

**Theorem 2.2.11.** *A state $\psi$ on a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is normal if and only if there is a positive trace-class operator $\rho$ on $\mathcal{H}$ such that $\psi(x) = tr(\rho x)$ for all $x \in \mathcal{A}$.*

For a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, A Banach space $\mathcal{A}_*$ is called the **predual** of $\mathcal{A}$ if the Banach dual $(\mathcal{A}_*)^*$ with norm topology coincides with $\mathcal{A}$ and with respect to weak-$*$ topology it coincides with ultra-weak topology of $\mathcal{A}$. In fact, Sakai [35] showed that a von Neumann algebra can be characterized in the class of $C^*$-algebras by the property of having a predual as a Banach space.

We give the explicit description of the predual of $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$. Since from a Theorem 2.1.7 we see that $\mathcal{A}_*$ is the some quotient space of $\mathcal{B}_1(\mathcal{H})$. Let $\mathcal{B}^{s.a.}(\mathcal{H})$ and $\mathcal{B}_1^{s.a.}(\mathcal{H})$ stand for the real linear space of all bounded self-adjoint operators and all trace-class self-adjoint operators on $\mathcal{H}$ respectively. Denote by $\mathcal{A}^{s.a.}$ the subset of all self-adjoint elements in $\mathcal{A}$. Let $\mathcal{A}_*^{s.a.}$ be the predual of $\mathcal{A}^{s.a.}$. We define an equivalence relation $\sim$ on $\mathcal{B}_1(\mathcal{H})$ by saying $\rho_1 \sim \rho_2$ if and only if $tr(\rho_1 x) = tr(\rho_2 x)$ for all $x \in \mathcal{A}$. We denote by $\mathcal{A}^\perp$ the closed subspace $\{\rho \in \mathcal{B}_1(\mathcal{H}) \ ; \ \rho \sim 0\}$. For $\rho \in \mathcal{B}_1(\mathcal{H})$, we denote by $\tilde{\rho}$ its equivalence class with respect to $\sim$ and $\|\tilde{\rho}\| = \inf_{\eta \sim \rho} \|\eta\|_1$. By $(\mathcal{A}^\perp)^{s.a.}$ we shall denote the set of all self-adjoint elements in $\mathcal{A}^\perp$. Clearly $(\mathcal{A}^\perp)^{s.a.}$ is a closed subspace of $\mathcal{B}_1^{s.a.}(\mathcal{H})$ and so one can make sense of the quotient space $\mathcal{B}_1^{s.a.}(\mathcal{H})/(\mathcal{A}^\perp)^{s.a.}$.

The following theorem determines the predual explicitly.

**Theorem 2.2.12.** *(i) There exists an isometric isomorphism*

$$\mathcal{A}_* \;\cong\; \frac{\mathcal{B}_1(\mathcal{H})}{\mathcal{A}^\perp} \;\cong\; \Omega_{\mathcal{A}},$$

*where $\Omega_{\mathcal{A}}$ denotes the space of all normal complex linear bounded functional on $\mathcal{A}$.*

*(ii) There exists an isometric isomorphism*

$$\mathcal{A}_*^{s.a.} \;\cong\; \frac{\mathcal{B}_1^{s.a.}(\mathcal{H})}{(\mathcal{A}^\perp)^{s.a.}} \;\cong\; \Omega_{\mathcal{A}^{s.a.}},$$

*where $\Omega_{\mathcal{A}^{s.a.}}$ denotes the space of all normal complex linear bounded functional on $\mathcal{A}^{s.a.}$.*

The canonical identification between $\mathcal{A}$ and $(\mathcal{B}_1(\mathcal{H})/\mathcal{A}^\perp)^*$ is given by, $x \to \psi_x$ where $\psi_x(\tilde{\rho}) = tr(\rho x)$. Moreover, an element $\tilde{\rho}$ of $\mathcal{B}_1(\mathcal{H})/\mathcal{A}^\perp$ is canonically associated with $\psi_{\tilde{\rho}}$ in $\Omega_{\mathcal{A}}$ where $\psi_{\tilde{\rho}}(x) = tr(\rho x), x \in \mathcal{A}$.

For quantum dynamical semigroups the condition of complete positivity is fundamental and it has very important mathematical and physical consequences. We give the brief introduction to completely positive maps.

## 2.3 Completely Positive Maps

Recall that a linear map $T$ between two unital $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to be **positive** if $T(x^*x) \geq 0$ in $\mathcal{B}$ for all $x \in \mathcal{A}$. A general element $x \in \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$ can be written in the form

$$\sum_{i,j=1}^{n} x_{ij} \otimes E_{ij},$$

where $E_{ij}$ is the $n \times n$ matrix with all entries 0 except 1 at the $(ij)^{th}$ place.
For $n \geq 1,\ 1 \leq i, j \leq n$, define the linear operator

$$\begin{aligned} T^{(n)} &: \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}) \to \mathcal{B} \otimes \mathcal{M}_n(\mathbb{C}) \\ T^{(n)}(x \otimes E_{ij}) &= T(x) \otimes E_{ij}. \end{aligned} \tag{2.2}$$

It is not necessary that $T^{(n)}$ be positive.

**Definition 2.3.1.** *Let $\mathcal{A}$ and $\mathcal{B}$ be $*$-algebras. A map $T : \mathcal{A} \to \mathcal{B}$ is called* ***n-positive*** *if $T^{(n)}$ as defined above is positive. If $T^{(n)}$ is positive for all $n \geq 1$ then $T$ is called* ***completely positive****.*

**Proposition 2.3.2.** *Let $T : \mathcal{A} \to \mathcal{B}$ be a completely positive linear map. Then for all $n \geq 1,\ (x_i)_{i=1}^n \subset \mathcal{A},\ (y_i)_{i=1}^n \subset \mathcal{B}$, we have*

$$\sum_{i,j=1}^{n} y_i^* T(x_i^* x_j) y_j \geq 0.$$

**Proposition 2.3.3.** *Let $(T_n)_{n\geq 1}$ be a sequence of completely positive maps $T_n : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Suppose for every $x \in \mathcal{A}$, the sequence $(T_n(x))_{n\geq 1}$ converges weakly. Then the map $T_n : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ defined by*

$$T(a) = \lim_{n\to\infty} T_n(a)$$

*is completely positive.*

Any $*$-homomorphism is a completely positive map, but converse is not true. The following theorem by Stinespring shows that completely positive maps essentially come from $*$-homomorphisms.

**Theorem 2.3.4. (Stinespring)** *For a $C^*$-algebra $\mathcal{A}$, let $T :\ \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a completely positive map. Then there exists another Hilbert space $\mathcal{K}$, a representation $\pi :\ \mathcal{A} \to \mathcal{B}(\mathcal{K})$*

*and* $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ *such that the set* $\{\pi(x)Vu \ : \ x \in \mathcal{A},\ u \in \mathcal{H}\}$ *is total in* $\mathcal{K}$ *and the map* $T$ *has the form*

$$T(x) = V^* \pi(x) V, \quad \textit{for all} \ \ x \in \mathcal{A}.$$

Such a triple $(\mathcal{K}, \pi, V)$ is called **Stinespring's triple** associated with $T$ and is unique in the sense that if $(\mathcal{K}', \pi', V')$ is another such triple then there is a unitary operator $\Gamma : \mathcal{K} \to \mathcal{K}'$ such that $\pi'(x) = \Gamma \pi(a) \Gamma^*$ and $V' = \Gamma V$. Furthermore, if $\mathcal{A}$ is a von Neumann algebra and $T$ is normal, $\pi$ can be chosen to be normal. Any positive map $T : \ \mathcal{A} \to \mathcal{B}$ is completely positive if either of $\mathcal{A}$ or $\mathcal{B}$ is abelian. We conclude on completely positive maps by stating the characterization theorem by K. Kraus for ultra-weakly continuous (i.e. normal) completely positive maps.

**Theorem 2.3.5.** (Kraus) *A linear map* $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ *is normal and completely positive if and only if it can be expressed in the form*

$$T(x) = \sum_{n=1}^{\infty} V_n^* x V_n$$

*where* $(V_n)_{n=1}^{\infty}$ *is a sequence in* $\mathcal{B}(\mathcal{K}, \mathcal{H})$ *such that the series* $\sum\limits_{n=1}^{\infty} V_n^* x V_n$ *converges strongly.*

In the semigroup theory, a class of operators called conditionally completely positive maps play an important role. We now introduce this notion, which is related to completely positivity.

**Definition 2.3.6.** *A linear map* $T$ *on a* $*$*-algebra* $\mathcal{A}$ *is called* ***conditionally completely positive*** (CCP) *map if the map* $T^{(n)}$ *defined as in* (2.2) *satisfies the following inequality*

$$T^{(n)}(x^*x) - x^*T^{(n)}(x) - T^{(n)}(x^*)x + x^*T^{(n)}(1)x \geq 0, \tag{2.3}$$

*for every* $n \geq 1$ *and* $x \in \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$.

The following proposition shows that every completely positive map is CCP.

**Proposition 2.3.7.** *A map* $T: \mathcal{A} \to \mathcal{A}$ *is* CCP *if and only if for each pair of finite sequences* $(x_i)_{i=1}^n, (y_i)_{i=1}^n$ *in* $\mathcal{A}$*, we have*

$$\sum_{i,j=1}^{n} y_i^* T(x_i^* x_j) y_j \geq 0, \text{ whenever } \sum_{i,j=1}^{n} x_i y_i = 0.$$

In chapter 3, we shall see that bounded CCP maps are the generator of uniformly continuous completely positive semigroups and the converse is also true.

## 2.4 Semigroups on Banach Spaces

For this section, $X$ stands for a complex Banach space. The notion of semigroup of bounded linear operators has its roots in the basic observation that the Cauchy functional equation $f(t+s) = f(t)f(s) : f(0) = 1$ has only continuous solutions of the form $e^{ta},\ a \in \mathbb{R}$. In general, the theory was developed by taking into account the Cauchy problem in infinite dimensional framework, that is find all the maps $\mathcal{T} : \mathbb{R}_+ \to \mathcal{B}(X)$ satisfying the functional equation

$$\begin{cases} \mathcal{T}_{t+s}(a) = \mathcal{T}_t(\mathcal{T}_s(a)) \text{ for all } a \in X \quad , \forall\ t, s \geq 0, \\ \mathcal{T}_0(a) = a. \end{cases} \tag{2.4}$$

**Definition 2.4.1.** *A family* $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ *of bounded linear operators on a Banach space* $X$ *is called* ***one-parameter semigroup*** *or simply* ***semigroup*** *on* $X$ *if it satisfies the functional equation* (2.4).

**Definition 2.4.2.** *The* ***infinitesimal generator*** *or simply* ***generator*** *of a semigroup* $(\mathcal{T}_t)_{t\geq 0}$ *is the linear operator* $G : X \to X$ *defined by*

$$Dom(G) = \left\{ x \in X \ ; \ such\ that\ \lim_{t\downarrow 0} \frac{1}{t}(\mathcal{T}_t - I)x\ exists \right\}$$

*and* $Gx = \lim_{t\downarrow 0} \frac{1}{t}(\mathcal{T}_t - I)x,\ x \in Dom(G)$. *We write* $\mathcal{T}_t = e^{tG}$ *and* $G = \frac{d}{dt}\Big|_{t=0} \mathcal{T}_t$, *whenever* $G$ *generates* $\mathcal{T}$.

**Definition 2.4.3.** *A semigroup* $(\mathcal{T}_t)_{t\geq 0}$ *on* $X$ *is called* ***uniformly continuous*** *(norm continuous) semigroup if the map*

$$\mathbb{R}_+ \ \ni \ t \to \mathcal{T}_t \ \in \ \mathcal{B}(X)$$

*is continuous with respect to the norm topology on* $\mathcal{B}(X)$.

**Theorem 2.4.4.** *A semigroup* $(\mathcal{T}_t)_{t\geq 0}$ *on* $X$ *is uniformly continuous if and only if the generator* $G$ *is bounded.*

### 2.4.1 Strongly Continuous ($C_0$)-Semigroups

To describe many important physical processes we come across unbounded operators and thus to describe the dynamics of these physical systems, uniform continuity is too strong requirement. So we study the semigroups with some weak continuity conditions.

**Definition 2.4.5.** *A semigroup* $(\mathcal{T}_t)_{t\geq 0}$ *on a Banach space* $X$ *is called* ***strongly continuous*** *semigroup if the maps*

$$\mathbb{R}_+ \ni t \to \mathcal{T}_t(x) \in X$$

*are continuous for every* $x \in X$. *Equivalently, we can say if the map* $t \to \mathcal{T}_t$ *is continuous with respect to the strong operator topology on* $\mathcal{B}(X)$.

Every strongly continuous semigroups $(\mathcal{T}_t)_{t\geq 0}$ is **quasi-bounded**, that is there exists constants $w \in \mathbb{R}$ and $M \geq 1$ such that for all $t \geq 0$

$$\|(\mathcal{T}_t)\| \leq Me^{wt}.$$

A Semigroup $(\mathcal{T}_t)_{t\geq 0}$ is called **isometric** or **contractive** if each $\mathcal{T}_t$ is so. For a strongly continuous contraction semigroup $(\mathcal{T}_t)_{t\geq 0}$, resolvent of the generator $G$ is given by Laplace transform of the semigroup, that is for $\mathrm{Re}\lambda > 0$,

$$\mathcal{R}(\lambda, G) = \int_0^\infty e^{-\lambda t}\mathcal{T}_t(x)\ dt.$$

**Lemma 2.4.6.** *For the generator $G$ of a strongly continuous semigroup $(\mathcal{T}_t)_{t\geq 0}$, the following properties hold:*

(*i*) *For every $t \geq 0$ and $x \in Dom(G)$, we have $\mathcal{T}_t(x) \in Dom(G)$ and*

$$\frac{d}{dt}\mathcal{T}_t(x) = \mathcal{T}_t(Gx) = G(\mathcal{T}_t(x)). \tag{2.5}$$

(*ii*) *For every $t \geq 0$ and $x \in X$, we have $\int_0^t \mathcal{T}_s(x)\ ds \in Dom(G)$ and*

$$\mathcal{T}_t(x) - x = G(\int_0^t \mathcal{T}_s(x)\ ds) = \int_0^t \mathcal{T}_s(Gx)\ ds,\ if\ x \in Dom(G). \tag{2.6}$$

**Theorem 2.4.7.** *For a strongly continuous semigroup $(\mathcal{T}_t)_{t\geq 0}$, the generator $G$ is a closed and densely defined linear operator that determines the semigroup uniquely.*

It is often seen that the results which are true for the generator, it is sufficient if we could prove the same for some core for the generator. So life becomes easy if we are able to identify a nice core for the generator. Proving a set $\mathcal{D}$ to be a core for $G$ it is equivalent to prove that $\mathcal{D}$ is dense in $Dom(G)$ in the graph norm

$$\|x\|_\Gamma := \|x\| + \|Gx\|.$$

Nelson in [28] gave a useful criteria for a subspace to be a core for the generator $G$.

**Proposition 2.4.8. (Nelson)** *Let $G$ be the generator of the strongly continuous semigroup $(\mathcal{T}_t)_{t\geq 0}$ on $X$. A subspace $\mathcal{D}$ of $Dom(G)$ which is dense in $X$ and invariant under the semigroup $(\mathcal{T}_t)_{t\geq 0}$ is a core for $G$.*

We now state the most important theorem in the theory of strongly continuous semigroups, which characterize the strongly continuous semigroups in terms of the generator. Hille-Yosida in [18], [40] proved it for the contraction semigroups, which then extended for the general case by Feller-Miyadera-Phillips. Lumer-Phillips in [26] reformulated this result in terms of dissipative operators.

**Theorem 2.4.9. (Hille-Yosida)** *A linear operator $G$ is a generator of a strongly continuous contraction semigroup $(\mathcal{T}_t)_{t\geq 0}$ on a Banach space $X$ if and only if $G$ is closed, densely defined and for every $\lambda > 0$, we have $\lambda \in \rho(G)$ and the resolvent of $G$ at $\lambda$ satisfies: $\|\mathcal{R}(\lambda, G)\| \leq \dfrac{1}{\lambda}$.*

The Lumer-Phillips characterization of strongly continuous semigroups in terms of dissipative operator is important because it does not require the explicit knowledge of resolvent. For the completion of hierarchy of generation theorems similar to Theorem 2.4.9, we are incorporating the following results.

Let $X^*$ be the Banach dual of $X$. Denote by $\langle x, x^* \rangle$ or $\langle x^*, x \rangle$, the value $x^*(x)$, where $x \in X$ and $x^* \in X^*$. For every $x \in X$, define the dual set $F(x) \subseteq X^*$ of $x$ by

$$F(x) := \{x^* \in X^* \ ; \ \langle x, x^* \rangle = \|x^*\|^2 = \|x\|^2\}$$

**Definition 2.4.10.** *A linear operator $G$ is called **dissipative** if for each $x \in X$ there exists $x^* \in F(x)$ such that $Re\langle Gx, x^*\rangle \leq 0$.*

**Remark 2.4.11.** *In particular, when $X$ is a Hilbert space, then $F(x) = x$ and a linear operator $G$ is dissipative if for each $x \in Dom(G)$, we have $\langle x, Gx\rangle \leq 0$, that is if $-G$ is a positive operator.*

**Theorem 2.4.12.** *A linear operator $G$ is dissipative if and only if for each $x \in Dom(G)$ and $\lambda > 0$, we have*

$$\|(\lambda I - G)x\| \geq \lambda\|x\|.$$

**Theorem 2.4.13. (Lumer-Phillips)** *A linear operator $G$ is a generator of a strongly continuous contraction semigroup $(\mathcal{T}_t)_{t\geq 0}$ on a Banach space $X$ if and only if $G$ is closed, densely defined, dissipative and for all $\lambda > 0$, $Range(\lambda I - G)$ is dense in $X$.*

Following theorems tells the important fact that how the convergence of strongly continuous semigroups is related to the convergence of generators as well as with the convergence of their resolvents.

**Theorem 2.4.14.** *Suppose $(\mathcal{T}_t^{(n)})_{t\geq 0}$ and $(\mathcal{T}_t)_{t\geq 0}$ are strongly continuous contraction semigroups on a Banach space $X$ with generators $G^{(n)}$ and $G$ respectively, then $(\mathcal{T}_t^{(n)})_{t\geq 0}$ converges strongly to $(\mathcal{T}_t)_{t\geq 0}$ if and only if $(\lambda I - G^{(n)})^{-1}$ converges strongly to $(\lambda I - G)^{-1}$.*

**Theorem 2.4.15. (Chernoff)** *Let $(\mathcal{T}_t^{(n)})_{t\geq 0}$ and $(\mathcal{T}_t)_{t\geq 0}$ are strongly continuous contraction semigroups on a Banach space $X$ with generators $G^{(n)}$ and $G$ respectively with a common core $\mathcal{D}$ such that $G^{(n)}x \to Gx$ for all $x \in \mathcal{D}$. Then $(\mathcal{T}_t^{(n)})_{t\geq 0}$ converges strongly to $(\mathcal{T}_t)_{t\geq 0}$.*

# Chapter 3

# Quantum Dynamical Semigroups and Quantum Stochastic Calculus

In the first part of this chapter we will discuss about quantum dynamical semigroups on $C^*$-algebra or von Neumann algebras. The possible structure of the generator of such semigroups has been completely characterized for norm-continuous semigroups in [25], [17] and [8]. Kato [22] and Davies [12], under some assumption showed that the generator of stongly continuous quantum dynamical semigroups on $\mathcal{B}(\mathcal{H})$ is of similar form. There are various attempts to understand the generator of strongly continuous quantum dynamical semigroups but still it is not well understood. We give the Chebotarev's construction of quantum dynamical semigroups from Lindbladian form which was developed in [4] and discuss the Chebotarev-Fagnola conditions [6] for such semigroups to be conservative. Details can seen in the expositroy article [16]. In the second part we briefly discuss theory of quantum stochastic calculus developed by Hudson and Parthasarthy and present the quantum stochastic dilations of completely positive semigroups. The details can be seen in [36, 29] and reference there in.

## 3.1 Quantum Dynamical Semigroups

In this section, we give the introduction to semigroups of completely positive maps and also discuss the sufficient conditions [6] for the semigroup to be identity preserving (i.e. conservative). Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the von Neumann algebra of bounded linear operators on $\mathcal{H}$.

**Definition 3.1.1.** *A* ***quantum dynamical semi-group*** *(QDS) on a $C^*$-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a semi-group $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ of completely positive maps on $\mathcal{A}$ with the following properties:*

*(i)* $\mathcal{T}_0(x) = x$, *for all* $x \in \mathcal{A}$.

*(ii)* $\mathcal{T}_t(I) \leq I$, *for all* $t \geq 0$.

*(iii)* $\mathcal{T}_t$ *is strongly continuous for all* $t \geq 0$.

*(iv) for each* $a \in \mathcal{A}$, *the map* $t \to \mathcal{T}_t(x)$ *is continuous with respect to strong topology on* $\mathcal{A}$.

In case of von Neumann algebra, the continuity conditions (*iii*) and (*iv*) change to ultra-weak continuity that is $\mathcal{T}_t$ are normal maps and for each $a \in \mathcal{A}$ the maps $t \to \mathcal{T}_t(a)$ must be continuous with respect to ultra-weak topology on $\mathcal{A}$. A QDS is called **Markov** or **Conservative** if $\mathcal{T}_t(I) = I$ for every $t \geq 0$. The generator of a QDS defined similarly as in the Definition 2.4.2 with existence of limit in respective topologies.

**Example 3.1.2.** *Let* $(\mathcal{S}_t)_{t\geq 0}$ *be a strongly continuous contraction semigroup on* $\mathcal{H}$. *The family of linear operators* $\mathcal{T}_t$ *defined by*

$$\mathcal{T}_t(x) = \mathcal{S}_t^* x \mathcal{S}_t$$

*forms a quantum dynamical semigroup. All the continuity properties of* $\mathcal{T}$ *follow from the strong continuity of* $\mathcal{S}_t$ *and the result 2.3.5.*

We call a QDS **uniformly continuous (norm-continuous)** if in addition to the conditions $(i)$ and $(ii)$, in $(iii)$ maps are continuous with respect to the norm topology. For uniformly continuous QDS, the generator is bounded conditionally completely positive map and has nice structure. The structure of uniformly continuous QDS on hyper-finite von Neumann algebras was characterized by Lindblad in [25] in term of the generator. The generator is called "Lindbladian" by many authors.

**Theorem 3.1.3. (Lindblad)** *A bounded operator $\mathcal{L}$ on a von Neumann algebra $\mathcal{B}(\mathcal{H})$ is the infinitesimal generator of a uniformly continuous QDS $(\mathcal{T}_t)_{t\geq 0}$ if and only if it can be written as*

$$\mathcal{L}(x) = \sum_{n=1}^{\infty} L_n^* x L_n + G^* x + xG \textit{ for all } x \in \mathcal{B}(\mathcal{H}),$$

*where $L_n$'s and $G$ are the elements of $\mathcal{B}(\mathcal{H})$. The series on the right side converges strongly and $-Re(G)$ generates a contraction semigroup.*
*In case QDS is unital, we have*

$$Re(G) = -\frac{1}{2}\sum_{n=1}^{\infty} L_n^* L_n.$$

We state the structure theorem for uniform continuous QDS on $C^*$-algebras which was proved by Christensen-Evens in [8].

**Theorem 3.1.4. (Christensen-Evans)** *Let $(\mathcal{T}_t)_{t\geq 0}$ be a uniformly continuous QDS on a $C^*$-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ with $\mathcal{L}$ as its generator. Then there exist a completely positive map $\Psi$ of $\mathcal{A}$ into the ultra-weak closure $\mathcal{A}''$ and an operator $k$ in $\mathcal{A}''$ such that the generator is given by*

$$\mathcal{L}(x) = \Psi(x) + k^* x + xk.$$

Since $\Psi$ is completely positive, by Stinespring Theorem 2.3.4, there exists a Hilbert space $\mathcal{K}$, a unital $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$, $L \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and a self-adjoint element $H$ of $\mathcal{A}''$ such that the generator $\mathcal{L}$ is given by

$$\mathcal{L}(x) = L^*\pi(x)L - \frac{1}{2}(L^*L - \mathcal{L}(1))x - \frac{1}{2}x(L^*L - \mathcal{L}(1)) + i[H, x], \text{ for all } x \in \mathcal{A}. \tag{3.1}$$

This representation is minimal in the sense that $\{(Lx - \pi(x)L)\xi : \xi \in \mathcal{H} \ , x \in \mathcal{A}\}$ is total in $\mathcal{K}$.

**Strongly Continuous Quantum Dynamical Semigroups:**

There is no complete characterization of the generator of a general strongly continuous QDS. The problem of constructing strongly continuous QDS with unbounded generator could be treated with the Theorem 2.4.9 at least in the case when the domain of the generator is an algebra so that conditional complete positivity makes sense. However in general the infinitesimal generator $\mathcal{L}$ may not makes sense but can be understood as an unbounded quadratic form on the Hilbert space $\mathcal{H}$.

The **predual semigroup** of a QDS $\mathcal{T}$ on $\mathcal{A}$ is the semigroup $(\mathcal{S}_t)_{t\geq 0}$ of operators in $A_*$ defined by $(\mathcal{S}_t(x_*))(x) = x_*(\mathcal{T}_t(x))$ for every $x \in \mathcal{A}$ and every $x_* \in \mathcal{A}_*$.

Davies in [11] constructed the minimal predual semigroup on the space of positive trace-class operators (density matrices) in some Hilbert space $\mathcal{H}$, a method similar to that of Kato, Chebotarev in [4] constructed the minimal QDS on $\mathcal{B}(\mathcal{H})$ by an iteration method. The only assumption on the $G$ and $L_n$ is the following:

**Assumption:** The operator $G$ is the infinitesimal generator of a $C_0$ contraction semigroup $S = (\mathcal{S}_t)_{t\geq 0}$ on $\mathcal{H}$. The domain of operators $(L_n)_{n=1}^{\infty}$ contains a core $\mathcal{D}$ of $G$. For all $u, v \in \mathcal{D}$, we have

$$\langle u, Gv\rangle + \langle Gu, v\rangle + \sum_{n=1}^{\infty}\langle L_n u, L_n v\rangle = 0. \tag{3.2}$$

**Theorem 3.1.5.** *For all $x \in \mathcal{B}(\mathcal{H})$, Consider the sesquilinear form $\mathcal{L}(x)$ on $\mathcal{H}$ with the domain $D \times D$ given by*

$$\langle u, \mathcal{L}(x)v\rangle = \langle u, xGv\rangle + \langle Gu, xv\rangle + \sum_{n=1}^{\infty}\langle L_n u, xL_n v\rangle$$

*Suppose the above assumption* (3.2) *holds, then there exists a minimal QDS* $(\mathcal{T}_t^{(min)})_{t\geq 0}$ *associated with this form, in the sense :*

$$\langle u, \mathcal{T}_t^{(min)}(x)v\rangle = \langle u, xv\rangle + \int_0^t \langle u, \mathcal{L}(\mathcal{T}_s^{(min)}(x))v\rangle ds \tag{3.3}$$

*for all* $t \geq 0, u, v \in D$ *and all* $x \in \mathcal{B}(\mathcal{H})$*. The QDS* $(\mathcal{T}_t^{(min)})_{t\geq 0}$ *is minimal in the sense that for any QDS* $(\mathcal{T}_t)_{t\geq 0}$ *which satisfies equation 3.3, we have*

$$\mathcal{T}_t^{(min)}(x) \leq \mathcal{T}_t(x) \text{ for all } x \in \mathcal{B}(\mathcal{H}),\ t \geq 0.$$

**Sketch of the proof:**

Equivalent condition for the semigroup $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ to satisfy the equation 3.3 is that for all $t \geq 0, u, v \in D$ and all $x \in \mathcal{B}(\mathcal{H})$,

$$\langle u, \mathcal{T}_t(x)v\rangle = \langle \mathcal{S}_t u, x\mathcal{S}_t v\rangle + \sum_{n=1}^{\infty}\int_0^t \langle L_n\mathcal{S}_{t-s}u, \mathcal{T}_s(x)L_n\mathcal{S}_{t-s}v\rangle ds.$$

Moreover, there exist a sequence $(\mathcal{T}_t^{(n)})_{t\geq 0}$ of linear contractions on $\mathcal{B}(\mathcal{H})$ defined by

$$\langle u, \mathcal{T}_t^{(0)}(x)v\rangle = \langle \mathcal{S}_t u, x\mathcal{S}_t v\rangle$$

$$\langle u, \mathcal{T}_t^{(n+1)}(x)v\rangle = \langle \mathcal{S}_t u, x\mathcal{S}_t v\rangle + \sum_{n=1}^{\infty}\int_0^t \langle L_n\mathcal{S}_{t-s}u, \mathcal{T}_s^{(n)})(x)L_n\mathcal{S}_{t-s}v\rangle ds$$

for all $t \geq 0, u, v \in D$ and all $x \in \mathcal{B}(\mathcal{H})$.

Furthermore the maps $(\mathcal{T}_t^{(n)})_{t\geq 0}$ are completely positive and normal and satisfy $\mathcal{T}_t^{(n)}(I) \leq I$ for every $n, t \geq 0$. The sequence $(\mathcal{T}_t^{(n)}(x))_{t\geq 0}$ is non-decreasing for every positive $x \in \mathcal{B}(\mathcal{H})$. Since the $(\mathcal{T}_t^{(n)})_{t\geq 0}$ are contractive, the $\lim_{n\to\infty}\langle u, \mathcal{T}_t^{(n)}(x)v\rangle$ exists. Setting $\langle u, \mathcal{T}_t^{(min)}(x)v\rangle = \lim_{n\to\infty}\langle u, \mathcal{T}_t^{(n)}(x)v\rangle$, the family $(\mathcal{T}_t)_{t\geq 0}^{min}$ satisfies the equation 3.3. It can be checked that for any QDS $(\mathcal{T}_t)_{t\geq 0}$ which satisfies equation 3.3, we have

$$\mathcal{T}_t^{(min)}(x) \leq \mathcal{T}_t(x)$$

for all $t \geq 0$.

**Corollary 3.1.6.** *Suppose the minimal QDS* $((\mathcal{T}_t^{(min)})_{t\geq 0}$ *constructed in the theorem 3.1.5 is conservative (Markov). Then it is the unique solution to the equation 3.3.*

**Conservative Quantum dynamical semigroup:**

In view of the corollary 3.1.6, one can give reasons to be interested in understanding the conservative property of a QDS. In a series of papers [5, 7, 6], Chebotarev and Fagnola simplified the sufficient conditions for the quantum dynamical semigroup to be conservative. We give the one of these conditions which appeared in [6].

**Theorem 3.1.7.** *In addition to assumption 3.2, suppose there exists a self-adjoint operator* $C$ *with domain coinciding with the domain of* $G$ *and a core* $D$ *for* $G$ *with the following properties:*

- $(i)$ *For all* $u \in Dom(G)$, *there exists a sequence* $(u_n)_{n\geq 0}$ *of elements of* $D$ *such that both* $(Gu_n)_{n\geq 0}$ *and* $(Cu_n)_{n\geq 0}$ *converge strongly.*
- $(ii)$ *The subspace* $L_n(D) \subseteq Dom(C)$ *for all* $n \geq 1$.
- $(iii)$ *There exists a positive self-adjoint operator* $\Phi$, *with* $Dom(G) \subseteq Dom(\Phi)$ *such that for all* $u \in D$, *we have*

$$-2Re\langle u, Gu\rangle = \sum_{n=1}^{\infty} \|L_n u\|^2 = \langle u, \Phi u\rangle \leq \langle u, Cu\rangle\,. \tag{3.4}$$

*(iv) There exists a positive constant k, such that for all $u \in D$, we have*

$$2Re\langle Cu, Gu\rangle + \sum_{n=1}^{\infty} \langle L_n u, CL_n u\rangle \leq k\,\langle u, Cu\rangle\,. \tag{3.5}$$

*Then the QDS $(\mathcal{T}_t^{(min)})_{t\geq 0}$ constructed in theorem 3.1.5 is conservative.*

The second way to understand the conservativity of QDS is by getting the knowledge about resolvent of the minimal QDS $(\mathcal{T}_t^{(min)})_{t\geq 0}$. Recall from Definition 2.1.13 that for $\lambda \in \rho(T)$, the resolvent of $T$ at $\lambda : \mathcal{R}_\lambda(T) = (\lambda I - T)^{-1}$. The resolvent of the minimal QDS $(\mathcal{T}_t^{(min)})_{t\geq 0}$ is characterized by the equation:

$$\langle u, \mathcal{R}_\lambda^{(min)}(x)v\rangle = \int_0^t e^{-\lambda s}\langle u, \mathcal{T}_s^{(min)}(x)v\rangle ds \tag{3.6}$$

with $x \in \mathcal{B}(\mathcal{H})$ and $u, v \in \mathcal{H}$.

## 3.2 Quantum Stochastic Calculus on Symmetric Fock Space

In this section we discuss about the quantum stochastic calculus on Boson Fock space developed by Hudson and Parthasarathy [20]. This gives a very powerful tool to study QDS on open systems by incorporating the noise in the Fock space.

### 3.2.1 Tensor Product of Infinitely Many Hilbert Spaces

Let $\{\mathcal{H}_i : i \in \mathcal{I}\}$ be the collection of Hilbert spaces indexed by $\mathcal{I}$ and $\{e_{n_i}^{(i)}\}$ be an orthonormal basis for $\mathcal{H}_i$. Let $S$ be a set of all sequences $\bar{n} = \{n_i\}$ of positive integers, that is the $k^{th}$ term in any sequence is the suffix of a basis vector of $\mathcal{H}_k$. Consider the vector space $W$ which is spanned by the finite linear combinations of the elements from the set

$$W_0 = \{e_{\bar{n}} := e^{(1)}_{n_1} \otimes e^{(2)}_{n_2} \otimes e^{(3)}_{n_3} \otimes \cdots \; : \; \bar{n} \in S\}.$$

Here, $u \in W$ if $u = \sum \eta(\bar{n}) e_{\bar{n}}$ such that $\eta(\bar{n}) : S \to \mathbb{C}$ is a map taking value zero for all sequences but finitely many. Define the inner-product between any two vectors of $W$, $u = \sum \eta(\bar{n}) e_{\bar{n}}$ and $v = \sum \zeta(\bar{n}) e_{\bar{n}}$ as

$$\langle u, v \rangle = \sum \overline{\eta(\bar{n})} \zeta(\bar{n})$$

**Definition 3.2.1.** *The completion of the vector space $W$ with respect to the metric given by above inner-product is the* ***tensor product*** *of Hilbert spaces $\{\mathcal{H}_i : i \in \mathcal{I}\}$. It is denoted by $\bigotimes_{i \in I} \mathcal{H}_i$ and the vector $e_{\bar{n}}$ is denoted by $\bigotimes_{i \in I} e^{(i)}_{n_i}$*

The set $W_0 = \{e_{\bar{n}} \; : \; \bar{n} \in S\}$ forms an orthonormal basis for $\bigotimes_{i \in I} \mathcal{H}_i$. The tensor product of infinite many Hilbert spaces defined above is in general non-separable. The other definition which has comparatively vast applications requires the choice of distinguished unit vectors $u_i$ in each $\mathcal{H}_i$. Let us choose an orthonormal basis $\{e^{(i)}_{n_i} \; : \; i \in I\}$ for $\mathcal{H}_i$ such that $e^{(i)}_1 = u_i$ for all $i \in I$.

**Definition 3.2.2.** *Let us consider the inner-product subspace $W'$ spanned by the orthonormal vectors $e_{\bar{n}} \in \bigotimes_{i \in I} \mathcal{H}_i$ such that $n_i = 1$ i.e. $e^{(i)}_{n_i} = u_i$ for all but finitely many $i \in I$. The completion of the inner-product space $W'$ is called* ***infinite tensor product*** *of the Hilbert spaces $\{\mathcal{H}_i \; : \; i \in I\}$ with respect to the stabilizing unit vectors $\{u_i\}$. Convention: $\mathcal{H}^{\otimes^0} = \mathbb{C}$, $\mathcal{H}^{\otimes^1} = \mathcal{H}$.*

**Symmetric Tensor Product**

In a given physical system of $n$ identical particles which are indistinguishable from one another, a transition may happen resulting in interchange of some physical properties, like position or momentum and such a change may not be possible to detect by any of the observable. Since the events concerning $n$ indistinguishable particles are described by the projection in $\mathcal{H}^{\otimes^n}$. Such transitions when occur in $i^{th}$ and $j^{th}$ particle then we should not distinguish between $\bigotimes_{i=1}^{n} P_i$ and $P_1 \otimes P_2 \otimes \cdots \otimes P_{i-1} \otimes P_j \otimes P_{i+1} \cdots \otimes P_{j-1} \otimes P_i \otimes P_{j+1} \otimes \cdots \otimes P_n$, which implies that we must consider projections on a subspace of $\mathcal{H}^{\otimes^n}$ which is invariant under the permutations. For a Hilbert space $\mathcal{H}$, consider $\mathcal{H}^{\otimes^n} = \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{\text{n times}}$ and an elementary element $u \in \mathcal{H}^{\otimes^n}$ is such that $u = u_1 \otimes u_2 \otimes \cdots \otimes u_n$. For $\sigma \in S_n$, the symmetric group on $n$ symbols, define a unitary map $U_\sigma : \mathcal{H}^{\otimes^n} \to \mathcal{H}^{\otimes^n}$ which is the linear extension of the map $U_\sigma(u) = \bigotimes_{i=1}^{n} u_{\sigma^{-1}(i)}$ on elementary elements. For

$$\sigma, \sigma' \in S_n, U_\sigma U_{\sigma'} = U_{\sigma\sigma'}.$$

Thus $\sigma \to U_\sigma$ is a group homomorphism from $S_n$ to $\mathcal{U}(\mathcal{H}^{\otimes^n})$.

**Definition 3.2.3.** *The closed subspace*

$$\mathcal{H}^{\otimes^n}_{sym} = \{u \in \mathcal{H}^{\otimes^n} \ : \ U_\sigma u = u \textit{ for all } \sigma \in S_n\}$$

*is called the $n$**-fold symmetric tensor product** in $\mathcal{H}^{\otimes^n}$.*

**Proposition 3.2.4.** *For the operator $T = \frac{1}{n!} \sum_{\sigma \in S_n} U_\sigma$, in $\mathcal{H}^{\otimes^n}$, $\mathcal{H}^{\otimes^n}_{sym}$ is an invariant subspace. Let $\{e_i \ : \ i = 1, 2, 3, \cdots\}$ be an orthonormal basis in $\mathcal{H}$, then the set*

$$\{(\frac{n!}{r_1 \cdots r_k!})^{\frac{1}{2}} T \bigotimes_{j=1}^{k} e_{i_j}^{\otimes^{n_j}} \ : \ i_1 < i_2 < \cdots < i_k,$$
$$n_j \geq 1 \textit{ for all } 1 \leq j \leq k, \ n_1 + n_2 + \cdots + n_k = n, \ k = 1, 2, \cdots, n\}$$

*is an orthonormal basis in $\mathcal{H}^{\otimes^n}_{sym}$. In particular, if dim $\mathcal{H} = N < \infty$, then dim $\mathcal{H}^{\otimes^n}_{sym} = \binom{N+n-1}{n}$.*

### 3.2.2 Symmetric Fock Space and Weyl Representation

For a Hilbert space $\mathcal{H}$, let $\mathcal{H}_{sym}^{\otimes n}$ be the $n$-fold symmetric tensor product. The Hilbert space

$$\Gamma_{sym}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_{sym}^{\otimes n}$$

is called **symmetric Fock space** over $\mathcal{H}$. For $u \in \mathcal{H}$, the element

$$e(u) = \bigoplus_{n \geq 0} \frac{1}{\sqrt{n!}} u^{\otimes n}$$

belongs to $\Gamma_{sym}(\mathcal{H})$ and is said to be the **exponential vector** associated with $u$. The vector $e(0) = 1 \oplus 0 \oplus 0 \cdots$ is called vacuum vector and is denoted by $\Phi$. Given $u, v \in \mathcal{H}$, the exponential vectors satisfy

$$\langle e(u), e(v) \rangle = \exp\langle u, v \rangle.$$

Let $S$ be a subset of $\mathcal{H}$, the collection of exponential vectors $\{e(u) \ : \ u \in S\}$ is a linearly independent set in $\Gamma_{sym}(\mathcal{H})$. Let $\mathcal{E}(S)$ be the subspace generated by the exponential vectors associated with vectors in $S$, then $\mathcal{E}(S)$ is dense in $\Gamma_{sym}(\mathcal{H})$ if $S$ is dense in $\mathcal{H}$. For $u \in \mathcal{H}$ and $U \in \mathcal{U}(\mathcal{H})$, the space of unitary operators in $\mathcal{H}$, consider the Euclidean group $E(\mathcal{H}) = \mathcal{H} \times \mathcal{U}(\mathcal{H})$ with composition

$$(u, U_1)(v, U_2) = (U_1 v + u, U_1 U_2).$$

Define the **Weyl operator** $W : E(\mathcal{H}) \to \mathcal{U}(\Gamma_{sym}(\mathcal{H}))$ by setting,

$$W(u, U)e(v) = \exp\{\frac{-1}{2}\|u\|^2 - \langle u, Uv \rangle\} e(Uv + u) \ \text{ for all } v \in \mathcal{H}. \tag{3.7}$$

The above representation is continuous with respect to the product norm topology on $\mathcal{H} \times \mathcal{U}(\mathcal{H})$ and strong operator topology on $\mathcal{U}(\Gamma_{sym}(\mathcal{H}))$.

**Theorem 3.2.5.** *Let $\mathcal{H}$ be a complex separable space and $\Gamma_{sym}(\mathcal{H})$ be the symmetric Fock space. The representation of the Euclidean group $E(\mathcal{H})$ in $\mathcal{B}(\Gamma_{sym}(\mathcal{H}))$ defined by associated Weyl operator:*

$$W : E(\mathcal{H}) \to \mathcal{B}(\Gamma_{sym}(\mathcal{H}))$$
$$(u, U) \to W(u, U)$$

*is strongly continuous, irreducible and unitary projective representation.*

On substituting $W(u) = W(u, I)$, where $I$ is the identity operator and for $u, v \in \mathcal{H}$ following are true:

$$W(u)W(v) = \exp(-i\ \operatorname{Im}\langle u, v\rangle)W(u + v)$$
$$W(u)W(v) = \exp(-2i\ \operatorname{Im}\langle u, v\rangle)W(v)W(u)$$
$$W(su)W(tv) = W((s + t))u) \text{ for all } s, t \in \mathbb{R}.$$

The last equation shows that every $u \in \mathcal{H}$ yields a one parameter unitary group $\{W(tu) : t \in \mathbb{R}\}$ and hence a bounded self-adjoint Stone generator $p(u)$, that is,

$$W(tu) = e^{-itp(u)} \quad t \in \mathbb{R}, \quad u \in \mathcal{H}.$$

For $U \in \mathcal{U}(\mathcal{H})$, the operator $\Gamma(U) := W(0, U)$ is called the **second quantization** of $U$. For $U, V \in \mathcal{U}(\mathcal{H}),\ u \in \mathcal{H}$, we have

$$\Gamma(U)e(v) = e(Uv)$$
$$\Gamma(U)\Gamma(V) = \Gamma(UV)$$
$$\Gamma(U)W(u)\Gamma(U)^{-1} = W(Uu).$$

For every bounded self-adjoint operator $H$ such that $\{U_t = e^{-itH} : t \in \mathbb{R}\}$ be a unitary group in $\mathcal{U}(\mathcal{H})$, there corresponds a unitary group $\{\Gamma(U_t) = e^{-it\lambda(H)} : t \in \mathbb{R}\}$ in $\Gamma_{sym}(\mathcal{H})$ and the stone generator $\lambda(H)$ is called the **differential second quantization** of $H$. Consider the following collection of operators, for $u \in \mathcal{H}$

$$q(u) = -p(iu),\ a(u) = \frac{1}{2}(q(u) + ip(u)),\ a^\dagger(u) = \frac{1}{2}(q(u) - ip(u))$$

and for a bounded operator $H$ in $\mathcal{H}$

$$\lambda(H) = \lambda(\frac{1}{2}(H + H^*)) + i\lambda(\frac{1}{2i}(H - H^*)).$$

The domain of the operators $a(u)$, $a^\dagger(u)$ and $\lambda(H)$ contains the space of exponential vectors $\mathcal{E}(H)$ and the following are true :

(*i*) $a(u)e(v) = \langle u, v\rangle e(v)$.

(*ii*) $\langle a^\dagger(u)e(v), e(w)\rangle = \langle e(v), a(u)e(w)\rangle = \langle u, w\rangle\langle e(v), e(w)\rangle$.

The operators $a(u)$, $a^\dagger(u)$ are respectively anti-linear and linear in $u$, $\lambda(H)$ is linear in $H$ and the following commutation relations hold:

(*i*) $[a(u), a(v)]e(w) = [a^\dagger(u), a^\dagger(v)]e(w) = 0$.

(*ii*) $[a(u), a^\dagger(v)]e(w) = \langle u, v\rangle e(w)$.

(*iii*) $[\lambda(H_1),\ \lambda(H_2)]e(u) = \lambda([H_1, H_2])e(u)$.

(*iv*) $[a(u), \lambda(H)]e(v) = a^*(H^*u)e(v)$.

(*v*) $[a^\dagger(u),\ \lambda(H)]e(v) = -a^\dagger(Hu)e(v)$.

**Proposition 3.2.6.** *Let $u \in \mathcal{H}$ and $H \in \mathcal{B}(\mathcal{H})$, then the operators $a(u)$, $a^\dagger(u)$, $\lambda(H)$ satisfy the following relations:*

(*i*) $a^\dagger(u)e(v) = \frac{\mathrm{d}}{\mathrm{d}t}e(v + tu)|_{t=0}$.

(ii) $\langle e(v), \lambda(H)e(w)\rangle = \langle v, Hw\rangle e^{\langle v,w\rangle}$.

(iii) $\langle a^\dagger(u_1)e(v),\ a^\dagger(u_2)e(w)\rangle = \{\langle u_1, w\rangle\langle v, u_2\rangle + \langle u_1, u_2\rangle\}e^{\langle v,w\rangle}$.

(iv) $\langle a^\dagger(u)e(v),\ \lambda(H)e(w)\rangle = \{\langle u, w\rangle\langle v, Hw\rangle + \langle u, Hw\rangle\}e^{\langle v,w\rangle}$.

(v) $\langle \lambda(H_1)e(v),\ \lambda(H_2)e(w)\rangle = \{\langle H_1 v, w\rangle\langle v, H_2 w\rangle + \langle H_1 u, H_2 w\rangle\}e^{\langle v,w\rangle}$.

In view of the following properties, the operators $a(u),\ a^\dagger(u),\ \lambda(H)$ are called **annihilation** operator associated with $u$, **creation** operator associated with $u$ and **conservation** operator associated with $H$ respectively.

(i) $a(u)e(0) = 0,\ a(u)v^{\otimes^n} = \sqrt{n}\langle u, v\rangle v^{\otimes^{n-1}}$, if $n \geq 1$.

(ii) $a^\dagger(u)v^{\otimes^n} = \dfrac{1}{\sqrt{n+1}}\sum_{k=0}^{n} v^{\otimes^k} \otimes u \otimes v^{\otimes^{n-k}}$.

(iii) $\lambda(H)v^{\otimes^n} = \sum_{k=0}^{n-1} v^{\otimes^k} \otimes Hv \otimes v^{\otimes^{n-k-1}}$.

### 3.2.3 Fundamental Processes

Consider a complex separable Hilbert space $\mathbf{k}$ with an orthonormal basis $\{e_i\}_{i=1}^N$. In quantum stochastical calculus the Hilbert space $\mathbf{k}$ represents the **noise space** associated to the physical system. Let $\mathcal{H} = L^2(\mathbb{R}_+, \mathbf{k})$ be the space of $\mathbf{k}$-valued square-integrable maps. This space can be seen as $L^2(\mathbb{R}_+) \bigotimes \mathbf{k}$ such that $u \otimes e_i(t) = u(t)e_i$. From this onward, $\Gamma(\mathcal{H})$ means the symmetric Fock space over $\mathcal{H} = L^2(\mathbb{R}_+, \mathbf{k})$.

For any partition $0 < s < t < \infty$, let $\mathbf{1}_{[0,s]},\ \mathbf{1}_{(s,t]}$ and $\mathbf{1}_{[t,\infty)}$

be the canonical orthogonal projections, where $\mathbf{1}$ represents the characteristic function. Let us denote the ranges of projections $\mathbf{1}_{[0,s]},\ \mathbf{1}_{(s,t]}$ and $\mathbf{1}_{[t,\infty)}$ on $\mathcal{H}$ by $\mathcal{H}_{s]},\ \mathcal{H}_{(s,t]}$ and $\mathcal{H}_{[t}$ respectively. Thus we have $\mathcal{H} = \mathcal{H}_{s]} \oplus \mathcal{H}_{(s,t]} \oplus \mathcal{H}_{[t}$ and any function $f$ in $\mathcal{H}$ decomposes as $f = f_{s]} \oplus f_{(s,t]} \oplus f_{[t}$, where $f_{s]} = \mathbf{1}_{[0,s]}f,\ f_{(s,t]} = \mathbf{1}_{(s,t]}f$ and $f_{[t} = \mathbf{1}_{[t,\infty)}f$. Therefore for any partition $0 < s < t < \infty$, the symmetric Fock space $\Gamma(\mathcal{H})$ over $\mathcal{H}$ can be written as the tensor product

$$\Gamma(\mathcal{H}) = \Gamma(\mathcal{H}_{s]}) \otimes \Gamma(\mathcal{H}_{(s,t]}) \otimes \Gamma(\mathcal{H}_{[t})$$

and the vacuum vector in the Fock space $\Gamma(\mathcal{H})$ can be written as $\Phi = \Phi_{s]} \otimes \Phi_{(s,t]} \otimes \Phi_{[t}$, where $\Phi_{s]}$, $\Phi_{(s,t]}$ and $\Phi_{[t}$ are the vacuum vectors in $\Gamma(\mathcal{H}_{s]})$, $\Gamma(\mathcal{H}_{(s,t]})$ and $\Gamma(\mathcal{H}_{[t})$ respectively. For the next subsection $\mathcal{H}$ is considered to be $L^2(\mathbb{R}_+, \mathbf{k})$, unless otherwise stated. However, the result are true for a general Hilbert space.

Let $\mathbf{h}_0$ be a Hilbert space and $\tilde{\mathcal{H}} = \mathbf{h}_0 \otimes \Gamma(\mathcal{H})$. For $0 < s < t < \infty$, let us define the following notations:

$$\tilde{\mathcal{H}}_{0]} = \mathbf{h}_0, \ \tilde{\mathcal{H}}_{s]} = \mathbf{h}_0 \otimes \Gamma(\mathcal{H}_{s]})$$
$$\tilde{\mathcal{H}}_{(s,t]} = \Gamma(\mathcal{H}_{(s,t]}), \ \tilde{\mathcal{H}}_{[t} = \Gamma(\mathcal{H}_{[t}).$$

Let $P_{s]}$, $P_{(s,t]}$ and $P_{[t}$ be the orthogonal projections such that the ranges of $\tilde{\mathcal{H}}$ under $P_{s]}$, $P_{(s,t]}$ and $P_{[t}$ are $\tilde{\mathcal{H}}_{s]}$, $\tilde{\mathcal{H}}_{(s,t]}$ and $\tilde{\mathcal{H}}_{[t}$ respectively. Let us consider $\mathcal{B} = \mathcal{B}(\tilde{\mathcal{H}}) \simeq \mathcal{B}_0 \otimes \mathcal{B}(\Gamma(\mathcal{H}))$, where $\mathcal{B}_0$ denotes the algebra of bounded of operators on $\mathbf{h}_0$. For $0 < s < t < \infty$, $\mathcal{B}$ can be written as $\mathcal{B} = \mathcal{B}_{s]} \otimes \mathcal{B}_{(s,t]} \otimes \mathcal{B}_{[t}$, where $\mathcal{B}_{s]} = \mathcal{B}_0 \otimes \mathcal{B}(\Gamma(\mathcal{H}_{s]}))$, $\mathcal{B}_{(s,t]} = \mathcal{B}(\tilde{\mathcal{H}}_{(s,t]})$ and $\mathcal{B}_{[t} = \mathcal{B}(\tilde{\mathcal{H}}_{[t})$. These von Neumann algebras are canonically embedded in $\mathcal{B}$. In other words, this increasing family of von Neumann algebras forms a **filtration** $\mathcal{B}_t$.

Let $\mathcal{D}_0 \subset \mathbf{h}_0$ and $\mathcal{M} \subset \mathcal{H}$ be two dense subspaces. The algebraic tensor product $\mathcal{D}_0 \otimes \mathcal{M}$ is a dense subspaces of $\tilde{\mathcal{H}}$ generated by all the vectors of the form $ue(f)$, $u \in \mathcal{D}_0$ and $f \in \mathcal{M}$. (We drop the tensor symbol between the elements for the ease of notations and calculations.)

**Definition 3.2.7.** *A family $\{L_t\}_{t\geq 0}$ of operators on $\tilde{\mathcal{H}}$ is said to be an $(\mathcal{D}_0, \mathcal{M})$-**adapted process** with respect to the filtration $\mathcal{B}_t$ if,*

(*i*) $D(L_t) \supset \mathcal{D}_0 \otimes \mathcal{E}(\mathcal{M}), \ \forall t \geq 0.$

(*ii*) *For every $u \in \mathcal{D}_0$ and $f \in \mathcal{M}$, we have $L_t ue(f_{t]}) \in \tilde{\mathcal{H}}_{t]}$ and $L_t ue(f) = L_t ue(f_{t]}) \otimes e(f_{[t}) \ \forall \ t \geq 0$.*

It is said to be regular, if in addition, for every $u \in \mathcal{D}_0$ and $f \in \mathcal{M}$, the map $t \to L_t ue(f)$ from $\mathbb{R}_+$ into $\tilde{\mathcal{H}}$ is continuous. An adapted process is called bounded, contractive, isometric, co-isometric or unitary if the operators $L_t$ 's are so.

Notice that operators $L \in \mathcal{B}_0$ and $T \in \mathcal{B}_{s]}$ can be identified with $L \otimes I_{\Gamma(\mathcal{H}_{t]})}$ and $T \otimes I_{\Gamma(\mathcal{H}_{[s})}$ on $\mathcal{B}$. Furthermore, given a operator $L \in \mathcal{B}_{s]}$, it represents the adapted process given by

$$L_t ue(f) = \begin{cases} P_{t]} L P_{t]} ue(f) & \text{if} \quad t \leq s \\ L_s ue(f_{s]}) \otimes e(f_{[s}) & \text{if} \quad t \geq s. \end{cases} \tag{3.8}$$

Let us introduce the **vacuum conditional expectation** $\mathbb{E}_0 : \mathcal{B} \to \mathcal{B}_0$, which is given by

$$\langle \mathbb{E}_0(X)u, v \rangle = \langle u\Phi, Xv\Phi \rangle, \ \forall u, v \in \mathbf{h}_0, \ X \in \mathcal{B}.$$

The **fundamental processes** $\{\Lambda^\mu_\nu \ : \ 0 \leq \mu, \nu < \infty\}$ associated with the orthonormal basis $\{e_j \ : \ j \geq 1\}$ are given by:

$$\Lambda^\mu_\nu(t) = \begin{cases} tI_{\tilde{\mathcal{H}}}, & \text{if } (\mu,\nu) = (0,0), \\ a(\mathbf{1}_{[0,t]} \otimes e_j), & \text{if } (\mu,\nu) = (0,j), \\ a^\dagger(\mathbf{1}_{[0,t]} \otimes e_i), & \text{if } (\mu,\nu) = (i,0), \\ \lambda(M_{\mathbf{1}_{[0,t]}} \otimes |e_i\rangle\langle e_j|), & \text{if } (\mu,\nu) = (i,j), \end{cases} \tag{3.9}$$

where $M_{\mathbf{1}_{[0,t]}}$ is the multiplication operator on $L^2(\mathbb{R}_+)$ by the characteristic function of the interval $[0,t]$. The processes $\Lambda^\mu_\nu(t)$ are defined on the space $\mathcal{E}(\mathcal{H})$ and notice that as earlier, the process $\Lambda^\mu_\nu(t)$ are identified with $I_{h_0} \otimes 1_{\Gamma(\mathcal{H}_{t]})}$ on $\tilde{\mathcal{H}}$.

With the above notations in mind, now we discuss quantum stochastic integration with respect to the processes defined in (3.9).

### 3.2.4 Quantum Stochastic Integration

First the integration of a simple adapted process is defined with respect to the fundamental process, then as a limiting case it is defined for arbitrary adapted processes.

**Definition 3.2.8.** *An $(\boldsymbol{h}_0, \mathcal{H})$-adapted process $L$ is said to be* ***simple*** *with respect to a partition $P = \{0 = t_0 < t_1 < \cdots < t < \cdots\}$ of $\mathbb{R}_+$, if $L_t = L_{t_{n-1}}$, whenever $t \in (t_{n-1}, t_n]$.*

The process $L_t$ remains simple with respect to any finer partition.
For a given $\mu, \nu \geq 0$, consider the simple process $X$ given by:

$$X_t = \sum_{j=1}^{n-1} L_{t_{j-1}}[\Lambda_\nu^\mu(t_j) - \Lambda_\nu^\mu(t_{j-1})] + L_{t_{n-1}}[\Lambda_\nu^\mu(t) - \Lambda_\nu^\mu(t_{n-1})],$$

whenever $t \in (t_{n-1},\ t_n]$. The process $X$ is called the **quantum stochastic integral** of the process $L$ with respect to $\Lambda_\nu^\mu$ and written as

$$X_t = \int_0^t L \ \mathrm{d}\Lambda_\nu^\mu = \int_0^t L_s \ \mathrm{d}\Lambda_\nu^\mu(s).$$

Observe that the definition of $X$ is independent of the partition with respect to which $L$ is simple. Given $u \in \mathbf{h}_0$ and $f \in \mathcal{H}$ the map $t \to X_t ue(f)$ is continuous, thus $X$ is a regular $(\mathbf{h}_0, \mathcal{H})$-adapted process. The properties of annihilation, creation and conservation operators which are listed in previous sub-section, in particularly Proposition 3.2.6 give rise to the following lemma.

**Proposition 3.2.9. (First Fundamental Lemma)** *For $\mu, \nu \geq 0,\ u, v \in \boldsymbol{h}_0$ and $f, g \in \mathcal{H}$, we have*

$$\langle ue(f), \int_0^t L_s \ d\Lambda_\nu^\mu(s) \ ve(g)\rangle = \int_0^t \overline{f_\mu(s)} g_\nu(s) \langle ue(f), L_s ve(g)\rangle \ ds. \tag{3.10}$$

Let $L$ and $M$ be two simple $(\mathbf{h}_0, \mathcal{H})$-adapted process with respect to the partition $0 = t_0 < t_1 \cdots < t_n = t$ of $\mathbb{R}_+$. Let $X_t = \int_0^t L_s \, \mathrm{d}\Lambda_\nu^\mu(s)$ and $Y_t = \int_0^t M_s \, \mathrm{d}\Lambda_\eta^\xi(s)$. Then for $u, v \in \mathbf{h}_0$ and $f, g \in \mathcal{H}$,

$$\begin{aligned}
&\langle X_t \; ue(f), Y_t \; ve(g)\rangle = \\
&\qquad \sum_{j=1}^{n} \langle L_{t_{j-1}} ue(f_{t_{j-1}]}), M_{t_{j-1}} ve(g_{t_{j-1}]})\rangle \langle \Lambda_\nu^\mu(j) \; e(f_{[t_j}), \Lambda_\eta^\xi(j) \; e(g_{[t_j})\rangle \\
&\qquad + \sum_{j=1}^{n} \langle X_{t_{j-1}} ue(f_{t_{j-1}]}), M_{t_{j-1}} ve(g_{t_{j-1}]})\rangle \langle e(f_{[t_j}), \Lambda_\eta^\xi(j) \; e(g_{[t_j})\rangle \\
&\qquad + \sum_{j=1}^{n} \langle L_{t_{j-1}} ue(f_{t_{j-1}]}), Y_{t_{j-1}} ve(g_{t_{j-1}]})\rangle \langle \Lambda_\nu^\mu(j) \; e(f_{[t_j}), e(g_{[t_j})\rangle.
\end{aligned} \tag{3.11}$$

From the equations (3.10) and (3.11) we have the following result:

**Proposition 3.2.10. (Second Fundamental Lemma)**

$$\begin{aligned}
\langle X_t \; ue(f), Y_t \; ve(g)\rangle &= \int_0^t \delta_\xi^\mu \overline{f_\nu(s)} g_\eta(s) \langle L_s ue(f), M_s ve(g)\rangle \; ds \\
&\quad + \int_0^t \overline{f_\xi(s)} g_\eta(s) \langle X_s ue(f), M_s ve(g)\rangle \; ds \\
&\quad + \int_0^t \overline{f_\nu(s)} g_\mu(s) \langle L_s ue(f), Y_s ve(g)\rangle \; ds.
\end{aligned} \tag{3.12}$$

The **quantum Ito formula** can be express as for all $\mu, \nu, \eta, \xi \geq 0$:

$$\mathrm{d}\Lambda_\nu^\mu \; \mathrm{d}\Lambda_\eta^\xi = \hat{\delta}_\nu^\xi \; \mathrm{d}\Lambda_\eta^\mu, \text{ where } \hat{\delta}_\nu^\xi = \begin{cases} 0 & \text{if } \xi = \nu = 0, \\ \delta_\nu^\xi & \text{otherwise.} \end{cases}$$

The function $\delta_\nu^\xi$ being the Kronecker delta function.

Using Second Fundamental Lemma (3.12) with $X = Y$ and $L = M$, we observe that following holds:

$$\|X_t ue(f)\|^2 = 2\mathrm{Re}\int_0^t \overline{f_\nu(s)} f_\mu(s)\langle X_s ue(f), L_s ue(f)\rangle\ ds + \int_0^t \|f_\nu(s)\|^2\|L_s ue(f)\|^2\ ds.$$

For general adapted processes, the integral is defined as the limit of integrals and the following result shows when such limit exists.

**Proposition 3.2.11.** *Let $L$ be a $(\boldsymbol{h}_0, \mathcal{H})$-adapted process satisfying the following: For given $u \in \boldsymbol{h}_0,\ f \in \mathcal{H}$*

(*i*) *the map $t \to L_t ue(f)$ is left continuous.*

(*ii*) $\sup_{0\le s\le t} \|L_s ue(f)\| < \infty$, *for each $t$.*

*Then there exists a sequence of simple $(\boldsymbol{h}_0, \mathcal{H})$-adapted processes $\{L^{(n)}\}$ such that*

$$\lim_{n\to\infty} L_t^{(n)} ue(f) = L_t ue(f),\ \ \forall\ t \ge 0$$

*and for any $\mu, \nu \ge 0$,*

$$s-\lim_{n\to\infty}\int_0^t L_s\ d\Lambda_\nu^\mu(s)\ \textit{exists on the domain}\ \boldsymbol{h}_0 \otimes \mathcal{E}(\mathcal{H}).$$

Notice that the strong limit, say $X_t$, is independent of the choice of approximating sequence. The limit $X_t$ is said to be the quantum stochastic integration of $L$ with respect $\Lambda_\nu^\mu$. For all such general processes, the first and second fundamental lemmas hold and can be proved similarly. We denote the space of all such integrable processes by $\mathbb{L}(\mathbf{h}_0, \mathcal{H})$. Since most of the time we deal with the family of process, so here we discuss the integrability of family of adapted process.

**Proposition 3.2.12.** *Let* $\{L^\mu_\nu\}$ *be a family in* $\mathbb{L}(\boldsymbol{h}_0, \mathcal{H})$ *satisfying, for all* $t \geq 0,\ \nu \geq 0,\ u \in \boldsymbol{h}_0,\ f \in \mathcal{H}$,

$$\int_0^t \sum_{\mu=0}^{\infty} \|L^\mu_\nu(s) ue(f)\|^2 \, d\gamma_f(s) < \infty, \tag{3.13}$$

*where* $\gamma_f(t) = \int_0^t (1 + \|f(s)\|^2) ds$. *Then there exists a regular* $(\boldsymbol{h}_0, \mathcal{H})$*-adapted process* $X$ *satisfying, for all* $t \geq 0, u \in \boldsymbol{h}_0, f \in \mathcal{H}$ :

(*i*) $\lim_{n\to\infty} \sup_{0 \leq s \leq T} \|X^{(n)}_s ue(f) - X_s ue(f)\| = 0,$ *where* $T < \infty,\ s \in [0, T]$ *and*

$$X^{(n)}_t = \sum_{0 \leq \mu,\nu \leq n} \int_0^t L^\mu_\nu(s) ue(f)\ d\Lambda^\mu_\nu(s).$$

(*ii*) $\|X_t ue(f)\|^2 = 2e^{\gamma_f(t)} \sum_{\nu \geq 0} \int_0^t \sum_{\mu \geq 0} \|L^\mu_\nu(s) ue(f)\|^2 d\gamma_f(s),$

Such a family $\{L^\mu_\nu\}$ is called **stochastically integrable** with respect to $\Lambda^\mu_\nu$ and its stochastic integral is defined to be:

$$X_t = \sum_{\mu,\nu \geq 0} \int_0^t L^\mu_\nu(s) ue(f)\ \mathrm{d}\Lambda^\mu_\nu(s).$$

and this is written as a differential equation:

$$\mathrm{d}X = \sum_{\mu,\nu \geq 0} L^\mu_\nu\ \mathrm{d}\Lambda^\mu_\nu.$$

Here we state that first fundamental lemma and second fundamental lemma hold for any two stochastically integrable families of process $\{L^\mu_\nu\}$ and $\{M^\mu_\nu\}$.

**Proposition 3.2.13.** *With notations in 3.12 and for* $\mu, \nu \geq 0,\ u, v \in \boldsymbol{h}_0$ *and* $f, g \in \mathcal{H}$, *we have:*

(*i*) (First Fundamental Lemma)

$$\langle ue(f), X_t ve(g)\rangle = \sum_{\mu,\nu\geq 0}\int_0^t \overline{f_\mu(s)}g_\nu(s)\langle ue(f), L_\nu^\mu(s)\ ve(g)\rangle\ ds. \tag{3.14}$$

(*ii*) (Second Fundamental Lemma)

$$\begin{aligned}\langle X_t\ ue(f), Y_t\ ve(g)\rangle = &\sum_{\mu,\nu\geq 0}\int_0^t \sum_{\xi\geq 0}\overline{f_\mu(s)}g_\nu(s)\langle L_\mu^\xi(s)\ ue(f), M_\nu^\xi(s)\ ve(g)\rangle\ ds\\ &+\sum_{\mu,\nu\geq 0}\int_0^t \overline{f_\mu(s)}g_\nu(s)\langle X_s ue(f), M_\mu^\nu(s)\ ve(g)\rangle\ ds\\ &+\sum_{\mu,\nu\geq 0}\int_0^t \overline{f_\mu(s)}g_\nu(s)\langle L_\nu^\mu(s)\ ue(f), Y_s ve(g)\rangle\ ds.\end{aligned} \tag{3.15}$$

**Proposition 3.2.14.** *Let* $\{L_\nu^\mu\}$ *be a family of bounded linear operators on* $\boldsymbol{h}_0$ *such that for each* $\nu\geq 0$ *there exists a constant* $c_\nu$ *satisfying:*

$$\sum_{\mu\geq 0}\|L_\nu^\mu u\|^2 \leq c_\nu^2\|u\|^2 \text{ for all } u\in\boldsymbol{h}_0.$$

*Then there exists a unique regular* $(\boldsymbol{h}_0,\mathcal{H})$*- adapted process* $X=\{X_t\}_{t>0}$ *satisfying the differential equation*

$$dX = \sum_{\mu,\nu\geq 0} X L_\nu^\mu\ d\Lambda_\nu^\mu,\quad X(0)=X_0\otimes 1, \tag{3.16}$$

*where* $X_0\in\mathcal{B}(\boldsymbol{h}_0)$.

The following result proves the existence of a unitary solution for a class of quantum stochastic differential equation (QSDE) studied and developed by Hudson and Parthasarathy know as HP (left) differential equation:

$$dU = \sum_{\mu,\nu \geq 0} U L^\mu_\nu \, \mathrm{d}\Lambda^\mu_\nu \quad U(0) = U_0 \otimes 1. \tag{3.17}$$

**Theorem 3.2.15.** *Let $\boldsymbol{h}_0$ be a complex separable Hilbert space with orthonormal basis $\{e_i\}_{i=1}^N$, $N \leq \infty$. Let $H \in \mathcal{B}(\boldsymbol{h}_0)$ be a self-adjoint operator and $\{L_i \; ; \; 1 \leq i \leq m\}, \{S^\mu_\nu \; ; \; \mu, \nu \geq 0\}$ are bounded operators on $\boldsymbol{h}_0$ such that $\sum_{\mu,\nu\geq 0} S^\mu_\nu \otimes |e_\mu\rangle\langle e_\nu|$ is a unitary operator in $\boldsymbol{h}_0 \otimes \mathbf{k}$. If the coefficients of differential equation* (3.17) *are as follows:*

$$L^\mu_\nu = \begin{cases} -(iH + \frac{1}{2}\sum_{k\geq 1} L^*_k L_k) & \textit{if } (\mu,\nu) = (0,0). \\ L_i & \textit{if } (\mu,\nu) = (i,0), \\ -\sum_{k\geq 1} L^*_k S^k_j & \textit{if } (\mu,\nu) = (0,j), \\ S^i_j - \delta^i_j & \textit{if } (\mu,\nu) = (i,j). \end{cases} \tag{3.18}$$

*Then there exists a unique unitary process $U$ satisfies:*

$$dU = \sum_{\mu,\nu\geq 0} U L^\mu_\nu \; d\Lambda^\mu_\nu, \quad U(0) = I \textit{ on } \tilde{\mathcal{H}}. \tag{3.19}$$

One can see that the solution $U$ of the left QSDE (3.19) with the coefficients as defined in (3.18) is an isometry. To prove that $U^*$ is an isometry, we consider a dual process associated with $U^*$ given by the time reversal operator and satisfies a left QSDE. Consider the **time reversal operator** $R_t$ on $L^2(\mathbb{R}_+, \mathbf{k})$ defined by

$$R_t(f)(s) := \begin{cases} f(t-s) & \text{if } s \leq t; \\ f(s) & \text{if } s > t. \end{cases} \tag{3.20}$$

Observe that $R_t$ is a self-adjoint unitary. Thus the second quantization $\Gamma(R_t)$ is so. For a bounded process $U$, define the dual process $\tilde{U}$ by

$$\tilde{U_t} := (1 \otimes \Gamma(R_t)) {U_t}^* (1 \otimes \Gamma(R_t)).$$

**Proposition 3.2.16.** *Let $U$ be a bounded process satisfying the QSDE* (3.19). *Then the dual process $\tilde{U}$ will satisfy the QSDE of the similar form given by,*

$$\tilde{U_t} = I + \sum_{\mu,\nu \geq 0} \int_0^t \tilde{U}_s L_\mu^{\nu *} \; d\Lambda_\nu^\mu(s).$$

The equation (3.19) can be interpreted as a Schrödinger equation in the presence of noise. Now we look at the Heisenberg picture of this equation. Let $U$ be a unitary process satisfying (3.19), for $X \in \mathcal{B}(\mathbf{h}_0)$ define a family of $*$-homomorphisms $(J_t)_{t \geq 0}$ by:

$$J_t : \mathcal{B}(\mathbf{h}_0) \to \mathcal{B}(\mathbf{h}_0 \otimes \Gamma(\mathcal{H}))$$

$$X \to U_t^* (X \otimes I) U_t.$$

Then $\{J_t(X)\}_{t \geq 0}$ is a regular $(\mathbf{h}_0, \mathcal{H})$-adapted process which satisfies the QSDE

$$dJ_t(X) = \sum_{\mu,\nu \geq 0} J_t \theta_\nu^\mu(X) \; d\Lambda_\nu^\mu(t), \quad J_0(X) = X \otimes I_{\Gamma(\mathcal{H})}, \tag{3.21}$$

where $\theta_\nu^\mu$ are the maps from $\mathcal{B}(\mathbf{h}_0)$ to itself, given by:

$$\theta_\nu^\mu(X) = \begin{cases} \sum_{k \geq 1} (S_i^k)^* [X, L_k] & \text{if } (\mu,\nu) = (i,0), \\ \sum_{k \geq 1} [L_k^*, X] S_j^k & \text{if } (\mu,\nu) = (0,j), \\ \sum_{k \geq 1} (S_i^k)^* X S_j^k - \delta_j^i X & \text{if } (\mu,\nu) = (i,j) \end{cases} \tag{3.22}$$

and

$$\theta_0^0(X) = i[H, X] - \frac{1}{2}\sum_{k\geq 1}(L_k^* L_k X + X L_k^* L_k - 2L_k^* X L_k).$$

The map $\theta_0^0$ is the generator of a QDS $(\mathcal{T}_t)_{t\geq 0}$ and the homomorphic co-cycle $J_t$ dilates $\mathcal{T}_t$ in the sense that

$$\langle u\Phi, J_t(X)v\Phi\rangle = \langle u, \mathcal{T}_t(X)v\rangle, \ \forall \ u, v \in \mathbf{h}_0 \text{ and } X \in \mathcal{B}(\mathcal{H}). \tag{3.23}$$

In other words, QDS $(\mathcal{T}_t)_{t\geq 0}$ is the vacuum expectation of the homomorphic co-cycle $(J_t)_{t\geq 0}$, implemented by the HP flow $U$ known as an **Hudson-Parthasarathy (HP) dilation** of the QDS $\mathcal{T}_t$.

**Definition 3.2.17.** ***L****et $\mathcal{A}$ be a unital $C^*$-algebra of $\mathcal{B}(\boldsymbol{h}_0)$. A family $(J_t)_{t\geq 0}$ of $*$-homomorphisms from $\mathcal{A}$ into $\mathcal{B}(\tilde{\mathcal{H}})$ is called an* ***Evans-Hudson (EH) flow*** *with* ***initial algebra*** *$\mathcal{A}$ if following conditions are satisfied:*

(*i*) $j_0(X) = X \otimes 1 \ \forall \ X \in \mathcal{A}$.

(*ii*) $J_t(X) \in \mathcal{B}_{t]} \ \forall \ X \in \mathcal{A}$.

(*iii*) *For $\mu, \nu \geq 0$, there exists $\mathcal{A}_0$ a unital dense $*$-subalgebra of $\mathcal{A}$ and the family of maps $\theta_\nu^\mu : \mathcal{A}_0 \to \mathcal{A}_0$ such that $(J_t(X))_{t\geq 0}$ is a regular $(\boldsymbol{h}_0, \mathcal{H})$-adapted process satisfying the differential equation* (3.21).

The maps $\{\theta_\nu^\mu : \mu, \nu \geq 0\}$ are called **structure maps** of the EH flow $(J_t)_{t\geq 0}$ and they satisfy the following **structure equations**:

(*i*) $\theta_\nu^\mu$ is linear for all $\mu, \nu \geq 0$.

(*ii*) Whenever $1 \in \mathcal{A}_0, \theta_\nu^\mu(1) = 0$ for all $\mu, \nu \geq 0$.

(*iii*) $\theta_\nu^\mu(X^*) = (\theta_\nu^\mu(X))^*$ for all $X \in \mathcal{A}_0$.

(*iv*) $\theta^\mu_\nu(XY) = \theta^\mu_\nu(X)Y + X\theta^\mu_\nu(Y) + \sum_{\mu,\nu\geq 0} \theta^\mu_\nu(X)\theta^\mu_\nu(Y)$, for all $X, Y \in \mathcal{A}_0$.

In equation (*iv*) the convergence of the series is in strong sense. The structure maps are said to satisfy **Mohari-Sinha regularity condition** if there exists constants $\alpha_j > 0$ and a family of maps $\{T^\mu_\nu \ : \ \mu \in N, \nu \geq 0\}$ in $\mathcal{B}(\mathbf{h}_0)$ such that for all $u \in \mathbf{h}_0$ , $X \in \mathcal{B}(\mathbf{h}_0)$

$$\begin{aligned} \sum_{i\in\mathbb{N}} \|T^\mu_\nu u\|^2 &\leq \alpha_j^2 \|u\|^2, \\ \sum_{i\in\mathbb{N}} \|\theta^\mu_\nu(X)u\|^2 &\leq \sum_{i\in\mathbb{N}} \|XT^\mu_\nu u\|^2. \end{aligned} \tag{3.24}$$

If there exists $n \in \mathbb{N}$ such that $\theta^\mu_\nu = 0$ whenever $max\{\mu, \nu\} \geq n$ and the maps $\theta^\mu_\nu$ satisfy structure equations then Mohari-Sinha regularity conditions are automatically satisfied. The following results tells us that such a family of homomorphisms give rise to an EH flow.

**Theorem 3.2.18.** *Let* $\{\theta^\mu_\nu : \mu, \nu \geq 0\}$ *be a family of maps from* $\mathcal{A}_0$ *into itself satisfying the structure equations and the regularity condition* (3.24)*, then there exists a unique EH flow* $\{J_t\}_{t\geq 0}$ *with initial algebra* $\mathcal{A}_0$ *such that for* $X \in \mathcal{A}_0$ *it satisfies the differential equation* (3.21)*.*

# Chapter 4

# Examples of Quantum Dynamical Semigroups

As an attempt to understand strongly continuous QDS, here we look at certain classes of Lindbladians on UHF $C^*$-algebras. Though, the Lindbladians are given as forms, we manage to solve the associated quantum stochastic differential equations and are able to construct the QDS which are conservative [34].

## 4.1 Operators on GNS space of UHF algebra

In this section, we discuss the construction of operators on the GNS space of UHF $C^*$-algebras which are the coefficients in the Lindbladian forms as well as in the associated HP type QSDE.

Consider the UHF $C^*$-algebra $\mathcal{A}$ as the $C^*$-inductive limit of the infinite tensor product of the matrix algebra $M_N(\mathbb{C})$,

$$\mathcal{A} = \overline{\bigotimes_{j \in \mathbb{Z}^d} M_N(\mathbb{C})}^{c^*}.$$

The algebra $\mathcal{A}$ can be interpreted as inductive limit of full matrix algebras. For $x \in M_N(\mathbb{C})$ and $j \in \mathbb{Z}^d$, $x^{(j)}$ denotes an element of $\mathcal{A}$ with $x$ in the $j^{th}$ component and identity everywhere else. We shall call the elements of the form $\prod_{i=1}^{k} x_i^{(l_i)}$, where $l_1, l_2, \cdots, l_k \in \mathbb{Z}^d$, to be simple tensor elements in $\mathcal{A}$. For a simple tensor element $x$ in $\mathcal{A}$, let $x_{(j)}$ be the $j^{th}$ component of $x$. Support '$supp(x)$' of $x$ is defined to be the subset $\{j \in \mathbb{Z}^d; x_{(j)} \neq I\}$. For a general element $x \in \mathcal{A}$ such that $x = \sum_{n=1}^{\infty} c_n x_n$ with simple tensor elements $x_n$ and complex coefficients $c_n$, define $supp(x) = \bigcup_{n \geq 1} supp(x_n)$. For any $\Delta \subset \mathbb{Z}^d$, let $\mathcal{A}_\Delta$ denotes the $*$-sub algebra generated by the elements of $\mathcal{A}$ with support in $\Delta$. For $j = (j_1, j_2, \cdots, j_d) \in \mathbb{Z}^d$, define $|j| = max\{|j_i| \, ; \, 1 \leq i \leq d\}$ and set $\Delta_n = \{j \in \mathbb{Z}^d; |j| \leq n\}$, $\partial\Delta_n = \{j \in \mathbb{Z}^d; |j| = n\}$. We say an element $x \in \mathcal{A}$ is local if $x \in \mathcal{A}_{\Delta_p}$ for some $p \geq 1$. We write $\mathcal{A}_{loc}$ for the dense $*$-algebra generated by local elements. The unique normalized trace $tr$ on $\mathcal{A}$ is given by $tr(x) = \frac{1}{N^n} Tr(x)$, for $x \in M_{N^n}(\mathbb{C})$, where $Tr$ denotes the matrix trace. The trace $tr$ is a faithful normal state on the elements of $\mathcal{A}$. The algebra $\mathcal{A}$ can be represented as vectors in the Hilbert space $\mathbf{h}_0 = L^2(\mathcal{A}, tr)$, the GNS Hilbert space for $(\mathcal{A}, tr)$, and as an operator in $\mathcal{B}(\mathcal{H})$ by left multiplication.

Consider a formal element of the type

$$r := \sum_{n=1}^{\infty} W_n \text{ such that } \sum_{n=1}^{\infty} \|W_n\| = \infty,$$

where each $W_n$ belongs to $\mathcal{A}_{\partial\Delta_n}$. Let us denote formally

$$\sum_{n=1}^{\infty} W_n^* \text{ by } r^*.$$

Now, if we set $\mathcal{C}_r(x) = [r, x] = \sum_{n=1}^{\infty} [W_n, x]$ for $x \in \mathcal{A}_{loc}$, clearly it is well defined since $[W_n, x] = 0$ for all $n > m$ when $x$ is in finite dimensional algebra $\mathcal{A}_{\Delta_m} \subseteq \mathcal{A}_{loc}$. Thus we have a densely defined linear operator $(\mathcal{C}_r, \mathcal{A}_{loc})$ in $\mathbf{h}_0$. In case, $\sum_{n=1}^{\infty} \|W_n\| < \infty$, the operator $(\mathcal{C}_r, \mathcal{A}_{loc})$ would be bounded.

**Lemma 4.1.1.** *Let $r$ be as above and $n \geq 1$. Consider the element $r_n = \sum_{k=1}^{n} W_k$ in $\mathcal{A}$ and define a bounded operator $\mathcal{C}_r^{(n)}$ on $\boldsymbol{h}_0$ by setting $\mathcal{C}_r^{(n)}(x) = [r_n, x] = \sum_{k=1}^{n} [W_k, x]$ for $x \in \mathcal{A}_{loc}$. Then for each $n \geq 1$, $\mathcal{A}_{\Delta_n}$ is an invariant subspace for $\mathcal{C}_r$ and $\mathcal{C}_r^{(n)}$. Also for $m \geq p$,*

$$\mathcal{C}_r|_{\mathcal{A}_{\Delta_p}} = \mathcal{C}_r^{(m)}|_{\mathcal{A}_{\Delta_p}} = \mathcal{C}_r^{(p)}|_{\mathcal{A}_{\Delta_p}}. \tag{4.1}$$

*Proof.* For $x$ is in $\mathcal{A}_{\Delta_n}$, $[W_k, x] = 0$ for $k > n$. Thus $[r, x] = [r_n, x] \in \mathcal{A}_{\Delta_n}$ and $\mathcal{A}_{\Delta_n}$ is an invariant subspace under $\mathcal{C}_r$ and $\mathcal{C}_r^{(n)}$. Now for $x \in \mathcal{A}_{\Delta_p}$ and $m \geq p$, it is easy to see that $\mathcal{C}_r(x) = \mathcal{C}_r^{(m)}(x) = \mathcal{C}_r^{(p)}(x)$. □

**Proposition 4.1.2.** *The operator $(\mathcal{C}_r, \mathcal{A}_{loc})$ is closable.*

*Proof.* We shall show that $\mathcal{A}_{loc} \subseteq Dom(\mathcal{C}_r^*)$ and for $x \in \mathcal{A}_{loc}$, $\mathcal{C}_r^*(x) = \mathcal{C}_{r^*}(x) = [r^*, x]$, thereby showing that the operator $\mathcal{C}_r^*$ is densely defined and therefore $(\mathcal{C}_r, \mathcal{A}_{loc})$ is closable. Indeed for $x \in \mathcal{A}_{loc}$, there exists $p \geq 1$ such that $x \in \mathcal{A}_{\Delta_p}$. Define $\Phi_x(y) := \langle x, \mathcal{C}_r(y)\rangle \, \forall y \in \mathcal{A}_{loc}$. For each $y \in \mathcal{A}_{loc}$, there exists $m$ such that $y \in \mathcal{A}_{\Delta_m}$. As $\{\mathcal{A}_{\Delta_n}\}$ is an increasing family of algebras, with no loss of generality, let us assume $m \geq p$. Then by definition and property of trace and Lemma 4.1.1,

$$\Phi_x(y) = tr(x^*\mathcal{C}_r(y)) = tr(x^*\mathcal{C}_r^{(m)}(y)) = tr([\mathcal{C}_{r^*}^{(m)}(x)]^* y) = \langle \mathcal{C}_{r^*}^{(p)}(x), y\rangle = \langle \mathcal{C}_{r^*}(x), y\rangle,$$

and thus

$$|\Phi_x(y)| \leq \|\mathcal{C}_{r^*}(x)\| \|y\|, \; \forall \; y \in \mathcal{A}_{loc}.$$

Thus $x \in Dom(\mathcal{C}_r^*)$ and

$$\mathcal{C}_r^*(x) = \mathcal{C}_{r^*}(x), \forall x \in \mathcal{A}_{loc}. \tag{4.2}$$

□

We denote by $\bar{\mathcal{C}_r}$, the closure of the densely defined, closable operator $\mathcal{C}_r$. Note here that for an operator $T$ on $\mathbf{h}_0$, $T^* = \bar{T}^*$, if $T$ is closable. Then by standard theorem of von Neumann, $\mathcal{C}_r^*\bar{\mathcal{C}_r}$ is a positive self-adjoint operator in $\mathbf{h}_0$ and $Dom(\mathcal{C}_r^*\bar{\mathcal{C}_r})$ is a core for $\bar{\mathcal{C}_r}$. Furthermore, the operator $G := -\frac{1}{2}\mathcal{C}_r^*\bar{\mathcal{C}_r}$ is closed, densely defined dissipative operator. Hence by Theorem 2.4.13 generates a $C_0$-contraction semigroup $\mathcal{S}_t$ on $\mathbf{h}_0$.

**Proposition 4.1.3.** *For $n \geq 1$, define the bounded operator $G^{(n)}$ on $\boldsymbol{h}_0$ by*

$$G^{(n)} := -\frac{1}{2}\mathcal{C}_{r^*}^{(n)}\mathcal{C}_r^{(n)}.$$

*Then each $\mathcal{A}_{\Delta_n}$ is invariant under $G^{(n)}$. Furthermore,*

$$G^{(m)}|_{\mathcal{A}_{\Delta_p}} = G^{(p)}|_{\mathcal{A}_{\Delta_p}} = G|_{\mathcal{A}_{\Delta_p}} \text{ if } m \geq p, \tag{4.3}$$

*Proof.* By Lemma 4.1.1, we have $\mathcal{A}_{\Delta_n}$ invariant under $\bar{\mathcal{C}_r}$ and $\mathcal{C}_r^{(n)}$ and for $m \geq p$, the identity $\mathcal{C}_r|\mathcal{A}_{\Delta_p} = \mathcal{C}_r^{(m)}|\mathcal{A}_{\Delta_p} = \mathcal{C}_r^{(p)}|\mathcal{A}_{\Delta_p}$ holds. By Proposition 4.1.2, $\mathcal{C}_r^*(x) = \mathcal{C}_{r^*}(x)$, for all $x \in \mathcal{A}_{loc}$, we have $\mathcal{C}_r^*|\mathcal{A}_{\Delta_p} = \mathcal{C}_{r^*}^{(m)}|_{\mathcal{A}_{\Delta_p}} = \mathcal{C}_{r^*}^{(p)}|_{\mathcal{A}_{\Delta_p}}$ and hence result follows. □

**Proposition 4.1.4.** *The subspace $\mathcal{A}_{loc}$ is a core for the operator $G$.*

*Proof.* It is enough to prove that the subspace $\mathcal{A}_{loc}$ is invariant under the semigroup $\mathcal{S}_t$. For a vector $x \in \mathcal{A}_{loc}$, there exists $n \geq 1$, such that $x \in \mathcal{A}_{\Delta_n}$. Now by Proposition 4.3, for any $k \geq 0$, $G^k(x) = (G^{(n)})^k(x) \in \mathcal{A}_{\Delta_n}$ and it follows that the series $\sum_{k\geq 0} \frac{t^k G^k x}{k!}$ converges strongly in $\mathcal{A}_{\Delta_n}$. Therefore, we have, $\mathcal{S}_t x = \mathcal{S}_t^{(n)} x = e^{tG^{(n)}}$ for $x \in \mathcal{A}_{\Delta_n}$. Thus, $\mathcal{S}_t$ leaves $\mathcal{A}_{loc}$ invariant and by Nelson's theorem [28], the core property follows. □

## 4.2 HP Flow Associated to the Lindbladian

Formally, let us define $\mathcal{L}$ by, for $a \in \mathcal{B}(\mathbf{h}_0)$,

$$\mathcal{L}(a) = \mathcal{C}_r^* a \mathcal{C}_r - \frac{1}{2}(aG + G^* a).$$

By definition of $\mathcal{C}_r$, it is clear that $\mathcal{L}$ is not densely defined.

We consider the sesquilinear form, Lindbladian $\mathcal{L}(X)$, for $X \in \mathcal{B}(\mathbf{h}_0)$, with the domain $\mathcal{A}_{loc} \times \mathcal{A}_{loc} \subseteq Dom(G) \times Dom(G)$ given by

$$\langle u, \mathcal{L}(X)v\rangle \equiv \langle u, XGv\rangle + \langle Gu, Xv\rangle + \langle \bar{\mathcal{C}}_r u, X\bar{\mathcal{C}}_r v\rangle. \tag{4.4}$$

By definition of $G$, it is clear that $\langle u, \mathcal{L}(I)v\rangle = \langle u, Gv\rangle + \langle Gu, v\rangle + \langle \bar{\mathcal{C}}_r u, \bar{\mathcal{C}}_r v\rangle = 0$.

Though by Chebotarev's iterative method in the Theorem 3.1.5 we can construct the minimal QDS, but the conservativity does not follow. Alternatively, let us look at the HP type QSDE associated to the Lindbladian (4.4). Let $\mathcal{H} = L^2(\mathbb{R}_+, \mathbb{C})$ and set $\tilde{\mathcal{H}} = \mathbf{h}_0 \otimes \mathcal{H}$. Recall that $\Gamma(\mathcal{H})$ denotes the symmetric Fock space on $\mathcal{H}$ and $\mathcal{E}$ is the space of all the exponential vectors in $\Gamma(\mathcal{H})$. Let $\mathcal{A}_{loc} \otimes \mathcal{E}$ be the linear span of $\{x \otimes e(f) \,:\, x \in \mathcal{A}_{loc},\ f \in L^2(\mathbb{R}_+, \mathbb{C})\}$. Then the set $\mathcal{A}_{loc} \otimes \mathcal{E}$ is a dense subspace of $\tilde{\mathcal{H}}$. Here the noise space is of one dimension.

**Theorem 4.2.1.** *Consider the HP type QSDE in* $\mathcal{A}_{loc} \otimes \mathcal{E}$

$$U_t = I + \int_0^t U_s G ds + \int_0^t U_s \bar{\mathcal{C}}_r a^\dagger(ds) - \int_0^t U_s \mathcal{C}_r^* a(ds), \tag{4.5}$$

*where* $a^\dagger, a$ *are creation and annihilation processes respectively. The QSDE* (4.5) *admits a unitary solution* $U_t$. *Moreover, the expectation semigroup* $(\mathcal{T}_t)_{t\geq 0}$ *of the homomorphic cocycle* $J_t(X) = U_t^*(X \otimes I)U_t$ *is the unique (minimal) semigroup associated with the formal Lindbladian* $\mathcal{L}$ *in* (4.4) *and is conservative.*

*Proof.* Recall that the UHF algebra $\mathcal{A}$ can be approximated by finite dimensional algebras, namely $A_{\Delta_n} = \prod_{\|j\| \leq n} M_N(\mathbb{C})$ and $\mathcal{A}_{loc} = \bigcup_{n=0}^{\infty} \mathcal{A}_{\Delta_n}$. For $n \geq 0$, consider the following QSDE in $\mathcal{A}_{loc} \otimes \mathcal{E}$,

$$U_t^{(n)} = I + \int_0^t U_s^{(n)} G^{(n)} ds + \int_0^t U_s^{(n)} \mathcal{C}_r^{(n)} a^\dagger(ds) - \int_0^t U_s^{(n)} \mathcal{C}_r^{(n)*} a(ds). \tag{4.6}$$

By Theorem 3.2.15, the QSDE 4.6 admits a unitary solution $U_t^{(n)}$ on $\mathbf{h}_0 \otimes \Gamma(\mathcal{H})$.
We now show that the operators $U_t^{(n)}$ satisfy some compatibility condition, that is for $n \geq m$,

$$U_t^{(n)}|_{\mathcal{A}_{\Delta_m}} = U_t^{(m)}|_{\mathcal{A}_{\Delta_m}}. \tag{4.7}$$

Here the symbol $T|_{\mathcal{A}_{\Delta_m}}$ means the restriction of $T$ to the subspace $A_{\Delta_m} \otimes \Gamma_{sym}$.
Since these operators $\mathcal{C}_r^{(m)}, \mathcal{C}_r^{(m)*}$ and $G^{(m)}$ leave $\mathcal{A}_{\Delta_m}$ invariant, the restriction $U_t^{(m)}|_{\mathcal{A}_{\Delta_m}}$ satisfies the following QSDE in $A_{\Delta_m} \otimes \mathcal{E}$,

$$U_t^{(m)}|_{\mathcal{A}_{\Delta_m}} = I|_{\mathcal{A}_{\Delta_m}} + \int_0^t U_s^{(m)}|_{\mathcal{A}_{\Delta_m}} G^{(m)}|_{\mathcal{A}_{\Delta_m}} ds \tag{4.8}$$

$$+ \int_0^t U_s^{(m)}|_{\mathcal{A}_{\Delta_m}} \mathcal{C}_r^{(m)}|_{\mathcal{A}_{\Delta_m}} a^\dagger(ds) - \int_0^t U_s^{(m)}|_{\mathcal{A}_{\Delta_m}} \mathcal{C}_r^{(m)*}|_{\mathcal{A}_{\Delta_m}} a(ds).$$

For $n \geq m$, consider the QSDE in $A_{\Delta_m} \otimes \mathcal{E}$,

$$U_t^{(n)}|_{\mathcal{A}_{\Delta_m}} = I|_{\mathcal{A}_{\Delta_m}} + \int_0^t U_s^{(n)}|_{\mathcal{A}_{\Delta_m}} G^{(n)}|_{\mathcal{A}_{\Delta_m}} ds \tag{4.9}$$

$$+ \int_0^t U_s^{(n)}|_{\mathcal{A}_{\Delta_m}} \mathcal{C}_r^{(n)}|_{\mathcal{A}_{\Delta_m}} a^\dagger(ds) - \int_0^t U_s^{(n)}|_{\mathcal{A}_{\Delta_m}} \mathcal{C}_r^{(n)*}|_{\mathcal{A}_{\Delta_m}} a(ds).$$

With reference to Lemma 4.1.1, equation (4.3) and Theorem 3.2.15, the unitary processes $U_t^{(n)}|_{\mathcal{A}_{\Delta_m}}$ and $U_t^{(m)}|_{\mathcal{A}_{\Delta_m}}$ satisfy the same QSDE in $A_{\Delta_m} \otimes \mathcal{E}$. Therefore, by uniqueness of solution in Theorem 3.2.15, (4.7) follows.

Define $U_t$ on $\mathcal{A}_{loc} \otimes \mathcal{E}$ by setting

$$U_t(x \otimes e(f)) = U_t^{(n)}(x \otimes e(f)) \quad \text{if} \quad x \in \ A_{\Delta_n}$$

and extending linearly. Since the family $U_t^{(n)}$ satisfies the compatibility condition (4.7), $U_t$ is well defined on $\mathcal{A}_{loc} \otimes \mathcal{E}$, and for $x \in \ A_{\Delta_m}$ we have

$$U_t(x \otimes e(f)) = U_t^{(m)}(x \otimes e(f)) = U_t^{(n)}(x \otimes e(f)), \forall n \geq m. \tag{4.10}$$

Hence $U_t^{(n)}$ converges strongly to $U_t$ on $\mathcal{A}_{loc} \otimes \mathcal{E}$ and $U_t$ extends to a contraction operator on $\mathbf{h}_0 \otimes \Gamma_{sym}$. As $\mathcal{A}_{loc} \otimes \mathcal{E}$ is dense in $\mathbf{h}_0 \otimes \Gamma_{sym}$, (4.10) gives that $U_t^{(n)}$ converges strongly to $U_t$ on $\mathbf{h}_0 \otimes \Gamma_{sym}$ as well and the limit $U_t$ is an isometry.
For $U_t^{(n)}$, consider the dual process $\tilde{U_t}^{(n)} = (1 \otimes \Gamma(R_t))U_t^{(n)^*}(1 \otimes \Gamma(R_t))$. Then by Proposition 3.2.16, $\{\tilde{U_t}^{(n)}\}$ satisfies the following QSDE in $\mathcal{A}_{loc} \otimes \mathcal{E}$,

$$\tilde{U_t}^{(n)} = I + \int_0^t \tilde{U_s}^{(n)} G^{(n)^*} ds + \int_0^t \tilde{U_s}^{(n)} \mathcal{C}_r^{(n)^*} a(ds) - \int_0^t \tilde{U_s}^{(n)} \mathcal{C}_r^{(n)} a^\dagger(ds). \tag{4.11}$$

The equation (4.11) is identical to (4.6) except that $\mathcal{C}_r^{(n)}$ is replaced by $-\mathcal{C}_r^{(n)}$. So similar arguments yield that the operators $\tilde{U_t}^{(n)}$ also satisfy the compatibility condition and converge strongly to an isometry and because $\tilde{U_t}^{(n)}$ and $\Gamma(R_t)$ are unitaries, the sequence $U_t^{(n)^*}$ of unitaries converges strongly and thus it must converge to $U_t^*$. Hence $U_t^*$ is an isometry, so $U_t$ is a unitary process.
It remains to prove that $U_t$ satisfies the QSDE (4.5). As $U_t$ is a unitary process, the quantum stochastic integral on the right-hand side of (4.5) makes sense. Thus, it is enough to establish that integrals on the right-hand side of (4.6) converge to integrals in

(4.5). For $xe(f) \in \mathcal{A}_{loc} \otimes \mathcal{E}$, we have

$$\| \int_0^t (U_s^{(n)} G^{(n)} - U_s G) ds (xe(f)) \| \leq \int_0^t \| (U_s^{(n)} G^{(n)} - U_s G)(xe(f)) \| ds,$$

hence by (4.3) and (4.10), it converges to 0. By estimates of quantum stochastic integrals [29], we have

$$\begin{aligned} &\| \int_0^t (U_s^{(n)} \mathcal{C}_r^{(n)} - U_s \mathcal{C}_r) a^\dagger(ds)(xe(f)) \|^2 \\ &\qquad \leq 2e^{\int_0^t (1+|f(s)|^2) ds} \int_0^t \| (U_s^{(n)} \mathcal{C}_r^{(n)} - U_s \mathcal{C}_r) xe(f) \|^2 (1 + |f(s)|^2) ds. \end{aligned}$$

Therefore, by (4.1) and (4.10),

$$\lim_{n \to \infty} \| \int_0^t (U_s^{(n)} \mathcal{C}_r^{(n)} - U_s \mathcal{C}_r) a^\dagger(ds)(xe(f)) \|^2 = 0.$$

Convergence of annihilation term follows from a simpler estimate and using (4.2), (4.1) and (4.10). Thus $U_t$ is a unitary solution to the QSDE (4.5).

Now let us consider the expectation semigroup $(\mathcal{T}_t)_{t \geq 0}$ of the homomorphic co-cycle $J_t(\cdot) = U_t^*(\cdot \otimes I) U_t$. As $U_t$ is a unitary process, the QDS $(\mathcal{T}_t)_{t \geq 0}$ is conservative minimal semigroup associated with the form (4.4). □

## 4.3 EH Flow Associated to the Lindbladian

In this section, we deal with the structure maps on $\mathcal{A}_{loc}$ in the UHF algebra. For $W_k \in \mathcal{A}_{\partial \Delta_k}$, define the operators:

$$\delta_k(X) = [X, W_k], \quad \delta_k^\dagger(X) = (\delta_k(X^*))^* = [W_k^*, X],$$

for every $X \in \mathcal{A}_{loc}$. Consider the Lindbladian:

$$\mathcal{L}(X) = \frac{1}{2}\sum_{k=1}^{\infty}\{W_k^*\delta_k(X) + \delta_k^{\dagger}(X)W_k\}, \text{ for all } X \in \mathcal{A}_{loc}. \tag{4.12}$$

Though each component $W_k^*\delta_k(.) + \delta_k^{\dagger}(,)W_k$ are bounded maps, $\mathcal{L}$ is unbounded due to presence of infinitely many components (like in [27]). For $n \geq 1$, define a bounded map $\mathcal{L}^{(n)}(X) = \frac{1}{2}\sum_{k=1}^{n}\{W_k^*\delta_k(X)+\delta_k^{\dagger}(X)W_k\}$, for all $X \in \mathcal{A}$. Note that for $X \in \mathcal{A}_{\Delta_n}$, $\delta_k(X) = \delta_k^{\dagger}(X) = 0$ and $\mathcal{L}^{(k)}(X) = \mathcal{L}^{(n)}(X)$ for every $k \geq n$. The operators $\mathcal{L}^{(n)}$ are CCP, bounded and the associated uniformly continuous QDS $(\mathcal{T}_t)_{t\geq 0}$ leaves the space $\mathcal{A}_{\Delta_n}$ invariant.
As an operator $-\frac{1}{2}\sum_{k=1}^{\infty} W_k^* W_k$ may have a trivial domain or not a generator of a $C_0$-semigroup on $\mathcal{H}$. Thus the HP equation associated to (4.12) on $\tilde{\mathcal{H}}$,

$$U_t = I + \int_0^t U_s(-\frac{1}{2}\sum_{k=1}^{\infty} W_k^*W_k)ds + \sum_{k=1}^{\infty}\int_0^t U_sW_ka_k^{\dagger}(ds) - \sum_{k=1}^{\infty}\int_0^t U_sW_k^*a_k(ds)$$

may not make sense.
Let us consider the EH type quantum stochastic differential equation with structure maps $\mathcal{L}$, $\delta_k$ and $\delta_k^{\dagger}$. In the following result we show that EH flow exists and give rise to conservative QDS.

**Theorem 4.3.1.** *There exist a homomorphic co-cycle $J_t : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\Gamma(\mathcal{H}))$ satisfying the Evans-Hudson equation, for $X \in \mathcal{A}_{loc}$,*

$$J_t(X) = X\otimes I + \int_0^t J_s(\mathcal{L}(X))ds + \sum_{k=1}^{\infty}\int_0^t J_s(\delta_k(X))a_k^{\dagger}(ds) + \sum_{k=1}^{\infty}\int_0^t J_s(\delta_k^{\dagger}(X))a_k(ds). \tag{4.13}$$

*The expectation semigroup $(\mathcal{T}_t)_{t\geq 0}$ of the homomorphic co-cycle $J_t$ is conservative minimal semigroup associated with the Lindbladian form* (4.12)*.*

*Proof.* For each $n$, we look at the following EH equation:

$$J_t^{(n)}(X) = X \otimes I + \int_0^t J_s^{(n)}(\mathcal{L}^{(n)}(X))ds + \sum_{k=1}^{n} \int_0^t J_s^{(n)}(\delta_k(X))a_k^\dagger(ds).$$

$$+\sum_{k=1}^{n} \int_0^t J_s^{(n)}(\delta_k^\dagger(X))a_k(ds). \tag{4.14}$$

By Theorem 3.2.18, there exists a homomorphic solution $J_t^{(n)} : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\Gamma(\mathcal{H}))$ which satisfies the EH equation (4.3), for all $X \in \mathcal{A}$. When $X \in \mathcal{A}_{loc}$, there exist $m$ such that $x \in \mathcal{A}_{\Delta_m}$ and as in HP case we can see that

$$J_t^{(n)}(X) = J_t^{(m)}(X), \text{ for all } n \geq m. \tag{4.15}$$

Let us define

$$J_t(X) = \lim_{n\to\infty} J_t^{(n)}(X) \text{ for all } X \in \mathcal{A}_{loc}.$$

To prove that $J_t$ satisfies the EH equation (4.13), we observe that for every $X \in \mathcal{A}_{loc}$, the stochastic integrals on the right hand side of equation (4.3) converge to that of equation (4.13). Since for $X \in \mathcal{A}_{\Delta_m}, \delta_k(X) = 0$ for all $k \geq m$ implies $\delta_k(X) \in \mathcal{A}_{\Delta_m}$. Therefore we have,

$$\sum_{k=1}^{\infty} \int_0^t J_s(\delta_k(X))a_k^\dagger(ds) - \sum_{k=1}^{n} \int_0^t J_s^{(n)}(\delta_k(X))a_k^\dagger(ds)$$
$$= \sum_{k=1}^{m} \int_0^t J_s(\delta_k(X))a_k^\dagger(ds) - \sum_{k=1}^{m} \int_0^t J_s^{(n)}(\delta_k(X))a_k^\dagger(ds). \tag{4.16}$$

Since $\delta_k(X) \in \mathcal{A}_{\Delta_m}$, by (4.15) $J_s(\delta_k(X)) = J_s^{(n)}(\delta_k(X)) = J_s^{(m)}(\delta_k(X))$ for $n \geq m$. Thus for any $X \in \mathcal{A}_{loc}$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\int_0^t J_s\delta_k(X)a_k^\dagger(ds) - \sum_{k=1}^{n}\int_0^t J_s^{(n)}\delta_k(X)a_k^\dagger(ds)\right) = 0.$$

Similar argument gives the convergence of annihilation and $dt$ term. Therefore $J_t : \mathcal{A}_{loc} \to \mathcal{A}'' \otimes \mathcal{B}(\Gamma(\mathcal{H}))$ is the homomorphic solution of QSDE (4.13) which extends to the UHF $C^*$-algebra $\mathcal{A}$ by boundedness of $J_t$ on the dense $*$-algebra $\mathcal{A}_{loc}$.

Now let us consider the expectation semigroup $(\mathcal{T}_t)_{t\geq 0}$ of the homomorphic co-cycle $J_t$. The QDS $(\mathcal{T}_t)_{t\geq 0}$ is conservative semigroup associated with the form (4.12) in the sense that for all $u, v \in \mathbf{h}_0$

$$\langle u, \mathcal{T}_t(X)v\rangle = \langle u, Xv\rangle + \int_0^t \langle u, \mathcal{L}(\mathcal{T}_s(X))v\rangle ds.$$

□

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