Ramsey Theory and Topological Dynamics

Vijay Singh Yadav

A dissertation submitted for the partial fulfilment of BS-MS dual degree in Science

Indian Institute of Science Education and Research Mohali

April 2018

Certificate of Examination

This is to certify that the dissertation titled **"Ramsey Theory and Topological Dynamics"** submitted by **Vijay Singh Yadav** (Reg. No. MS13130) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Prof. Kapil H. Paranjape   Dr. V. R. Srinivasan   Dr. Chetan T. Balwe
(Supervisor)

Dated: April 20, 2018

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr.Chetan Tukaram Balwe at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

Vijay Singh Yadav
(Candidate)

Dated: April 20, 2018

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Chetan Tukaram Balwe
(Supervisor)

Acknowledgment

First of all, I would like to express my sincere gratitude to my supervisor Dr. Chetan Tukaram Balwe for providing me an opportunity to work under his guidance and for his constant support throughout the duration of the project. I would also like to thank my thesis committee members, Prof. Kapil Hari Paranjape and Dr. Varadharaj R. Srinivasan for giving their valuable suggestions to work. I am grateful to my parents for their guidance, encouragement and support. I would also like to extend my gratitude to my friends for their support.

Vijay Singh Yadav

Notation

Here are some notations that we will be using throughout this Ramsey theory:

$\mathbb{N} = \{1, 2, 3, \cdots\}$ = set of positive integers.

$|X|$ = cardinality of set X.

$[n] = \{1, \cdots, n\}$ defined for $n \in \mathbb{N}$.

$[X]^k = \{Y : Y \subset X, |Y| = k\}$.

When $X = [n]$ we say $[n]^k = \{Y : Y \subset \{1, \cdots, n\}, |Y| = k\}$

Abstract

Van Der Waerden's theorem says that "If the positive integers are partitioned into two classes then at least one of those classes must contain arbitrarily long arithmetic progression." A more generalized version of this theorem can be said in the way that "If the set of positive integers are partitioned into r classes then at least one of the class must contain an arithmetic progression of arbitrary finite length." We will study the proof of this theorem with Ramsey Theory and Topological Dynamics.

Contents

Chapter 1

Introduction

Ramsey theory is named after Frank Plumpton Ramsey, he is mainly known for his major contribution in this area before his early death at age 26 in 1930.And later the theory was developed by Erdos.
In the second chapter we have proved the Ramsey's theorem and Compactness principle.The classical problem in Ramsey theory is the party problem, which asks the minimum number of guests R(m,n) that must be invited so that at least m will know each other or at least n will not know each other . Here, R(m,n) is called a Ramsey number. We will begin with this example with $m = 3$ and $n = 3$ as a glimpse for Ramsey theory.Then we have proved the Ramsey's Theorem for graph ,which gives the proof for the existence of minimum such number, what we shall call Ramsey number.Then we give a generalized version of Ramsey theorem for graph, which is known as Ramsey's Theorem hypergraph , that deals with the coloring of $[n]^k$. Then we come to the compactness principle which allows us to go from n sufficiently large to larger set $\mathbb{N}$, an $r-$ coloring of $[n]$ or $[n]^k$ has certain property then any r-coloring of larger set $\mathbb{N}$ or $[\mathbb{N}]^k$ has the property.
In 1927 B.L. Van Der Waerden published a proof of the theorem which says:"For any finite partition of $\mathbb{N}$ there is a partition which contains arithmetic progression of arbitrary finite length." In the chapter 3 we have given a proof of the theorem of Van Der Waerden's using Ramsey theory.
In chapter 4 we have proved the theorem of Van Der Wearden using the topological Dynamics.The approach to problems we are going to use is slightly different ,we first translate a combinatorial problem into a problem in dynamical systems and then by proper use of techniques in dynamical systems ,we try to solve the restated problem. Hillel Furstenberg is widely known for his contributions in 1970's for developing the

connections between Ramsey theory and topological dynamics and then many mathematicians has elaborated this connection. We can define the dynamical systems with different properties on the underlying space and by the map which we shall call the transformation of the states. We look for the topological properties, which is the field of topological dynamics.
In chapter 5, the main result is the theorem of Hindman's which says: "For any finite partition of $\mathbb{N}$ there is some C_j which contains an IP -set."
In the last chapter we have shown the IP-version of Van Der Waerden's theorem.

Chapter 2

Ramsey Theory

2.1 Puzzle problem

We will start with a "puzzle problem" and this problem , we can consider as the first nontrivial example of Ramsey theory.

Example 1. *In any collection of six people either three of them mutually know each other or three of them mutually do not know each other.*

Here we will assume that the relation of "knowing" is symmetric, that means if A and B are two persons and if A knows B then B knows A.But we will not assume transitivity ; that is if A,B,C are three persons and A knows B and B knows C then A may or may not know C.We will give a proof of this example.
Let A,B,C,D,E,F are six people. Consider the relation of A with rest of 5 persons.Now by the Pigeon-hole principle we can say that either A must know at least three of them or not know at least three of them .Suppose A knows B,C,D. Then if B and D also knows each other then we got three people A,B,D who knows each other.Or if none of B,C,D knows each other ,then again we got three people who do not know each other. Consider the other case :suppose A does not not three of them,let say D,E,F are those three people.Now if D,F are also unknown to each other then we got a set of three people who do not know mutually each other.And if D,E,F know mutually each other then we got a set of three people who mutually know each other. so we are done.

Definition 2.1.1. An $r-coloring$ of a set S is a map

$$\chi : S \longrightarrow [r].$$

For some element $s \in S$, we will call $\chi(s)$ the color of s. A set $T \subseteq S$ is said to be monochromatic under χ if χ is constant on T.

Now we will introduce arrow notation that will be used throughout this Ramsey theory:

Definition 2.1.2. We write

$$n \longrightarrow (l)$$

if given any $2-coloring$ of $[n]^2$, there is a set $T \subseteq [n], |T| = l$ so that $[T]^2$ is monochromatic.

Definition 2.1.3. We write $n \longrightarrow (l_1, \cdots, l_r)$ if, for every $r-coloring$ of $[n]^2$, there exists $i, 1 \leq i \leq r$, and a set $T \subseteq [n], |T| = l_i$ so that $[T]^2$ is colored i.

In case $l_1 = \cdots = l_r = l$ we use the shorthand

$$n \longrightarrow (l)_r.$$

That is , every $r-$ coloring of $[n]^2$ yields a monochromatic $[l]^2$. If the number of colors r is not indicated it is assumed to be 2. Thus $n \longrightarrow (l), n \longrightarrow (l)_2$, and $n \longrightarrow (l, l)$ denote the same thing.

Definition 2.1.4. We will denote the Ramsey Function by $R(l_1, \cdots, l_r)$, which denotes the minimal n such that

$$n \longrightarrow (l_1, \cdots, l_r).$$

2.2 Ramsey's Theorem - For graph

Theorem 1. Ramsey Theorem - For graph: The Ramsey function R is well defined , that is for all $l_1, \cdots, l_r$ there exists n so that

$$n \longrightarrow (l_1, \cdots, l_r).$$

We first give the proof of this theorem for case $r = 2$.

Proof For proving this theorem we will be using a double induction on l_1 and l_2. We know that $R(l, 2) = R(2, l) = l$ so, first step of induction is true. Now assume by induction that $R(l_1, l_2 - 1)$ and $R(l_1 - 1, l_2)$ exists.We claim that $n = R(l_1, l_2 - 1) + R(l_1 - 1, l_2) = R(l_1, l_2)$.
Claim: $R(l_1, l_2 - 1) + R(l_1 - 1, l_2) \longrightarrow (l_1, l_2)$.

Proof: Fix a 2-coloring χ of $[n]^2.n = R(l_1, l_2 - 1) + R(l_1 - 1, l_2)$.
Fix one element $x \in [n]$ and set

$$L_x = \{y \in [n] : \chi(x, y) = 1\}.$$

$$M_x = \{y \in [n] : \chi(x, y) = 2\}.$$
$$= [n] - L_x - \{x\}.$$

Then $|L_x| + |M_x| = n - 1$ so that either

a) $|L_x| \geq R(l_1-, l_2)$. or

b) $|M_x| \geq R(l_1, l_2 - 1)$.

Assume the case (a) first. Apply the definition of R, and we say either there exists $T \subseteq L_x, |T| = l_2$ such that $[T]^2$ is colored 2 or there exists $S \subseteq L_x, |S| = l_1 - 1$ so that $[S]^2$ is colored 1. If we take $S^* = S \cup \{x\}$. Then since $S \subseteq L_x$ all $\{x, s\}, s \in S$ are colored 1. Then $|S^*| = l_1$ and $[S^*]^2$ is colored 1, so we are done for this case.Case b)is symmetric.
The proof for $l_1, l_2, \cdots, l_r$ follows same as the above argument , here we can use induction on $l_1, l_2, \cdots, l_r$ to show that :

$$2 + \sum_{i=1}^{r} R(l_1, l_2, \cdots, l_i - 1, \cdots, l_r) - 1 \rightarrow (l_1, l_2, \cdots, l_r)$$

so we are done.

2.3 Ramsey's Theorem :For hypergraph

We now consider coloration of $[n]^k$, where k is an arbitrary integer. This generalizes the case $k = 2$.

Definition 2.3.1. $n \longrightarrow (l_1, \cdots, l_r)^k$ if, for every $r-$ coloring of $[n]^k$ there exists $i, 1 \leq i \leq r$, and a set $T, |T| = l_i$ so that $[T]^k$ is colored i.
In the case $l_1 = \cdots = l_r = l$ we use the shorthand

$$n \longrightarrow (l)_r^k.$$

we say in this case that every $r-$ coloring of $[n]^k$ yields a monochromatic $[l]^k$. If the number of colors r is not indicated it is assumed to be 2. Thus $n \longrightarrow (l)^k, n \longrightarrow (l)_2^k$, and $n \longrightarrow (l, l)^k$ are identical relations.If k is not given it is also assumed to be 2.
The Ramsey function for $k-$ sets is indicated by R_k :

$$R_k(l_1, \cdots, l_r) = \min\{n_0 : for \;\; n \geq n_0, n \longrightarrow (l_1, \cdots, l_r)^k\},$$

$$R_k(l, r) = \min\{n_0 : for \;\; n \geq n_0, n \longrightarrow (l)_r^k\},$$

$$R_k(l) = \min\{n_0 : for \;\; n \geq n_0, n \longrightarrow (l)^k\}.$$

Theorem 2. Ramsey's theorem: The function R is well defined; that is , for all $k, l_1, \cdots, l_r$ there exists n_0 so that , for $n \geq n_0$,

$$n \longrightarrow (l_1, \cdots, l_r)^k.$$

Proof We will give the proof using induction. The case for $k = 1$ is done in Ramsey's theorem -for graph.So the first step of induction is true.Now let $\chi : [n]^k \longrightarrow \{1, \cdots, r\}$ be an $r-$ coloring . By induction hypothesis, let

$$t = R_{k-1}(l, r).$$

now we choose some arbitrary elements $y_1, \cdots, y_{k-2} \in [n]$ and denote

$$S_{k-2} = [n] \setminus \{y_1, \cdots, y_{k-2}\}$$

Now we can define points y_i and sets S_i as follows:
(i) If S_i is defined , then choose any $y_{i+1} \in S_i$
(ii) If y_{i+1} is defined then divide $S_i \setminus \{y_{i+1}\}$ into equivalence classes by

$$x \equiv y \Leftrightarrow (\forall T \subseteq \{y_1, \cdots, y_{i+1}\}, |T| = k - 1)$$

$$\chi(T \cup \{x\}) = \chi(T \cup \{y\})$$

and choose S_{i+1} equal to the maximal class.
So the number of these equivalence classes is atmost $r^{\binom{i+1}{k-1}}$. Because equivalence class has been determined by the color of $\binom{i+1}{k-1}$ sets.
It is clear from the construction that

$$S_{i+1} \subseteq S_i \setminus \{y_{i+1}\} \qquad (2)$$

and

$$|S_{i+1}| \geq \frac{(|S_i| - 1)}{r^{\binom{i+1}{k-1}}}$$

from this relation

$$u_{i+1} = \frac{u_i - 1}{r^{\binom{i+1}{k-1}}} \qquad , u_{k-2} = n - (k-2).$$

surely $n = 2r^{\sum_{i=k-1}^{t-1} \binom{i+1}{k-1}}$ suffices. Next consider $y_1, \cdots, y_t$ sequence. Assume that $1 \leq i_1 < \cdots < i_{k-1} < s \leq t$. now from (2) and (i)

$$y_s \in S_{s-1} \subseteq S_{i_{k-1}+1}.$$

Further by definition of equivalence relation:

$$\chi(y_{i_1}, \cdots, y_{i_{k-1}}, y_s) = \chi(y_{i_1}, \cdots, y_{i_{k-1}}, x) \qquad \forall x \in S_{i_{k-1}+1} \qquad (3).$$

And we can say that (3) is true for $x = y_r$ where $i_{k-1} < r < t$. It follows that we can define the coloring χ^* of $(k-1)$ elements subsets of $\{y_1, \cdots, y_t\}$ by the condition :

$$\chi^*(y_{i_1}, \cdots, y_{i_{k-1}}) = \chi(y_{i_1}, \cdots, y_{i_{k-1}}, y_s) \qquad \forall i_{k-1} < s \leq t. \qquad (4)$$

Now here we are almost done.By the induction hypothesis and the choice of t the sequence $y_1, \cdots, y_t$ has a subsequence $d_1, \cdots, d_l$ which is monochromatic under χ^*, that is :
Every subset of $\{d_1, \cdots, d_l\}$ with $(k-1)$ element, has the same color ,say red , under χ^*. Then for all sequences with indices $1 \leq j_1 < \cdots < j_{k-1} \leq l$ we will have

$$\chi(d_{j_1}, \cdots, d_{j_{k-1}}, d_{j_k}) = \chi^*(d_{j_1}, \cdots, d_{j_{k-1}}) = red.$$

This follows from (4). So we have to found an subset of $[n]$ with $l-$ elements such that its all subsets with $k-$ elements, have the same color under our original coloring χ. This completes the proof.

2.4 Compactness Principle

Now we will just state the "Tychonoff theorem" which says that :

Theorem 3. An arbitrary product of compact spaces is compact in the product topology.

Definition 2.4.1. Let $H = (V, E)$ be a hypergraph and $W \subseteq V$. And the restriction of H to W is denoted by H_W , is a hypergraph $H_W = (W, E_W)$,

$$E_W = \{X \in E : X \subseteq W\}.$$

Theorem 4. Let $H = (V, E)$ be a hypergraph where all $X \in E$ are finite (but V need not be). Suppose that , for all $W \subseteq V, W$ finite,

$$\chi(H_W) \leq r.$$

Then

$$\chi(H) \leq r.$$

Proof Let T be the set of all functions $f : V \to [r]$. We topologize T by giving $[r]$ the discrete topology and giving T the induced function space topology.In other words , for all $x_1, \cdots, x_n \in V, y_1, \cdots, y_n \in [r]$.
We will show that T is homeomorphic to direct product of $|V|$ copies of $[r]$.That means there is a homeomorphism :

$$\phi : T \to \prod_{i \in |V|} [r]$$

.
we will show that this is a homeomorphism.

- ϕ is continuous.
  consider
  $$\phi : T \to \prod_{i\in|V|}[r]$$
  given by
  $$f \mapsto (f(a_1), f(a_2), f(a_3), \cdots)$$

  take $([r]^{|V|}, \tau)$ where τ is the induced function topology on $[r]^{|V|}$.Where
  $$S_{x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_n} = \{f : f(x_i) = y_i\}$$
  is a basis foe the topology on $[r]^{|V|}$ and is both open and closed.
  so ϕ will be continuous because
  $$\phi^{-1}(f(a_1), f(a_2), \cdots) = f$$
  and this f is in the basis of the topology. Hence it is open.Hence ϕ is continuous.

- Now we will show that ϕ^{-1} is continuous.
  take
  $$\phi^{-1} : \prod_{i\in|V|}[r] \to T$$
  given by
  $$(y_i)_{i\in X} \mapsto g$$
  where
  $$g : V \mapsto [r]$$
  define as
  $$x_i \mapsto y_i$$
  and g are in the basis. Hence g^{-1} will be open. Hence ϕ^{-1} is open.
  Hence $\phi : \prod_{i\in|V|}[r] \to T$ is a homeomorphism.Hence T is compact, because it is the direct product of $|V|$ copies of $[r]$ and $[r]$ is compact because it is a finite set with discrete topology.Hence by Tychnoff theorem T is compact.

Now for every finite $W \subseteq V$ we denote F_W to be the set of functions $f \in T$ such that no $X \in E, X \subseteq W$ is monochromatic. The set F_W is the collection of those functions that are $r-$ colorations when restricted to W. Since each F_W is the union of a finite number of slices $S_{w_1,\cdots,w_n,y_1,\cdots,y_n}(W = \{w_1, \cdots, w_n\})$ hence is closed.And each $F_W \neq \phi$ because , we assumed that , there is an $r-$ coloring of each finite set W. Clearly , if $W \subseteq W', F_W \supseteq F'_W$. We can apply this on sets ,if $W_1, \cdots, W_m$ are finite subsets of V then

$$F_{W_1} \cap \cdots \cap F_{W_m} \supseteq F_{W_1 \cup \cdots W_m}.$$

Now because $W_1, \cdots, W_m$ are finite subsets so $W_1 \cup \cdots \cup W_m$, so their finite union will again be finite, so $F_{W_1 \cup \cdots \cup W_m} \neq \phi$. Thus $\{E_W : W \subseteq V, W finite\}$ is a finite family of closed sets which satisfies the finite intersection property that means any finite intersection of the F_W is nonvoid. And in a compact topological space, if a family of closed sets $\mathfrak{F}$ satisfies the finite intersection property then $\cap \mathfrak{F} \neq \phi$; that means there exists $f : V \to [r], f \in F_W$ for all $W \subseteq V, W$ finite. This f is the desired coloring , for if $X \in E$, so X is finite, $f \in F_X$, and therefore X is not monochromatic under f. So we are done.

Chapter 3

Van Der Waerden's Theorem

3.1 Van Der Waerden's Theorem

Theorem 5. If we partition the positive integers into two classes then there will be at least one class which must be containing arbitrarily long arithmetic progressions.

A more generalized version of the above statement can be said in the way: If we partition the positive integers into r- classes then there will be at least one class which must be containing arbitrarily long arithmetic progressions.

We can make two modifications :

- For each k we will partition only a finite initial segment of integers (depending on k) so in this we are forcing at least one class to contain an arithmetic progression of k terms.
- Second, we are allowing the partition of the sets of integers into r classes instead of just two.

Thus modified statement is as follows:

There exists an positive integer $W(k, r)$ for all positive integers k and r so that , if we partition the set of integers $1, 2, \cdots, W(k, r)$ into r classes, then there will be at least one class which will contain a $k-$ term arithmetic progression.

3.1.1 An Example

We will give an example that will clearify the statement of the theorem:

Example 2. $W(3,2)$

To motivate the proof of the general theorem, we first examine a small cases. Let us consider the case $k = 3, r = 2$. We claim that we can take $W(3,2) = 325$. For this, assume that integers $\{1, 2, \cdots, 325\} = [1, 325]$ are arbitrarily partitioned into two classes(because here r = 2) . Now divide them into 65 blocks of length 5, that is ,

$$[1, 325] = [1, 5] \cup [6, 10] \cup \cdots \cup [321, 325],$$

which we say that block B_1 consisting the numbers from 1 to 5 and block B_2 consisting the numbers from 6 to 10 and so on. Since we have to split these integers into $r = 2$ classes, that is , they are 2-colored , there are just $2^5 = 32$ possible ways to 2-color a block B_i. Thus , out of the first 33 blocks B_i there will be two blocks that will be 2-colored in exactly the same way (by Pigeon-Hole principle), say B_{11} and B_{26}. Now focus at the 2-coloring of $B_{11} = \{51, 52, 53, 54, 55\}$. Of the first three elements of B_{11} that is $\{51, 52, 53\}$, at least two of them must have the same color say x and $x + d$. Since x and $x + d$ belong to $\{51, 52, 53\}$, $x + 2d$ belongs to B_{11}. (This is why we choose B_i to have length of 5.) If $x + 2d$ has the same color x (and $x + d$), we are done. Thus we may assume that it has the other color. But if the integer $205 \in B_{41}$ is red then 55,130,205 is one colored and 51, 128,205 is another colored.So we got arithmetic progression of length 3.

3.1.2 l-equivalence

Definition 3.1.1. Consider $[0, l]^m$: we define $(m + 1)$ l-equivalence classes of $[0, l]^m$. We call $(x_1, x_2, \cdots, x_m)$, $(x_1^{'}, \cdots, x_m^{'}) \in [0, l]^m$ l-equivalent if they agree up through their last occurrences of l.

Example 3. *We will discuss one example of this l-equivalence: take*

$$l = 4, m = 2$$

that is $[0, 4]^2 = \{0, 1, 2, 3, 4\}^2 = \{0, 1, 2, 3, 4\} \times \{0, 1, 2, 3, 4\}$ *Then according to the definition of l-equivalence there will be 3 equivalence classes :* $\{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1$

, $(2,1),(2,2),(2,3),(3,0),(3,1),(3,2),(3,3)\}$ *second equivalence class will be* $\{(4,0),(4,1),(4,2),(4,3$
the third equivalence class will be $\{(4,4)\}$

	0	1	2	3	4
0	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)
1	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)
2	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)
3	(3,0)	(3,1)	(3,2)	(3,3)	(3,4)
4	(4,0)	(4,1)	(4,2)	(4,3)	(4,4)

Definition 3.1.2. S(l,m): For any $r, \exists N(l,m,r)$ so that for any function $\beta : [1, N(l,m,r)] \to [1,r]$ there always exists positive integers $a, d_1, \cdots, d_m$ such that $\beta(a + \sum_{i=1}^{m} m_i d_i)$ remains constant on each l-equivalence class of $[0,l]^m$.

Example 4. *We will discuss this definition by an example:*
$l = 4, m = 2, r = r.$
there exists $N(4,2,r)$ *so that for any function* $C : [1,(4,2,r)] \to [1,r]$, *there exists positive integers* $a, d_1, d_2, \cdots, d_m$ *such that*

$$C(a + 0d_1 + 0d_2) = C(a + 0d_1 + 1d_2) = \cdots = C(a + 1d_1 + 0d_2) = \cdots$$

and

$$C(a + 4d_1 + 0d_2) = C(a + 4d_1 + 1d_2) = \cdots = C(a + 4d_1 + 3d_2) = constant.$$

and

$$C(a + 4d_1 + 4d_2) = constant.$$

Claim: $S(l,1)$ is equivalent to Van Der Waerden's Theorem.

Proof Consider $[0,l]^m = [0,l]^1$ simply 2 equivalence classes will be $\{0,1,\cdots,l-1\}$ and $\{l\}$ there exists $N(l,m,r) = N(l,1,r)$ so that for any function

$$C : [1, N(l,1,r)] \to [1,r]$$

there exists positive integers $a, d_1, \cdots, d_m$ so that

$$C(a + 0d_1) = C(a + 1d_1) = \cdots = C(a + (l-1)d_1)$$

and that is l-term arithmetic progression.

3.1.3 Van Der Waerden's Theorem: Generalized Proof

Theorem 6. $S(l, m)$ holds for all $l, m \geq 1$.

Proof Note that $S(1, 1)$ is true.Because consider $[o, 1]^1$ here equivalence classes will be $\{0\}$ and $\{1\}$. For $\{0\}$:

$$C(a + 0d_1) = constant.$$

and

$$C(a + 1d_1) = constant.$$

Now first we will prove

(i) $S(l, m) \implies S(l, m + 1)$.

Proof Assume induction hypothesis. and let
$T = N(l, m, r)$ and $T' = N(l, 1, r^T)$
Now we claim that we can choose :

$$N(l, m + 1, r) = T(T' + 1).$$

so let

$$\phi : [1, (T' + 1)T] \to [1, r]$$

be an $r-$ coloring . This induces a coloring

$$\phi' : [1, T'] \to [1, r^T].$$

by $\phi'(k) = \phi'(k')$ if and only if $\phi(kT + j) = \phi(k'T + j)$ for $j \in (0, T]$ So clearly ϕ' is well defined.
Now by induction hypothesis on T' there exists a' and d' such that

$$\phi'(a' + xd') = constant$$

on $x \in [0, l - 1]$ or $x = l$.
Now we can apply induction hypothesis $S(l, m)$ to the interval $[a'T + 1, (a' + 1)T]$

length of the interval $= T - 1:$ there exists numbers $a, d_1, \cdots, d_m$ such that

$$a + \sum_{i=1}^{m} x_i d_i \in [a'T + 1, (a' + 1)T]$$

for $x_i \in [0, l]$. And

$$\phi(a + \sum_{i=1}^{m} x_i d_i) = constant$$

on l-equivalence classes $(m + 1)$. We set $d'_i = d_i$ for $i \in [i, T]$ and $d'_{m+1} = d'T$.
Then we have to show that

$$\phi(a + \sum_{i=1}^{m+1}) = constant \qquad (1)$$

on l-equivalence classes $(m + 2)$. First in the class where $x_{m+1} = l$ there is just one element namely $(l, l, \cdots, l)$. In other classes x_{m+1} we gets the values $0, 1, \cdots, l - 1$. Let us consider such a fixed class. By the choice of a' and d'

$$\phi'(a' + x_{m+1}d') = \phi'(a')$$

for $x_{m+1} = 0, \cdots, l - 1$. Consequently by definition of ϕ'

$$\phi((a' + x_{m+1}d')T + j) = \phi(a'T + j) \qquad (2)$$

for $j \in [0, T]$ and $x_{m+1} \in [0, l - 1.]$
But the choice of numbers a and d_i we have:

$$a + \sum_{i=1}^{m} x_i d_i = a'T + j$$

for some $j = 1, \cdots, T$.

$$\phi(a + \sum_{i=1}^{m} x_i d_i) = constant \qquad (3)$$

in l-equivalence classes. Therefore (1) follows from (2) and (3):

$$a + \sum_{i=1}^{m+1} x_i d'_i = x_{m+1}d'T + a + \sum_{i=1}^{m} x_i d_i = (a' + x_{m+1}d')T + j$$

. Hence we are done.

implication (ii)$S(l,m)\forall \implies S(l+1,1)$

Our claim is that we will be able to choose

$$N(l+1,1,r) = N(l,r,r) + q.$$

So consider the coloring

$$\phi : [1, N(l,r,r) + q] \to [1,r]$$

by induction hypothesis there exists numbers $a, d_1, \cdots, d_m$ such that

$$a + \sum_{i=1}^{r} x_i d_i \leq N(l,r,r)$$

for $x_i \in [0,l]$. and

$$\phi(a + \sum_{i=1}^{r} x_i d_i) = constant \qquad (4)$$

on l-equivalence classes $(r+1)$. Now by pigeon hole principle there exists numbers w and y, $0 \leq w < y \leq r$ such that

$$\phi(a + \sum_{i=1}^{w} l d_i) = \phi(a + \sum_{i=1}^{y} l d_i) \qquad (5)$$

Now denote

$$a' = a + \sum_{i=1}^{w} l d_i$$

and

$$d' = \sum_{i=w+1}^{y} d_i$$

There exists 2 $(l+1)-$ equivalence classes (since $m=1$): $[0,\cdots,l]$ and $\{l+1\}$. The latter is singleton so that for sure $\phi(a'+xd')$ is constant.
What remains to show is that

$$\phi(a' + xd') = constant \qquad (6)$$

for $x = 0, \cdots, l$. Now by (5) we have

$$\phi(a' + 0d') = \phi(a + \sum_{i=1}^{w} ld_i) = \phi(a + \sum_{i=1}^{y} ld_i) = \phi(a' + ld').$$

In the l-equivalence class
$\{(x_1, \cdots, x_r) : x_i = l\}$ for $i \leq w$ and $x_j \in [0, l-1]$ otherwise; (4) is valid . So for any $x \in [0, l-1]$ we have :

$$\phi(a' + 0d') = \phi(a + \sum_{i=1}^{w} ld_i + \sum_{i=w+1}^{r} 0d_i)$$

$$= \phi(a + \sum_{i=1}^{w} ld_i + \sum_{i=w+1}^{y} xd_i + \sum_{i=y+1}^{r} 0d_i)$$

$$= \phi(a' + xd').$$

Hence (6) is proved . Hence the whole induction step is proved.

Chapter 4

Van der Waerden's theorem : proof with Topological Dynamics

4.1 Topological Dynamics

In this section we will be proving the theorem of van Der Waerden's with the help of topological Dynamics.

In this section X is a compact metric space and T be a homeomorphism from X to X.

Definition 4.1.1. A point $x \in X$ is said to be recurrent for the T if for some $n_k \to \infty, T^{n_k}x \to x$.

Definition 4.1.2. We will call (X, G) a dynamical system if X is compact space and G be a group of homeomorphisms from X to X. If G is a cyclic group $G = \{T^n\}$ then we denote the corresponding system by (X, T).

Definition 4.1.3. A dynamical system (X, G) is said to be minimal if no proper closed subset $Y \subset X$ is invariant by all the transformations of G.

Definition 4.1.4. We will say that a dynamical system (X, G) is homogeneous if there exists a group G' of homeomorphisms of X commuting with the transformations of G and such that (X, G') is minimal.More generally let us call a closed subset $A \subset X$ homogeneous with respect to the system (X, G) if there is a group G' of homeomorphisms of X commuting with G and such that G' leaves A invariant , and (A, G') is minimal.

lemma 1. Let (X, G) be a dynamical system with X a compact metric space. Then (X, G) is minimal if and only if for every $\epsilon > 0$ there exists a finite set of transformations $S_1, S_2, \cdots, S_N \in G$ such that for any $x, y \in X$, $\min_i d(S_i x, y) < \epsilon$.

Proof Assume that (X, G) is minimal.We will show that for any $\epsilon > 0$ there exists a finite set of transformations such that $\min_i d(S_i x, y) < \epsilon$ Let V be an open subset of X, ,consider $\cup_{S \in G} S^{-1}V$ is an open set , G-invariant set, and it is equal to all of X. Consider the compliment of this set, that will be closed , and that will be inside X so that contradicts the assumption that (X, G) is minimal.Hence $\cup_{S \in G} S^{-1}V$ is open set and equal to X. And this is G-invariant because its complement is G-invariant.Now since X is compact ,a finite sub -covering covers X. Letting V over a finite cover of X by sets of diameter $< \epsilon$ we get the condition of lemma.
Now we will prove the converse , Because X is compact so a finite subcover covers X that means $\cup_{S \in G} S^{-1}V = X$ and complement of $\cup_{S \in G} S^{-1}V$ will be empty set.that means there is no proper closed subset contained in X. Hence (X, G) is minimal.

proposition 1. Let (X, T) be a dynamical system with X a compact metric space and let A be a homogeneous closed subset of X. Suppose that for every $\epsilon > 0$ we can find $x, y \in A$ and $n \geq 1$ with $d(T^n x, y) < \epsilon$, then for every $\epsilon > 0$ we can find $z \in A$ with $d(T^n z, z) < \epsilon$, for some $n \geq 1$.

Proof As assumed in the statement that for every $\epsilon > 0$ we can find some $n \geq 1$ with $d(T^n x, y) < \epsilon$ for some pair of points $x, y \in A$.
claim: The point y can in fact be chosen arbitrarily in A.
Because it is given that A is homogeneous , so by the definition of homogeneous subset ,we take , let G be the group acting minimally on A that commutes with T , and let $S_1, S_2, \cdots, S_N \in G$ satisfy:

$$\min_i d(S_i x, y) < \epsilon/2 \qquad x, y \in A \qquad (1).$$

This is true because of previous lemma. Now we choose x_0, y_0 and n_0 so that

$$d(T^{n_0} x_0, y_0) < \delta \qquad (2)$$

where δ is so small that $d(x, x') < \delta$ implies all $d(S_j x, S_j x') < \epsilon/2$ this is true by the continuity of S_j .
Because of the definition of S_j it is clear that S_j are continuous and we know that

allS_j commutes with T, so from (2)we get $d(S_j T^{n_0} x_0, S_j y_0) = d(T^{n_0} S_j x_0, S_j y_0) < \epsilon/2$ (3), and combining this with (1) we obtain

$$\min_i d(T^{n_0} S_j x_0, y) < \epsilon$$

for any $y \in A$. After establishing this we choose an arbitrary point $z_0 \in A$ and we find $z_1 \in A, n_1 \in N$ with

$$d(T^{n_1} z_1, z_0) < \epsilon/2 \qquad (4).$$

Repeat the process for z_1, finding $z_2 \in A, n_2 \in N$ with

$$d(T^{n_1} z_1, z_0) < \epsilon_2 \qquad (5)$$

where ϵ_2 is so small that $(i)\epsilon_2 < \epsilon/2$, (ii) if we replace z_1 by $T^{n_2} Z_2$, the inequality of (4) is still valid. Now we will proceed inductively. Assume $z_0, z_1, \cdots, z_r$ have been chosen in A as well $n_1, n_2, \cdots, n_r \in N$ and $\epsilon_2, \cdots, \epsilon_r$ with $\epsilon_j < \epsilon/2$ and

$$d(T^{n_j} z_j, z_{j-1}) < \epsilon_j, \qquad j = 1, 2, \cdots, r \qquad (6).$$

we find $\epsilon_{r+1} < \epsilon/2$ so that (6) is valid when z_r is replaced by a point whose distance from it is less than ϵ_{r+1}. Then find $z_{r+1} \in A$ and $n_{r+1} \in N$ with

$$d(T^{n_{r+1}} z_{r+1}, z_r) < \epsilon_{r+1}.$$

So from all these we finally get that

$$d(T^{n_j + n_{j-1} + \cdots + n_{i+1}} z_j, z_i) < \epsilon/2 \qquad (7)$$

whenever $i < j$. Since A is compact we have for some pair $i < j, d(z_i, z_j) < \epsilon/2$, and this together with (7) gives

$$d(T^n z_j, z_j) < \epsilon$$

for $n = n_j + n_{j-1} + \cdots + n_{i+1}$. Hence claim is proved, hence proposition.

Definition 4.1.5. Upper semi continuous function: A function $f : X \to \mathbb{R} \cup \{-\infty, \infty\}$ is said to be upper semi continuous at $x_0 \in X$ if for every $\epsilon > 0$, there exists open neighborhood U of x_0 such that $f(x) \leq f(x_0) + \epsilon \quad \forall x \in U$.

proposition 2. Under the hypotheses of proposition 1 we can find a point $x \in A$ which is recurrent for (X, T).

Proof Let $F(x) = \inf_{n \in \mathbb{N}} d(T^n x, x)$. We will check that $F(x)$ is upper semi continuous:
Let U be an open neighborhood around x_0. Take a sequence of points $\{x_n\}$ which converges to x_0. Now because the transformations T^i are continuous so we can say $\{T^n x_n\} \to \{T^n x_0\}$. From the real analysis we can say that $d(T^n x_n, x_n) \to d(T^n x_0, x_0)$ hence ,

$$|d(T^n x_n, x_n) - d(T^n x_0, x_0)| < \epsilon \quad \forall n \geq \mathbb{N}.$$

From here we conclude that

$$|d(T^n x_n, x_n)| - |d(T^n x_0, x_0)| < \epsilon.$$

From here ,

$$d(T^n x_n, x_n) < d(T^n x_0, x_0) + \epsilon.$$

Taking inf on both sides,

$$\inf(d(T^n x_n, x_n)) < \inf(d(T^n x_0, x_0)) + \epsilon.$$

Hence we proved that $F(x)$ is upper semi continuous.
Now from the proposition 1 we conclude that $F(x)$ is not bounded from below on A. Take $x_0 \in A$ be a point of continuity when F is restricted to A. Assume that $F(x_0) > 0$.Then $F(x) > \delta > 0$ in an open subset V of A. Now because A is homogeneous so for some finite set of transformations S_i which commute with T, we will get

$$A \subset \cup_{i=1}^{\mathbb{N}} S_i^{-1} V.$$

Now take $\eta > 0$ be such that $d(x_1, x_2) < \eta$ implies $d(S_i x_1, S_i x_2) < \delta$ for all S_i. Then on each $S_i^{-1} V$, $F(x) \geq \eta$. Since F is not bounded from below on A we must have $F(x_0) = 0$. Hence x is recurrent point for (X, T).

Theorem 7. Let X be a compact metric space and $T_1, T_2, \cdots, T_p$ commuting homeomorphisms of X. Then there exists a point $x \in X$ and a sequence $n_k \to \infty$ with $T_i^{n_k} x \to x$ simultaneously for $i = 1, 2, \cdots, p$.

Proof Let G be the group generated by $T_1, T_2, \cdots, T_p$. Because as given in statement that $T_1, T_2, \cdots, T_p$ are commuting homeomorphisms so by the definition of homogeneous subset , if necessary we can suppose without loss of generality that (X, G) is a homogeneous system.Now we will proceed with induction. The case $p = 1$ follows from proposition (1) of this chapter.Now by the induction hypothesis assume the theorem is true for $(p-1)$ transformations and suppose $T_1, T_2, \cdots, T_p$ are p commuting homeomorphisms . Now form the p-fold product

$$X^{(p)} = X \times X \times \cdots \times X$$

and let $\triangle^{(p)}$ denote the diagonal consisting of $p-$ tuples $(x, \cdots, x)$. Let

$$T = T_1 \times T_2 \times \cdots \times T_p$$

on $X^{(p)}$. Note that G acts $X^{(p)}$ by $T_i \times T_i \times \cdots T_i$ and that $\triangle^{(p)}$ is a homogeneous subset of $(X^{(p)}, T)$.
Claim: $A = \triangle^{(p)}$ satisfies the hypothesis of the proposition 1 with respect to T.
Proof: let's take $R_i = T_i T_p^{-1} \qquad i = 1, 2, \cdots, p-1$. and suppose that $R_i^{n_k} x \to x$ for $i = 1, 2, \cdots, p-1$. For any $\epsilon > 0$ and appropriate n we will have

$$d(T_1^n \times \cdots \times T_p^n Z^*, x^*) < \epsilon.$$

with

$$Z^* = (T_p^{-n} x, T_p^{-n} x, \cdots, T_p^{-n} x).$$

$$x^* = (x, \cdots, x).$$

Hence $A = \triangle^{(p)}$ satisfies the hypothesis of proposition (1) of this chapter; so by proposition (1) we conclude that the system (X^p, T) and the subset $\triangle^{(p)}$ yields the theorem .So there exists a point $x \in X$ and a sequence $n_k \to \infty$ with $T_i^{n_k} x \to x$ simultaneously for $i = 1, 2, \cdots, p$. Hence we are done.

Theorem 8. Let (X, G) be a homogeneous dynamical system and $T_1, T_2, \cdots, T_p$ p commuting transformations in G. If U is any nonempty open set in G , there exists $n \geq 1$ with

$$T_1^n U \cap T_2^n \cap \cdots \cap T_p^n U \neq \phi.$$

Proof By the previous theorem we can say that ,since (X, G) is homogeneous U contains some x with $T_i^{n_k} x \to x, \qquad i = 1, 2, \cdots, p$ and as soon as $T_i^{n_k} x \in U$ for all

i we obtain an element which is common in all $T_1^n U, T_2^n U, \cdots, T_p^n U$. So intersection of all these will be nonempty .Hence we are done.

Theorem 9. (Van Der Waerden.) For any finite partition $N = C_1 \cup C_2 \cup \cdots \cup C_r$ there is a C_j containing arithmetic progression of arbitrary finite length.

Proof let $N = C_1 \cup C_2 \cup \cdots \cup C_r$ be a given partition of $\mathbb{N}$, and let $\wedge = \{1, 2, \cdots, r\}$ and form $\Omega = \wedge^Z$, the space of $\wedge-$ valued sequences. Define the shift $S : \Omega \to \Omega$ by $S\omega(n) = \omega(n+1)$. We endow Ω with a metric

$$d(\omega, \omega') = \inf\{\frac{1}{k+1} : \omega(n) = \omega'(n) \qquad for \qquad |n| < k\}.$$

Then Ω is compact and S is a homeomorphism of Ω. Let $\xi \in \Omega$ be defined by

$$\xi(n) = j \Leftrightarrow n \in C_j \qquad if \qquad n > 0,$$

$$\xi(n) = 1 \qquad n \leq 0.$$

Let X be the set of limit points of the sequence $\{S_\xi^n\}_{n=1,2,3,\ldots}$. Clearly we will have $SX = X$. Now set $T_i = S^i$ for $i = 1, 2, \cdots, p$ and apply theorem 6 of this chapter to $(X, T_1, T_2, \cdots, T_p)$. And according to the theorem 6 there will exists some $\eta \in X$ with

$$d(T_1^n \eta, \eta) < \frac{1}{2}, d(T_2^n \eta, \eta) < \frac{1}{2}, \cdots, d(T_p^n \eta, \eta) < \frac{1}{2}$$

But this means that $\eta(0) = \eta(n) = \eta(2n) = \cdots = \eta(pn)$. Finally choose m with $d(S^m \xi, \eta) < 1/(pn+1)$. We then have

$$S^m \xi(0) = S^m \xi(n) = S^m \xi(2n) = \cdots = S^m \xi(pn).$$

And implies

$$\xi(m) = \xi(m+n) = \xi(m+2n) = \cdots = \xi(m+pn).$$

But this says that $C_{\xi(m)}$ contains an arithmetic progression of length $p+1$. Hence we are done.

Chapter 5

Proximality, Idempotents and Hindman's theorem

5.1 Proximality, Idempotents and Hindman's theorem

Definition 5.1.1. For the dynamical system (X, T) ,we say a pair of points $x_1, x_2 \in X$ are proximal if for some sequence $\{n_k\} \in \mathbb{N}$

$$d(T^{n_k}x_1, T^{n_k}x_2) \to 0$$

Now we will illustrate this notion by taking $X = \wedge^{\mathbb{Z}}$, the space of all $\wedge$ - valued sequences.And T be the shift transformation on X.

$$T : X \to X$$

defined as

$$T(\omega(n)) = \omega(n+1)$$

we adopt a metric on X:

$$d(\omega, \omega') = \inf\{ \frac{1}{k+1} : \omega(n) = \omega'(n) for |n| \leq k\}$$

so that two sequences are close if they agree on a large interval centered about 0.We will say that two sequences $\omega_1, \omega_2 \in X$ are proximal if and only if there are arbitrary long interval $(n_k, n_k + l_k) \subset \mathbb{N}$ with

$$\omega_1(n) = \omega_2(n) \qquad\qquad n \in \cup(n_k, n_k + l_k)$$

Definition 5.1.2. A subset $H \subset \mathbb{N}$ is called an IP-sequence if there exists a sequence $p_1, p_2, p_3, \cdots, p_n, \cdots$ of elements in H such that H consists of all finite sums

$$\{p_{i_1} + p_{i_2} + \cdots p_{i_n}, \qquad i_1 < i_2 < \cdots < i_n, \qquad n = 1, 2, 3, \cdots\}$$

5.1.1 An IP-sequence is an approximation to a semigroup.

Proof Let $p_1, p_2, p_3 \in H$ where H is an IP-sequence.
we need to show that :

$$p_1 + (p_2 + p_3) = (p_1 + p_2) + p_3$$

consider the left hand side : $(p_2 + p_3)$ will be in H because H is an IP-sequence. so now $p_1 + (p_2 + p_3)$ that will again be in H, because H is an IP-sequence.
Similarly $(p_1 + p_2) + p_3$ will be in H.
Hence

$$p_1 + (p_2 + p_3) = (p_1 + p_2) + p_3.$$

Here IP refers for the "Idempotence". They also stands for "Infinite-dimensional parallelepiped" which gives another description of IP- sets.

5.1.2 Infinite Dimensional Parallelepiped:

Definition 5.1.3. set of sums $\{p_{i_1} + p_{i_2} + \cdots p_{i_n}, \qquad i_1 < i_2 < \cdots < i_n, \qquad n = 1, 2, 3, \cdots\}$ together with 0 form an infinite dimensional parallelepiped:

$$\{0, p_1\} \cup \{p_2, p_2 + p_1\} \cup \{p_3, p_3 + p_1\} \cup \{p_3 + p_2 + p_1\} \cup \cdots$$

5.1.3 An analysis of the dynamical system (X,T):

Let X^X denotes the set of all mappings continuous or not from X to X.

Claim: X^X is compact in product topology.

Proof We will show that X^X is homeomorphic to direct product of $|X|$ copies of X.That means there is a homeomorphism :

$$\alpha : X^X \to \prod_{i \in X} X$$

.

we will show that this is a homeomorphism.

- α is continuous.
  consider

  $$\alpha : X^X \to \prod_{i \in X} X$$

  given by

  $$f \mapsto (f(a_1), f(a_2), f(a_3), \cdots)$$

  take (X^X, τ) where τ is the induced function topology on X^X.Where

  $$S_{x_1, x_2, \cdots, y_1, y_2, \cdots} = \{f : f(x_i) = y_i\}$$

  is a basis foe the topology on X^X and is both open and closed.
  so α will be continuous because

  $$\alpha^{-1}(f(a_1), f(a_2), \cdots) = f$$

  and this f is in the basis of the topology. Hence it is open.Hence α is continuous.

- Now we will show that α^{-1} is continuous.
  take

  $$\alpha^{-1} : \prod_{i \in X} X \to X^X$$

  given by

  $$(y_i)_{i \in X} \mapsto g$$

  where

  $$g : X \mapsto X$$

define as

$$x_i \mapsto y_i$$

and g are in the basis. Hence g^{-1} will be open. Hence α^{-1} is open.
Hence $\alpha : \prod_{i \in X} X \to X^X$ is a homeomorphism.Hence X^X is compact(by Tychonoff theorem).

Claim: X^X forms a semigroup.

Proof take $f_1, f_2, f_3 \in X^X$ note that $f_1 \circ (f_2 \circ f_3) : X \mapsto X$. similarly $(f_1 \circ f_2) \circ f_3 : X \mapsto X$. Hence

$$(f_1 \circ f_2) \circ f_3 = f_1 \circ (f_2 \circ f_3)$$

Hence X^X forms a semigroup.

Claim: The one sided multiplication

$$\psi_{f_0} : X^X \to X^X$$

given by

$$f \mapsto f f_0$$

is continuous for any f_0.

Proof consider

$$\psi_{f_0}^{-1}(S_{x,y}) = \{f \in X^X : \psi_{f_0}(f) \in S_{x,y}\}$$

.

$$= \{f \in X^X : f f_0 \in S_{x,y}\}$$

.

$$= \{f \in X^X : f f_0(x) = y\}$$

.

$$= S_{(f_0(x),y)}$$

and this is open.Hence $\psi_{f_0} : X^X \to X^X$ is a continuous map.

Claim: The left multiplication

$$\phi_{f_0} : X^X \to X^X$$

given by

$$f \mapsto f_0 f$$

is continuous if f_0 is continuous.

Proof consider

$$\phi_{f_0}^{-1}(S_{x,y}) = \{f \in X^X : \phi_{f_0}(f) \in S_{(x,y)}\}$$

.

$$= \{f \in X^X : f_0 \circ f \in S_{(x,y)}\}$$

.

$$= \{f \in X^X : f_0 \circ f(x) = y\}$$

.

$$= \{f \in X^X : f_0(f(x)) = y\}$$

take U to be open neighbourhood of y in X then

$$= \{f \in X^X : f_0(f(x)) \in U\}$$

.

$$= \{f \in X^X : f(x) \in f_0^{-1}(U)\}$$

but given that f_0 is continuous hence $f_0^{-1}(U)$ is open.Hence

$$\{f \in X^X : f(x) \in f^{-1}(U)\}$$

is open.Hence ϕ_{f_0} is continuous.

- Now take $E = cl\{T^n : n \in \mathbb{Z}\} \subseteq X^X$.
  we say $f \in E$ if and only if for all $x_1, x_2, \cdots, x_s \in X, \epsilon > 0, \exists n \in \mathbb{N}$ such that

$$\rho(f(x_i), T^n(x_i)) < \epsilon \qquad\qquad 1 \leq i \leq s.$$

  E is a closed subset of a compact X^X space , Hence E is compact.

Claim: E is closed under composition.

Proof Set $f, g \in E$. We need to show that $fg \in E$
for some $n \in \mathbb{N}$,

$$\rho(f(g(x_i)), T^n(g(x_i)) < \frac{\epsilon}{2} \qquad 1 \leq i \leq s$$

.$\exists\delta > 0$ so that $\rho(x, y) < \lambda \implies \rho(T^n x, T^n y) < \frac{\epsilon}{2}$ for some $m \in \mathbb{Z}$,

$$\rho(g(x_i), T^m(x_i) < \delta \qquad 1 \leq i \leq s$$

. Hence

$$\rho(T^n(g(x_i)), T^n T^m(x_i) < \frac{\epsilon}{2}$$

and so $\rho(fg(x_i), T^{n+m}(x_i) \ < \epsilon$. Hence $fg \in E$.

- E is called the enveloping semigroup of (X, T). E is a compact semigroup for which multiplication on one side is continuous.

lemma 2. If E is a compact semigroup for which one-sided multiplication $x \mapsto xx_0$ is continuous, then E contains an idempotent, i.e., an element u with $u^2 = u$.

Proof Let $\mathcal{A}$ denote the family of compact semigroups $A \subseteq E$. $\mathcal{A} \neq \phi$ as $E \in \mathcal{A}$. Let A be the minimal subset of E satisfying (i)$AA \subset A$ (ii)A is compact. The existence of such a minimal set A is given by Zorn's lemma.(If $\mathcal{C} \subseteq \mathcal{A}$ is a chain then $\cap\mathcal{C} \in \mathcal{A}$. As all $A \in \mathcal{C}$ are compact hence $\cap\mathcal{C} \neq \phi$.)
Take $u \in A$ then Au is compact because given that right multiplication is continuous .So $AuAu \subset Au$. But we have assumed that A is minimal so, $Au = A$. In particular for some $v \in A$ we will have $vu = u$.Take $A' = \{v \in A | vu = u\}$. By one sided continuity A' is closed , and clearly $A'A' \subset A'$. So $A' = A$ whence $u^2 = u$.

lemma 3. If E be the enveloping semigroup of a dynamical system(X, T) and $u \in E$ is an idempotent, then for every point $x \in X$, x and ux will be proximal.

Proof As in the discussion above for the topology on X^X if $u \in E$ and $\epsilon > 0$ and $x_1, x_2, x_3, \cdots, x_m$ are points of X, then there is n such that $d(T^n x_i, ux_i) < \epsilon$ for $i = 1, 2, ..., m$. In particular we can find n with

$$d(T^n x, ux) < \epsilon$$

and because multiplication from the left is continuous in an enveloping semigroup so,we will have

$$d(T^n ux, u^2 x) < \epsilon$$

But we know from the above lemma that $u^2 = u$, $u^2 x = ux$ so we get

$$d(T^n ux, ux) < \epsilon$$

from (i) and (ii) above we have

$$d(T^n x, T^n ux) < 2\epsilon.$$

Hence x and ux will be proximal.

proposition 3. Let (X, T) be a dynamical system, $x \in X$, and let Z be the set of limit points of the forward orbit $\{T^n x\}_{n \in \mathbb{N}}$. If Y is any minimal set in Z, then there is a point $y \in Y$ such that x and y are proximal.

Proof From the definitions of E and Z we see that $Ex = Z$. Let $F = \{s \in E | sx \in Y\}$, then $Fx = Y$. F will be closed. and since $EF \subset F$, $F^2 \subset F$. By lemma 2, F contains an idempotent u. So $ux \in Y$ and proposition follows from the lemma 3 .

5.1.4 Hindman's Theorem

We now consider the setup:

- $\wedge = \{1, 2, \cdots, r\}$ finite set.
- $\Omega = \wedge^{\mathbb{Z}}$ = space of $\wedge -$ valued sequences.
- Ω is endowed with the metric for which it is compact.

- The transformation :
$$S : \Omega \mapsto \Omega.$$
defined as
$$S\omega(n) = \omega(n+1).$$
(Ω, S) be a dynamical system.

- we wish to characterize the points of Ω which belongs to minimal sets for the system (Ω, S).

- we will define block as a finite ordered sequence of elements of $\wedge$. Denote as:
$$b = b(1)b(2)\cdots b(l) \in \wedge^l.$$
whose length is l. b is said to occur in b' at $(t+1)$ if
$$b(i) = b'(i+1) \qquad 1 \leq i \leq |b|$$
$|b|$ is the length of b.

- Ω_0 is an infinite sequence in Ω.

proposition 4. Let $\omega_0 \in \Omega$. ω_0 belongs to the minimal set of (Ω, S) if and only if it has the following property : If b is a block in ω_0 then $\exists$ a length $L = L(b)$ such that b occurs in Ω_0 provided the length of b' is atleast L.

Proof Let X be orbit closure of ω_0. i.e. $X = \overline{\{S^n\omega_0\}} \subset \Omega$.
$\Rightarrow$ Suppose $\omega_0 \in X$ and X is a minimal set.
$\Leftrightarrow$ if $\xi \in X$ then $\omega_0 \in \{\overline{S^n\xi}\}$
$\Leftrightarrow d(S^i\xi, \omega_0) < \epsilon$
$\Leftrightarrow$ for any word b occurring in ω_0, the word occurs in ξ.
But since $\xi \in \overline{\{S^n\omega_0\}}$, any word b' which occurrs in ξ also occurs in ω_0. Hence any word b' of length $L(b)$ that occurs in ξ has the word b occurring in it.

$\Leftarrow$ Now we will prove the other direction, we are using lemma 1 of chapter 4, here with $G = \{S^n\}$. Suppose b occurs in ω_0 at place $(t+1)$. Choose $\epsilon > 0$ so that if $\omega \in \Omega$ and $d(\omega, \omega_0) < \epsilon$ then b occurs in ω at place $(t+1)$. By lemma 1 of chapter 4, $\exists$ some sequence of powers of S take $S, S^2, S^3, \cdots, S^L$ has the property that for any $\omega \in X$ some $d(S^i\omega, \omega_0) < \epsilon$. Hence X is minimal.

Theorem 10. {Hindman's Theorem} For any finite partition $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there is some C_j containing a IP-set.

Proof Consider $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ and form $\Omega = \wedge^{\mathbb{Z}}$ (space of $\wedge-$ sequences) where $\wedge = \{1, 2, \cdots, r\}$. Define ω_1 by

$$\omega_1(n) = \begin{cases} j & \text{if } n > 0 \text{ and } n \in C_j \\ 1 & \text{if } n \leq 0 \end{cases}$$

now by the proposition 2 of this chapter there is a point $\omega_0 \in \Omega$ which is proximal to ω_1 such that $\omega_0 \in$ minimal set of (Ω, S). Let $\omega_0(0) = j_0$. We shall show that the set C_{j_0} contains an IP-set.
since ω_0 and ω_1 are proximal so from the discussion from the begining of this section we say that there are arbitrary large blocks occur in both ω_0 and ω_1 at identical places.Now we will use the condition of proposition 3 which is valid for ω_0. Begin with the block $b_0 = \{j_0\}$. b_0 occurs in any block of length $L(b_0)$ in ω_0 and we can find such a block that occurs at same positive position in ω_1 and ω_0. Let p_1 be that simultaneous position of b_0 in ω_0 and ω_1 in two such blocks. Now consider the block

$$b_1 = \omega_0(0)\omega_0(1) \cdots \omega_0(p_1).$$

any sufficiently long block in ω_0 has b_1 occurring in it and can find such a block in ω_1 and ω_0 at the same position. Let $p_2 > 0$ be such position. Then from these we will have

$$\omega_1(p_2)\omega_1(p_2 + 1) \cdots \omega_1(p_2 + p_1) = \omega_0(p_2)\omega_0(p_2 + 1) \cdots \omega_0(p_2 + p_1).$$

$$= \omega_0(0)\omega_0(1) \cdots \omega_0(p_1)$$

this gives us

$$\omega_1(p_2 + p_1) = \omega_0(p_2 + p_1) = \omega_0(p_1) = j_0.$$

$$\omega_1(p_2) = \omega_0(p_2) = j_0 = \omega_0(0).$$

$$\omega_1(p_1) = \omega_0(p_1) = j_0.$$

From here we conclude that p_1, p_2 and $p_1 + p_2 \in C_{j_0}$. We now construct of $\{p_1, p_2, \cdots, p_n, \cdots\}$ inductively. Assume that we have found $p_1, p_2, \cdots, p_n$ such that for any $p_\alpha = p_{i_1} +$

$\cdots + p_{i_s}, \qquad i_1 < i_2 < \cdots < i_s \leq n$. we have $\omega_1(p_\alpha) = \omega_0(p_\alpha) = j_0$. Form the block

$$b_n = \omega_0(0)\omega_0(1)\cdots\omega_0(p_\alpha).$$

b_n occurs in ω_0 whose length is atleast $L(b_n)$ and there are such blocks which occurs in both ω_0 and ω_1 at the same positive place.And let p_{n+1} be that positive position. Then from here we conclude that

$$\omega_1(p_{n+1})\omega_1(p_{n+1}+1)\cdots\omega_1(p_{n+1}+(p_1+\cdots+p_n))$$

$$= \omega_0(p_{n+1})\omega_0(p_{n+1}+1)\cdots\omega_0(p_{n+1}+(p_1+\cdots+p_n))$$

$$= \omega_0(0)\omega_0(1)\cdots\omega_0(p_1+\cdots+p_n).$$

This gives

$$\omega_o(p_{n+1}+p_\alpha) = \omega_0(p_{n+1}+p_\alpha) = j_0 = \omega_0(p_\alpha)$$

as required . We now have for any $i_1 < i_2 < \cdots < i_s, \qquad p_{i_1} + p_{i_2} + \cdots + p_{i_0} \in C_{j_0}$. So we are done.

5.1.5 Hindman's theorem implies proposition 0.6

In the previous section we proved Hindman's theorem with the help of proposition 0.6. The goal of this section is to prove the proposition 0.6 with the help of Hindman's theorem. We will introduce some terminology here:

- $\mathfrak{F}$ =set of finite subsets of $\mathbb{N}$. $\alpha, \beta, \cdots$ are elements of $\mathfrak{F}$. Here $\alpha + \beta$ denotes the union of α and β when α and β are disjoint.
  An IP- system of integers is a set $\{n_\alpha\}$ that satisfies

  $$n_{\alpha+\beta} = n_\alpha + n_\beta.$$

  While an IP- system of commuting transformations is a set T^α indexed by $\alpha \in \mathfrak{F}$ and satisfying

  $$T_{\alpha+\beta} = T_\alpha T_\beta.$$

-

Definition 5.1.4. Sub IP-systems: Let us say that $\{\hat{n_\alpha}\}$ is a sub IP-system of $\{n_\alpha\}$, determined by $\{\alpha_j\}_{j\in\mathbb{N}}$ if α_j's are disjoint and

$$\hat{n_j} = n_{\alpha_j} \qquad j \in \mathbb{N}.$$

- Consider $\mathfrak{F}$ as directed set with partial order:

$$\alpha < \beta \Leftrightarrow \max\alpha < \max\beta.$$

If $\{x_\alpha\}_{\alpha\in\mathfrak{F}} \subset X$ a metric space we write

$$\lim x_\alpha = x.$$

If $\{x_\alpha\}_{(\mathfrak{F},<)}$ converges to x as a net i.e. to any $\epsilon > 0$ there is some α_0 such that for all $\alpha_0 < \alpha$

$$d(x_\alpha, x) < \epsilon.$$

We will state this proposition which will be helpful in provin the next lemma:

proposition 5. If $\{x_\alpha\}_{\alpha\in\mathfrak{F}} \subset X$ a compact metric space then there is a sub IP-net $\{\hat{x_\alpha}\}_{\alpha\in\mathfrak{F}} \subset X$ that converges.

lemma 4. If $\{T^\alpha\}$ is an IP-system of continuous mappings of a compact metric space $X, y \in X$, there is a sub IP-system $\{\hat{T^\alpha}\}$ such that

$$\lim \hat{T^\alpha} y = x$$

exists, moreover

$$\lim \hat{T^\alpha} x = x.$$

Proof We see that first part is clear from the above proposition.Now assume that (i) holds and that $\epsilon > 0$ is given. Choose α_0 so that for $\alpha > \alpha_0$

$$d(\hat{T^\alpha} y, x) < \frac{\epsilon}{2}.$$

Because it is given that $\hat{T^\alpha}$ is continuous , there is a $\delta = \delta(\alpha)$ such that if

$$d(z, x) < \delta$$

then

$$d(\hat{T}^{\alpha}z, \hat{T}^{\alpha}x) < \frac{\epsilon}{2}.$$

Now find $\beta > \alpha$ so that

$$d(\hat{T}^{\beta}y, x) < \delta.$$

then

$$d(\hat{T}^{\alpha}\hat{T}^{\beta}y, \hat{T}^{\alpha}x) < \frac{\epsilon}{2}$$

And because $\hat{T}^{\alpha}$ is a sub IP-system so ,

$$d(\hat{T}^{\alpha}\hat{T}^{\beta}y, \hat{T}^{\alpha}x) = d(\hat{T}^{\alpha+\beta}y, \hat{T}^{\alpha}x) < \frac{\epsilon}{2} \qquad (i).$$

since $\alpha > \alpha_0, \beta > \alpha$ hence $\alpha + \beta > \alpha_0$ so

$$d(\hat{T}^{\alpha+\beta}y, x) < \epsilon/2 \qquad (ii)$$

from (i) and (ii)

$$d(\hat{T}^{\alpha}x, x) < d(\hat{T}^{\alpha}x, \hat{T}^{\alpha}\hat{T}^{\beta}y) + d(\hat{T}^{\alpha+\beta}y, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

so

$$d(\hat{T}^{\alpha}x, x) < \epsilon.$$

Hence we are done.

proposition 6. Let (X, T) denote the dynamical system and $y \in X$, let Z be the set of limit points of the forward orbit $\{T^n\}_{n\in\mathbb{N}.}$ If M is any minimal set in Z, there is a point $y \in Y$ such that x and y are proximal.

Proof Discussion:If $y \in X$ is given let M be any minimal set contained in the positive orbit closure of y. Our goal is to construct an IP-system $\{n_{\alpha\}}$ such that

$$\lim d(T^{n_\alpha}y, M) = 0 \qquad (1).$$

Once we have constructed the IP-system $\{n_{\alpha\}}$ then the proof is completed by applying lemma 4 of this chapter to obtain a sub IP-system $T^{\hat{n}_\alpha}$ such that

$$\lim T^{\hat{n}_\alpha}y = x$$

and

$$\lim T^{\hat{n}_\alpha} x = x \qquad (2).$$

Here we see that from (1), $x \in M$, while by (2) x and y are proximal . So in this way we will be done.
Now to construct $\{n_\alpha\}$ so that (1) holds, let n_1 satisfy

$$d(T^{n_1} y, M) < 1.$$

Having found n_α, $\max \alpha \leq m$ such that

$$d(T^{n_\alpha} x, M) < 1/\max \alpha \qquad (3)$$

use the fact $TM = M$ to find a neighborhood U of M so that for all $z \in U$,

$$d(T^j z, M) < 1/(m+1), \qquad 0 \leq j \leq n_{(1,2,\cdots,m)}$$

and then use the fact that M is in the positive orbit closure of y to find n_{m+1} with $T^{n_{m+1}} y \in U$. this extends (3) to all α with $\max \alpha \leq m+1$, and thus by induction $\{n_\alpha\}$ is completely constructed.So by the previous lemma and by (2) x and y are proximal.

Chapter 6

The IP version of Van Der Waerden's theorem

In this section we will state one lemma and theorem that will be useful for IP-version of Van Der Wearden's theorem:

lemma 5. Let (T^α, X) be an IP-system of continuous maps of a compact metric space X and suppose that $A \subset X$ is closed and

$$\lim T^\alpha A \supset A,$$

where we use the Hausdorff topology on closed sets.Then

$$\inf_{a \in A, \alpha \in \mathfrak{F}} d(T^\alpha a, a) = 0.$$

Now we will state a theorem which will allow us to prove the IP-version of Van Der Waerden's theroem.

Theorem 11. Let $S_1^\alpha, S_2^\alpha, \cdots, S_k^\alpha$ be k-commuting IP-systems of homeomorphisms of X, a compact metric space , that all commute with a group G that acts minimally on X. If $U \subset X$ is any open set , there is some $\alpha_0 \in \mathfrak{F}$ such that

$$\cap_{i=1}^k S_i^{\alpha_0} U \neq \phi.$$

6.1 IP -version of Van Der Waerden's theorem:

Theorem 12. For any finite partition $N = C_1 \cup C_2 \cup \cdots \cup C_r$, there is a C_j such that for any $l = 1, 2, 3, \cdots$ there is a number $d \in C_j$ and a number e such that the arithmetic progression $e + id, 0 \leq i \leq l$, is contained in C_j.

Proof Let consider a finite partition $\{C_1, C_2, \cdots, C_r\}$ of $\mathbb{N}$. Now introduce $\Omega = \wedge^Z, \wedge = \{1, 2, \cdots, r\}$; where ω is $\wedge-$ sequences let S denote the shift , and define

$$\omega_1(n) = j \qquad C_j,$$

$$= 1 \qquad n \leq 0$$

Now according to proposition 2 of chapter 5 , we say that there is a point $\omega_0 \in \Omega$ proximal to Ω_1 and such that ω_0 belongs to a minimal set M of (Ω, S). Let $\omega_0(0) = j_0$; we shall show that the set C_{j_0} fulfills the condition of theorem .

Now by Hindman's theorem of chapter 5 in we recall that C_{j_0} contains an IP - set $\{n_\alpha\}$. Now for any k, apply theorem 10 of this chapter to the system $T^{in_\alpha}, 1 \leq i \leq k$ and the open set

$$U = \{\omega \in M : \omega(0) = j_0\},$$

which is non-empty .We thus find some α_0 with :

$$\cap_{i=1}^{k} T^{in_{\alpha_0}} U \neq \phi.$$

This gives a set of n for which $\omega_0(n + in_\alpha) = j_0$ and which intersects any sufficiently large interval. Since ω_1 is proximal to ω_0 we can find a long interval along which ω_0 and ω_1 agree and we deduce that

$$\omega_1(n + id) = j_0$$

for $d = n_{\alpha_0} \in C_{j_0}$ and $1 \leq i \leq k$. this completes the proof.

Bibliography

[Dmi14a] Dmitry Kleinbock, Brandeis University, *kra appendices*, summer 2014.

[Dmi14b] ______, *kra lectures*, summer 2014.

[Hil78] Hillel Furstenberg, Benjamin Weiss, *Topological dynamics and combinatorial number theory*.

[Jam] James R. Munkres, *Topology*.

[Ron] Ronald L. Graham, Bruce L. Rothschild , Joel H. Spencer, *Ramsey theory*, second edition ed., A Wiley Interscience Publication.

[Wal] Walter Rudin, *Principles of mathematical analysis*.

[Hil78] [Ron] [Dmi14b] [Jam] [Wal] [Dmi14a]