ESTIMATING THE TEMPERATURE OF THERMAL QGP SYSTEMS

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A dissertation submitted for the partial fulfilment of BS-MS dual degree in Science

Under the guidence of

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April 2018

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Certificate of Examination

This is to certify that the dissertation titled **"Estimating the Temperature of Thermal QGP Systems"** submitted by **Anjaly S Menon** (Reg. No. MS13151) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: 20.04.2018

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Satyajit Jena at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Anjaly S Menon
(Candidate)

Dated: April 20, 2018

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Satyajit Jena
(Supervisor)

Acknowledgement

First and foremost I praise and thank the almighty for being the unfailing source of support, comfort and strength throughout the completion of my project work.

I would like to thank Prof. Debi Prasad Sarkar, Director of Indian Institute of Science Education and Research Mohali and Prof. Jasjeet Singh Bagla, Head of the Department of Physical Sciences, for the infrastructure, library facilities, research facilities and for exposing myself to the working environment of a full–fledged research institute. I would like to also thank Prof. N. Sathyamurthy, former Director, Indian Institute of Science Education and Research Mohali.

It's an honour and proud privilege in expressing my deepest and sincere gratitude to Dr. Satyajit Jena, Assistant Professor, Department of Physical sciences, Indian Institute of Science Education and Research Mohali, for his kindness and support, for accepting my request to work under his guidance, for enabling me in the pursuit of my career.

I express my warm appreciation and most respectful regards to Prof. Charanjit Singh Aulakh and Dr. Ketan Patel, my thesis committee members for their valuable suggestions and comments during the committee meetings.

I am so obliged and extremely thankful to Rohit Gupta, for patiently teaching me everything from basics. He helped me throughout this project by providing me with necessary intellectual input in most needful times. Without his help and effort, this project work would be incomplete.

I would like to thank my lab members, Anjali Krishnan, Nishat Fiza, Tasha Gautam, Shahina Ali, Akhil Bharadwaj, Asish Moharana, Shubham Varma, Neeraj Maan and Kartik Joshi for making the lab lively and lending a hand at times when I needed.

I express my enormous love and affection to my family members, for their unconditional love and support which I always received. Finally, I would like to express my love and care to my friends especially Arathy, Neelima and my classmates for their unselfish, loyal, true benevolent concern for me.

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Contents

List of Figures

Abstract

Ultra-relativistic heavy ion collisions at RHIC (Relativistic Heavy Ion Collider) and LHC (Large Hadron Collider) has made it possible to study the state of matter produced under extreme conditions of temperature and energy density known as Quark Gluon Plasma (QGP). To explore and investigate the possibility of formation of this new deconfined state of only quarks and gluons, tools of Statistical thermodynamics can be used. We focus on the study of transverse momentum distributions which has proven to be an effective probe for understanding the properties of systems produced in relativistic heavy ion collisions. Our objective is to find an accurate distribution function to approximate the identified particle spectra. Since no function exactly describes the transverse momentum spectra, finding a correct distribution function to describe the spectra is of great interest to present day particle physics community. The classical description of high energy collisions uses statistical models that are based on Boltzmann-Gibbs distribution (BG). In spite of its great success BG statistical mechanics is not completely universal. A class of physical ensembles involving long-range interactions and long-time memories can hardly be treated within this classical framework. Such systems are analyzed using a generalized non-extensive statistical thermodynamics known as Tsallis statistics. We will be examining both Boltzmann-Gibbs and Tsallis statistical approaches in detail. An extension of Tsallis statistics applicable in high energy physics is explored here in order to study the phase space of particles produced from both soft as well as hard scattering processes occurring in heavy ion collisions using a generalized Tsallis distribution. We carried out analysis for an invariant yield of pions. ROOT, data analysis framework has been used along with MINUIT class for fitting. Results have shown that this new generalized approach gives a successful explanation in a consistent way as compared to earlier approaches. Fit details including the values of temperature and other relevant parameters are also given. Also, the connection of new parameters to physics is explored, and generalized thermodynamics for relativistic particles is derived.

Chapter 1

Introduction

Particle physics is all about studying the nature of fundamental constituents of matter that makes up this universe. Our knowledge of these constituents is necessary to understand the laws that shape this universe. In 20^{th} century, particle physicists formulated a series of equations that describe all fundamental forces except gravitational force and classify entire known elementary particles in a consistent way, that is known as Standard Model.

This model makes several predictions based on the theoretical calculations and high energy experiments occurring at particle accelerators all over the world, aimed at confirming these predictions. It is well known that several physics phenomena have been explained successfully by this model mutually working with the experimental data from detectors and also, there are evidences showing that the standard model built is in excellent accord with almost all current data.[GGS99],[Nov00]

Even though standard model is currently the best description, it does not explain the complete picture. What happened to antimatter after the Big Bang?, What is dark matter? are some of the questions which standard model is unable to explain. The reasons for considering it as an approximation to the actual story and, the hints which indicate that it fails at arbitrarily short distances are presented in [GGS99]. Physicists are looking for answers to the above-mentioned questions, and several other phenomena need to be investigated. One of the important questions physicists concerned about is how did the entire universe come into existence.

Big Bang theory gave an explanation that all matter formed in an explosive event 13.7 million years ago and the universe began as a fireball with extremely high temperature and energy density. The observational evidences for Big Bang and the profound discovery of accelerating the expansion of universe are discussed in [AP12],[Mac04]. Next question that caught the attention of scientists was what kind of matter existed at those earliest moments of universe.

Theoretical calculations based on Lattice QCD (Quantum chromodynamics) framework, first predicted a new state called Quark Gluon Plasma (QGP) to exist at these sufficiently higher temperatures or baryon densities. Later, heavy ion collision experiments performed at RHIC made it possible to reach energy densities above critical values predicted by Lattice QCD, for the establishment of QGP and, thus the droplets of the matter that filled the universe microseconds after the Big Bang was recreated in the laboratories[JW05, GVWZ04, KH04a]. A brief discussion on the basic understanding of QCD and the formation of QGP will be provided in coming chapters.

Considering QGP to be a thermally equilibrated state in which quarks and gluons are deconfined from hadrons and are free to move over nuclear, rather than nucleonic volumes, tools from statistical thermodynamics can be used to compute its thermodynamic properties [Ris04]. Review of processes that drives to the thermalization of QGP and evidence for the creation of thermal QGP systems is done in [Str13].

We focus on the study of transverse momentum distributions which has proven to be an effective probe for understanding the properties of systems produced in relativistic heavy ion collisions. Our objective is to find a distribution function to approximate the published data on transverse momentum spectra of identified particles in collider experiments. Many attempts were done in the past, and the results were not satisfying. We begin with a short review of developments happened on this path of finding a distribution that can explain particle spectra with good accuracy and, later a solution to the problem is proposed using a generalized form of Tsallis distribution, called Pearson distribution.

The classical description of high energy collisions uses statistical models that are based on

Boltzmann-Gibbs distribution (BG). In spite of its great success BG statistical mechanics is not completely universal. A class of physical ensembles involving long-range interactions and long-time memories can hardly be treated within this classical framework.

Such systems are analyzed using a generalized non-extensive statistical thermodynamics known as Tsallis statistics. In this thesis work, We will be examining both Boltzmann-Gibbs and Tsallis statistical approaches in detail. The domains in which BG and Tsallis formalism are valid, and the rationale behind the need for a generalized Tsallis statistical mechanics will also be discussed.

For analysis, we have taken the data of invariant yield of π^- produced in Au-Au and Pb-Pb collisions at several collision energies and the temperature corresponding to the thermal system produced, is extracted from the fit details. This procedure is performed for BG, Tsallis and Pearson distributions such that a comparison can be made. The accuracy of fits is determined by examining the Chi-square values associated with each fit.

A derivation which shows that the Pearson distribution is a generalization to both BG and Tsallis distribution is also presented. Our expected behavior is that this new approach using Pearson density will give better fits and successful physical explanation in a consistent way as compared to earlier approaches. There are parameters other than temperature in the Pearson distribution, for which the physical connection is to be extracted.

The outline of this thesis is following. In chapter 2, we will introduce some basics about Quantum Chromodynamics. Chapter 3 starts with the discussion of why do we study heavy ion collisions. The theoretical background behind the formation of QGP is also given. Further, review of developments happened in the area of QGP research is done. In chapter 4, we will be presenting the idea of using statistical tools to analyze the transverse momentum data and the motivation for doing this. The earlier approaches which were used for describing the particle spectra are also explained in this chapter. These approaches include Boltzmann-Gibbs and Tsallis statistical framework.

It is known that till now no function could approximate the particle spectra without any deviation. Various drawbacks and difficulties of above-mentioned formalism will be discussed in chapter 4. Then, we will introduce our proposal to the problem using a generalized distribution called Pearson distribution. The mathematical formulation of this function along with the derivation which allows us to write Pearson function as an extended form of Tsallis distribution is also presented. Chapter 5 will include the plots, where comparison of BG, Tsallis and Pearson statistics in the context of describing transverse momentum spectra of identified particles is performed. The fit details including the *chi-square* values and extracted parameters will also be given in this chapter.

Chapter 2

Quantum Chromodynamics

For several years, the question of why an atomic nucleus sticks together without falling apart remained unanswered by physicists. However, it was known before the 1970s itself that the nucleus is composed of protons and neutrons and that the protons are positively charged, and neutrons are electrically neutral. The framework of electromagnetic interactions was understood completely by that time, which suggested that the protons would repel each other when they are close and, the nucleus would fly apart due to this repulsive interaction. But this was never observed, and a new physics phenomenon was needed to explain this unusual stability of atomic nuclei. This new force that is stronger than electric repulsion among protons and, binds the nuclei together is known as strong nuclear force or color force.

The modern quantum theory that describes this strong nuclear interactions is Quantum Chromodynamics (QCD). It is a non-abelian SU(3) gauge theory with color charges as the generators of the gauge group. In discussing the basic properties of QCD, the most important role is played by the QCD Lagrangian since it gives the complete description of the theory. The expression for QCD Lagrangian is given in 2.1, Here m_j and q_j are the mass and quantum field of the quark of j^{th} flavor, and A is the gluon field, with spacetime indices μ and ν and color indices a, b, c. The numerical coefficients f and t guarantee SU(3) color symmetry. Aside from the quark masses, the one coupling constant g is the only free parameter of the theory.

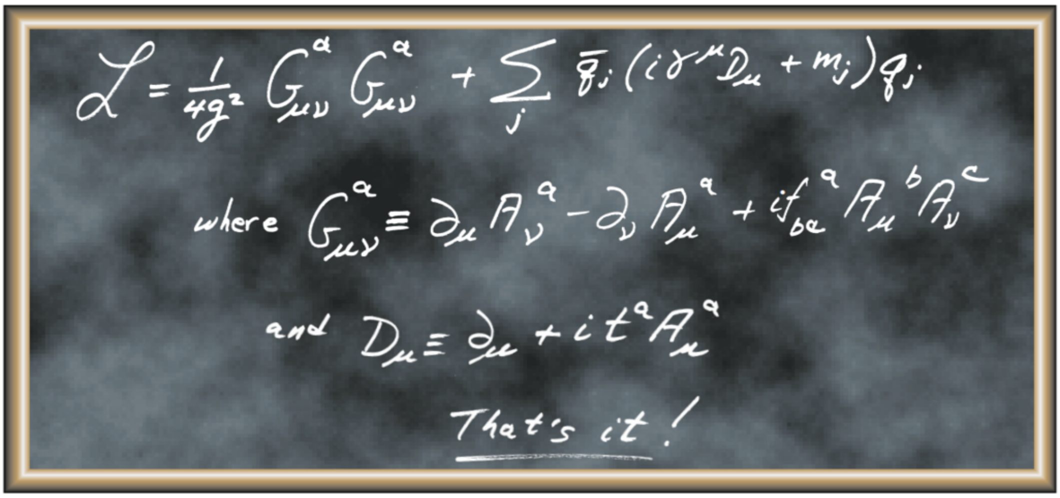


Figure 2.1: This picture is taken from the paper *QCD Made Simple* written by *Frank Wilczek*

2.1 Quarks and QCD

When probed at short wavelengths that are million times smaller than that of an atom, strong force can overcome all other fundamental interactions such as gravitational, weak and electromagnetic forces. However, it affects only at short distances and can be observed at two distance scales. On a larger scale, it is the force that binds the neutrons and protons together inside a nucleus. If we try to understand it on a small scale, the strong force will act as a binding force between quarks, carried by gluons, to form protons and neutrons and other hadron particles.

Since protons and neutrons with which nucleus is made of, are themselves combinations of quarks and are elementary particles, these are considered as fundamental objects in strong interactions. They come in 6 different varieties or flavors as up, down, strange, charmed, bottom, and top (u, d, s, c, b, t). Among these, only up and down quarks play a remarkable role in the structure of ordinary matter that we see in the universe. This is because of the finding that the protons and neutrons are composed of up and down quarks.

Just like electrons, quarks are also spin-1/2 particles without structure according to the understanding till now. But instead of electrical charge, they carry color charge. In addition to this color charge, they carry fractional electrical charge also ($+2e/3$ for the u, c, and t quarks, and $-e/3$ for the d, s, and b quarks). It has been observed experimentally that the quarks never exist in isolation. There are two reasons for this finding.

Firstly, this can be explained on the basis that, in nature, only integral charged particles are found to be isolated, not fractionally charged ones.

Another explanation comes from the concept of color confinement in QCD. Due to this, particles that are physically observed in nature are formed by the combinations of quarks. And, these combinations are made in such a way that all the particles show color neutrality. In another way, we can say that only color-singlet states can propagate over macroscopic distances and, the only stable color singlets are quark-anti-quark pairs, which are mesons, and three-quark states, called baryons.

2.2 Gluons and QCD

The next class of fundamental particles in QCD, which act as mediators in strong interactions are gluons. As we said earlier, quarks do not interact among each other; their interaction is mediated by gluon particles. Gluons act as a carrier in strong interaction because of their color charge. In contrast to electric charge, a scalar quantity in QED, the color charge is a quantum vector charge which follows the addition law of vectors just like angular momentum in quantum mechanics.

There are three basic color charge states which form a basis in three-dimensional complex vector space. Although the masses of different flavors of quarks are different, the QCD theory shows complete symmetry with respect to these three colors and this color symmetry is described by the Lie group SU(3).

Gluons are spin one, massless particles like photons in QED, and respond appropriately to the presence of color charges. Even though there is only one photon, there are eight type of gluons in QCD, which can be understood by the SU(3) color symmetry. Also, It has been observed that the quarks can change their color state to another under absorption and emission of quarks. So, gluons must also carry themselves a color charge. This is in complete contrast with QED where photons are electrically neutral. The consequence of this feature is that gluons show self-interactions.

2.3 Asymptotic Freedom

Now, let us understand how the strength of strong force between the quarks varies with the distance between them. It is known that the strength of interaction in QCD is given by its coupling constant, which is defined as follows.

$$\alpha_s = \frac{g^2}{4\pi} \text{ Where, g is the Coupling constant} \tag{2.1}$$

This is analogous to the fine structure constant in quantum electrodynamics. One of the remarkable characteristics of QCD that describes above fact is known as Asymptotic freedom. This special feature of QCD was discovered in 1973 and was a breakthrough for the theory of strong interactions. It states that the interaction of quarks decrease as the quarks come close to each other. The converse is also true when the quarks move apart; the interactions become stronger when the distance increases.

In other words, we can say that the coupling strength between quarks and gluons depends on the energy of the particles when they encounter for an interaction. The higher the energy transfer between them, the lower the coupling strength and hence the weaker the force. Due to this phenomenon, QCD becomes a free theory at large scales.

There are experimental evidences for this behavior of hard gluons experiencing weak interaction whereas soft gluons with strong interaction. This property of asymptotic freedom has been well verified experimentally, and the summary of it is given in [HM00]. In heavy ion collisions, soft hadrons are those which participate in interactions involving little momentum transfer, and on the other hand, particles produced with high transverse momentum and energy are called hard hadrons.

On the other hand, at an increasing distance, the coupling between quarks becomes so strong that it is impossible to isolate a quark from a hadron. This property is called color confinement, due to which the color charged particles like quarks and gluons could never be observed as free and they have to assemble together to form hadrons to be directly observed. This has been verified in Lattice QCD calculations but, not proven mathematically from first principles because of the coupling constant that makes it non-perturbative.

This is clearly visible if we take a look at the form of QCD potential. Then, it is the effect of the second term in the expression which is linear in r, makes the interaction potential to increase with distance, at large distances. A comparison with QED will indicate that the potential will decrease with increase in distance since there is only inverse term in the QED potential. It is also evident that the difference between QED and QCD is of importance only at short distances. Interaction potentials in QCD and QED are related to distances in the following way.

$$V(r) \approx \frac{\alpha}{r} + \sigma r \tag{2.2}$$

$$V(r) \approx \frac{-e^2}{r} \tag{2.3}$$

The description of asymptotic freedom given above is the interpretation of the mathematical results that were obtained by the renormalization procedure of QCD. The relation which shows the dependence of coupling constant up to some constant (α_s) on the exchanged four-momentum (Q) between the interacting partons is;

$$\alpha_s(|Q^2|) = \frac{12\pi}{(11n - 2f)\ln(Q^2/\Lambda^2)} \tag{2.4}$$

Where, $|Q^2|$ is the square of the exchanged four-momentum (energy scale), n is the number of colors in QCD (equal to 3), f is the number of quark flavours (equal to 6) and $\Lambda_{QCD} \sim 300MeV/c$ is the QCD scale parameter, a constant calculated from experimental data. A key property of QCD is that $11n - 2f > 0$. As a consequence, α_s decreases with increasing energy scale (decreasing distance), or asymptotic freedom is followed .

Current understanding of asymptotic freedom is displayed in Fig2.2. It shows the agreement between theoretical predictions and the experimental data measured in collider experiments.

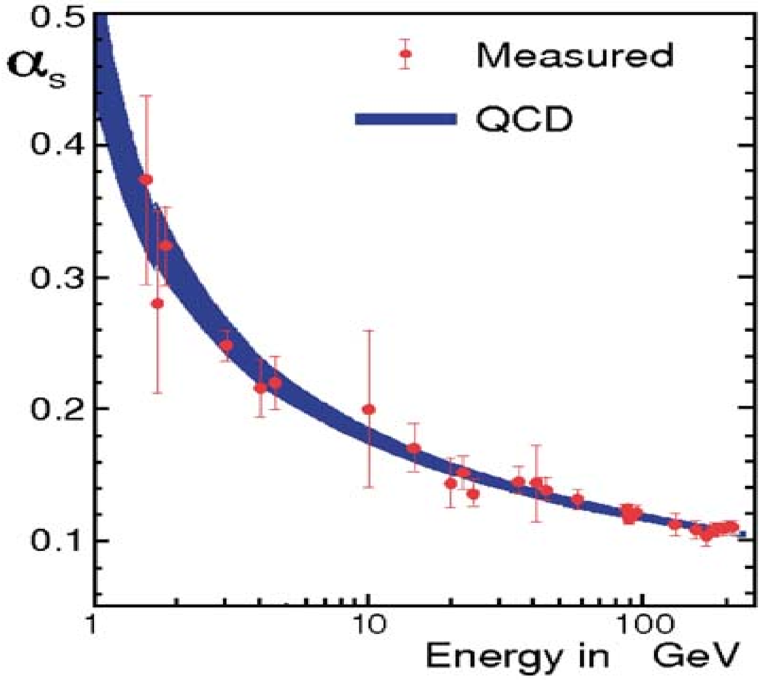


Figure 2.2: The value of QCD coupling constant (α_s) as a function of the energy Q is shown in the figure. The solid line represents the theoretical prediction and dotted one gives the experimentally measured data.
Reference: http://inside.hlrs.de/_old/htm/Edition_01_12/article_14.html

Chapter 3

QGP and Dynamics of Heavy Ion Collisions

3.1 Why do we study HIC?

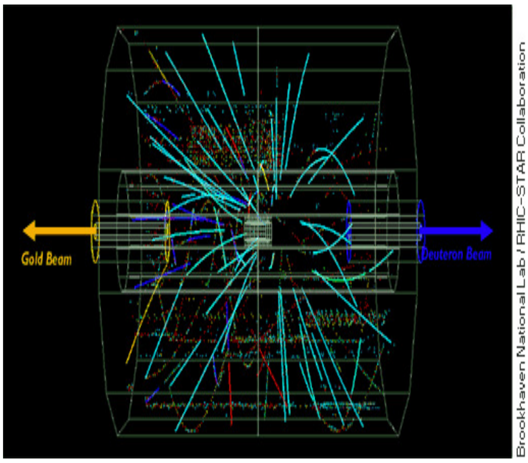
The history of heavy ion collision dates back to 1986 when CERN began to accelerate large nuclei containing many neutrons and protons in the Super Proton Synchrotron (SPS). At CERN, particles are made to collide at velocities close to the speed of light by making use of largest and extremely complex scientific instruments such as particle accelerators and detectors. These accelerators push the beams of particles to higher energies, and subsequently, these beams are collided with each other or with targets which are at rest. The detectors are then used to record the results from these collisions.

Over the past few decades, the domain of relativistic heavy ion collisions has been examined in detail on both theoretical and experimental side. And, it is understood from its descriptions that the collisions of nuclei are difficult to comprehend, with distinct stages that investigate different aspects of the theoretical framework, QCD. Therefore it is important to find answers to the following questions: why do we study heavy ion collisions? What are the new insights that we are looking for? What are the new physics phenomena that we have explored till now?

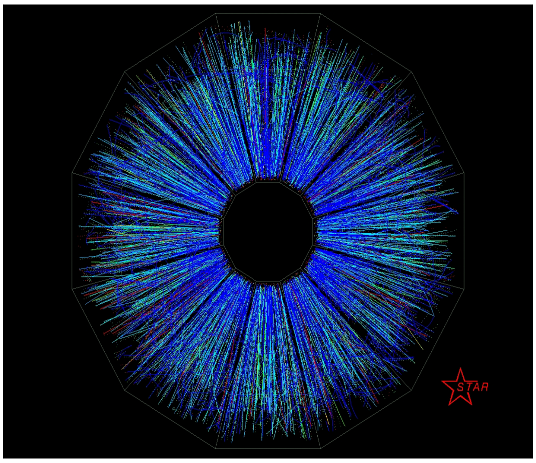
This chapter is devoted to the answers to above questions and, further the theoretical framework of QGP formation and review of recent works to understand it will also be discussed.

The intrinsic purpose behind starting this new era of heavy ion collision experiments is to study the basic building blocks of matter, the fundamental particles that constitute this universe. In collider experiments, one important characteristic we need to note is the large amount of energy that is involved. Such large fraction of the energy, when deposited in a small space, for a short duration of time, energy density can be very high.

There are various experiments all over the world that span large range of nucleus-nucleus center of mass energies, like AGS accelerator at BNL which uses 5 GeV, the CERN heavy ion beams at the SPS accelerator with collisions performed at 17 GeV and there is Relativistic Heavy Ion Collider (RHIC) at BNL, where energies are in the order of 200 GeV.[Pai09]. An image of the collision between gold and deuteron beams is displayed in Fig 3.1a. Also, the RHIC view of colliding Gold nuclei in the STAR experiment, that shows the appearance of droplets of quark-gluon plasma is given in Fig 3.1b.



(a) Collision between deuteron and gold nuclei



(b) RHIC View of gold nuclei collision as captured by the Solenoidal Tracker At RHIC.

Figure 3.1: Collisions of Gold nuclei in the STAR experiment at RHIC, creating a fireball in which the quark-gluon plasma briefly appears. Its properties are reconstructed from particle tracks captured in STAR's Time Projection Chamber. https://phys.org/news/2010-01-jetting-quark-gluon-plasma.html

Let us consider RHIC, where the collisions primarily involve gold nuclei, one of the heaviest common elements present. This was the first machine in the world to collide heavy ions at such ultra-relativistic energies, and the expectation was that the nuclear matter would melt at these energies to form a new state called Quark Gluon Plasma (QGP). Creating droplets of this state and studying phase transition of nuclear matter to this new asymptotically free state, under the conditions of very high temperature and density was the foremost aim behind constructing RHIC at BNL and LHC CERN. [STA05] And, there are numerous experimental and theoretical results, which provide evidence for the actual production of QGP during these collisions. [JW05],[GVWZ04],[KH04a],[Tan08],[LSZ04, BRv18]

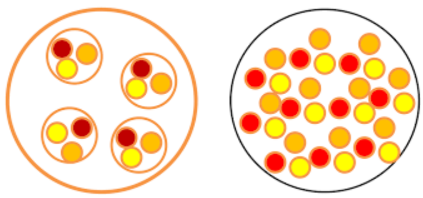
The theoretical understanding of this new state is obtained by Lattice QCD calculations whereas experimental approach involves the use of heavy ion collision experiments and analysis of its data. Before QGP was produced in collider experiments, lattice QCD calculations predicted the existence of this state and even the estimate of critical energy densities required for its establishment was also present. Then, significant advances in modern technology made it possible to collide nuclei each other and reach those conditions required for forming this new state of matter in laboratories.

3.2 Formation of QGP

Understanding QGP is of great interest to both cosmologists and particle physicists and is a rapidly growing field nowadays even with the challenges involved in its studies. This is mainly because QGP studies can give insights into the primordial matter that existed during the earliest moments of the universe.

It has been speculated by Big Bang theory that the universe began as a fireball with extremely high temperature and density, and the ordinary matter existed that time was not made of protons and neutrons, but was in a plasma state of only quarks and gluons. Therefore, by producing QGP, heavy ion experiments were recreating a small Big Bang in an artificial environment. Just like QGP is proposed to exist in the early universe, It is also possible that it exists at the core of neutron stars. In neutron stars, as compared to heavy ion collisions, the temperature is very less, and the density is extremely high.

Figure 3.2: A nucleus at normal (left) and high density (right).



Let us now try to understand the conceptual basis for the QGP formation. We know that traditionally nuclear physics is the study of nuclear matter at zero temperature and densities of the order of atomic nuclei or energy densities as $\epsilon \approx 0.16 GeV/fm^3$. It is the advent of accelerators which enabled us to investigate the matter at temperatures and densities several orders of magnitude higher. At these extreme conditions, individual hadrons lose their identity, and the matter is best described in terms of only quarks and gluons.

What happens to the hadronic system upon increasing its density is schematically depicted in Fig 3.2. At low density, one particular quark in a hadron knows its partner quark. Whereas as density increases, one quark would be unable to identify the quark which was its partner at low density, due to the interpenetration of hadrons. This can happen at high temperature also.

As the temperature increases or the energy given to the system increases, more hadrons will be created, and further, the increase in density will lead to more interpenetration. At some point in time, the interaction between quarks within a single hadron and that between quarks in different hadrons will be of equal strength. Then, the boundary that separates quarks in different hadrons will vanish, i.e., the system no longer has the hadronic identity. Only quarks will be there along with gluons, which is the phase of matter called QGP. Model calculations have shown that this transition happens beyond a critical energy density, $\epsilon_c \approx 1 GeV/fm^3$, or temperature $T_c \approx 200 MeV$.

In the hadronic state, the quarks are confined inside the hadrons whereas, in the QGP state, quarks are free to move over a nuclear volume rather than a nucleonic volume and are no longer confined. Therefore, we can think of this as a confinement to deconfinement transition. At low temperature, nuclear matter stays in confined state, and at high temperature, it goes to QGP state, which is the deconfined state of strongly interacting nuclear matter. But, this can not be considered as a thermodynamic transition, in which the energy or its derivatives must have a singularity at the transition point. In [Cha12], this transition is regarded as an analog of Mott transition in atomic physics.

In Mott transition, when Coloumb potential that holds electron and ion together inside neutral atom gets screened by other charges, the potential becomes short ranged. Then, at very high density, an electron can no longer feel the binding force of ion and will be free. Thus the normal insulating matter becomes conducting at higher densities. Quantum chromodynamic analogue of above transition is deconfinement of quarks, where quarks cannot bind to hadrons due to the screening of color potential.

3.3 Phases of QCD

It is known that nuclear matter can take different phases at different temperatures and densities. These various phases and phase transitions between them can be displayed in phase diagrams. Exploring this phase diagram and identifying several phases of matter in the phase space is one of the main challenges of nuclear and high energy physics and continue to be intensively studied both theoretically and experimentally.

A sketch of the QCD phase diagram is given in Fig 3.3b. Essentially there are three phases in the figure. Firstly, there is the hadron gas phase at low temperature and density, where both the vacuum in which we live in and normal nuclear matter are located. Secondly, the quark-gluon plasma phase is represented at high temperature (Cabibbo and Parisi 1975), which we described above in detail. Thirdly, the color superconductor phase is expected to appear at high density and low temperature, where quarks are believed to form Cooper pairs, leading to color superconductivity (Barrois 1977; Bailin and Love 1984).[Gub]

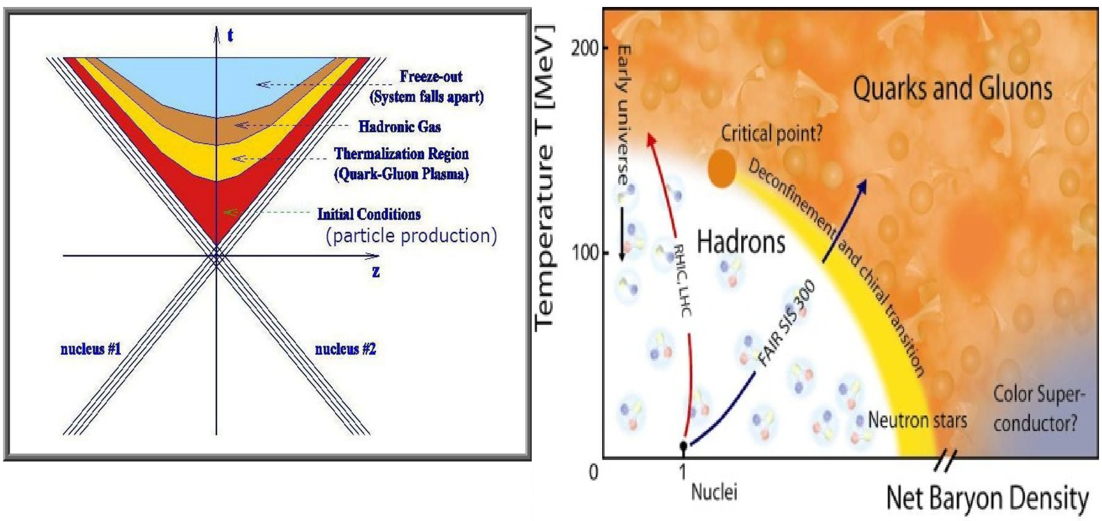
The important point to be noted is that many features of this phase diagram are not understood till now. Specifically, points of high chemical potential and low temperature where the three phases will meet. This is because perturbative theory breaks down here. The most interesting is the region of transition between QGP and hadron gas, which has been investigated in past years. As we described earlier, large energy densities in the nuclear collisions allow exploring matter at extreme conditions. Therefore, one of the major goals of studying heavy ion collisions is to understand the physical aspects of this phase diagram and its connection to the early universe.

3.4 Space Time Evolution of Matter

We have already discussed some of the objectives of collider experiments. In order to fulfill these, it is important to understand the time sequence of processes in an expanding strongly interacting hadronic systems. Ultra relativistic nucleus-nucleus collisions have recreated the environment by colliding heavy ions and the Fig 3.3a depicts the collision of two nuclei in (t, z) plane. Two Lorentz contracted nuclei approach each other with a velocity of light and collide at (t=0,z=0).

Figure 3.3

(a) Space time picture (b) Phase transitons



A nucleus-nucleus collision at relativistic energy passes through different stages. A schematic picture of various distinct stages are shown in Fig 3.4 . A fireball is created in the collision process. The produced fireball has such a high density and temperature that apparently all partons (quarks and gluons) reach equilibrium very rapidly (over a time scale of less than 1 fm/c). After the particle production occurs, QGP phase starts where equilibrium is achieved locally and thermally since no new particles are being produced here.

The basic steps of the evolution of a heavy ion collision can be described with a single sentence: at first collision of Lorentz contracted heavy ions occur, that leads to a pre-equilibrium phase, and is followed by QGP phase which expands and cools until hadronization occurs, and eventually long-lived particles arrive the detectors.

3.4.1 Pre-equilibrium Stage

This is the first stage in the evolution of heavy ion collision where a fireball is created in a highly excited state. The fireball is not in equilibrium and its constituents collide repeatedly to establish a state of local equilibrium. An important parameter here is the thermalization time. It is defined as the time taken by the system to establish local equilibrium.

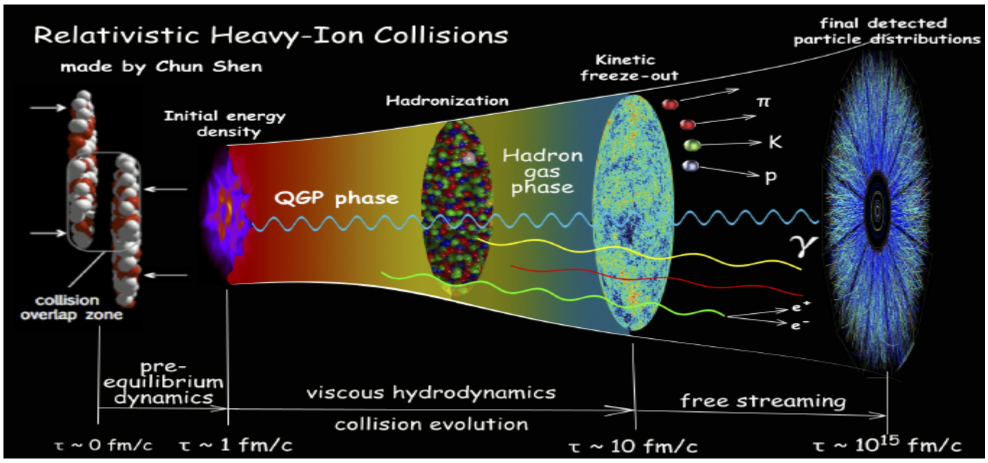


Figure 3.4: Evolution of strongly interacting system produced in heavy ion collisions

3.4.2 Expansion Stage followed by Hadronization

In the equilibrium or the thermalized state, the system has a thermal pressure which acts against the surrounding vacuum. Then the system undergoes collective (hydrodynamic) expansion. As a result of this expansion, density (energy density) of the system decreases, and subsequently, it will cool. Consequently, when the density and temperature become smaller than the critical density and critical temperature that has been predicted from theory, a phase transition occurs. This can itself be considered as a single stage where the partons (quarks and gluons) will convert to hadrons. This is called hadronization.

Sudden decrease in system entropy over a small temperature interval is a special feature of this stage. Since a decrease in total entropy is not plausible, it implies that the fireball will expand rapidly, while the temperature remains approximately constant. If the transition is

1st order, there will be a mixed phase, where QGP and hadronic resonance gas exist together. In 2nd or cross-over, there would not be mixed phase. Eventually, all the partonic matter will be converted into hadronic matter.

3.4.3 Freeze-out

As QGP, the hadronic matter will also be in thermal equilibrium. Constituent hadrons will collide each other to maintain local equilibrium. The system will continue to expand and cool. At some point, chemical equilibrium will stop, and an era of binary collision will begin, and particles will be freezed out finally. There are two types of freeze out-chemical and kinetic freeze out.

A stage will come when inelastic collisions, in which hadron changes identity, become too small to keep up with expansion. This stage is called chemical freeze-out. Hadron abundances will remain fixed after the chemical freeze-out. However, local equilibrium can still be maintained due to elastic collisions, and the system will cool and expand with fixed hadron abundances.

Eventually a stage will arrive when average distance between the constituents will be larger than their strong interaction range. Collisions between the constituents will be so infrequent that local thermal equilibrium can not be maintained. At this point the hydrodynamic description will break down. The hadrons decouple or freeze-out. It is called kinetic freeze-out. The hadrons detected in the detector are those from the freeze-out surface.

Let us discuss the collision process with an increase in energy. In very low energy collisions, nucleus as a whole interact. As we increase energy, nucleons inside the nucleus start to interact, and new particles will be produced. Quarks inside the nucleons will begin to interact on further increase in collision energy. Here also we can observe the production of new particle species.

Chapter 4

Statistical Approach to Thermal QGP Systems

4.1 QGP and Hadronic Gas in Ideal Limit

We described asymptotic freedom in the previous chapter as, the decrease in strength of strong nuclear interactions with the momentum transfer or the energy involved in the process. Therefore, this simplifies high-temperature QCD in such a way that, at large energies, we can assume the system to be a gas of free quarks and gluons, with very weak interactions with each other. This assumption is actually a valid starting point, and the reasoning behind this approximation is discussed in [Wil00].

One important consequence of these quarks interacting weakly when probed at large enough energies is that it allows using perturbative QCD to understand systems at these ranges. Then the problems arise for examining systems at smaller energies, where the perturbation theory will fail due to the higher value of coupling constant. Therefore, we need to use other techniques.

Application of statistical thermal models to the experimental measurements is one of the methods used since the 1950s. In this, QGP is described using statistical mechanics as a free relativistic Parton gas, which is the simplest system of strongly interacting particles in QCD. This idea was formulated by Heinz Koppe in 1948, and subsequently, the generalization was done by Enrico Fermi.[Fer50]

They implemented approaches using Fermi-Dirac or Bose-Einstein or Stefan-Boltzmann statistics to investigate particle production, formation and decay of resonances and temporal and thermal evolution of the interacting systems in thermal medium and under equilibrium in the final state of interaction. Further, Koppe estimated the equilibrium concentrations of each type of the produced particles. Fermi model is a reliable description in energy ranges comparable to that of cosmic rays only. It breaks down at lower energies. Later, several models were proposed, and Hagedorn was the first who systematically analyzed high-energy phenomena using all tools of statistical physics.[Taw14],[Hag65],[NUWW04]

While explaining the formation of QGP in the last section, we discussed that the quarks and gluons inside hadrons and those in QGP state as produced in collider experiments shows completely different properties. The basic difference is that the former, as opposed to the later, can be treated as a macroscopic system.

We know that a macroscopic system is generally characterized by state variables like number density (n), pressure (p), energy density (ϵ), temperature (T), a chemical potential (μ), etc. The equilibrium thermodynamic properties of hadronic systems obtained using statistical models can also be characterized by thermodynamic parameters that are mentioned above. This will finally give insights about the dynamics of the systems in terms of these state variables.

4.2 Relativistic Kinetic Theory

In this means of statistical description, we will use certain aspects of relativistic kinetic theory. In kinetic theory, a macroscopic system is generally studied in terms of the distribution function $f(x, p)$. $f(x, p)d^3xd^3p$ is the average number of particles in small volume d^3x, with momenta between p and $p + dp$.

The particle content in the volume element is large enough to apply the concepts of statistical physics. Most of the discussion is based on [SRGW]. Imagine a system of large number of relativistic particles with mass m, momenta **p** and energies cp^0 where, c is the speed of light and

$$p^0 = \sqrt{p^2 + m^2c^2}$$

All the macroscopic quantities required for the thermodynamic description of the system can be derived by making use of the distribution function introduced above. The particle four flow is defined as the 1^{st} moment of the distribution function.

$$N^{\mu}(x) = \int d^3p \frac{p^0}{p^{\mu}} f(x,p) \tag{4.1}$$

The time and space $(\mu = 1, 2, 3)$ components of this four flow will give particle density and particle flow respectively. They are given as follows where, particle flow involves a new quantity, velocity defined as

$$u = \frac{\vec{p}}{\vec{p}^0} \tag{4.2}$$

$$N^0(x) = \int d^3p f(x,p) \tag{4.3}$$

$$N^i(x) = \int d^3p \frac{p^i}{p^0} f(x,p) \tag{4.4}$$

$$= \int d^3p u^i f(x,p), i = 1, 2, 3 \tag{4.5}$$

The 2^{nd} moment of the distribution function is a tensor called energy-momentum tensor. The definition of this tensor is given in Eqn4.6.

$$T^{\mu\nu}(x) = \int \frac{d^3p}{p^0} p^{\mu} p^{\nu} f(x,p) \tag{4.6}$$

We know that μ and ν are Lorentz indices and can take any value from 0,1,2 and 3. Different components of this tensor give rise to quantities with different physical meaning. Time component of the tensor gives energy density whereas space component is called momentum flow which can be realized as pressure tensor. The mixed components give energy flow and momentum density based on the choice of indices.

$$\begin{aligned}
\text{Energy density, } T^{00}(x) &= \int \frac{d^3p}{p^0} f(x,p) \\
\text{Energy flow, } T^{0i}(x) &= \int \frac{d^3p}{p^0} u^i p^\mu p^\nu f(x,p) \\
\text{Momentum density, } T^{i0}(x) &= \int \frac{d^3p}{p^i} f(x,p) \\
\text{Momentum flow, } T^{ij}(x) &= \int \frac{d^3p}{p^i} u^j f(x,p)
\end{aligned} \tag{4.7}$$

The next quantity that can be derived is entropy flow which is also a four-vector and is denoted by S^μ.

$$S^\mu(x) = -\int \frac{d^3p}{p^0} p^\mu f(x,p)[log f(x,p) - 1] \tag{4.8}$$

The most simple description of relativistic heavy ion collisions is provided by hydrodynamics. Since the initial interactions among the particles in collision fireball are sufficiently strong to establish local thermal equilibrium rapidly, and then to maintain it over an evolution time, the resulting matter can be treated as a relativistic fluid undergoing collective, hydrodynamic flow. Hydrodynamic calculations for Au+Au collisions and a comparison between model and experimental data is discussed in [KH04b]. Even though some problems remain, from the equilibrium stage to the kinetic freeze-out, all the processes can be modeled by hydrodynamics, and it is in good agreement with the data from experiments. Since our focus is on statistical models, let us not go in detail to the hydrodynamic models.

Let us summarize the above discussion before going into detail to various statistical approaches. The aim of statistical models is to derive the equilibrium properties of a macroscopic system from the measured yields of the constituent particles. We realized that QGP exists in the early universe and in neutron stars, both of which are not accessible for study. But it is believed that the study of products of collisions in heavy ion experiments can yield information about the QGP phase.

But the challenge here is that the droplets of QGP exists only for microseconds and are produced in very small amounts. The only way we can analyze them is by focusing on the properties of the final state particles produced in the nucleus-nucleus collisions. The measurable quantities that can be used for this analysis are pseudo rapidity, transverse momentum, transverse energy, azimuthal angle, elliptic flow, multiplicity, etc. Among these variables available, we will be investigating the properties of the distribution of transverse momentum of identified particles produced in the collisions.

The discussion of applying relativistic thermodynamics to these particles starts with BG approach that uses exponential distribution and, followed by its generalization called Tsallis approach. Later, a new approach using generalized Tsallis distribution will be introduced.

4.3 Boltzmann Gibbs Statistical Framework

The initial attempt, undertaken by people to characterize the behavior of particle spectra produced in relativistic heavy ion collisions was using Boltzmann-Gibbs statistical distribution. This assumption of various hadrons being distributed according to BG statistics is valid considering the nature of thermalized ensemble at equilibrium.

Also, the fact that both Fermi-Dirac and Bose-Einstein distributions would reduce to BG statistics at temperatures that are sufficiently high gives a firm foundation to the approximation. This is because the energy at which heavy ions undergo collision is very large in high energy experiments and subsequently the thermal systems being produced will also have a larger value of temperature.

The conventional form of distribution function used in BG approach for fitting to momentum spectra is as following.

$$\frac{1}{2\pi p_T}\frac{d^2N}{dp_Tdy} = \frac{gVm_T}{(2\pi)^3}exp\left(-\frac{m_T}{T}\right) \tag{4.9}$$

Using the framework of relativistic kinetic theory, physically relevant quantities like number density, energy density and pressure can be obtained as,

$$n = \frac{1}{(2\pi)^3}\int d^3p exp\Big(-\frac{E-\mu}{T}\Big)$$
$$\epsilon = \frac{1}{(2\pi)^3}\int d^3p E exp\Big(-\frac{E-\mu}{T}\Big) \tag{4.10}$$
$$P = \frac{1}{(2\pi)^3}\int d^3p \frac{p^2}{3E} exp\Big(-\frac{E-\mu}{T}\Big)$$

Where m_T and p_T are the transverse mass and transverse momentum of particle respectively. The transverse mass is given as $\sqrt{p_T^2 + m^2}$. Also, g is the spin degeneracy factor, V is the volume of the system, y is rapidity, and T is the temperature. For several years, researchers used this function for approximating the measured transverse momentum and invariant yield data of identified particles. From the fit parameters, the temperature is extracted which is in agreement with the known values from theoretical models.

Calculation that gives the justification for approximating particle production spectra by BG formalism at large enough temperatures is discussed here. We know that the expression for average number of particles in the s^{th} state of a statistical system is as follows;

$$n_s = \frac{1}{e^{\beta(\epsilon_s - \mu)} \pm 1} \tag{4.11}$$

If the number of particles in the system is constant, the constraint which determines μ is the following equation.

$$\sum_s n_s = \frac{1}{e^{\beta(\epsilon_s - \mu) \pm 1}} = N \tag{4.12}$$

Where, upper and lower sign refers to the case of bosons and fermions respectively. When we look at classical limit which is defined by high temperature, the higher energy states will be mostly occupied and the relation $\epsilon_s << \mu$ will be obeyed.

For keeping N fixed, the following relation must be satisfied.

$$e^{\beta\epsilon_s - \mu} >> 1$$

When this is satisfied, the functional form for the number fo particles will become exponential like or BG distribution as given in Eqn4.13 Therefore, BG distribution function has fundamental significance in describing P_T spectra at high temperatures. Bose as well as

Fermi-Dirac distributions can always be written as an infinite sum of Boltzmann distribution.

$$n_s = e^{\beta(\epsilon_s - \mu)} \tag{4.13}$$

The Fig 4.1 represents the variation of the probability of each microstate or the population of particles to occupy each state in a thermal system at equilibrium, as a function of energy. The functional form of probability is also given, which is the standard Boltzmann factor. Taking QGP also as a thermal system, we can make the following analogy. Energy in the classical thermal system can be regarded as transverse momentum (P_T) measured from the collider experiments. A parallel connection can be made between the probability of each state and the measured invariant yield of particles, which gives the rate of change of the number of particles with respect to rapidity variable, in a fixed P_T window or vice versa. This is because by definition invariant yield is the normalized joint probability distribution of particle densities and rapidities.

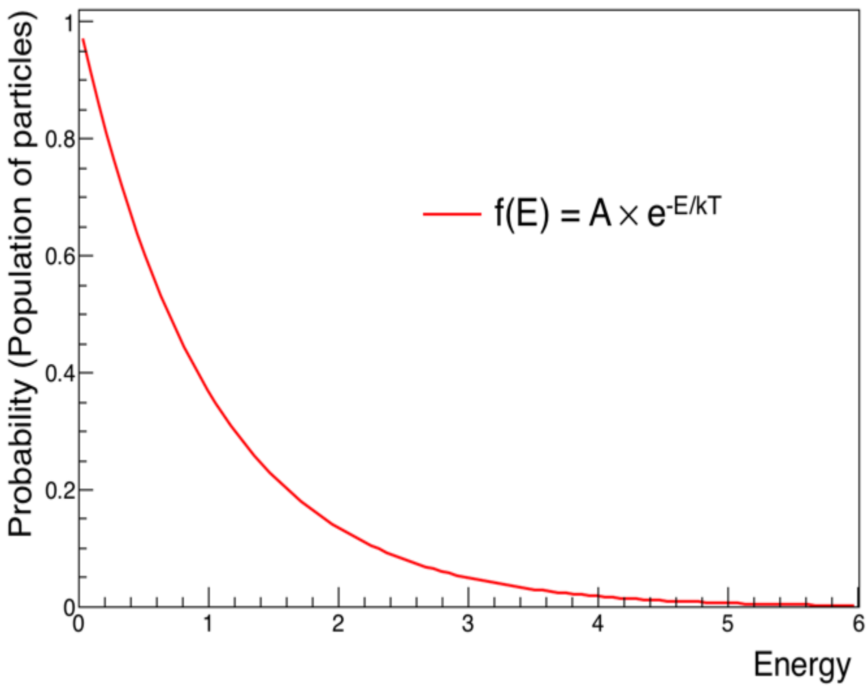


Figure 4.1: Variation of probability of each microstate in a classical statistical ensemble with the energy.

In order to make a comparison with the entropy, we will define in Tsallis statistics, let us introduce Boltzmann entropy initially. For a system of classical particles with discrete microstates defining its macroscopic state, the functional form of BG entropy or Gibbs entropy is as follows,

$$S_{BG} = \sum_i p_i ln(p_i) \tag{4.14}$$

Where, p_i is the probability of thermodynamical system to be in i^{th} microstate. Though this expression for entropy remains valid even when the system is far away from equilibrium, need for a generalization to Boltzmann-Gibbs theory arised due to its lack of universality. Here, universality refers to the fact whether the formalism is valid for all the phenomena [Abe01]. In the literature, many studies have been performed using exponential distribution to describe heavy ion collision dynamics.

A method for extracting temperature of the system from the transverse momentum (p_T) spectra of the emitted particles is presented in [BNC$^+$16], [SLH$^+$07], [BCC$^+$16]. An exponential Boltzmann-type fit to the p_T spectra gives a measure of the temperature. It is also important to test whether the approximation used is valid in the small volumes encountered in collision fireballs. A study of criterion for the applicability of thermodynamics in heavy ion collisions is performed in [EH92]. Analysis has shown that using continuous calculations may possibly be invalid for the case of hadronic resonance gas.

Now let us discuss the domains of applicability of BG statistical mechanics. The first property that should be referred here is mixing, the exponential mixing. This is associated with the extensivity of BG thermodynamics and the energy dependence of distribution function, the Boltzmann factor. Extensivity means that the entropy, thermodynamic potentials and similar quantities proportional to the number of microscopic elements of the system. Second thing concerns about the complexity of structure and what kind of structure is responsible for exponential mixing. It is known that the smooth phase space with Euclidean like structure obeys BG framework.

Next thing to consider is the physical background that leads to extensive statistical mechanics. These include short-range interactions and short-term memories (in Markovian processes). Therefore, systems with long-range interactions, multifractally structured phase space, long time memories, etc. can hardly be treated within this BG framework. So we need a generalization. This led to the foundation of a new breakthrough in statistical mechanics, known as non-extensive statistical mechanics.

In the context of high energy collisions, it was the deviation of experimental data from the BG function, which led physicists to think of a new approximating function. Even though p_T spectra fitted to BG distribution gave satisfying results, it was not accurate. Also, one can think of the possibility that extreme conditions in RHIC may lead to long-range color interactions. As a consequence, mixing might no longer be simply exponential, and the thermodynamic quantities may obey non-extensive statistical mechanics.

In the ref:[AL09], it is told that particle production at high p_T is sensitive to properties of the hot and dense matter in the nuclear collisions and therefore the transverse momentum spectra will be sensibly affected by non-extensive statistical effects. The non-extensive generalization to BG theory is known as Tsallis statistical thermodynamics, and the next section is devoted to the developments happened on the basis of this framework.

4.4 Tsallis Statistical Framework

The generalization of Boltzmann-Gibbs theory known as non-extensive statistical mechanics was initially constructed based on an entropy which was proposed by Tsallis in 1988. The functional form of this proposed entropy, known as Tsallis entropy is in such a way that it converges to BG entropy in a specific limit of its q-parameter. The parameter other than temperature in the Tsallis distribution is q, which gives the extent of non-extensivity in the thermodynamical system.

There exists a large number of physical systems involving long-range interactions and phase space of complex microscopic dynamics that violate BG statistical mechanics and standard thermodynamics. Some of the processes including such systems are ferromagnetism, solar neutrinos, black holes, cosmology, high energy collisions of particles and many more [Abe01]. We will be investigating the case of systems produced in heavy ion collisions.

Let us begin by the definition of non-extensive Tsallis entropy that was suggested by Tsallis.

$$S_q = -k \sum_i p_i^q ln_q(p_i) \tag{4.15}$$

$$= -k \sum_i p_i^q \frac{p_i^{1-q} - 1}{1-q} \tag{4.16}$$

$$= -k \sum_i \frac{p_i - p_i^q}{1-q} \tag{4.17}$$

$$= k \frac{1 - \sum_i p_i^q}{1-q} \tag{4.18}$$

Parallel to the definition of q-exponential which will be introduced later, its inverse also exists as q -logarithm which is used in deriving Tsallis entropy above. The q-logarithm is defined as follows:

$$ln_q(p_i) = \begin{cases} ln(p_i), & \text{if } p_i \geq 0, q = 1 \\ \frac{p_i^{1-q}-1}{1-q}, & \text{if } p_i \geq 0, q \neq 1 \\ undefined, & \text{if } p_i \leq 0 \end{cases}$$

It is obvious from the definition that the Tsallis entropy, in the limit $q \rightarrow 1$ gives standard Boltzmann-Gibbs entropy. When it approaches 1, we can say that the system stays in equilibrium and no longer exhibits non-extensivity.

It was observed that BG approximation to transverse momentum spectra fails at lower and higher momentum ranges. Further numerous studies happened on the implementation of Tsallis statistics to particle production spectra. The thermodynamical aspects of this formalism along with its foundations and applications are discussed in [TB04]. The Tsallis statistical distribution used for fitting to transverse momentum data can be obtained from BG distribution by replacing the exponential in BG function by q-exponential[CW12] which is defined as,

$$exp_q(x) = [1 + (q-1)x]^{\frac{1}{q-1}} \tag{4.19}$$

And the distribution function used for fitting to particle spectra is as following where, m_T, p_T, T, g, and V have the same meaning as in BG distribution and y is the rapidity variable.

$$\frac{1}{2\pi p_t}\frac{d^2N}{dp_T dy} = \frac{gVm_t}{(2\pi)^3}\left[1+(q-1)\frac{m_t-\mu}{T}\right]^{-\frac{q}{q-1}} \tag{4.20}$$

The definition of Tsallis statistical version of Fermi-Dirac and Bose-Einstein distributions along with corresponding entropy functionals are given in [CW12]. As we defined number density, energy density, pressure, etc. in BG formalism, similar can be done using non-extensive relativistic kinetic theory. The only difference is that the Tsallis distribution function will be raised to power q.

In standard thermodynamics, we are familiar with the following constraints on a total number of particles, N and energy, E in the system. Given the distribution function f_i,

$$N = \sum_i f_i \tag{4.21}$$

$$E = \sum_i f_i E_i \tag{4.22}$$

Tsallis statistics can be used to explain systems where temperature fluctuations are present around some initial value T_0. In such cases, the q parameter, which tells about non-extensivity in system, can be connected to variance of temperature [WW00] [WW12a] as:

$$q-1 = \frac{Var(T)}{\langle T\rangle^2} \tag{4.23}$$

In order to have physical significance to the formalism, the basic requirement is to check whether the distribution function obeys fundamental thermodynamic relations among the thermodynamic variables like pressure (P), volume (V), temperature (T), number density (n) and energy density (ϵ). In [CW12], they have shown that for Tsallis statistics to be thermodynamically valid, the above constraints must be redefined in following way with function raised to a power of q.

$$
\begin{aligned}
N &= \sum_i f_i^q \\
E &= \sum_i f_i^q E_i
\end{aligned}
\tag{4.24}
$$

A proper definition of entropy four current demanding positive entropy production according to the second law of thermodynamics is given in [BM12]. The Tsallis entropy derived from this will have following functional form.

$$
S_T = -\sum_i f_i^q ln_q f_i - f_i \tag{4.25}
$$

By extremizing above entropy under the constraints given in Eqn4.24, we can derive the distribution function, f_i. This method is called Lagrange Multiplier's variational principle. The variational equation is,

$$
\frac{\delta}{\delta f_i}\left[S_T + \alpha(N - \sum_i f_i^q) + \beta(E - \sum_i f_i^q E_i)\right] = 0 \tag{4.26}
$$

$$
\begin{aligned}
\frac{\partial S_T}{\partial f_i} &= -\sum_i \left[f_i^q \frac{\partial ln_q f_i}{\partial f_i} + ln_q f_i q f_i^{q-1} - 1\right] \\
&= -\sum_i ln_q f_i q f_i^{q-1} - 1 \\
&= -\sum_i \frac{1 - f_i^{1-q}}{1-q} q f_i^{q-1} \\
&= -\sum_i \frac{q}{q-1}(f_i^{q-1} - 1)
\end{aligned}
\tag{4.27}
$$

Using this derivative of entropy with respect to f_i in eqn4.26, we get

$$
\begin{aligned}
-\sum_i \left[\frac{q}{q-1}(f_i^{q-1} - 1) - \alpha q f_i^{q-1} - \beta E_i q f_i^{q-1}\right] &= 0 \\
\sum_i q f_i^{q-1}\left[\frac{-1}{q-1} - \alpha - \beta E_i\right] &= -\frac{q}{q-1}
\end{aligned}
\tag{4.28}
$$

Extracting the distribution function f_i from above expression,

$$
\begin{aligned}
f_i^{q-1} &= \frac{1}{1 + (q-1)(\alpha + \beta i)} \\
f_i &= \left[1 + (q-1)\frac{E_i - \mu}{T}\right]^{\frac{-1}{q-1}}
\end{aligned}
\tag{4.29}
$$

Therefore, f_i is derived and is equivalent to Tsallis distribution function. Now we have to show that the function is thermodynamically consistent. In order to check thermodynamic consistency of the formalism, we need to prove that the relations among thermodynamic parameters are indeed obeyed. Here, we will prove the following relation between pressure (P), and number density (n) that is familiar from the laws of thermodynamics.

$$\left.\frac{\partial P}{\partial \mu}\right|_T = n \tag{4.30}$$

Using the first law of thermodynamics, the thermodynamical definition for pressure is as follows.

$$P = \frac{-E + TS + \mu N}{V} \tag{4.31}$$

Taking derivative of pressure with respect to chemical potential (μ), we get

$$\frac{\partial P}{\partial \mu}|_T = \frac{1}{V}\Big[-\frac{\partial E}{\partial \mu} + T\frac{\partial S}{\partial \mu} + S\frac{\partial T}{\partial \mu} + \mu\frac{\partial N}{\partial \mu} + N\Big] \tag{4.32}$$

$$= \frac{1}{V}\Big[-\frac{\partial E}{\partial \mu} + T\frac{\partial S}{\partial \mu} + \mu\frac{\partial N}{\partial \mu} + N\Big] \tag{4.33}$$

$$= \frac{1}{V}\Big[N + \sum_i \frac{-T}{q-1}\Big([1 + (q-1)\frac{E_i - \mu}{T}]^{\frac{1}{q-1}}\Big)\frac{\partial f_i^q}{\partial \mu} + \frac{Tq(1-f_i)^{q-1}}{q-1}\frac{\partial f_i}{\partial \mu}\Big] \tag{4.34}$$

Now, let us calculate the derivative of f_i and f_i^q with respect to q,

$$\frac{\partial f_i}{\partial \mu} = \frac{1}{T}\left[1 + (q-1)\frac{E_i - \mu}{T}\right]^{\frac{q}{1-q}} \tag{4.35}$$

$$\frac{\partial f_i^q}{\partial \mu} = \frac{q f_i^{q-1}}{T}\left[1 + (q-1)\frac{E_i - \mu}{T}\right]^{\frac{q}{1-q}} \tag{4.36}$$

Putting this in eqn4.32,

$$
\begin{aligned}
\frac{\partial P}{\partial \mu}\bigg|_T = \frac{1}{V}\sum_i \Bigg[& N - \frac{E_i q}{T\left[1+(q-1)\frac{E_i-\mu}{T}\right]}\left[1+(q-1)\frac{E_i-\mu}{T}\right]^{\frac{q}{1-q}} \\
& + \frac{q}{q-1}\left[1+(q-1)\frac{E_i-\mu}{T}\right]^{\frac{q}{1-q}}\left[1-\frac{1}{1+(q-1)\frac{E_i-\mu}{T}}\right] \\
& + \frac{\mu q}{T\left[1+(q-1)\frac{E_i-\mu}{T}\right]}\left[1+(q-1)\frac{E_i-\mu}{T}\right]^{\frac{q}{1-q}}\Bigg]
\end{aligned}
$$

By cancelling the terms in the above expression,

$$
\begin{aligned}
&= \frac{1}{V}\Bigg[N + \sum_i \frac{q}{T}\left[1+(q-1)\frac{E_i-\mu}{T}\right]^{\frac{2q-1}{1-q}}\Big(\mu - E_i\Big) \\
&\qquad + \sum_i \frac{q}{T}\left[1+(q-1)\frac{E_i-\mu}{T}\right]^{\frac{2q-1}{1-q}}\Big(E_i - \mu\Big)\Bigg] \\
&= \frac{N}{V} \\
&= n
\end{aligned}
$$

Hence, it has been proven that the Tsallis Boltzmann formalism is thermodynamically consistent. We have shown only one thermodynamical relation and in a similar manner other relations can also be shown to be satisfied.

From the entire discussion, it is evident that the non-extensive Tsallis approach provides better fits and, explains heavy ion collisions more appropriately as compared to the standard Boltzmann and power-law approaches. The figure representing the plots of P_T spectra of π^+ and π^- particles fitted to Tsallis function is taken from [DBSB07], where satisfying agreement between the experimental data and function is established.

Yet this is not the complete story. It is known that the exponential function or standard BG theory can only take care of soft P_T region of the hadronic spectra where the particles produced will have small transverse momenta. Whereas, QCD calculations have shown that the explanation of spectra of particles produced in hard scattering processes can be provided by power law functions. Our interest lies in finding a single distribution function which describes the whole region of P_T spectra (both soft and hard part)in a unified manner.

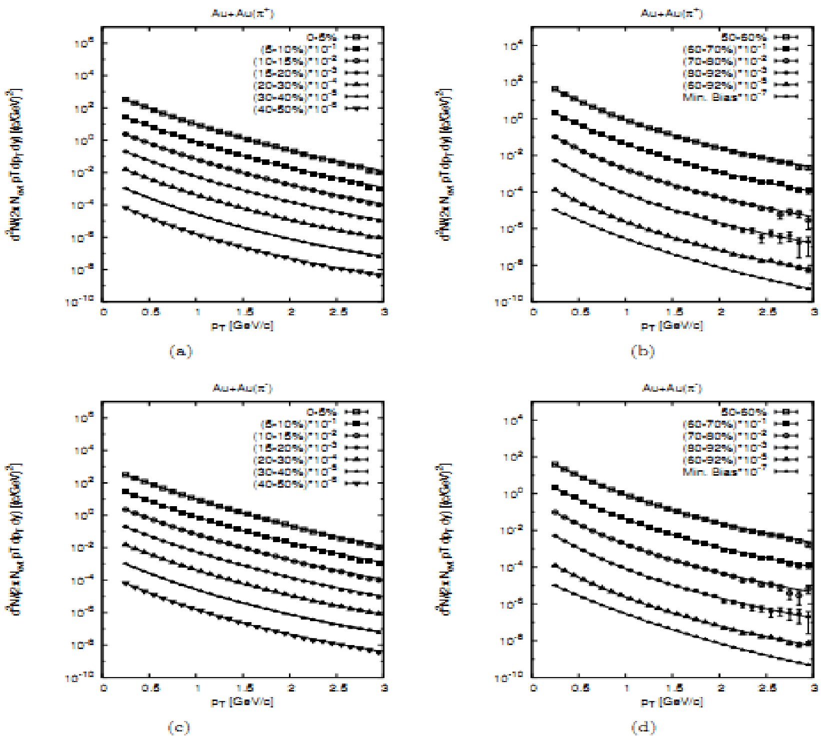


Figure 4.2: *Plots of transverse momentum spectra of* π^+ *and* π^- *produced in Au+Au collisions at* $\sqrt{s} = 200GeV$ *at different centralities. The filled symbols represent the experimental data points. The solid curves provide the fits on the basis of nonextensive approach*

Even though Tsallis approach does this [DBSB07], it has been shown in a recent paper [SLL17] that the Tsallis form of standard distribution can describe only the transverse momentum spectrum of particles produced in soft excitation process. A clear deviation from the data is observed in the higher momentum ranges of the spectra. Therefore, can some function accurately describe the entire range of spectra in an integrated way is still an open question.

Since perturbative QCD can be used to describe hard scattering processes, it is possible to extract the form of P_T spectra at these regions. And, the calculations suggest that the spectra will have the form of inverse power law which is expressed as; [SLL17],[A$^+$82],[WW12b], [BMS17],[MV77],[Mic79].

$$f(p_t) = \frac{1}{N}\frac{dN}{dp_t} = Ap_t\left(1 + \frac{p_t}{p_0}\right)^{-n} \tag{4.37}$$

Where p_0 and n are fitting parameters and A is the normalization constant related to free parameters. This QCD inspired formula was proposed by Hagedorn to describe data of invariant cross-section of hadrons as a function of P_T. Now, our proposal to the question above stated is to combine inverse power-law and Tsallis distribution using Pearson distribution. This statistical approach is discussed in detail in the next section.

4.5 Pearson statistical Framework

4.5.1 Mathematical Background

Pearson distribution was first proposed by Karl Pearson in 1895 [Pea95] and subsequently modified in 1901 and 1916. He proposed to classify a distribution function based on first four moments related to mean, standard deviation, skewness, and kurtosis of the distribution. Moments are defined for specifying the shape of any probability distribution.

The first moment or the mean locates the center of the distribution whereas variance gives the spread or dispersion in the data about the mean. Other two are called shape parameters, among which skewness gives the degree of asymmetry in the distribution around the mean and kurtosis specifies the relative peakedness or flatness of the distribution. Characterization of any statistical data involves the specification of skewness and kurtosis.

Gaussian, beta, gamma, inverse-gamma, exponential, Student's t-distribution are all special cases in Pearson distribution and belong to Pearson family of the curve. Due to this reason, it is considered as the most general distribution and has been used in many different fields like geophysics, biostatistics, and financial marketing. It is a family of continuous probability distributions, whose densities $p(x)$ satisfy the following differential equation.[Pol79]

$$\frac{1}{p(x)}\frac{dp(x)}{dx} + \frac{a+x}{b_0+b_1x+b_2x^2} = 0 \tag{4.38}$$

where the parameters a, b_0, b_1, b_2 can be related to first four central moments as follows:

$$a = b_1 = \frac{m_3(m_4+3m_2^2)}{10m_2m_4-18m_2^3-12m_3^2} \tag{4.39}$$

$$b_0 = \frac{m_2(4m_2m_4-3m_3^2)}{10m_2m_4-18m_2^3-12m_3^2} \tag{4.40}$$

$$b_2 = \frac{2m_2m_4 - 6m_2^3 - 3m_3^2}{10m_2m_4 - 18m_2^3 - 12m_3^2} \tag{4.41}$$

Here, m_1, m_2, m_3 and m_4 are the central moments with $m_1 = 0$. Pearson curves are classified into 12 different types based on the root of the quadratic equation in the denominator of differential equation. Therefore, Pearson criteria which will decide the type of distribution is the sign of discriminant of the quadratic equation which is expressed as,

$$k = \frac{b_1^2}{4b_0b_2} \tag{4.42}$$

A table including different types of Pearson distribution along with Pearson criteria and condition on parameters can be found in Ref [PLKP03]. Our task is to solve the differential equation4.38 to find the form of Pearson density.
By doing seperation of variables we get,

$$p(x) = C' \exp \int -\frac{P(x)}{Q(x)} dx \tag{4.43}$$

$$= C' \exp \int -\frac{a_0 + a_1x}{b_0 + b_1x + b_2x^2} dx \tag{4.44}$$

We can express the quadratic equation in the following form,

$$b_0 + b_1x + b_2x^2 = b_2(x - \alpha)(x - \beta) \tag{4.45}$$

$$p(x) = C \exp \int \frac{a_0 + a_1x}{(x - \alpha)(x - \beta)} dx \tag{4.46}$$

$$= C \exp \int \frac{m}{x - \alpha} + \frac{n}{x - \beta} dx \tag{4.47}$$

Where m and n have following definition.

$$m = -\frac{a_0 + a_1\alpha}{\beta - \alpha} \qquad n = -\frac{a_0 + a_1\beta}{\beta - \alpha} \tag{4.48}$$

After integration,

$$p(x) = C \exp ln|x - \alpha|^m + ln|x - \beta|^n \tag{4.49}$$

$$= C|x - \alpha|^m|x - \beta|^n \tag{4.50}$$

A general solution can be written as in eqn.4.51 where C is a normalization constant and e, f, g and h are free parameters.

$$p(x) = C(g + x)^h (e + x)^f \tag{4.51}$$

4.5.2 Pearson Function as an Extension to Tsallis

We will now show that the Pearson function can be expressed as an extended version of Tsallis distribution. It is easy to see that the Pearson distribution converges to exponential when the numerator, $P(x)$ and denominator, $Q(x)$ in Eqn 4.43 becomes constant and unity respectively. In similar way, we can derive the limit of Pearson parameters at which it will reduce to Normal or Gaussian distribution. For this, $P(x)$ has to be of linear form and $Q(x)$ has to be unity. Since Pearson density reduces to exponential at some limit, it is obvious that we can find a relation between Tsallis, which is a generalized Boltzmann and the Pearson function.

The Eqn4.51 can be rewritten in following form by doing simple algebra.

$$p(x) = B\left(1 + \frac{x}{g}\right)^h \left(1 + \frac{x}{e}\right)^f \tag{4.52}$$

Up to some normalization constant $B = Ce^f g^h$. Now if we replace $g = \frac{T}{q-1}$, $h = -\frac{q}{q-1}$, $f = -n$ and $e = p_0$ we will get:

$$p(x) = B\left(1 + (q-1)\frac{p_T}{T}\right)^{-\frac{q}{q-1}} \left(1 + \frac{p_T}{p_0}\right)^{-n} \tag{4.53}$$

where,

$$B = C\frac{1}{(p_0)^n}\left(\frac{T}{q-1}\right)^{-\frac{q}{q-1}} \tag{4.54}$$

Now, we can try to fit the particle spectra with this function.

$$\frac{1}{2\pi p_T}\frac{d^2N}{dp_T dy} = B\left(1 + (q-1)\frac{p_T}{T}\right)^{-\frac{q}{q-1}} \left(1 + \frac{p_T}{p_0}\right)^{-n} \tag{4.55}$$

Hence, it is inferred that the Pearson distribution is a generalised form of Tsallis distribution and can shown to be having two parts. In reference to Eqn4.37, the inverse power law term in the above equation can be considered as the hard scattering part in the extended Tsallis form of distribution. In the limit $n = -1$ and $p_0 = 0$ we can recover Tsallis statistics apart from the normalization factor.

Chapter 5

Analysis and Results

5.1 Fitting Details

The plots in Fig 5.1 present the main body of results obtained in this work where, a comparison between BG, Tsallis and Pearson statistical approaches in describing the transverse momentum spectra is demonstrated. It shows the degree of agreement between measured data and the results attainable by the approaches based on Statistical Thermodynamics. In the plots, symbols represent the experimentally measured data of transverse momentum, and solid lines represent the results fitted by BG, Tsallis and Pearson distribution functions. ROOT, CERN data analysis framework has been used along with MINUIT class for fitting. The analysis was done for the transverse momentum data of π^- particles produced in Au-Au and Pb-Pb collisions, and the collision energies we selected for study included 7.7 GeV, 11.5 GeV, 19.6 GeV, 39.0 GeV, 200.0 GeV and 2760 GeV.

The goodness of Pearson approach over other approaches is determined by looking at the *chi-square* values of each fit. *chi-square* goodness of fit test is used to find out how the observed value is significantly different from the expected value and to compare the observed sample distribution with the expected probability distribution. Using the ROOT framework, we have obtained A table including the Chi-square values of Boltzmann, Tsallis, and Pearson functions fitted to P_T spectra at several energies is given in Table 5.1.

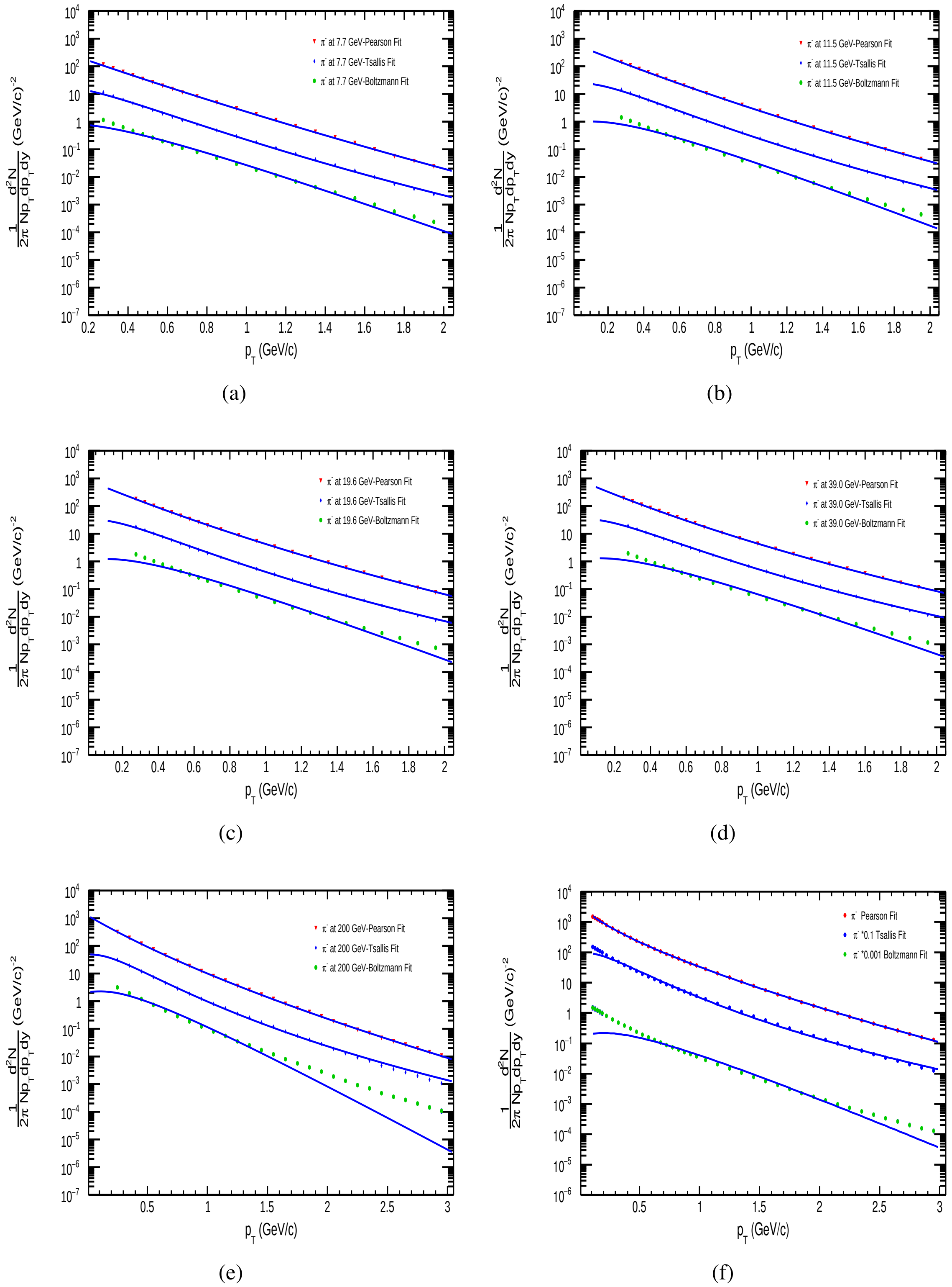


Figure 5.1: The transverse momentum data of π^- particles produced at collision energies of (a) 7.7 GeV, (b) 11.5 GeV, (c) 19.6 GeV, (d) 39.0 GeV (e) 200 GeV and (f) 2760 GeV fitted with Boltzmann (Eqn4.9), Tsallis (Eqn4.20) and Pearson distribution function (Eqn4.52).

Collision Energy in GeV	χ^2/NDF (Boltzmann)	χ^2/NDF (Tsallis)	χ^2/NDF (Pearson)
7.7	8.38446	1.33471	0.552933
11.5	9.80142	2.39404	0.142277
19.6	10.9543	0.422782	0.0810845
39.0	10.5826	0.349578	0.0050
200	346.41	14.13	2.0832
2760	24.4878	2.27986	0.0417518

Table 5.1: The χ^2/NDF values of transverse momentum data of π^- particles fitted to Boltzmann, Tsallis and Pearson functions at various collision energies is given.

Distribution	Energy (GeV)	T (MeV)	q	f	e	h	g
Boltzmann	7.7	161.209	–	–	–	–	–
Boltzmann	11.5	164.578	–	–	–	–	–
Boltzmann	39.0	175.545	–	–	–	–	–
Boltzmann	200.0	176.590	–	–	–	–	–
Boltzmann	2760.0	245.873	–	–	–	–	–
Tsallis	7.7	112.177	1.05798	–	–	–	–
Tsallis	11.5	106.611	1.07134	–	–	–	–
Tsallis	39.0	106.520	1.08761	–	–	–	–
Tsallis	200.0	111.984	1.09085	–	–	–	–
Tsallis	2760.0	118.895	1.11631	–	–	–	–
Pearson	7.7	14.59	1.00123	789.504	12.5658	−810.299	11.8127
Pearson	11.5	163.29	1.03710	15.9308	50.1280	−27.9524	4.40135
Pearson	39.0	221.70	1.182	8.54571	6.77728	−6.49152	1.21747
Pearson	200.0	381.298	1.016	4.5193	1.2502	−62.152	27.949
Pearson	2760.0	371.270	1.0355	1.04663	0.170752	−29.1514	10.4583

Table 5.2: The parameters extracted from Bolzmann, Tsallis and Pearson fits to the momentum data at various collision energies is given.

The parameters extracted from the fits are given in the Table **??**. Boltzmann distribution is parametrized by only one parameter, which is temperature (T). Tsallis framework includes another parameter called q-parameter apart from the temperature, which gives the extent of non-extensivity in the distribution. In all the systems that we have studied, the parameter q is greater than 1 and hence shows sub-additivity [WW02]. The proposed approach using Pearson distribution comprises of four free parameters in addition to the temperature and q-parameter.

5.2 Discussion

We have fitted P_T spectra of π^- particles produced in most central (0-5) collisions at energy of $7.7, 11.5, 19.6, 39.0, 200$ and 2760 GeV. It is clearly visible by merely observing the plots itself that the Pearson fits are better compared to Boltzmann and Tsallis fits. This can be confirmed from the Table 5.1, where the χ^2/NDF values of all the fits are displayed. It can be seen that at all the energies, χ^2/NDF values are minimum for Pearson fits. As we expect, these are large for Boltzmann fits, and that of Tsallis fits are intermediate.

Statistically speaking, one possible reason for obtaining best fit using Pearson distribution is the presence of higher order moments as parameters. Most of the distribution function uses mean and standard deviation as parameters. In case of Pearson, skewness, and kurtosis are also utilized for parametrization. Further, as we already discussed that Tsallis distribution is not fitting higher p_t values which is the tail part of the distribution. And the tail of a distribution is sensitive to higher order moments so a distribution function which depends on higher moments can fit tail part more precisely.

Several two-component models have been formulated to fit p_T spectra at low p_T as well as high p_T but interestingly Pearson distribution has never been considered. In this work, our objective was to find a single function which can describe both soft and hard scattering regions of particle spectra in a unified manner. Our proposed distribution function is shown to have two parts where the first part is an inverse power law and the second part is similar to Tsallis function. Hence, we have combined inverse power law and Tsallis framework and, it has been observed that the unified function is fitting spectra accurately. Further, the thermodynamical connection among the extracted parameters and generalized thermodynamics for relativistic particles based on this approach is yet to be discovered.

Appendix A

ROOT: Data Analysis Framework

ROOT is an object oriented data analysis frame work developed by physicists. It provides a very composite and fast platform for researchers to perform complicated and rigorous calculations. Its significant features are an advanced user interface for visualization and interactive data analysis and an interpreter for the C++ programming language.

One of its important features is the visualization of things. Hence we need to incorporate different visualization techniques such as graphs, plots and histograms. One needs to satisfy editing techniques also. That will help to modify the graphs and allows one to modify it in such a way that it is more efficient, accurate and perfect. We may need to save these outputs in a minimal space and use it for some further calculations.

Another important feature is fitting data. This is where I have made use of ROOT framework for this study. In experimental physics, analysing data is the most significant task, where we need to compare measurements to theoretical models. When we say models, they are any functions which can give predictions of the measured data. One may have to fit the data with predicted functions or may have to fit several functions to it. All these features are satisfied by ROOT. It has some well defined classes which enable the user to perform effective analysis.[Ref] Here, some of the basic libraries or classes are briefly discussed.

1. TF1 : It enable the user to define one dimensional functions in the program. Here the integral function is available as predefined and we used that for our calculations.

2. TFile : It is class dedicated for saving data to some file and reading data from a file.

3. TH1F : This class is for plotting one dimensional histograms. We need to specify the number of bins needed and the range of x axis.

4. TH2F : This one is also for plotting purpose, dedicated for two dimensional histograms. In a two dimensional histogram we need to specify number of bins in x and y axes. It is a good tool to save large number of one dimensional histograms in a compact form. It saves a lot of time while writing program and visualization is also good. We can get the projections along any axis and profile of the histogram in any direction, all these things are associated with this.

5. TGraph : For the visualization of graphs we need to use this class. We can modify the plots in different colour, different point markers, different style, and also labeling and plotting multiple graphs in one chart is also included.

6. TRandom : This class is very special for simulation studies. It is a powerful tool which helps the user to create random numbers which satisfy the user provided conditions. Different options are available even in this class, were T Random3 is very good for best randomness. Because the computer generated random numbers are not exact random number, while they are pseudo random numbers.

7. TCanvas : As the name indicates it gives a canvas for plotting all sorts of visualization methods. We can add multiple graphs in one canvas by dividing it into desired number of parts.

A code for fitting collected transverse momentum data with Boltzmann, tsallis and pearson distribution functions was written using C language and was compiled using ROOT framework. TMinuit class was used for fitting.

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