Eulerian Walker On A Square Lattice

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A dissertation submitted for the partial fulfilment of BS-MS dual degree in Science

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Certificate of Examination

This is to certify that the dissertation titled **Eulerian Walker On A Square Lattice** submitted by **Kushalpal Kaur (Reg. No. MS13146)** for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Sudeshna Sinha   Dr.Abhishek Chaudhuri   Dr. Rajeev Kapri
(Supervisor)

Dated: April 19, 2018

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr.Rajeev Kapri at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Kushalpal Kaur
(Candidate)

Dated: April 19, 2018

In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr.Rajeev Kapri
(Supervisor)

Acknowledgment

After an intensive period of one year, today is a special day and writing this note of thanks is the finishing touch on my dissertation. It has been a period of intense learning and I would like to express my gratitude to all those who have been helpful in this span of time. To commence with, I pay my obeisance to Almighty, who always bestowed upon me the zeal, courage and inspiration and a solid bunch of people around me who are consistent source of positivity and happiness in my life.

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List of Figures

Abstract

In this thesis, we study the asymptotic shape of the region visited by Eulerian walkers in a square lattice using monte carlo simulations. For a single walker, this region was found to be a perfect circle. We extended the study for two Eulerian walkers that start their walks from two different origins on the lattice. Our preliminary study suggests that the shape of the region is likely to be circular if both walkers rotate the direction of arrows on the lattice in the same sense. The shape of the region changes to elliptical if the two walkers rotate the direction of arrows on the lattice in the opposite sense.

Contents

Chapter 1

Introduction

Over the past few decades, the nature of complex systems in various fields of science became apparently clear with the introduction of concepts of stochastic process and self-organized criticality(SOC) in Physics and Mathematics. The study of stochastic or random process gained momentum due to various complicated dynamical phenomena of nature exhibiting its features. Many phenomena like earthquakes, economies of stock exchange market and population growth model exhibit properties of stochastic processes. The concepts of SOC and stochastic process has been applied to various diverse fields of science because of it's efficiency in explaining complex phenomena.

1.1 Stochastic Process in Physics

The role of probability theory and stochastic methods in physics is profound. Physically, a stochastic variable or a random variable is an object ,say X, defined by

- a set of possible values (called range, set of states, sample space, phase space)
- probability over this set[1].

This set can be discrete:- like heads or tails in a coin toss, or continuous in given interval such as:- potential difference between end points of electrical resistance $(-\infty, \infty)$.

In nature, we encounter various phenomena in which some quantity varies with time in a very irregular and complicated way, for instance, the motion of random walkers in medium. The position of a walkers vary continuously with time. In such cases, averaging over suitable time period could give useful information. This averaging could be done in two different ways. One way is to average over the ensemble and another way is to average over some time interval. In thermodynamic limit, these two averages coincide. For better justification, we can consider trajectory of a random walk. We can observe large number of non-interacting random walkers and average the result. Similarly, we can observe single random walker and take time averages over many time intervals.

If the trajectory of one walk does not affect the trajectory of other walk, time average will be equal to the ensemble average and the process is called ergodic. If a stochastic process possess the property of ergodicity then a large collection of large random samples from that process must represent the average statistical properties of the entire process.

1.2 Patterns in Randomness

It is observed that catastrophic events in nature such as earthquakes, flooding, volcanic eruptions and the prices in stock market also follow some pattern and laws. $1/f$ noise[5] and fractals observed in nature which are self driven to their state can be said to withhold property of SOC because the outlook of their state is complex and the system carries large scale variations. Critical or rather scale-invariant behavior is interesting because it is universal i.e seemingly different systems share the same critical properties such as critical exponents and amplitude ratios. Thus the problem of explaining the statistical features of complex systems can be understood as problem of explaining the underlying laws, and more specifically the values of exponents.

1.3 Random Walks

In the past decades, the study of random walks gained momentum because of its efficiency in explaining complex process. The concept of Random walks emerged with efforts of biostatistician Karl Pearson in 1905. Modelling of random walks in different dimensions helps to understand nature of stochastic process. Many theories regarding complex and fluctuating processes are focused on randomness involved in these systems because when the nature of randomness is determined the behavior of these systems can be predicted to an extent. Random walk model can be used to the explain various phenomena with large variations like fluctuating prices in stock market, the path traced by a molecule in liquid or gas.

1.3.1 Random walk in d-Dimensions

Let us define a random walk in 1 dimensional lattice. The walker starts from origin and takes a step of size l towards either left or right directions with equal probability. Let a_i represents the displacement of the walker at i^{th} step. Then its average displacement after N steps is given by

$$\langle R_N \rangle = \left\langle \sum_{i=1}^{N} a_i \right\rangle = 0, \tag{1.1}$$

as all steps are uncorrelated. Since, this does not gives any important information, we calculate the average square displacement

$$\langle R_N^2 \rangle = \left\langle \left(\sum_{i=1}^{N} a_i \right) \left(\sum_{j=1}^{N} a_j \right) \right\rangle, \tag{1.2}$$

which can be written as

$$\langle R_N^2 \rangle = \left\langle \sum_{i=j=1}^{N} a_i^2 \right\rangle + \left\langle \sum_{i \neq j}^{N} a_i a_j \right\rangle, \tag{1.3}$$

The second term on the R.H.S. is 0 as the i^{th} and j^{th} steps are uncorrelated. Therefore, we have

$$\langle R_N^2 \rangle = Nl^2 \sim N^{2v}, \tag{1.4}$$

where l is the step size and v is the exponent. Therefore at distance N the walker's displacement is of the order $\sqrt{N}$. For a random walk, $v = 1/2$. The size exponent $v = 1/2$ is true also in higher dimensions.

In Fig. 2.1 we plot the mean square displacement $< R^2 >$ for a random walker of length N=100 on a square lattice and on fitting the data we found the value of exponent v is equal to the value stated above.

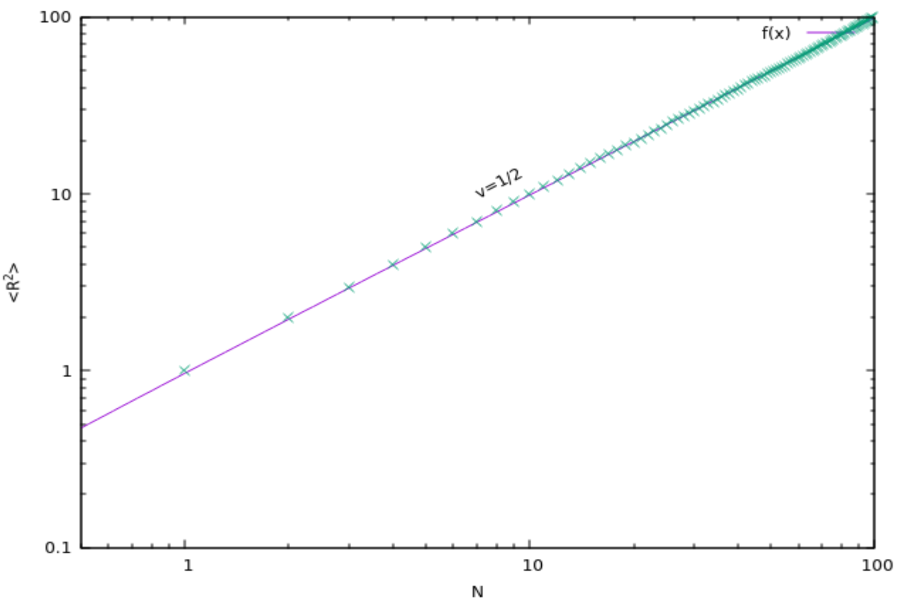

Figure 1.1: Mean Square Displacement $< R_N^2 >$ as function of number of steps N for random walker on Square lattice

1.4 Known Literature and Preliminaries

In this work, we aim to study the asymptotic shape of the region visited by different Eulerian Walkers (which is Rayleigh Pearson Random Walker) on Square lattice .

Let us give introduction to models that use a variant of random walk models. The algorithms and cluster formed by systems of these models have somewhat similarity with the Eulerian walker model.

1.4.1 Eden Growth Model

Eden growth model is a stochastic model that is used to explain the growth of tumor cells[11]. A cluster is defined as composition of connected sites, two sites are considered connected only if they are first neighbors. Cluster growth occurs by an empty site becoming occupied at the end of each time interval. The process starts with a germ which can be a line or a point. Now in case of square lattice a cell of a planar square grid is labeled as infected. Then any one of the four possible adjacent cells is randomly and independently chosen to be infected. The process continues until a cluster is formed. The region occupied and cluster observed has a compact core and rough surface as shown in Fig 1.2. Hence this model studies the random processes and is helpful in analysing cluster formation in growth model where randomness is present dynamically.

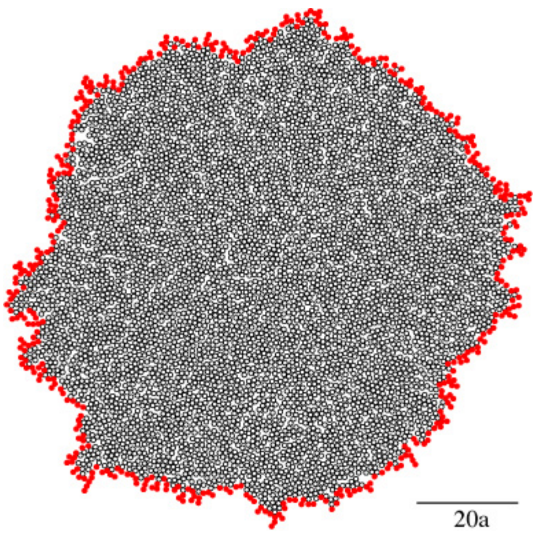

Figure 1.2: Eden cluster with 6000 particles[11]
This figure is taken from $https : //www.google.co.in/search?q = edengrowthmodeldcr0sourcelnmstbmischsaXvedimgrcYWUJsyGUm3jTOM$

1.4.2 Diffusion limited aggregation

Another clusterable random walk observed is the diffusion-limited aggregation (DLA)[10]. It is the stochastic process where particles moving randomly and undergoing Brownian motion come together to form aggregates or clusters. The cluster formed by DLA is a fractal as shown in Fig 1.3. Particles are diffused in the medium before getting attached to the cluster. In DLA also the process begins with a fixed seed as in Eden model. The nature of cluster formed depends on the type of seed, surface area for movement, attachment rules to the cluster. This theory was proposed in 1981. Some

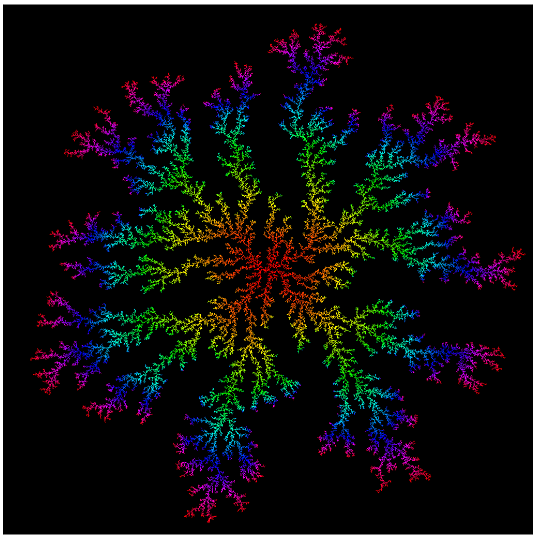

Figure 1.3: Cluster of DLA[10].
This figure is taken from
$http://statslab.cam.ac.uk/\ jpm205/quantum-gravity.html$

clusters are also observed in internal diffusion limited Aggregation(IDLA) which was introduced by Diaconis and Fulton in 1991[2] and they studied the aymptotic shape of occupied region for an interacting lattice system. In this model, the particles are dropped at the origin of lattice each at a time. Every successive particle performs independent random walk until it sticks or encounters a site which is previously not occupied. The cluster of stuck particles is asymptotically circular as particles diffuse from interior of occupied cluster and are more likely to stop at an unoccupied site which is closer to origin as contrast to DLA were particles stick to the extreme ends.

1.4.3 Sandpile Model: Dynamical system of SOC

Sandpile model is the first discovered model which exhibits self-organised criticality . It was introduced by Per Bak, Chao Tang, and Kurt Weisenfled in 1987[3]. This model can be studied on infinite square, square lattices and directed graphs. The sandpile model of Per Bak and its many extensions was helpful to find the new concept of self-organized criticality, which is now a useful concept to study complex systems. Per Bak proposed that the local interactions between the elements of a system could spontaneously reach the critical point and then the systems becomes self-organised after reaching this criticality[3]. He explained it using dynamics involved in sandpile. If we consider, sand running from the top of an hourglass to the bottom we observe that it gets accumulated grain by grain. Eventually, the the pile grows and reaches a point where it is so unstable that the next grain can cause pile to collapse and cause an avalanche. When a collapse occurs, the base widens, and the sand starts to pile up again, and the mound is formed once again to cause avalanche. It is through this series of avalanches of various sizes that the sand pile, a complex system of millions of tiny elements, maintains overall stability. Fig 1.4 shows mounds of various sizes which helps to stabilize the system.

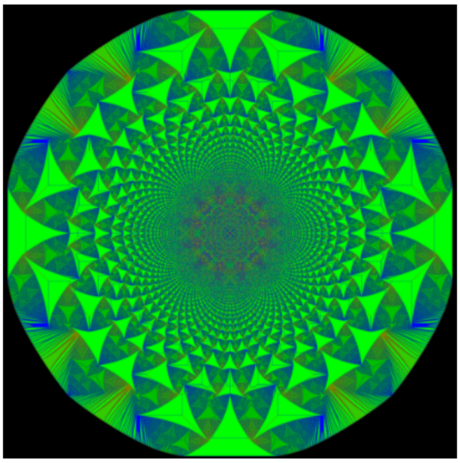

Figure 1.4: Four million grains of sandpile dropped on infinite grid. Colors represent different number of grains at each vertex[8]
This figure is taken from $https://mathmunch.org/2012/11/12/sandpiles-prime-pages-and-six-dimensions-of-color/$

1.4.4 Rotor Router Model

The rotor-router growth model is similar in many ways to the abelian sandpile model introduced by Per Bak[3]. The rotor-router model is a deterministic analogue of random walk and internal DLA. The cluster shape in rotor router aggregation model is circular. In this model, there is a rotor placed at each site in the integer lattice pointing to either of directions randomly:- north, south, east or west. A particle starts at the origin and the rotor at the particles current location rotates by 90 degrees clockwise, and then the particle moves a step ahead in the direction of the new direction. This happens every time particle takes a step. In rotor-router aggregation, we start with n particles at the origin; each particle in turn performs rotor-router walk until it reaches a site not occupied by any other particles. A cluster A_n is formed that has a occupied region proportional to $n^{1/4}$. Fig 1.5 shows a cluster formed through rotor-router walk by 270,000 particles.

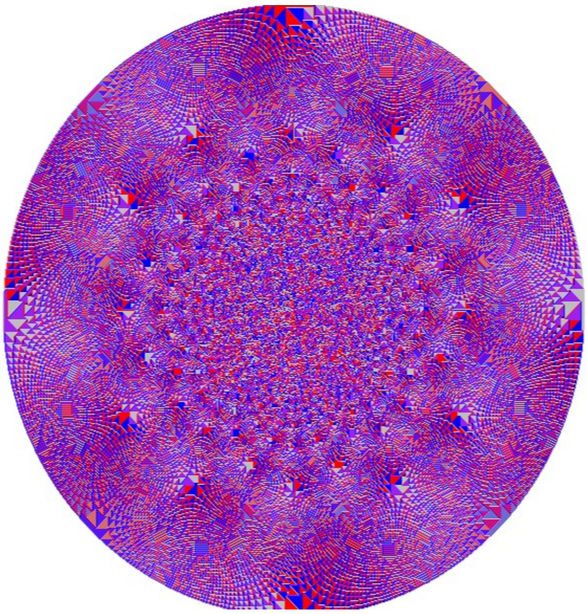

Figure 1.5: Rotor-Router aggregate of 270,000 particles[9]
This figure is taken from $http://yuvalperes.com/router/router.html$

1.5 Plan of thesis

The goal of this thesis work is to look at dynamics of Eulerian walker on square lattice. The plan of thesis is as follows:

- Looking at random walk model on square lattice
- Obtaining useful relationship by analyzing Eulerian walker[6] on square lattice.
- Study the dynamics of two Eulerian walker together on square lattice and comparing it with dynamic of single Eulerian walker

Chapter 2

Model

The Eulerian walker model on a two dimensional square lattice is defined as follows:

- Generate 2D lattice with random number at each lattice site.
- With each random number associate an arrow which can point to any one of the four directions denoted by N, S, E and W.
- Walker starts from the origin.
- The walker detects the arrow, rotates the arrow 90 degrees anti-clockwise and moves one step ahead in the new arrow direction.

Thus the motion of the walker is affected by the medium and it in turns affects the medium in which it moves. The initial configuration of directions of arrows are generated randomly and independently using a pseudo random number generator.
The aim of this work is to study the asymptotic shape of the region visited by the different Euler walkers which grows with the length N of the walk. We study the following cases:

- Case 1: A single anticlockwise Eulerian walker starting from origin O.

- Case 2: Two Euler walkers starting from origins O and O' that are separated by a distance d. Both walkers rotate the arrows in the anticlockwise direction while moving on lattice.

- Case 3: Two Eulerian walkers starting from origins O and O' separated by distance d. But now one walker rotates the arrows in the clockwise direction and other rotates in the anticlockwise direction.

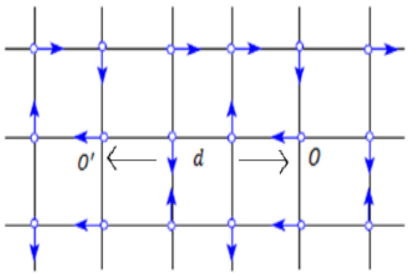

Figure 2.1: 2D lattice with randomly oriented direction where two Eulerian walkers are situated at O and $O^{'}$ which are separated distance d

For a single Euler walker, the asymptotic shape of the cluster formed by sites visited is found to be circular as shown in Fig 2.2. The cluster of region visited by walker shown in Fig 2.2 is irregular in shape but when it was averaged over 10^6 different initial arrow configurations some regularities were observed. The shape of the visited site was found to be a perfect circle. Fig. 2.3 shows the plots for $\overline{n_N}(x)$= constant where isopleths for $\overline{n_N}(x)$= 10, 20, 30, 40, 50 are plotted. For Case 2, the shape was again found to be circular and for case 3 the shape was found to be elliptical, arched towards upward direction. Fig. 2.4 and Fig. 2.5 shows plots of line $\overline{n_N}(x)$= constant for above 2 mentioned cases respectively. Both of these plots were observed after 10^6 different initial arrow configurations on 2D lattice.

It is clear from these figues that the sites closer to origin are visited more often,

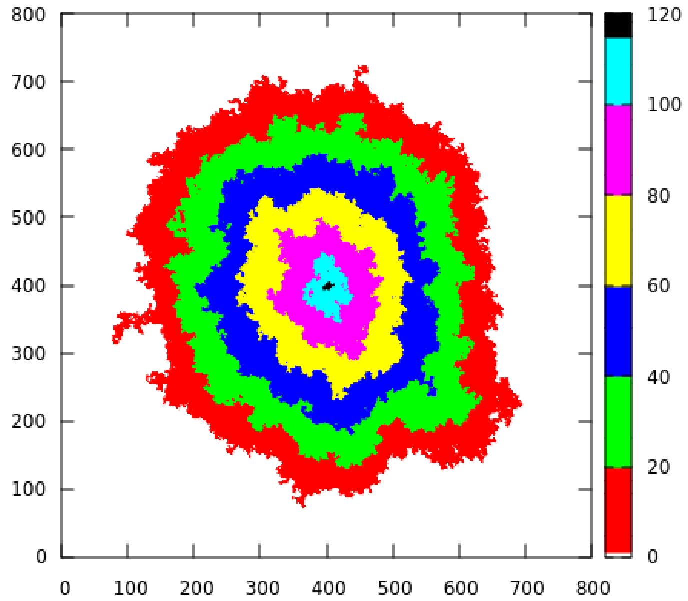

Figure 2.2: A single walk of case 1 with 10^7 steps

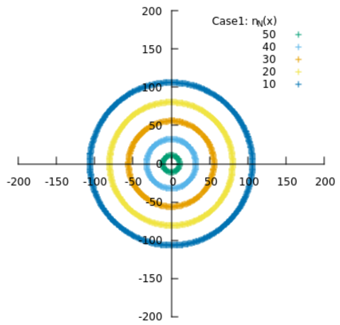

Figure 2.3: Contour of constant number of visits as indicated $\overline{n_N}(x)$:Case 1

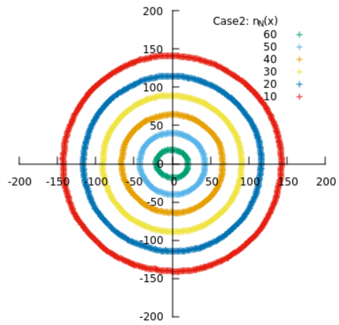

Figure 2.4: Contour of constant number of visits as indicated $\overline{n_N}(x)$:Case 2

therefore for $|x_2| > |x_1|$, x_1 will be visited more oftenly initially where x_2 and x_1 are distances from origin. As N tends to infinity the difference between the number of

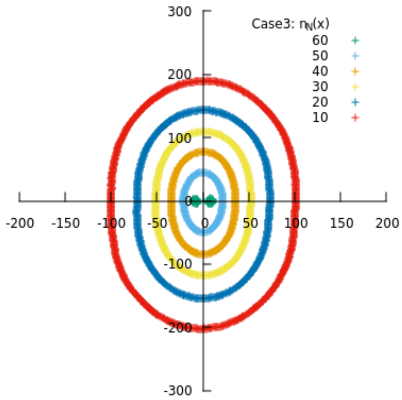

Figure 2.5: Contour of constant number of visits $\overline{n_N}(x)$ as indicated :Case 3

times sites at distances x_1 and x_2 are visited becomes constant. Hence

$$\overline{n_N}(x_1) - \overline{n_N}(x_2) = aN^{1/3}[F(y_1) - F(y_2)] = Constant \tag{2.1}$$

Also

$$\overline{n_N}(|x|) = aN^{\frac{1}{3}} F\left(\frac{|x|}{bN^{\frac{1}{3}}}\right) \tag{2.2}$$

is the scaling function[7] for average number of visits to the site x. $\overline{n_N}(|x|)$ is proportional to $N^{1/3}$ and is dependent on $|x|$ which is distance from the origin. $F(y)$ is the linear function of y. Therefore,

$$F(y) = \begin{cases} 1 - y, & for 0 \leq y \leq 1. \\ 0, & otherwise. \end{cases} \tag{2.3}$$

because as the distance of the site increases it is visited less number of times. Fig 2.6 shows the scaling of $\overline{n_N}(x)$ for different walks of $N = 10^6, 10^7, 10^8$ averaged over different initial arrow configurations. As the number of steps increases, the scaled data approaches the scaling form as mentioned in Eqn. 2.7.

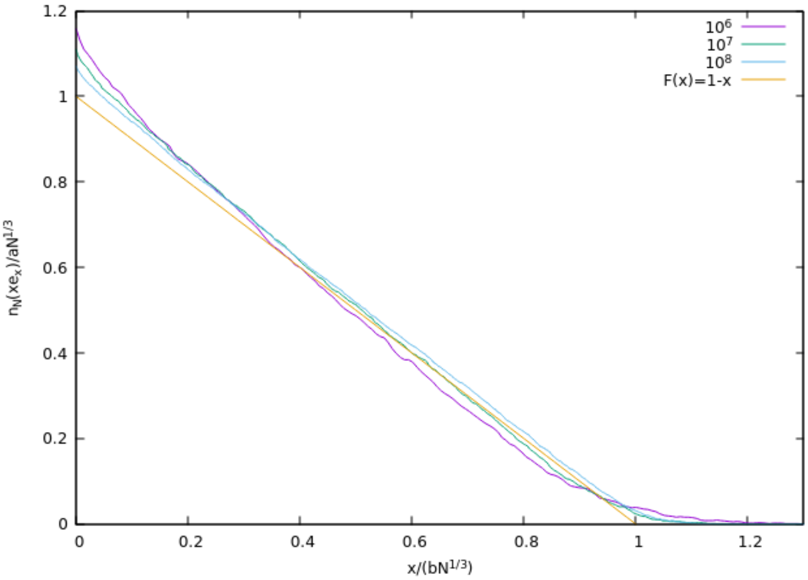

Figure 2.6: $\overline{n_N}(x)$ as a function of distance x from origin for different values of N along with scaling function

Chapter 3

Results and Discussions

In the previous chapter, the shape of cluster was observed through large statistical averaging. Now to analyze the shape of cluster we will use concept of moments and Fourier series to carry out the statistical analysis.

3.1 Statistical Analysis

The fundamental task in statistical analysis is to characterize the variations of data set generated in simulations. One of the methods is to determine the moments of the data set. The nature of the moments help us to determine the distribution functions and patterns of the data set. While observing a single anticlockwise walker on a 2D square lattice, the standardized fourth moment ratios was found to be 3.002 for Case 1 and 3.6 for Case 2. As four directions are equivalent in case of circular symmetry therefore for a cluster of circular shape following relation will hold

$$\frac{\langle x^4 \rangle}{\langle x^2 y^2 \rangle} = \frac{\langle y^4 \rangle}{\langle x^2 y^2 \rangle}, \tag{3.1}$$

And for case 3 it was found that

$$\frac{\langle x^4 \rangle}{\langle x^2 y^2 \rangle} = 4.2, \tag{3.2}$$

and

$$\frac{\langle y^4 \rangle}{\langle x^2 y^2 \rangle} = 48.2, \tag{3.3}$$

This shows that this data set is not generating cirular shape as two moments are not related, therefore we calculate the other moments

$$\langle x^2 \rangle = 485.76, \langle y^2 \rangle = 1090.5851, \tag{3.4}$$

and found that

$$\frac{\langle x^2 \rangle}{b^2} + \frac{\langle y^2 \rangle}{a^2} = 0.999, \tag{3.5}$$

where x=$Rcos\theta$, y=$Rsin\theta$ and R is distance from origin. For case 3, b=22.0418 and a=46.703 with 0.06% and 0.02% error respectively and $\langle x^2 \rangle$ and $\langle y^2 \rangle$ are calculated at fixed R of an elliptical contour.

The analysis of fourth moment can be linked to kurtosis which is standardized fourth moment. kurtosis for a distribution of random variable X is defined as

$$Kurt[X] = \frac{E[(X-\mu)^4]}{(E[X-\mu]^2)^2}, \tag{3.6}$$

where μ is the mean and σ is the standard deviation.

It describes the shape of the distribution's tail to its overall shape. As kurtosis is the expectation value of random variable minus mean value raised to the fourth power, therefore the values that are closer to the mean will contribute nothing to the kurtosis.

Therefore, the only data values that contribute to the kurtosis are those outside the region of the peak i.e. outliers. That is why, kurtosis is high for Case 3.

When, the mean square displacement $\overline{R^2}_N$ of a single anticlockwise Eulerian walker was calculated as a function of N, it was found that $\overline{R^2}_N$ grows as N^{2v} were $v = 1/3$. Fig 3.1 shows the plot of $\overline{R^2}_N$ versus N for a walk of 10^6 steps averaged over 10^6 different initial realizations. For a single walker, Euler walk was observed after large number of steps. In the initial stage, walker behaves like random walker as shown in Fig 3.1. Same is the observed for case 2 and case 3 as shown in Fig 3.2 and Fig 3.3 respectively. Fig 3.4 shows the plot of R_N^2 as a function of N for all cases.

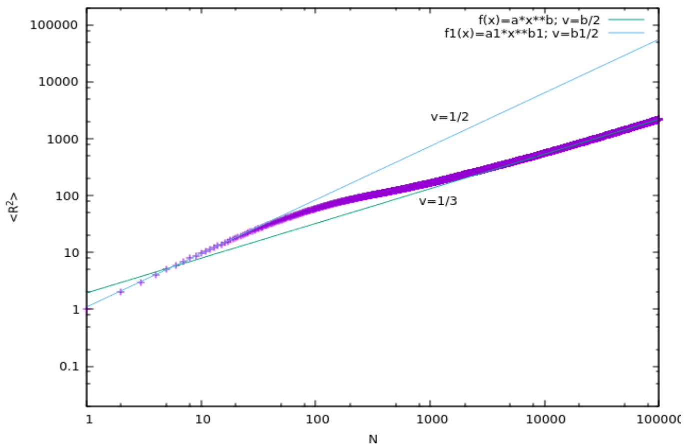

Figure 3.1: Mean Square Displacement $\overline{R^2}_N$ as function of N : Case 1

It is clear from initial values that single euler walker converges to euler regime slowly as compared to cases 2 and 3. Fig. 3.5 shows same contours for case 1 and case 2. It is also clear from the plot that values of v is slightly higher for walks in the cases 2 and 3. As v increases, the region spanned also increases. Therefore case 2 and 3 cannot show perfect Euler behavior. It is clear from values of v in Fig 3.4 that these walks are shifting towards random nature. The analysis of the shape of the rings for the contours of the constant average number of visits $\overline{n_N}(x)$ was also done through

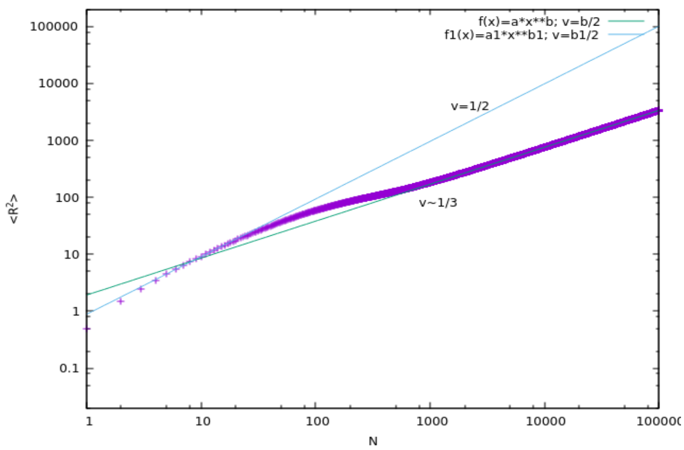

Figure 3.2: Mean Square Displacement $\overline{R^2}_N$ as function of N : Case 2

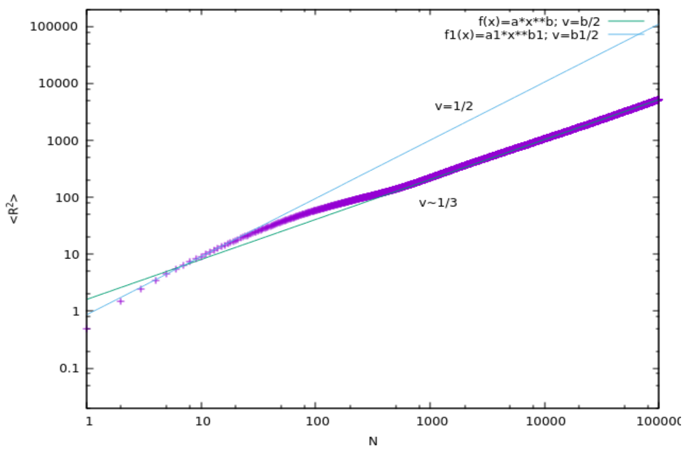

Figure 3.3: Mean Square Displacement $\overline{R^2}_N$ as function of N : Case 3

Fourier analysis. For this the shape of the rings for large N was defined as

$$f(\theta) = \lim_{N\to\infty} \frac{r_N(\theta)}{N^{1/3}} \tag{3.7}$$

where $r_N(\theta)$ is the angle dependent radius for which $0 \leq \theta \leq 2\pi$. If the shape of cluster is a perfect circle than $f(\theta) =$ constant because as N tends to infinity, $r_N(\theta)$ tends to $N^{1/3}$, otherwise $f(\theta)$ is the periodic function of θ which can be expressed in

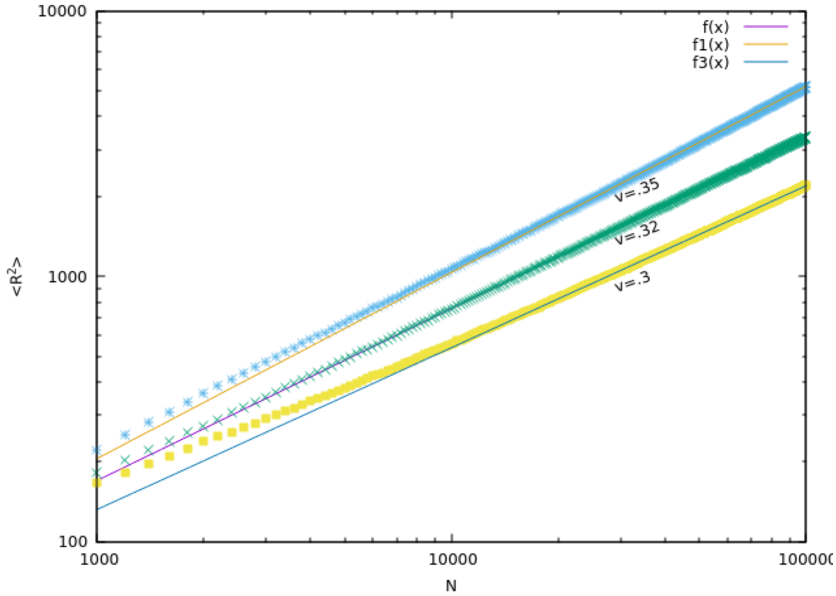

Figure 3.4: Mean square Displacement $\overline{R^2}_N$ as function of N for all 3 cases

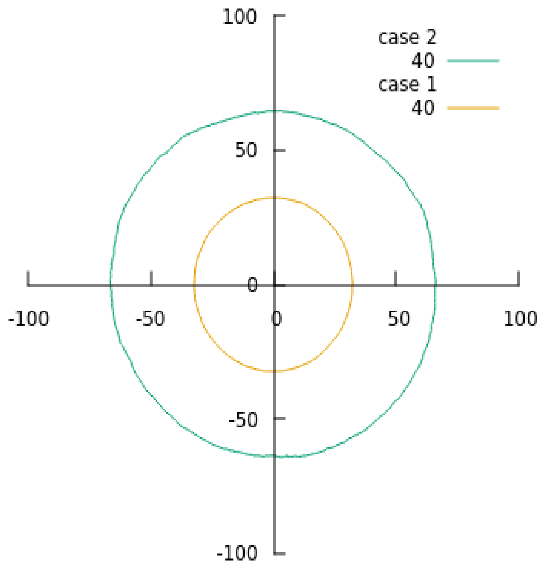

Figure 3.5: Contour of constant number of visits n_N for case 1 and case 2 on 2D lattice

terms of Fourier cosine series.

$$f(\theta) = \sum_{m=0}^{\infty} a_{4m} Cos(4m\theta) \tag{3.8}$$

Due to fourfold symmetry of the shape, this series will have terms terms with m=4s (s= 0,1,2...). A circular shape is implied by vanishing of all the modes $a_{4m}s'$ for all $m \neq 0$. A normalized amplitude of the fourth Fourier mode is given by

$$A(r) = \frac{\sum_j \overline{n_N}(x_j) Cos(4\theta_j)}{\sum_j \overline{n_N}(x_j)} \tag{3.9}$$

The summation j was done over all the lattice points whose Euclidean distance from the origin lies between $r-1$ and r and θ_j is the angle that vector x_j makes with positive x-axis.

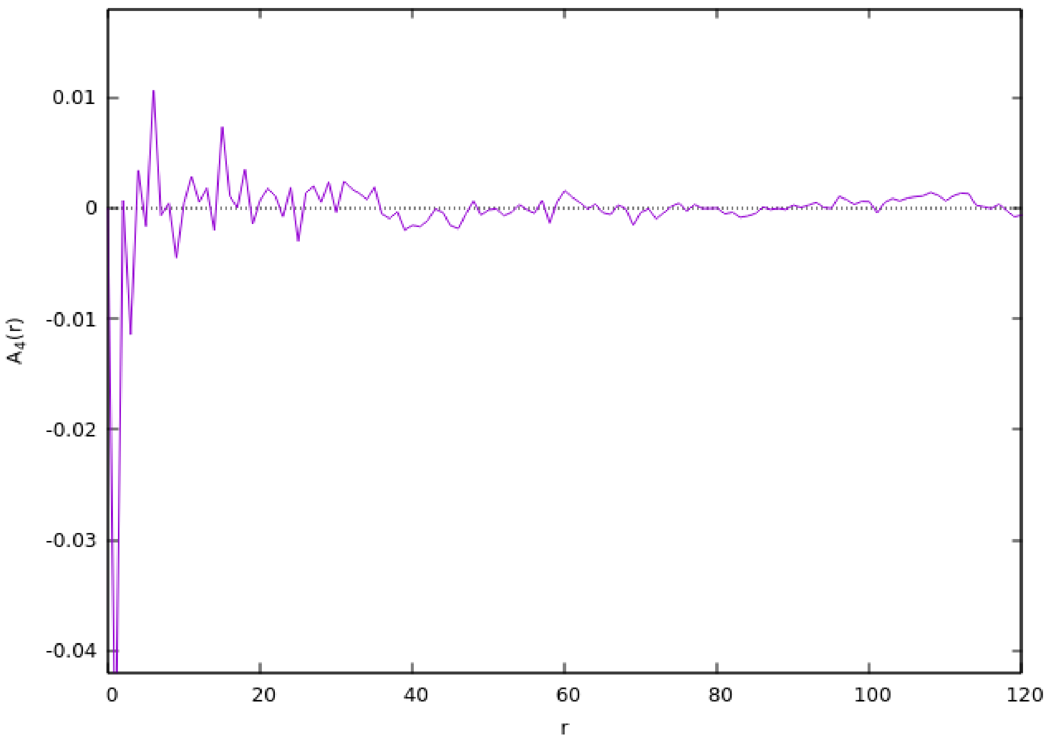

Figure 3.6: $A_4(r)$ as a function of r

Fig.3.6 shows $A_4(r)$ as function of r for a single euler walk of 10^6 steps averaged over 10^6 different initial realization. It was observed that these fluctuations do not tend to zero even with this large statistical averaging. These fluctuations arise because lattice points are non-uniformly distributed along the ring within the radius r-1 and r. These fluctuations are studied in mathematics under Gauss Circle problem and are said to be of number-theoretic origin.

The method adopted to study the shape of rings was carried out by fitting the data and observing deviations from the mean values. Fig 3.7 and Fig 3.8 shows a zoomed section for the plot of contour $\overline{n_N}(x)$ =1 and $\overline{n_N}(x)$ =40 respectively for case 1 and similarly Fig 3.9 and Fig 3.10 show contours of case 2. It can also be seen from the figure that the contours at farther distance are rough as compared to contours at closer distance from origin.

Along with this the deviation of distance of the points on the line of $\overline{n_N}(x)$ =constant

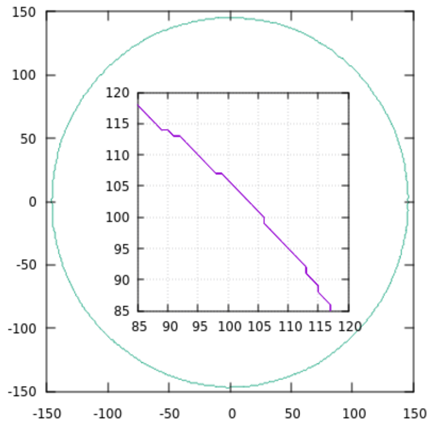

Figure 3.7: Close up region for contour of $\overline{n_N}(x) = 1$:Case 1

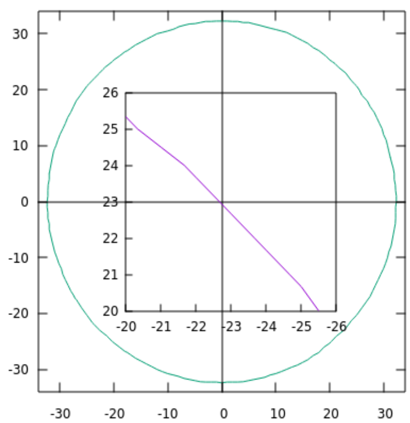

Figure 3.8: Close up region for contour of $\overline{n_N}(x) = 40$:Case 1

with mean radius $< R >$ was calculated and it was observed that these deviations

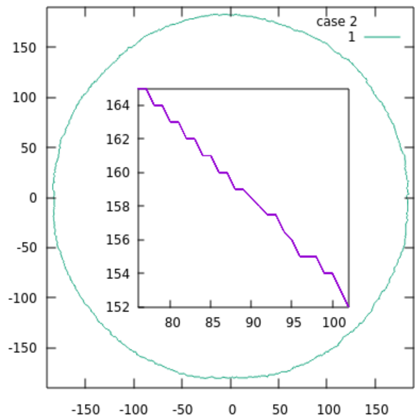

Figure 3.9: Close up region for contour of $\overline{n_N}(x) = 1$:Case 2

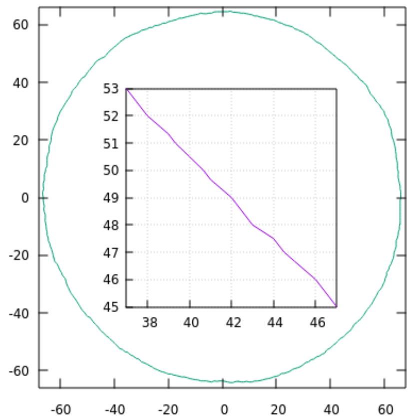

Figure 3.10: Close up region for contour of $\overline{n_N}(x) = 40$:Case 2

are of the order 10^{-4} for Case 1 and the pattern observed is someway linked to the symmetry of cluster. The deviations decreases as $< R >$ increases as shown in Fig 3.11. The probable reason can be that as number of steps increases the walk becomes clusterable; therefore the cluster will have more specific symmetry and shape at larger radius. It was also observed that these deviations have different pattern for Case 3 as shown in Fig.3.12 where elliptical cluster was observed and these deviations also decrease as radius increases. This pattern can also be linked to the symmetry in observed cluster.

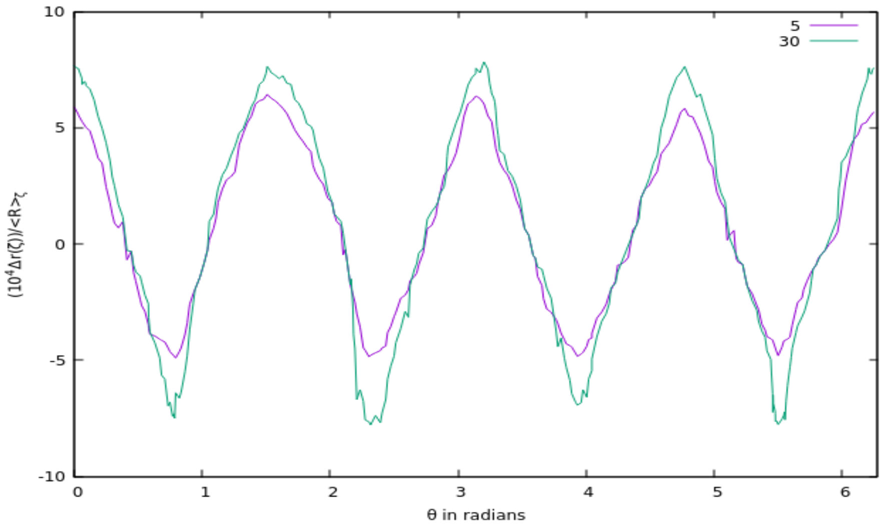

Figure 3.11: Deviation of distance of the points on the line $\overline{n_N}(x)$ =constant as indicated with mean radius $< R >$: Case 1

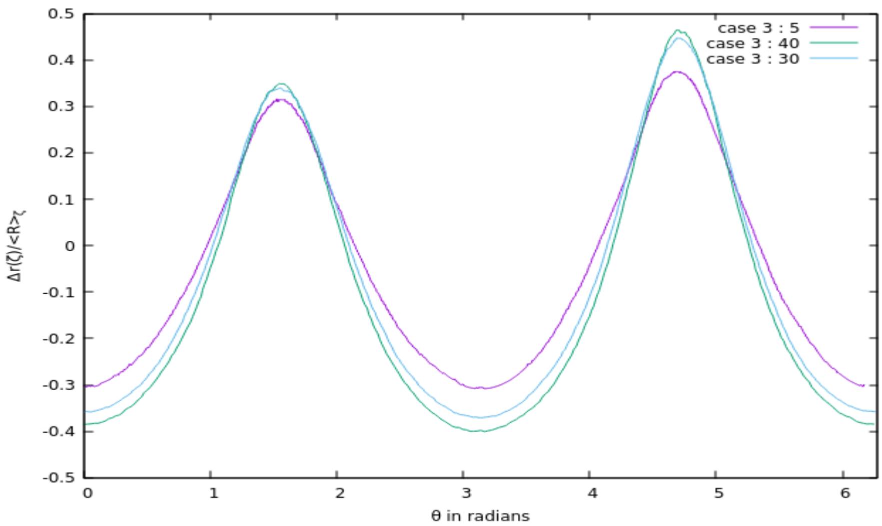

Figure 3.12: Deviation of distance of the points on the line $\overline{n_N}(x)$ =constant as indicated with mean radius $< R >$: Case 3

Chapter 4

Summary

In the present work, the main aim was to determine the shape of region visited by different Eulerian walkers moving on two -dimensional square lattice. Initially, randomness was introduced in the system by generating random directions on the lattice using pseudo random number generator. In this work mainly three cases were studied :- A single Eulerian walker which detected the direction; rotated it 90 degree anticlockwise and then moved one step ahead in new direction. In the second case, two Eulerian walker traversed the lattice by rotating directions in same sense that is in anticlockwise direction and lastly two Eulerian walkers were studied who rotated directions in the opposite sense; one of which moved by rotating directions 90 degree clockwise and other moved by rotating directions 90 degree anticlockwise. The region spanned was found to be circular after large sample averaging of 1 million samples for Eulerian walkers observed in Case 1 and 2. It was observed after large number of steps that the cluster becomes circular. For Case 2, if we have close look at the contours closer to the origin they don't seem to be circular and are little elliptical in nature. As the walkers have been studied on 2D square with different origins separated by particular distance, the distance can be changed to see what happens. For a single walk, a elliptical cluster was observed arched in the horizontal direction but a large sample averaging is required for prominent result. For Eulerian walkers in Case 3 it was found that the cluster shape was elliptical arched in the vertical direction. This shape was observed after sample averaging of 1 million samples. In this case also, the distance can be varied and some useful observations can be obtained. For a single walk, a circular shape was observed at particular distance but again a large statistical averaging is required for prominent results. To understand the particular orientation of the cluster; either vertical or horizontal, some other analysis linked to combined moments and correlation functions can be somewhat helpful .

Bibliography

[1] N.G.Van Kampan. *Stochastic Process in Physics and Chemistry*. Revised and enlarged edition,Elsevier, New York (2003).

[2] P. Diaconis, W . Fulton. *A Growth Model, A Game, An Algebra, Lagrange inversion, and Characteristic Classes*. Rend. Sem. Mat. Univ. Pol. Torino ,Vol. 49, 1 (1991).

[3] Per bak. *How nature works: The Science of Self Organized Criticality*. Springer-Verlag New York, Inc (1996).

[4] J.C. Tanner. *Delays on two Lane Road*. Journal of the Royal Statistical Society. Series B (Methodological) Vol. 23, No. 1 (1961), pp. 38-63.

[5] Misako Takayasu and Hideki Takayasu. *1/f noise in a Traffic Model*. Volume 01, Issue 04 (1993).

[6] V. B. Priezzhev, Deepak Dhar 2, Abhishek Dhar and Supriya Krishnamurthy. *Eulerian Walkers as a model of Self-Organised Criticality*. Phys. Rev. Lett. 77 (1996) 5079 (2012).

[7] R. Kapri and D. Dhar. *Asymptotic shape of the region visited by an Eulerian walker*. Phys. Rev. E. **80**, 051118 (2009).

[8] https://mathmunch.org/2012/11/12/ sandpiles-prime-pages-and-six-dimensions-of-color/

[9] http://yuvalperes.com/router/router.html

[10] http://statslab.cam.ac.uk/ jpm205/quantum-gravity.html

[11] https://www.google.co.in/search?q=eden+growth+modeldcr=0source=lnmstbm=ischsa=Xved=0
$1280bih = 694imgrc = YWUJsyGUm3jTOM$:

Appendix A

Useful code for Simulation of Eulerian Walker on Square Lattice

The following program is a section from my code which is helpful for generating lattice with random direction and carrying out Eulerian walk on 2D lattice with size ND×MD.

```
int SEED;
SEED=1234;
srand(SEED);
for(l=0;l<SAMPLES;l++){
for(j=0;j<ND;j++){
for(i=0;i<MD;i++){
lattice[j][i]=0;
}
}
for(j=0;j<ND;j++){
for(i=0;i<MD;i++){
nvisit[j][i]=0;
```

```
}
}
for(j=0;j<ND;j++){
for(i=0;i<MD;i++){
lattice[j][i]=rand()%4;
}
}
nvisit[xx][yy]=1;
for(k=0;k<STEPS;k++){
lattice[xx][yy]=(lattice[xx][yy]+1)%4;
RN=(lattice[xx][yy]);
switch(RN){
case 0: xx=xx+1;
break;
case 1: yy=yy+1;
break;
case 2: xx=xx-1;
break;
case 3: yy=yy-1;
break;
}
nvisit[xx][yy]+=1;
}
}
```

The C library function void srand 'seeds' the random number generator used by the function rand(). It initializes the sequence of pseudo-random numbers when rand() function is called. The seed value determines a particular sequence of random numbers and it is useful to fix the seed while carrying out the simulations so that we get reproducible data.