Shellable Posets

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Certificate of Examination

This is to certify that the dissertation titled **"Shellable Posets"** submitted by **Nayana Shibu Deepthi** (Reg. No. MS13127) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 20, 2018

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Chanchal Kumar at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Chanchal Kumar
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Abstract

The concept of shellability is an easy tool to verify whether the corresponding simplicial complex is Cohen-Macaulay or not. This dissertation aims at the detailed study of shellability and its generalization to the nonpure case, based on the established work of Bj*ö*rner and Wachs. Some of the fundamental properties of nonpure shellability are taken into consideration.

We begin the report with a brief introduction to some of the basic notions of commutative algebra and certain rudimentary topological results. To each simplicial complex, we associate a quotient ring called the Stanley-Reisner ring whose algebraic properties are firmly related to the combinatorial properties of the simplicial complex. The study of topological properties of shellable simplicial complex shows that it has the homotopy type of a wedge of spheres of certain dimensions.

Along with the fundamental ideas and properties of posets, this work also elaborate on the Möbius function, Möbius inversion and the order complexes associated with posets. Shellability of a partially ordered set is studied by considering the order complex associated with it.

The method of lexicographic shellability in its general form is introduced along with a detailed example of a nonpure lexicographically shellable poset, the k-equal partition lattice. Finally, we exploit an easy computation of Betti numbers of the k-equal partition lattice through the study of standard tableaux of hook shape.

Contents

Chapter 1

Introduction

Combinatorial commutative algebra is a moderately new, quickly developing branch of mathematics. As the name infers, it lies at the convergence of two more settled fields, commutative algebra and combinatorics, and as often as possible uses strategies for one field to address issues emerging in the other. Richard Stanley was the first mathematician who introduced the application of commutative algebraic strategies to crack combinatorial problems. Stanley converted the difficult conjectures in algebraic combinatorics into statements from commutative algebra and proved them by means of homological techniques.

This dissertation aims at providing a detailed study on shellability of simplicial complexes and spotlights on the generalization of the same to the nonpure case. We begin the report with a brief introduction to some of the basic notions of commutative algebra and certain rudimentary topological results. The second chapter of this report is on *Stanley-Reisner rings*, where we see that to each simplicial complex Δ on n vertices, we associated a quotient ring $K[\Delta]$ of the polynomial ring $K[x_1, x_2, ..., x_n]$ over a field K, called the *Stanley-Reisner ring* of Δ in such a manner that the combinatorial properties of Δ are related with the algebraic properties of the $K[\Delta]$.

At that point we went ahead to a definite study on partially ordered sets, the tool that formalizes and generalizes the concepts of ordering elements of a set. In the third chapter, along with the fundamental ideas and properties of posets, we also elaborate on the Möbius function, Möbius inversion and the order complexes associated with posets.

In the last two sections of the report we build up a few of the fundamental properties of the idea of nonpure shellability. The report closes with the extension of the technique of lexicographic shellability for posets from pure case to the general case and with a detailed section on *k-equal partition lattice*, a case of a nonpure lexicographically shellable poset.

1.1 Basics of Commutative Algebra

Here we will have a look at the essential definitions and results from commutative algebra which will be utilized all through our investigation in this report. All the results stated in this section can be found in the standard texts on commutative algebra; like Atiyah-Macdonald [1], Miller-Sturmfels [11], Herzog-Hibi [7] and Villarreal [15].

Definition 1.1.1. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and R be a commutative ring with identity. If R has a decomposition $R = \bigoplus_{i=0}^{\infty} R_i$ as direct sum of additive subgroups R_i such that $R_iR_j \subseteq R_{i+j}$ for all $i, j \geq 0$, then the ring R is called a *graded ring*. The set $\{R_i \colon i \in \mathbb{N}\}$ is called the *grading* and each R_i is said to be the i^{th} *homogeneous* (or *graded*) *component* of R.

The homogeneous component R_0 is a subring of the graded ring R and every R_i is an R_0-module. Any element $a \in R$ has an unique expression, $a = \sum_{i=0}^{\infty} a_i$, where $a_i \in R_i$ and $a_i = 0$ for all but finitely many $i \in \mathbb{N}$. This decomposition of each $a \in R$ is known as the *homogeneous decomposition* of a and each $a_i \in R_i$ is called the i^{th} *homogeneous* (or *graded*) *component* of a with degree i. An ideal I of R is said to be *graded* if I is generated by the homogeneous elements.

Proposition 1.1.2. *Let I be an ideal of the graded ring $R = \bigoplus_{i=0}^{\infty} R_i$. Then the following statements are equivalent.*

1. *For every $a \in I$, all its homogeneous components are in I.*

2. *The ideal I has a decomposition, $I = \bigoplus_{i=0}^{\infty}(I \cap R_i)$.*

3. *The ideal I is a graded ideal of R.*

Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring. An R-module M is said to be *graded* if it has a decomposition, $M = \bigoplus_{i=0}^{\infty} M_i$ as additive subgroups of $(M, +)$ such that $R_i M_j \subseteq M_{i+j}$ for all $i, j \geq 0$. Every x in M has an unique homogeneous decomposition, $x = \sum_{i=0}^{\infty} x_i$, where $x_i \in R_i$ and $x_i = 0$ for all but finitely many $i \in \mathbb{N}$. A submodule N of M, generated by homogeneous elements is said to be a *graded submodule* of M. Let M and M' be two graded R-modules. An R-module homomorphism $\phi\colon M \longrightarrow M'$ is *graded* if $\phi(M_i) \subseteq M'_i, \forall\, i$.

Proposition 1.1.3. *Let R be a graded ring and M be a graded R-module. Suppose N is an R-submodule of M, then the following conditions are equivalent.*

1. *For every $x \in N$, all the homogeneous components of x belongs to N.*
2. $N = \bigoplus_{i=0}^{\infty} (N \cap M_i)$.
3. *N is a graded submodule of M.*

Theorem 1.1.4. *Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring. R is Noetherian if and only if R_0 is Noetherian and R is generated as an R_0-algebra by finitely many homogeneous elements of positive degree, that is, we have $R \simeq R_0[x_1, \ldots, x_n]$.*

Theorem 1.1.5. *Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded Noetherian ring and $M = \bigoplus_{i=0}^{\infty} M_i$ be a finitely generated graded R-module. Then each M_i is finitely generated R_0-module.*

Let R be a commutative ring. The *Krull dimension* of R, denoted by $\dim R$ is defined as

$$\dim R = \sup \big\{ r \colon \text{there exists a chain of length } r \text{ in } \operatorname{Spec}(R) \big\}.$$

For R-module M, the dimension, $\dim M$ is defined as

$$\dim M = \sup \big\{ r \colon \text{there exists a } r\text{-chain of prime ideals in } \operatorname{supp}(M) \big\}.$$

Here, $\operatorname{supp}(M) = \{P \in \operatorname{Spec}(R) \colon M_P \neq 0\}$. For a nonzero ring and a nonzero R-module M, $\operatorname{Spec}(R)$ and $\operatorname{supp}(M)$ are both nonempty and therefore both, $\dim(R)$ and $\dim(M)$ are non-negative integers or ∞. By convention, dimension of a zero module is -1.

Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring. Let us consider R_0 to be Artinian and $R = R_0[x_1, \ldots, x_m]$ with $\deg(x_i) = e_i \geq 1$. Thus R_0 is also Noetherian and R becomes a Noetherian ring. Let $M = \bigoplus_{i=0}^{\infty} M_i$ be a finitely generated graded R-module. Then M is Noetherian and each M_i is a finitely generated R_0-module. Hence, *length* $\ell_{R_0}(M_i)$ of M_i is finite. We associate a numerical function $\mathbb{H}\big(M,\ \_\big)$, called the *Hilbert function* of M, defined as $\mathbb{H}\big(M,\ i\big) = \ell_{R_0}(M_i)$. The *Hilbert series* of M is defined as $\mathbb{F}\big(M,\ t\big) = \sum_{i=0}^{\infty} \mathbb{H}\big(M,\ i\big)t^i$.

Theorem 1.1.6. *Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded Noetherian ring such that $R = R_0[x_1, x_2, \ldots, x_m]$, where $x_i \in R_i$, $e_i \geq 1$ and R_0 is an Artinian ring. Suppose $M = \bigoplus_{i\in\mathbb{Z}} M_i$ is a nonempty finitely generated graded R-module. Then we have the following results.*

1. *The Hilbert series $\mathbb{F}\big(R,\ t\big)$ is a rational function of the form*

$$\mathbb{F}\big(R,\ t\big) = \frac{f_R(t)}{\prod_{i=1}^{m}(1 - t^{e_i})}, \qquad f_R(t) \in \mathbb{Z}[t].$$

2. *The Hilbert series of the module M is of the form*

$$\mathbb{F}\big(M,\ t\big) = \frac{f_M(t)}{\prod_{i=1}^{m}(1 - t^{e_i})}.$$

Here, $f_M(t) \in \mathbb{Z}[t, t^{-1}]$ or $f_M(t) \in \mathbb{Z}[t]$ if $M = \bigoplus_{i=0}^{\infty} M_i$.

Remark. Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded Noetherian ring with the above mentioned properties. Let $\deg x_i = e_i = 1$ for every i. Then

$$\mathbb{F}(R,\ t) = \frac{f_R(t)}{(1 - t^m)} \quad \text{and} \quad \mathbb{F}(M,\ t) = \frac{f_M(t)}{(1 - t^m)}.$$

Proposition 1.1.7. *Let* $R = \bigoplus_{i=0}^{\infty} R_i$ *be a graded ring such that we have* $R = R_0[x_1, \ldots, x_m]$ *with deg* $x_i = 1$ *and* R_0 *be an Artinian ring. Let* M *be a nonzero finitely generated graded* R*-module. Then there exists a unique polynomial* $P_M(x) \in \mathbb{Q}[x]$ *of degree* $\leq m-1$ *such that* $P_M(i) = \mathbb{H}\big(M,\ i\big) = \ell_{R_0}(M_i)$ *for* $i \gg 0$. (The unique polynomial associated with the Hilbert function is called the *Hilbert polynomial* of M.)

Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let I be an ideal of R such that $\sqrt{I} = \mathfrak{m}$. Clearly, $\frac{R}{I}$ is Artinian. Let M be a finitely generated R-module. By induction on i, we prove that $\ell_{\frac{R}{I}}\left(\frac{M}{I^iM}\right)$ is finite. Hence, we get a numerical function, $\mathcal{HS}\big(M,\ i\big) = \ell_{\frac{R}{I}}\left(\frac{M}{I^iM}\right)$. This numerical function is called the *Hilbert-Samuel function* of M with respect to the ideal I. We define the *Hilbert-Samuel series* as

$$\chi_I^M(t) = \sum_{i=0}^{\infty} \mathcal{HS}\big(M,\ i\big)t^i.$$

The Hilbert-Samuel function is of a polynomial type and the polynomial associated with it, $P_M^I(x)$ is called the *Hilbert-Samuel polynomial* of M with respect to the ideal I. The degree of $P_M^I(x)$ is independent of the choice of $\mathfrak{m}$-primary ideal I. Let $d(M)$ be the degree of the Hilbert-Samuel polynomial of M with respect to the ideal I.

Definition 1.1.8. Let $(R, \mathfrak{m})$ be a Noetherian local ring. For a finitely generated nonzero R-module M, we define *Chevalley dimension*, $\delta(M)$ as

$$\delta(M) = \inf\{r \colon \text{there exists } a_1, \ldots, a_r \in \mathfrak{m} \text{ with } \ell\left(\frac{M}{\langle a_1, \ldots, a_r\rangle M}\right) < \infty\}.$$

By convention, let $\delta(M) = -1$ if $M = 0$.

Now we are in a position to state the dimension theorem.

Theorem 1.1.9 (Dimension theorem)**.** *Let* $(R, \mathfrak{m})$ *be a local Noetherian ring and* M *be a finitely generated* R*-module. Then*

$$\dim(M) = d(M) = \delta(M).$$

Let R be a Noetherian ring and M be a nonzero finitely generated R-module with $\dim(M) = d$. A set of d elements $a_1, \ldots, a_d$ in $\mathfrak{m}$ such that $\ell\Big(\dfrac{M}{\langle a_1, \ldots, a_d \rangle M}\Big) < \infty$ is called a *system of parameters*. By the dimension theorem, we conclude that every nonzero module has a system of parameters.

Definition 1.1.10. Let M be a nonzero finitely generated R-module and $a_1, \ldots, a_r$ be elements of a proper ideal of R. If a_i is a nonzero divisor of $\dfrac{M}{\langle a_1, \ldots, a_{i-1} \rangle M}$ for $1 \leq i \leq r$, then the sequence $a_1, \ldots, a_r$ is called a *M-regular sequence* or a *M-sequence* of length r.

Remark. Every M-regular sequence can be extended to a system of parameters for M.

An M-sequence $\{a_1, \ldots, a_r\}$ is said to be *maximal*, if $\{a_1, \ldots, a_r, a\}$ is not an M-sequence for any $a \in R$.

Proposition 1.1.11. *Let I be a proper ideal of R such that $IM \neq M$. Then every maximal M-sequence of elements in I has the same number of elements.*

Definition 1.1.12. The number of elements in a maximal M-sequence of an ideal I is called the *I-depth* of M, denoted as $\operatorname{depth}(I, M)$.

Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated R-module. The depth of the module M, $\operatorname{depth}(M)$ is given by the $\mathfrak{m}$-depth, $\operatorname{depth}(\mathfrak{m}, M)$ of M.

Proposition 1.1.13.

1. *Let M be a finitely generated R-module and I be an ideal of R such that $IM \neq M$. Let $\{a_1, \ldots, a_l\}$ be an M-sequence in I. Then*

$$\operatorname{depth}\bigg(I, \frac{M}{\langle a_1, \ldots, a_l \rangle M}\bigg) = \operatorname{depth}(I, M) - l.$$

2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be nonzero finitely generated R-module. Then*

$$\operatorname{depth}(M) \leq \min\Big\{ \dim\Big(\frac{R}{P}\Big) : P \in \operatorname{Ass}(M)\Big\},$$

where $\mathrm{Ass}(M)$ *is the set of all associated primes of* M*. Therefore we have,*

$$\operatorname{depth}(M) \leq \dim(M).$$

Definition 1.1.14. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated R-module. Then M is said to be *Cohen-Macaulay* if either $M = 0$ or $\operatorname{depth}(M) = \dim(M) = \dim\left(\frac{R}{P}\right)$ for all $P \in \mathrm{Ass}(M)$.

Proposition 1.1.15 (Depth Lemma)**.** *Let* R *be a Noetherian ring and* $M_i,\ i = 1, 2, 3$ *be finitely generated* R*-modules. Suppose* I *is an ideal of* R *such that* $IM_i \neq M_i$ *for* $i = 1, 2, 3$ *and there is a short exact sequence* $0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$*. Then one of the following inequality holds.*

1. $\operatorname{depth}(I, M_1) = \operatorname{depth}(I, M_2) < \operatorname{depth}(I, M_3)$.
2. $\operatorname{depth}(I, M_2) = \operatorname{depth}(I, M_3) \leq \operatorname{depth}(I, M_1)$.
3. $\operatorname{depth}(I, M_1) - 1 = \operatorname{depth}(I, M_3) < \operatorname{depth}(I, M_2)$.

We now associate numerical invariants, called *Betti numbers*, to finitely generated modules over Noetherian local rings or finitely generated graded modules over standard polynomial rings.

Let R be a commutative ring. A *complex* $\mathcal{F}$ of R-modules is a sequence of modules F_i and homomorphisms $\phi_i \colon F_i \longrightarrow F_{i-1}$ such that $\phi_i \circ \phi_{i+1} = 0$ for all i. Then the R-module

$$\mathcal{H}_i(\mathcal{F}) = \frac{\mathrm{Ker}(\phi_i \colon F_i \longrightarrow F_{i-1})}{\mathrm{Im}(\phi_{i+1} \colon F_{i+1} \longrightarrow F_i)}$$

is called the i^{th} *homology module.* This measures the extent of deviation of the complex from being exact.

Definition 1.1.16. Let R be a commutative ring and M be an R-module. Then the complex

$$\mathcal{F} \colon \quad \cdots \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \longrightarrow 0$$

of free R-modules such that $\mathrm{Coker}(\phi_1) = M$ and $\mathcal{F}$ is exact except at the 0^{th} position. Then the complex $\mathcal{F}$ is called a *free resolution* of M and the image $\mathrm{Im}(\phi_i)$ of ϕ_i is called the i^{th} *syzygy module* of M.

If we have a free resolution $\mathcal{F}$ with $F_{n+1} = 0$ and $F_i \neq 0$ for $0 \leq i \leq n$, then $\mathcal{F}$ is said to be a *finite free resolution* of length n. Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring and $M = \bigoplus_{i=0}^{\infty} M_i$ be a graded R-module. A free resolution $\mathcal{F}$ of M with graded homogeneous maps ϕ_i of degree 0 is called the *graded free resolution.* Consider a free or graded complex

$$\mathcal{F}\colon \quad \cdots \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \longrightarrow 0$$

over a polynomial ring $R = K[x_1, x_2, \ldots, x_n]$. If $\phi_i(F_i)$ is a subset of $\big\langle X_1, \ldots, X_n \big\rangle F_{i-1}$, then $\mathcal{F}$ is said to be *minimal.*

Theorem 1.1.17 (Hilbert syzygy theorem)**.** *Let $R = K[x_1, x_2, \ldots, x_n]$ be a polynomial ring. Then every finitely generated R-module has a finite graded free resolution of the length $\leq n$.*

Definition 1.1.18. Let $R = K[x_1, x_2, \ldots, x_n]$ be a polynomial ring and the free complex

$$\mathcal{F}\colon \; 0 \longrightarrow F_i \xrightarrow{\phi_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \longrightarrow 0$$

be a minimal free resolution of finitely generated $\mathbb{N}^n$-graded module M such that $F_i = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} R[-\mathbf{a}]^{\beta_{i,\mathbf{a}}}$. Then the invariant $\beta_{i,\mathbf{a}}$ is called the i^{th}*Betti number* of M in degree $\mathbf{a}$. It measures the minimum number of generators required in degree $\mathbf{a}$ for any i^{th} syzygy module of M.

The Betti numbers of an ideal I in a polynomial ring $R = K[x_1, .., x_n]$ has the information regarding the homological structure of the quotient ring $\dfrac{R}{I}$.

1.2 Elementary Topological Results

Let Δ be a simplicial complex on the vertex set $[n] = \{1, 2, \ldots, n\}$ (simplicial complexes will be dealt in detail in the second chapter). Depending on the topological space associated with the simplicial complex, we can characterize simplicial homology which formalizes the number of holes of a given dimension in the simplicial complex.

Let Δ be a simplicial complex on the vertex set $[n]$. Let $\mathcal{C}_i$ be the free abelian group generated by the i dimensional faces of Δ, that is, we have, $\mathcal{C}_i = \bigoplus_{\substack{A\in\Delta \\ \dim A = i}} \mathbb{Z}A$. We set $\mathcal{C}_i = 0$ for $i < 0$ and $i > \dim \Delta$. The group $\mathcal{C}_i$ is called the *i-chain group* of Δ with coefficients from $\mathbb{Z}$. The chain complex $\mathcal{C}_*(\Delta)$ of Δ over $\mathbb{Z}$ is of the form

$$\mathcal{C}_*(\Delta)\colon\ 0 \longrightarrow \mathcal{C}_n \xrightarrow{\partial_{n-1}} \mathcal{C}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{C}_1 \xrightarrow{\partial_0} \mathcal{C}_0 \longrightarrow 0,$$

where $\partial_i\colon \mathcal{C}_{i+1} \longrightarrow \mathcal{C}_i$ is a homomorphism given by

$$\partial(A) = \sum_{k=1}^{i+1} (-1)^{k-1}\Big(A - \{j_k\}\Big)$$

for $A = \{j_1, \ldots, j_{i+1}\} \in \Delta$. It can be verified that $\partial_i \circ \partial_{i+1} = 0\ \forall i$. The homomorphism $\partial_i\colon \mathcal{C}_{i+1} \to \mathcal{C}_i$ is called the *boundary operator*. The elements in $\mathrm{Ker}(\partial_i)$ are called *i-cycles* of Δ and elements in $\mathrm{Im}(\partial_{i+1})$ are *i-boundaries* of Δ. As $\partial_i \circ \partial_{i+1} = 0\ \forall i$, we have $\mathrm{Im}(\partial_{i+1}) \subset \mathrm{Ker}(\partial_i)$. The i^{th} homology group $\mathcal{H}_i(\Delta)$ of the simplicial complex Δ is defined as the quotient group

$$\mathcal{H}_i(\Delta) = \frac{\mathrm{Ker}(\partial_{i-1}\colon \mathcal{C}_i \longrightarrow \mathcal{C}_{i-1})}{\mathrm{Im}(\partial_i\colon \mathcal{C}_{i+1} \longrightarrow \mathcal{C}_i)}.$$

Remark. The rank of i^{th} homology group of the simplicial complex Δ gives the measure of the number of i-dimensional holes in Δ. This number is known as the i^{th} *Betti number* of Δ.

Proposition 1.2.1. *1. Let Δ be a simplicial complex and $\mathcal{H}_i(\Delta)$ be the i^{th} homological group of Δ. Then $\mathcal{H}_i(\Delta) = 0$ for $i < 0$ and $i > \dim(\Delta)$.*

2. Suppose the simplicial complex Δ is connected. Then $\mathcal{H}_0(\Delta) \cong \mathbb{Z}$.

Definition 1.2.2. Let Δ be a simplicial complex of dimension $d-1$ on the vertex set $[n]$. Let f_i be the number of i-dimensional faces of Δ. Then the *Euler characteristics* $\chi(\Delta)$ of Δ is defined as

$$\chi(\Delta) = \sum_{i=0}^{d-1} (-1)^i f_i.$$

Theorem 1.2.3 (Euler-Poincaré Theorem)**.** *For a simplicial complex Δ of dimension $d-1$,*

$$\chi(\Delta) = \sum_{i=0}^{d-1} (-1)^i \beta_i(\Delta).$$

For a simplicial complex Δ, we define a *reduced chain complex* with coefficients from $\mathbb{Z}$ as the following chain complex (with coefficients from $\mathbb{Z}$)

$$\tilde{\mathcal{C}}_*(\Delta)\colon\ 0 \longrightarrow \mathcal{C}_n \xrightarrow{\partial_{n-1}} \mathcal{C}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{C}_1 \xrightarrow{\partial_0} \mathcal{C}_0 \xrightarrow{\varepsilon} \mathcal{C}_{-1} \longrightarrow 0,$$

where $\mathcal{C}_{-1} = \mathbb{Z}$ and $\varepsilon\colon \mathcal{C}_0 \longrightarrow \mathcal{C}_{-1}$ is defined as $\varepsilon\Big(\sum\limits_i a_i\{i\}\Big) = \sum\limits_i a_i$. The i^{th} *reduced homology group* of the simplicial complex is defined as $\tilde{\mathcal{H}}_i(\Delta) = \mathcal{H}_i(\Delta)$, if $i \geq 1$ and $\operatorname{rank}\big(\tilde{\mathcal{H}}_0(\Delta)\big) = \operatorname{rank}\big(\mathcal{H}_0(\Delta)\big) - 1$.

Given a simplicial complex Δ and a chain complex

$$\mathcal{C}_*(\Delta)\colon\ 0 \longrightarrow \mathcal{C}_n \xrightarrow{\partial_{n-1}} \mathcal{C}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{C}_1 \xrightarrow{\partial_0} \mathcal{C}_0 \longrightarrow 0,$$

with coefficients from $\mathbb{Z}$, we define *cochains* as $\mathcal{C}^k = Hom_{\mathbb{Z}}\big(\mathcal{C}_k, \mathbb{Z}\big)$. The map $\delta_n\colon \mathcal{C}^{n-1} \longrightarrow \mathcal{C}^n$ is called a *coboundary map* and is given by

$$\delta_n(\phi)\left(\sum_{\dim F_i = n} n_i F_i\right) = \sum_{\dim F_i = n} n_i \delta(\phi)(F_i),$$

where $\phi \in \mathcal{C}^{n-1}$ and $F_i \in \Delta$. Hence we have $\delta(\phi)(F) = \phi(\partial F)$ for any $F \in \Delta$. Since $\partial_i \circ \partial_{i+1} = 0$, we have $\delta_{i+1} \circ \delta_i = 0,\ \forall i$. Thus we have, $\operatorname{Im}(\delta_i) \subseteq \operatorname{Ker}(\delta_{i+1})$. The i^{th} *cohomology group* of Δ is defined as the quotient

$$\tilde{\mathcal{H}}^i\big(\Delta,\ \mathbb{Z}\big) = \frac{\operatorname{Ker}(\delta_{i+1})}{\operatorname{Im}(\delta_i)}.$$

Theorem 1.2.4. *If $\tilde{\mathcal{H}}_k\big(\Delta,\ \mathbb{Z}\big)$ is a free abelian group, then*

$$\tilde{\mathcal{H}}_k\big(\Delta,\ \mathbb{Z}\big) \cong \tilde{\mathcal{H}}^k\big(\Delta,\ \mathbb{Z}\big).$$

Now, we give a brief introduction to homological theories for the topological spaces. The following results are taken from Croom[5]. For more details on algebraic topology, we refer to Bredon[3], Rotman[12] and Satya Deo[6]. Let X be a topological space and A be a subspace of X. Let H be a function such that for each $i \in \mathbb{Z}$, $H_i(X, A)$ is an abelian group, called the *i-dimensional relative homology group* of X modulo A. For $A = \emptyset$ the group $\mathcal{H}_i(X, \emptyset)$ is the i-homology group $\mathcal{H}_i(X)$. Let (X, A) and (Y, B) be two pairs and $f \colon X \longrightarrow Y$ be an admissible map with $f(A) \subset B$. We define a map $*$ such that for each $i \in \mathbb{Z}$, it determines a homomorphism

$$f_i^* \colon H_i(X, A) \longrightarrow H_i(Y, B),$$

called the *homomorphism induced by f* in dimension i. To each (X, A) and $i \in \mathbb{Z}$, the function ∂ assigns a homomorphism

$$\partial \colon H_i(X, A) \longrightarrow \mathcal{H}_{i-1}(A),$$

called the *boundary operator*. The homology theory consists of the functions $H, *$ and ∂ as defined above and they satisfies the following axioms, called the *Eilenberg-Steenrod Axioms*.

1. (*The Identity Axiom*) If $\iota \colon (X, A) \longrightarrow (X, A)$ is the identity map, then the induced homomorphism $\iota^* \colon H_i(X, A) \longrightarrow H_i(X, A)$ is the identity isomorphism for each $i \in \mathbb{Z}$.

2. (*The Composition Axiom*) If the map $f \colon (X, A) \longrightarrow (Y, B)$ and $g \colon (Y, B) \longrightarrow (Z, C)$ are admissible maps, then

$$(g \circ f)_i^* = g_i^* \circ f_i^* \colon H_i(X, A) \longrightarrow H_i(Z, C)$$

for each $i \in \mathbb{Z}$.

3. (*The Commutativity Axiom*) If $f \colon (X, A) \longrightarrow (Y, B)$ is an admissible map and $g \colon A \longrightarrow B$ is the restriction of f, then for each $i \in \mathbb{Z}$, the following diagram commutes.

$$\begin{tikzcd} H_i(X, A) \arrow[r, "f_i^*"] \arrow[d, "\partial"] & H_i(Y, B) \arrow[d, "\partial"] \\ \mathcal{H}_{i-1}(A) \arrow[r, "g_i^*"] & \mathcal{H}_{i-1}(B) \end{tikzcd}$$

4. (*The Exactness Axiom*) If $\iota \colon A \longrightarrow X$ and $\kappa \colon (X, \emptyset) \longrightarrow (X, A)$ are inclusion maps, then the homology sequence

$$\cdots \to \mathcal{H}_p(A) \xrightarrow{\iota^*} \mathcal{H}_p(X) \xrightarrow{\kappa^*} \mathcal{H}_p(X, A) \xrightarrow{\partial} \mathcal{H}_{p-1}(A) \to \ldots$$

is exact.

5. (*The Homotopy Axiom*) If the maps $f, g\colon (X, A) \longrightarrow (Y, B)$ are homotopic, then the induced homomorphisms f_i^* and g_i^* are equal for each $i \in \mathbb{Z}$.

6. (*The Excision Axiom*) If U is an open subset of X with $\overline{U} \subset A$, then the inclusion map $e\colon (X - U, A - U) \longrightarrow (X, A)$ induces an isomorphism
$$e_i^*\colon H_i(X - U, A - U) \longrightarrow \mathcal{H}_i(X, A)$$
for each $i \in \mathbb{Z}$. The map e is called the *excision* of U.

7. (*The Dimension Axiom*) If X is a space with a single point, then for each nonzero value of i,
$$\mathcal{H}_i(X) = \{0\}.$$

The algebraic invariants, such as; homology or cohomology groups, of the topological spaces can be computed using an algebraic tool known as the *Mayer-Vietoris sequence*. The strategy of this tool is to partition the entire space into subspaces whose homology or cohomology groups can be easily computed. We can implement Mayer-Vietoris sequence in all the homology theories that satisfies the Eilenberg-Steenrod Axioms.

Theorem 1.2.5. *Let X be a topological space and suppose U and V are subspaces of X, whose interiors may not be disjoint such that $U \cup V = X$. Then there exists a long exact sequence*
$$\cdots \to \mathcal{H}_i(U \cap V) \to \mathcal{H}_i(U) \oplus \mathcal{H}_i(V) \to \mathcal{H}_i(U \cup V) \xrightarrow{\delta} \mathcal{H}_{i-1}(U \cap V) \to \dots$$

Example 1.2.6. Let us consider $X = \mathcal{S}^1$ and compute the homology group of $\mathcal{S}^1$ by the application of Mayer-Vietoris sequence. Let us partition the space $\mathcal{S}^1$ as given in the diagram (Figure 1).

Consider the Meyer-Vietoris sequence
$$\begin{tikzcd}
\cdots & \mathcal{H}_{i-1}(U) \oplus \mathcal{H}_{i-1}(V) \arrow[l] & \mathcal{H}_{i-1}(U \cap V) \arrow[l] \\
\mathcal{H}_i(\mathcal{S}^1) \arrow[urr, "\delta"] & \mathcal{H}_i(U) \oplus \mathcal{H}_i(V) \arrow[l] & \mathcal{H}_i(U \cap V) \arrow[l] \\
\mathcal{H}_{i+1}(\mathcal{S}^1) \arrow[urr, "\delta"] & \mathcal{H}_{i+1}(U) \oplus \mathcal{H}_{i+1}(V) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

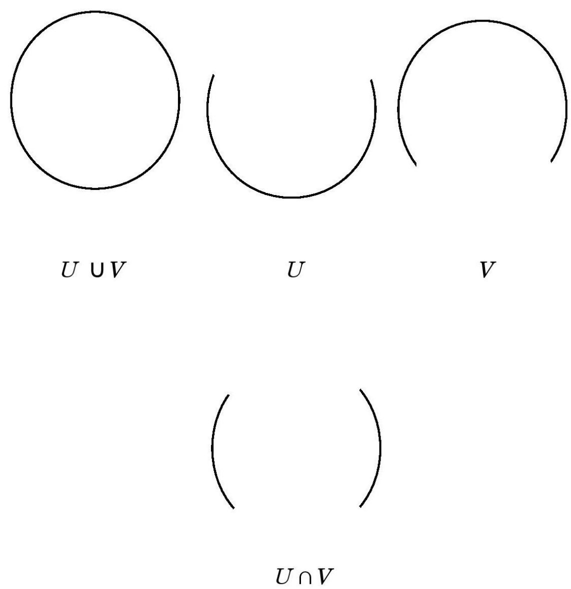


Figure 1

Along with the exactness property of the Mayer-Vietoris sequence and the fact that U and V are contractible and $U \cap V$ is a disjoint union of two contractible spaces, we have $\mathcal{H}_i(\mathcal{S}^1) = 0$ for all $i > 1$. The homology group $\mathcal{H}_0(U \cap V) \backsimeq \mathbb{Z} \oplus \mathbb{Z}$ and $\mathcal{H}_0(U) \oplus \mathcal{H}_0(V) \backsimeq \mathbb{Z} \oplus \mathbb{Z}$. Therefore the Mayer-Vietoris sequence reduces to the sequence

$$0 \longrightarrow \mathcal{H}_1(\mathcal{S}^1) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathcal{H}_0(\mathcal{S}^1) \longrightarrow 0$$

where $\alpha \colon \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$ is defined by $(a, b) \mapsto (a+b, a+b)$. We have that the homology groups $\mathcal{H}_1(\mathcal{S}^1)$ and $\mathcal{H}_0(\mathcal{S}^1)$ are isomorphic to the $\mathrm{Ker}(\alpha)$ and $\mathrm{Coker}(\alpha)$ respectively. For the map α, $\mathrm{Ker}(\alpha) = \big\{(a, -a) \colon a \in \mathbb{Z}\big\} \backsimeq \mathbb{Z}$ and $\mathrm{Coker}(\alpha) = \frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathrm{Im}(\alpha)} \backsimeq \mathbb{Z}$. Thus we have

$$\mathcal{H}_i(\mathcal{S}^1, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } \ i = 0, 1, \\ 0 & \text{if otherwise.} \end{cases}$$

Remark. If the topological space X is the wedge of two spaces X_1 and X_2 then we have

$$\mathcal{H}_i(X) \cong \mathcal{H}_i(X_1) \oplus \mathcal{H}_i(X_2) \text{ for } i \geq 1.$$

Chapter 2

Stanley-Reisner Rings

This chapter deals with a finite *simplicial complex* (a combinatorial object), to which one can associate an algebraic structure called *Stanley-Reisner ring*. Further, we study relationship between combinatorial invariants of simplicial complexes (such as **f**-*vectors*) and algebraic invariants of the corresponding Stanley-Reisner rings (such as **h**-*vectors*). We also emphasis upon a special class of simplicial complexes called *pure shellable complexes* and study their properties.

2.1 Simplicial Complexes

We shall start with recalling basic definitions and properties of *simplicial complexes*.

Definition 2.1.1. Let $V = [n] = \{1, 2, \ldots, n\}$ and Δ be a subset of the powerset $\mathcal{P}([n])$ of $[n]$. We say that Δ is a *simplicial complex* on the *vertex set* V if whenever $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$. In other words, a simplicial complex Δ on the vertex set $[n]$ is a subset of $\mathcal{P}([n])$, which is closed under taking subsets.

Any $F \in \Delta$ is called a *face* of Δ and its *dimension* is given by $\dim F = |F| - 1$. A face with dimension i is said to be an *i-face*. The *dimension* of a simplicial complex Δ is given by

$$\dim \Delta = \max\{\dim F : F \in \Delta\}.$$

The maximal faces of Δ under inclusion are called *facets* of Δ. A simplicial complex Δ is determined by its facets. If $F_1, F_2, \ldots, F_r$ are facets of Δ, we say that the simplicial complex Δ is *generated by* the facets $F_1, F_2, \dots, F_r$ and we write $\Delta = \langle F_1, F_2, \dots, F_r \rangle$. The simplicial complex generated by a single facet of dimension d is called a *d-simplex*.

Example 2.1.2. Let $V = [4]$ be a vertex set. Consider the simplicial complex $\Delta = \big\{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}\big\}$. Then facets of Δ are given by $\{4\}, \{1, 2\}, \{1, 3\}$. Hence, $\Delta = \big\langle \{4\}, \{1, 2\}, \{1, 3\} \big\rangle$. Clearly, $\dim \Delta = 1$.

Example 2.1.3. Let $V = [5]$ be a vertex set. Consider the simplicial complex Δ,

$$\big\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{1, 2, 3\}\big\}.$$

The facets of Δ are given by $\{1, 2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}$. Hence, we can write $\Delta = \big\langle \{1, 2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\} \big\rangle$ and $\dim \Delta = 2$.

For any non-empty simplicial complex Δ, the emptyset $\emptyset \in \Delta$ is a face of Δ with dimension -1. The 0-dimensional faces of Δ are called *vertices* and 1-dimensional faces are called the *edges* of Δ.

Remark. Finite simplicial complexes can be conveniently represented by diagrams. In fact, simplicial complex in Example 2.1.2 is represented by the following diagram (Figure 1).

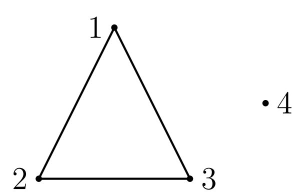


Figure 1

Similarly, the simplicial complex in Example 2.1.3 is represented by the diagram (Figure 2).

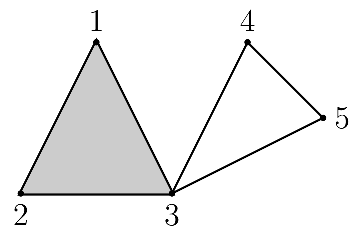


Figure 2

To every finite simplicial complex Δ, one can associate a topological space, called its *geometric realization* $|\Delta|$. If X is a topological space and suppose there exists a simplicial complex Δ such that X is homeomorphic to $|\Delta|$, then we say that X is a *triangulable space* and Δ is a *triangulation* of X.

Example 2.1.4. Let $\Delta = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\big\}$ be a simplicial complex on the vertex set [3]. Then Δ is represented by a triangle as shown in the following diagram (Figure 3).

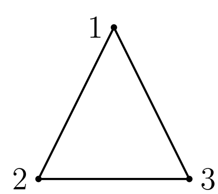


Figure 3

Clearly, simplicial complex Δ gives a triangulation of the unit circle $S^1 = \{z \in \mathbb{C} \colon \mid z \mid = 1\}$.

2.2 Combinatorial Invariants

Let Δ be an arbitrary simplicial complex on a (finite) vertex set V with dimension $d-1$. Let f_i be the number of i-faces in Δ. For a non-empty simplicial complex Δ, the empty face $\emptyset \in \Delta$. Thus, we have $f_{-1} = 1$. Also, $f_0 = |V|$. Consider the d-tuple $\mathbf{f}(\Delta) = (f_0, f_1, \ldots, f_{d-1})$.

The d-tuple $\mathbf{f}(\Delta)$ is called the $\mathbf{f}$-*vector* of Δ. This is an important combinatorial invariant of Δ. We illustrate $\mathbf{f}$-*vector* of a simplicial complex in the following example.

Example 2.2.1. Consider a simplicial complex Δ as given in the diagram (Figure 4). Clearly, $f_{-1} = 1,\ f_0 = 5,\ f_1 = 6,\ f_2 = 1$ and Δ is a simplicial complex of $\dim 2$. Thus $\mathbf{f}$-vector of Δ is given by $\mathbf{f}(\Delta) = (5, 6, 1)$.

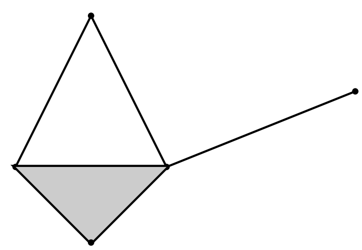


Figure 4

Another important invariant associated to a finite simplicial complex is its *Euler characteristics.*

Definition 2.2.2. Let Δ be a finite simplicial complex on a vertex set V. Let $\mathbf{f}(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ be the $\mathbf{f}$-vector of Δ. Then the *Euler characteristics* $\chi(\Delta)$ of Δ is given by

$$\chi(\Delta) = \sum_{i=0}^{d-1} (-1)^i f_i.$$

Also, the *reduced Euler characteristic* $\widetilde{\chi}(\Delta)$ of Δ is given by

$$\widetilde{\chi}(\Delta) = \sum_{i=-1}^{d-1} (-1)^i f_i \ = \ \chi(\Delta) \ - \ 1.$$

A characterization of $\mathbf{f}$-vector of a (finite) simplicial complex was obtained by Joseph Kruskal[10] and Gyula Katona[8], independently. We shall state the result of Kruskal and Katona without proof. In order to state this theorem, we first recall the notion of *Macaulay expansion* of a positive integer.

Lemma 2.2.3. *Each positive integer α has a unique expansion*

$$\alpha = \binom{\alpha_i}{i} + \binom{\alpha_{i-1}}{i-1} + \cdots + \binom{\alpha_k}{k},$$

where i is a positive integer with $\alpha_i > \alpha_{i-1} > \cdots > \alpha_k \geq k \geq 1$. This decomposition is called the Macaulay expansion of α.

Example 2.2.4. Consider $\alpha = 25$ and $i = 3$. The greatest integer in the required form lesser than or equal to 25 is $20 = \binom{6}{3}$. The largest integer of the form $\binom{\alpha_2}{2}$ smaller than or equal to $5 = 25 - \binom{6}{3}$ is $3 = \binom{3}{2}$. The largest integer of the form $\binom{\alpha_1}{1}$ lesser than or equal to $2 = 25 - \binom{6}{3} - \binom{3}{2}$ is $2 = \binom{2}{1}$. Hence the Macaulay expansion of 25 with respect to 3 is given by

$$25 = \binom{6}{3} + \binom{3}{2} + \binom{2}{1}.$$

Definition 2.2.5. Let $\alpha = \binom{\alpha_i}{i} + \binom{\alpha_{i-1}}{i-1} + \cdots + \binom{\alpha_k}{k}$ be the Macaulay expansion of α with respect to i. Then we define the symbol $\alpha^{\langle i\rangle}$ and $\alpha^{(i)}$ by

$$\alpha^{\langle i\rangle} = \binom{\alpha_i + 1}{i+1} + \binom{\alpha_{i-1}+1}{i} + \cdots + \binom{\alpha_k + 1}{k+1},$$

$$\alpha^{(i)} = \binom{\alpha_i}{i+1} + \binom{\alpha_{i-1}}{i} + \cdots + \binom{\alpha_k}{k+1}.$$

As in Example 2.2.4., we have $\alpha = 25 = \binom{6}{3} + \binom{3}{2} + \binom{2}{1}$. Therefore,

$$25^{\langle 3\rangle} = \binom{7}{4} + \binom{4}{3} + \binom{3}{2} = 35 + 4 + 3 = 42$$

and

$$25^{(3)} = \binom{6}{4} + \binom{3}{3} + \binom{2}{2} = 15 + 1 + 1 = 17.$$

Now we are in a position to state the celebrated theorem of Kruskal-Katona.

Theorem 2.2.6 (Kruskal-Katona Theorem)**.** *Let $\mathbf{f} = (f_0, \ldots, f_{d-1})$ be a sequence of positive integers. Then the following conditions are equivalent :*

1. There exists a simplicial complex Δ *with* $\mathbf{f}(\Delta) = \mathbf{f}$.

2. $f_{j+1} \leq f_j^{(j+1)}$ *for* $0 \leq j \leq d-2$.

Example 2.2.7. Consider the simplicial complex given in Example 2.2.1. The $\mathbf{f}$-vector is given as $\mathbf{f}(\Delta) = (5, 6, 1)$. Let us compute $f_0^{(1)}$ and $f_1^{(2)}$ to see whether $f_{j+1} \leq f_j^{(j+1)}$ for $j = 0, 1$. Macaulay expansion of $f_0 = 5$ and $j = 1$ is $5 = \binom{5}{1}$ and that of $f_1 = 6$ and $j = 2$ is $6 = \binom{4}{2}$. Thus $f_0^{(1)} = \binom{5}{2} = 10 \geq f_1$ and $f_1^{(2)} = \binom{4}{3} = 4 \geq f_2$.

2.3 Stanley-Reisner Rings

Richard Stanley associated a commutative ring (or a K-algebra) $K[\Delta]$ to every finite simplicial complex Δ. He demonstrated that the combinatorial properties of Δ are intimately related to the algebraic properties of the K-algebra $K[\Delta]$.

Definition 2.3.1. Let Δ be a simplicial complex on the vertex set $V = [n]$. Let K be a field and $R = K\ [x_1, x_2, \ldots, x_n]$ be the standard polynomial ring over K. For every subset $A \subseteq [n]$, we associate a square-free monomial $\mathbf{x}_A = \prod\limits_{i \in A} x_i$. The monomial ideal I_Δ generated by square-free monomials $\mathbf{x}_F$ such that $F \notin \Delta$ is called the *face ideal* or *Stanley-Reisner ideal* of the simplicial complex Δ.

The square-free monomial $\prod\limits_{i \in F} x_i$ corresponding to the minimal non-face F of Δ is a minimal generator for I_Δ. Thus,

$$I_\Delta = \big\langle \mathbf{x}_F \ : \ \text{F is minimal non-face in } \Delta \big\rangle.$$

The face ideal I_Δ of a simplicial complex Δ is a radical ideal. Therefore, I_Δ can be written as intersection of prime ideals.

Proposition 2.3.2. *The primary decomposition of face ideal* I_Δ *of a simplicial complex* Δ *is given by*

$$I_\Delta = \bigcap_{F \in \Delta} P_{\overline{F}}$$

where $P_{\overline{F}}$ *is the prime ideal generated by all* x_i *such that* $i \notin F$. *That is,*

$$P_{\overline{F}} = \big\langle x_i \colon i \notin F \big\rangle.$$

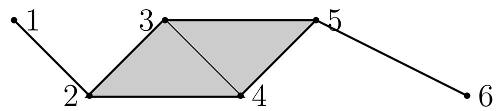


Figure 5

Example 2.3.3. Consider a simplicial complex Δ as given in the diagram (Figure 5). Clearly the Stanley-Reisner ideal I_Δ of Δ is given by

$$I_\Delta = \langle X_1X_3, X_1X_4, X_1X_5, X_1X_6, X_2X_5, X_2X_6, X_3X_6, X_4X_6 \rangle$$

By primary decomposition of I_Δ,

$$I_\Delta = \langle X_3, X_4, X_5, X_6 \rangle \cap \langle X_1, X_5, X_6 \rangle \cap \langle X_1, X_2, X_6 \rangle \cap \langle X_1, X_2, X_3, X_4 \rangle.$$

Definition 2.3.4. Let Δ be a finite simplicial complex on the vertex set $[n]$. Let K be a field and $R = K\ [x_1, x_2, \ldots, x_n]$ be the polynomial ring over the field K. The quotient ring $K[\Delta] = \dfrac{R}{I_\Delta} = \dfrac{K\ [x_1, \ldots, x_n]}{I_\Delta}$ is called the *Stanley-Reisner ring* of Δ.

Proposition 2.3.5. *Let Δ be a finite simplicial complex on the vertex set $[n]$ and $K[\Delta]$ be the corresponding Stanley-Reisner ring. The Krull dimension of $K[\Delta]$ is given by*

$$\dim K[\Delta] = \dim \Delta + 1.$$

Proof. Krull dimension of the ring $K[\Delta]$ is given by

$$\dim K[\Delta] = \sup \left\{ \dim \left(\frac{K[x_1, \ldots, x_n]}{P_{\overline{F}}} \right) : F \text{ is a facet of } \Delta \right\}.$$

Let $F = \{i_1, i_2, \ldots, i_t\}$. Then, $\dfrac{K[x_1, \ldots, x_n]}{P_{\overline{F}}} \simeq K[x_{i_1}, \ldots, x_{i_t}]$ and $\dim \left(\dfrac{K[x_1, \ldots, x_n]}{P_{\overline{F}}} \right) = t = |F|$. Therefore, we write

$$\dim K[\Delta] = \sup \left\{ \dim F + 1 : F \text{ is a facet of } \Delta \right\}.$$

Hence, we get that $\dim K[\Delta] = \dim \Delta + 1$. $\square$

2.4 Hilbert Series of $\boldsymbol{K}[\Delta]$

Let Δ be a finite simplicial complex on the vertex set $[n]$. For the polynomial ring $R = K[x_1, \ldots, x_n]$ over any field K,

$$K[x_1, \ldots, x_n]_{\mathbf{a}} = \begin{cases} c\mathbf{x}^{\mathbf{a}} & \text{if } \mathbf{a} \in \mathbb{N}^n,\ c \in K \\ 0 & \text{if } a_i < 0 \ \text{ for some } i, \end{cases}$$

where $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$, $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$. This shows that the polynomial ring R is $\mathbb{N}^n$-graded. The face ideal I_Δ of Δ is generated by monomials. Hence I_Δ is also $\mathbb{N}^n$-graded. Therefore, the quotient $\dfrac{R}{I_\Delta}$ has $\mathbb{N}^n$-grading given by

$$\left(\frac{R}{I_\Delta}\right)_{\mathbf{a}} = \frac{R_{\mathbf{a}}}{\left(I_\Delta\right)_{\mathbf{a}}},\ \forall\, \mathbf{a} \in \mathbb{N}^n.$$

Hence, the Stanley-Reisner ring $K[\Delta]$ of the simplicial complex Δ inherits a natural $\mathbb{N}^n$-grading.

Proposition 2.4.1. *Let Δ be a finite simplicial complex and $K[\Delta]$ be the associated Stanley-Reisner ring. Then,*
(a) $\dim_K \big(K[\Delta]\big) = 1 \iff \mathbf{x}^{\mathbf{a}} \notin I_\Delta \iff \operatorname{supp}(\mathbf{a}) \in \Delta$, *where support of* $\mathbf{a} \in \mathbb{N}^n$ *is given by* $\operatorname{supp}(\mathbf{a}) = \{i\colon a_i > 0\}$.
(b)The (coarse) Hilbert series $\mathbb{F}\big(K[\Delta],\ t\big)$ of the Stanley-Reisner ring $K[\Delta]$ is given by

$$\mathbb{F}\big(K[\Delta],\ t\big) = \sum_{i=0}^{d} f_{i-1}\left(\frac{t}{1-t}\right)^i.$$

Proof. (a) Let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$. Then

$$\begin{aligned} \mathbf{x}^{\mathbf{a}} \in I_\Delta \quad &\iff \quad X_G \text{ divides } \mathbf{x}^{\mathbf{a}} \text{ for some } G \notin \Delta. \\ &\iff \quad a_i \geq 1 \text{ for } i \in G,\ \text{where } G \notin \Delta. \\ &\iff \quad G \subseteq \operatorname{supp}(\mathbf{a}),\ \text{where } G \notin \Delta. \\ &\iff \quad \operatorname{supp}(\mathbf{a}) \notin \Delta \quad (\text{as } G \notin \Delta). \end{aligned}$$

(b) Let $\mathbf{t} = (t_1, \ldots, t_n)$ and for $\mathbf{a} \in \mathbb{N}^n, \mathbf{t}^{\mathbf{a}} = t_1^{a_1} t_2^{a_2} \ldots t_n^{a_n}$. Then the (fine) Hilbert series of the Stanley-Reisner ring $K[\Delta]$ can be computed as follows.

$$
\begin{aligned}
\mathbb{F}\big(K[\Delta],\ \mathbf{t}\big) &= \sum_{\substack{\mathbf{a}\in\mathbb{N}^n,\\ \mathrm{supp}(\mathbf{a})\in\Delta}} \mathbf{t}^{\mathbf{a}} \\
&= \sum_{F\ \in\ \Delta} \left(\sum_{\substack{\mathbf{a}\in\mathbb{N}^n,\\ \mathrm{supp}(\mathbf{a})=F}} \mathbf{t}^{\mathbf{a}} \right) \\
&= \sum_{F\in\Delta} \left(\prod_{i\in F} \frac{t_i}{1-t_i} \right).
\end{aligned}
$$

We compute the (coarse) Hilbert series $\mathbb{F}\big(K[\Delta],\ t\big)$ of the Stanley-Reisner ring $K[\Delta]$ by putting $t_i = t,\ \forall\ i$ in the (fine) Hilbert series of $K[\Delta]$. Thus, we get

$$
\begin{aligned}
\mathbb{F}\big(K[\Delta],\ t\big) &= \sum_{F\in\Delta} \left(\frac{t}{1-t}\right)^{|F|} \\
&= \sum_{i=0}^{d} \left(\sum_{\substack{F\in\Delta,\\ |F|=i}} \left(\frac{t}{1-t}\right)^{i} \right) \\
&= \sum_{i=0}^{d} f_{i-1} \left(\frac{t}{1-t}\right)^{i} \\
&= 1 + f_0\left(\frac{t}{1-t}\right) + f_1\left(\frac{t}{1-t}\right)^{2} + \cdots + f_{d-1}\left(\frac{t}{1-t}\right)^{d}.
\end{aligned}
$$

$\square$

For a finite simplicial complex Δ, the Hilbert function of its Stanley-Reisner ring $K[\Delta]$ can be obtained from the Hilbert series.

Proposition 2.4.2. *The Hilbert function for the Stanley-Reisner ring $K[\Delta]$ of a finite simplicial complex Δ is given by*

$$
\mathbb{H}\big(K[\Delta],\ n\big) = \begin{cases} 1 & \textit{if } n = 0 \\ \sum\limits_{i=0}^{d-1} f_i \binom{n-1}{i} & \textit{if } n > 0. \end{cases}
$$

The Hilbert polynomial $P_{K(\Delta)}(n)$ of the Stanley-Reisner ring $K[\Delta]$ is given by

$$P_{K(\Delta)}(n) = \sum_{i=0}^{d-1} f_i \binom{n-1}{i}.$$

Clearly, $P_{K(\Delta)}(n) = \mathbb{H}\big(K(\Delta),\ n\big)$ for $n > 0$. We note that for $n = 0$, $P_{K(\Delta)}(0) = \sum\limits_{i=0}^{d-1}(-1)^i f_i = \chi(\Delta)$. From this we can infer that the Hilbert function and the Hilbert polynomial of the Stanley-Reisner ring $K[\Delta]$ are the same $\forall\ n\ \geq\ 0$ if and only if the Euler characteristic $\chi(\Delta) = 1$.

2.5 *h*-vector of a Simplicial Complex

Let Δ be a finite $(d{-}1)$-dimensional simplicial complex on the vertex set $[n]$. The Hilbert series of the corresponding Stanley-Reisner ring $K[\Delta]$ can be written as

$$\mathbb{F}\big(K[\Delta],\ t\big) = \sum_{i=0}^{d} f_{i-1}\left(\frac{t}{1-t}\right)^i = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d},$$

where $h_i\ \in \mathbb{Z}$. The $(d+1)$-tuple $\mathbf{h}(\Delta) = (h_0, h_1, \ldots, h_d)$ is called the $\mathbf{h}$-*vector* of the simplicial complex Δ.

Lemma 2.5.1. *Let Δ be a $(d-1)$-dimensional simplicial complex. The $\mathbf{f}$-vector and $\mathbf{h}$-vector of the simplicial complex are related to each other. This relation is given by*

$$h_j = \sum_{i=0}^{j}(-1)^{j-i}\binom{d-i}{j-i} f_{i-1} \quad \big(0 \leq j \leq d-1\big),$$

and

$$f_{j-1} = \sum_{i=0}^{j}\binom{d-i}{j-i} h_i \qquad \big(0 \leq j \leq d\big).$$

Proof. The Hilbert series of the Stanley-Reisner ring $K[\Delta]$ is given by

$$\mathbb{F}\big(K[\Delta],\ t\big) = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}.$$

By equating $\mathbb{F}\big(K[\Delta],\ t\big) = \sum_{i=0}^{d} f_{i-1}\left(\dfrac{t}{1-t}\right)^{i}$, we get

$$\sum_{i=0}^{d} f_{i-1}\left(\frac{t}{1-t}\right)^{i} = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}. \qquad (*)$$

Therefore,

$$\sum_{j=0}^{d} h_j t^j \;=\; \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i}.$$

On comparing the coefficients of t^i, we get

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1}.$$

Now in order to get f_j in terms of h_i we substitute $t = \dfrac{r}{1+r}$ in (*).

$$\sum_{j=0}^{d} h_j \left(\frac{r}{1+r}\right)^{j} \;=\; \sum_{i=0}^{d} f_{i-1}\left(\frac{r^i}{(1+r)^d}\right).$$

Thus, $\sum_{j=0}^{d} h_j r^j (1+r)^{d-j} \;=\; \sum_{i=0}^{d} f_{i-1} r^i$. On comparing the coefficients of r^j, we get

$$f_{j-1} \;=\; \sum_{i=0}^{d} \binom{d-i}{j-i} h_i.$$

This completes the proof. $\square$

Corollary 2.5.2. *From the relation between* $\mathbf{f}$*-vector and* $\mathbf{h}$*-vector of a simplicial complex* Δ *with dimension* $(d-1)$*, we get the following identities.*

$h_0 = 1$,
$h_1 = f_0 - d$,
$h_d = (-1)^{d-1}\tilde{\chi}(\Delta)$ and
$f_{d-1} = \sum_{i=0}^{d} h_i$.

Recall that a graded commutative ring R with identity is Cohen-Macaulay if its Krull dimension $\dim(R)$ is same as the depth, $\mathrm{depth}(R)$. That is, R is Cohen-Macaulay if $\dim(R) = \mathrm{depth}(R)$.

Proposition 2.5.3. *Let K be an infinite field. If the Stanley-Reisner ring $K[\Delta]$ of the simplicial complex Δ is Cohen-Macaulay then the $\mathbf{h}$-vector $\mathbf{h}(\Delta) = (h_0, h_1, \dots, h_d)$ of Δ satisfies the inequality*

$$0 \ \leq \ h_i \ \leq \ \binom{h_i - d + i - 1}{i} \quad \text{for } 0 \leq i \leq d.$$

Proof. Let the Stanley-Reisner ring $K[\Delta]$ be Cohen-Macaulay. This implies that $\mathrm{depth}\Big(\mathfrak{M}, \ K[\Delta]\Big) = d$, where $\mathfrak{M} = \bigoplus\limits_{i=1}^{\infty} K[\Delta]_i$. Therefore there exists a regular sequence in $\mathfrak{M}$, say $y_1, \dots, y_d$ with degree equals 1. We can form an exact sequence

$$0 \longrightarrow K[\Delta](-1) \xrightarrow{y_1} K[\Delta] \longrightarrow \frac{K[\Delta]}{\langle y_1 \rangle} \longrightarrow 0.$$

Using this exact sequence, we compute the Hilbert series of $\dfrac{K[\Delta]}{\langle y_1 \rangle}$ as

$$\mathbb{F}\left(\frac{K[\Delta]}{\langle y_1 \rangle}, \ t\right) = (1-t) \ \mathbb{F}\big(K[\Delta], \ t\big).$$

On repeating this process, the Hilbert series of $\dfrac{K[\Delta]}{\langle y_1, \dots, y_d \rangle}$ is given by

$$\mathbb{F}\left(\frac{K[\Delta]}{\langle y_1, \dots, y_d \rangle}, \ t\right) = (1-t)^d \ \mathbb{F}\big(K[\Delta], \ t\big) = h_0 + h_1 t + \cdots + h_d t^d.$$

Let $\overline{R} = \dfrac{K[\Delta]}{\langle y_1, \dots, y_d \rangle}$. From the above expression, we get that $h_i = \dim_K(\overline{R}_i) \geq \ 0$. We know that $\overline{R}$ is generated over K by $n-d$ elements of degree 1. Recall that the number of homogeneous monomials in $x_1, x_2, \dots, x_n$ of degree i is given by $\binom{n-1+i}{i}$. Hence, we get

$$0 \ \leq \ h_i \ = \ \dim_K\left(\overline{R}_i\right) \ \leq \ \binom{h_i - d + i - 1}{i}.$$

□

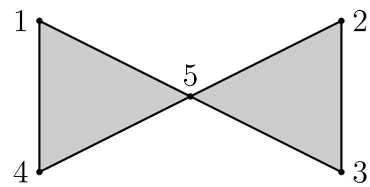


Figure 6

Example 2.5.4. Consider the simplicial complex Δ represented as in the Figure 6. The $\mathbf{f}$-vector and $\mathbf{h}$-vector of this simplicial complex is $\mathbf{f}(\Delta) = (5, 6, 2)$ and $\mathbf{h}(\Delta) = (1, 2, -1, 0)$. As $h_2 = -1 < 0$, the Stanley-Reisner ring associated with the given simplicial complex is not Cohen-Macaulay.

2.6 Shellable Simplicial Complex

Let Δ be a finite simplicial complex and K be any field. The simplicial complex Δ is Cohen-Macaulay over K if the Stanley-Reisner ring $K[\Delta]$ associated with Δ is a Cohen-Macaulay ring. The property of Cohen-Macaulayness depends on the characteristic of the field K. For instance, the minimal triangulation of $\mathbb{RP}^n$ is Cohen-Macaulay if characteristic of the field K, $\mathrm{char}(K)$ is not 2. When $\mathrm{char}(K) = 2$ then Δ is not Cohen-Macaulay. We now introduce a class of simplicial complex Δ with the property that the associated Stanley-Reisner rings $K[\Delta]$ are Cohen-Macaulay over any field K.

Definition 2.6.1. A pure simplicial complex Δ of dimension $d-1$ is called a *shellable simplicial complex* if there exists a linear ordering of facets say $F_1, F_2, \ldots, F_t$ such that the subcomplex $\langle F_1, F_2, ..., F_{i-1}\rangle \bigcap \langle F_i\rangle$ is generated by a non-empty subset of maximal proper faces of F_i with dimension $d-2$, for $2 \leq i \leq t$. The linear ordering $F_1, F_2, \ldots, F_t$ is called a *shelling* of Δ.

For a finite simplicial complex Δ, let us denote the subcomplex generated by the facets $F_1, F_2, \ldots, F_i$ as

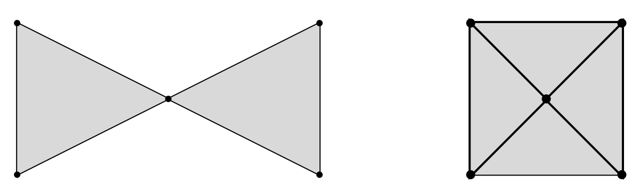
$$\Delta_i = \big\langle F_1, F_2, \ldots, F_i \big\rangle.$$

Theorem 2.6.2. *Let Δ be a pure simplicial complex of dimension $d-1$. Let $F_1, F_2, \ldots, F_t$ be a linear ordering of its facets. Then the following are equivalent:*

1. *Δ is shellable.*
2. *The set $\{F : F \in \Delta_i$ and $F \notin \Delta_{i-1}\}$ has a unique minimal element. That is, in each shelling step a unique minimal face has been introduced.*
3. *For all i, j; $1 \leq j < i \leq t$, there exists some k such that $1 \leq k < i$ and some $i_k \in F_i - F_j$ such that $F_i - F_k = \{i_k\}$.*

Proof. 1. $\Longrightarrow$ 2. Let the facet $F_i = \{i_1, i_2, \ldots, i_m\}$ and the subcomplex $\Delta_{i-1} \bigcap \langle F_i \rangle = \langle i_1, i_2, ..., i_{j-1}, i_{j+1}, ..., i_m \rangle$ such that $1 \leq j \leq r \leq m$. Therefore, we get that $\{i_1, i_2, \ldots, i_r\}$ is the unique minimal element of the set $\{F : F \in \Delta_i$ and $F \notin \Delta_{i-1}\}$.
2. $\Longrightarrow$ 3. Consider a facet G such that it is the unique minimal element of the set $\{F \colon F \in \Delta_i$ and $F \notin \Delta_{i-1}\}$. Then we have $G \not\subset F_j$, $\forall\ j < i$. Hence, $\exists\ i_k \in G - F_j$. Since $G \subseteq F_i$, we have $i_k \in F_i - F_j$. Thus there exists some k such that $1 \leq k \leq i-1$ and $\{i_k\} = F_i - F_k$.
3. $\Longrightarrow$ 1. Let $F \in \langle F_i \rangle \bigcap \Delta_{i-1}$. Then $F \subset F_j$ for some $j < i$. We have $i_k \in F_i - F_j$. Thus, $F_i - \{i_k\}$ is the maximal proper face of $\langle F_i \rangle$ and $F_i - \{i_k\} \in \langle F_i \rangle \bigcap \Delta_{i-1}$ such that $F \in F_i - \{i_k\}$. Therefore by the definition of shellability of a simplicial complex, Δ is shellable. $\square$

Example 2.6.3. Consider the finite simplicial complexes represented in diagrams given below (Figure 7). The first simplicial complex, (a) is not a shellable simplicial complex. The second simplicial complex, (b) is a pure shellable simplicial complex.



(a) (b)

Figure 7

Proposition 2.6.4. *Every shellable simplicial complex is Cohen-Macaulay over any field.*

Proof. Let us look at a brief sketch of the proof. Let Δ be a $(d-1)$-dimensional shellable simplicial complex on vertex set $[n]$. Let $F_1, F_2, .., F_t$ be a shelling for Δ and $P_{\overline{F}_i}$ be the associated prime for each of the facets F_i.
Recall that for any two graded ideals I_1 and I_2 of a polynomial ring R there exists a short exact sequence

$$0 \longrightarrow \frac{R}{I_1 \cap I_2} \longrightarrow \frac{R}{I_1} \bigoplus \frac{R}{I_2} \longrightarrow \frac{R}{I_1 + I_2} \longrightarrow 0.$$

If $\dfrac{R}{I_1}$ and $\dfrac{R}{I_2}$ are Cohen-Macaulay of dimension d and $\dfrac{R}{I_1+I_2}$ is Cohen-Macaulay of dimension $d-1$, then from the *depth lemma* we conclude that $\dfrac{R}{I_1 \cap I_2}$ is a Cohen-Macaulay ring of dimension d. This observation along with induction on the number of facets will be used to prove the proposition.
If $t = 1$, then the Stanley-Reisner ring $K[\Delta]$ is a polynomial ring which is already Cohen-Macaulay. Now assume that the proposition holds for $t-1$. Let $I = \bigcap\limits_{i=1}^{t-1} P_{\overline{F}_i}$ and $J = P_{\overline{F}_t}$. Let K be any field and $R = K[X_1, \ldots, X_n]$. Consider the exact sequence

$$0 \longrightarrow \frac{R}{I \cap J} \longrightarrow \frac{R}{I} \bigoplus \frac{R}{J} \longrightarrow \frac{R}{I + J} \longrightarrow 0.$$

Now by induction hypothesis we can see that the ring $\dfrac{R}{I}$ is Cohen-Macaulay. Since $\dfrac{R}{J}$ is a polynomial ring, it is also Cohen-Macaulay. We also prove that $\dfrac{R}{I+J}$ is Cohen-Macaulay. Now by using the observation stated above, we deduce that $K[\Delta] = \dfrac{R}{I \cap J}$ is a Cohen-Macaulay ring of dimension d. Hence the simplicial complex Δ is Cohen-Macaulay. $\square$

Consider a shellable simplicial complex Δ. Let $F_1, F_2, \ldots, F_t$ be a shelling of Δ. For $2 \leq j \leq t$, $\Delta_{j-1} \bigcap \langle F_j \rangle$ is generated by r_j

maximal proper faces of F_j. We take $r_1 = 0$. Then, the **h**-vector $\mathbf{h}(\Delta) = (h_1, \ldots, h_d)$ is given by

$$h_i = |\{j \colon r_j = i\}|.$$

This is known as the Mac-Mullen characterization of **h**-vectors of a pure shellable simplicial complex.

Example 2.6.5. Consider the pure shellable simplicial complex Δ represented by the diagram (Figure 8). Let $F_1 = \{1, 2, 3\}$, $F_2 = \{1, 2, 4\}$, $F_3 = \{1, 3, 4\}$, $F_4 = \{2, 3, 4\}$ be a shelling of Δ. Clearly, $r_1 = 0$, $r_2 = 1$, $r_3 = 2$, $r_4 = 3$. Therefore, $h_i = 1$ for $0 \leq i \leq 3$.

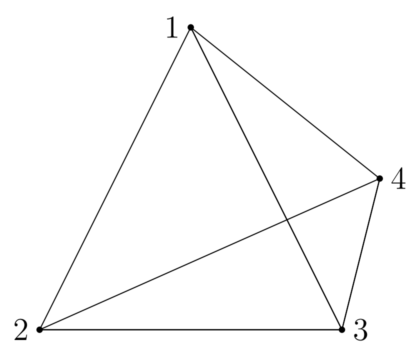


Figure 8

Chapter 3

Partially Ordered Sets

This chapter introduces the notion of *partially ordered set* (*poset*) which formalizes and generalizes the concepts of ordering or sequencing the elements of a set. Along with various properties of a poset, the chapter highlights Möbius function and its properties.

3.1 Basic Concepts

This section covers some of the basic definitions and properties of partially ordered sets.

Definition 3.1.1. Let P be a finite set. Let $\leq : P \times P \longrightarrow P$ be a relation defined on P which satisfies the following properties.

1. For every $x \in P,\ x \leq x$ (*Reflexivity*).
2. For $x, y \in P,\ x \leq y$ and $y \leq x \implies x = y$ (*Anti-symmetry*).
3. For $x, y, z \in P,\ x \leq y$ and $y \leq z \implies x \leq z$ (*Transitivity*).

The relation $\leq$ on P is called a *partial ordering* on P. The set P with a partial ordering $\leq$, denoted as $(P, \leq)$, is called a *partially ordered set* (or simply a *poset*).

Example 3.1.2. Given below are some examples of posets.

1. The set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ with usual ordering is a poset.

2. For $n \in \mathbb{N}$, let $C(n)$ denotes the set $\{0, 1, 2, \ldots, n\}$. Then $C(n)$ is a poset with usual ordering of natural numbers.

3. Let $n \in \mathbb{N}$ and $[n] = \{1, 2, 3 \ldots, n\}$ be the set of first n positive integers. By convention, we take $[0] = \emptyset$. We define a poset $(B(n), \leq)$ as the set of all subsets of $[n]$ with an ordering $\leq$ defined as $A \leq B$ if $A \subseteq B$ for any $A, B \in B(n)$. The poset $(B(n), \leq)$ is called a *Boolean poset*.

4. Let $\mathbb{P} = \mathbb{N} - \{0\}$ be the set of positive integers. For $n \in \mathbb{P}$, let $D(n)$ be the set of all positive divisors of n. For any two elements $d_1, d_2 \in D(n)$, the relation $\leq$ is defined as $d_1 \leq d_2$ if and only if d_1 divides d_2. Hence the set $D(n)$ along with the given relation $\leq$ is a poset.

5. The set $(L(n, q), \leq)$ of all subspaces of n-dimensional vector space $\mathbb{F}_q^n$ over $\mathbb{F}_q$ ordered by inclusion is a poset.

6. For $n \in \mathbb{P}$ consider the poset $\big(\prod(n), \leq\big)$ of all partitions of $[n]$ with ordering $\leq$ defined by refinement. For example, consider $n = 3$. Then, $\prod(3) = \{123\ ,\ 12|3\ ,\ 13|2\ ,\ 23|1\ ,\ 1|2|3\}$, where $ab|c$ denotes the partition $\big\{\{a, b\}, \{c\}\big\}$ and $a|b|c$ denotes $\big\{\{a\}, \{b\}, \{c\}\big\}$.

Recall that by a *partition* of a finite set A we refer to a set $\mathfrak{B}$ of subsets of A such that all elements of $\mathfrak{B}$ are non-empty, mutually disjoint and union of all elements of $\mathfrak{B}$ equals the entire set A. In other words, $\mathfrak{B} = \{B_1, \ldots, B_t\}$ is a partition of A, if $B_i \neq \emptyset,\ B_i \cap B_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{t} B_i = A$. Let $\mathfrak{B}$ and $\mathfrak{C}$ be two partitions of the set A. If every block of $\mathfrak{C}$ is contained in a block of $\mathfrak{B}$, then $\mathfrak{B}$ is said to be a *refinement* of the partition $\mathfrak{C}$.

In a poset $(P, \leq)$, we say that x and y are *comparable* if either $x \leq y$ or $y \leq x$. Otherwise, we say that x and y are *incomparable*. A poset is called a *totally ordered set* if every pair of elements are comparable.

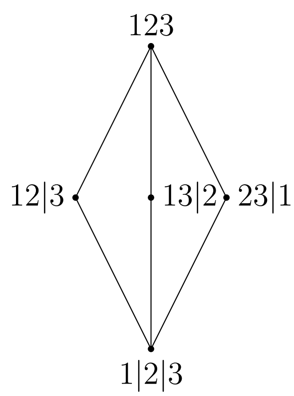
Let $a, b \in (P, \leq)$. If $a \leq b$ and $a \neq b$, we write $a < b$ (or $b > a$). For $a, b \in (P, \leq)$ with $a \leq b$, we set $[a, b] = \{c \in P\colon\ a \leq c \leq b\}$. The subset $[a, b]$ is called a *closed interval*. Also, $(a, b) = \{c \in P\colon a < c < b\}$ is called an *open interval*.

Definition 3.1.3. Let $(P, \leq)$ be a finite poset. Consider any two elements $a,\ b$ in P such that $a < b$. If there exists no other element c in P such that $a < c < b$, then we say that b *covers* a. In a finite poset $(P, \leq)$, the cover relation is given by

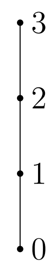
$$\mathrm{Cov}(P, \leq) = \big\{(a,b) \in P \times P \ : \ b \text{ covers } a\big\}.$$

It is clear that the cover relation determines the poset. Plotting the cover relation of a finite poset gives its *Hasse diagram* where each element of P is represented as a vertex and line segments are drawn from x to y if y covers x. In a Hasse diagram, usually larger elements are placed above the smaller elements.

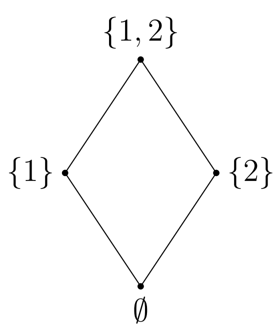
Example 3.1.4. Lets have a look at the Hasse diagrams of some of the familiar posets.



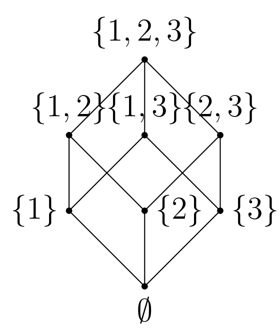
$\prod(3)$



$C(3)$



$B(2)$



$B(3)$

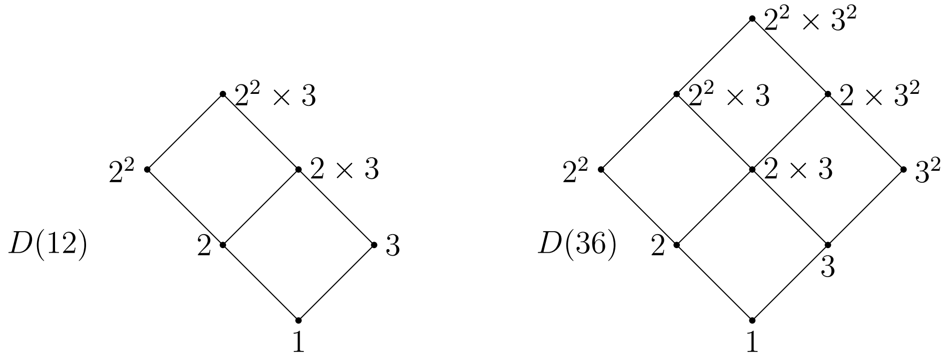


Figure 1

Definition 3.1.5. Let $(P, \leq)$ be a poset. Let C be a subset of P. If any two elements of C are comparable then C is called a *chain*. Equivalently, C is called a *totally ordered subset* of P. Let $\mathcal{T}$ be a subset of $(P, \leq)$. If no two distinct elements of $\mathcal{T}$ are comparable, then $\mathcal{T}$ is called an *antichain*.

If a chain C has n elements say $a_1, a_2, \ldots, a_n$, then these elements can be arranged in an increasing order; $a_1 < a_2 < \cdots < a_{n-1} < a_n$. In this case, chain C is of length $n - 1 = |C| - 1$. The length $\ell(P)$ of the poset P is

$$\ell(P) = \max\{\ell(C) \,:\, C \text{ is a chain of } P\}.$$

A chain C is said to be *saturated* if there is no element $c \in P - C$ satisfying $a < c < b$ for any $a, b \in C$ such that $C \cup \{c\}$ is a chain. Hence, the chain $a_1 < a_2 < \cdots < a_n$ is *saturated* if and only if a_i covers a_{i-1} for $i = 2, \ldots, n$. If a chain C in the poset $(P, \leq)$ is inclusion wise maximal, then it is said to be a *maximal* chain.

Definition 3.1.6. If every maximal chain of a poset $(P, \leq)$ has the same length n, then the poset $(P, \leq)$ is called a *graded poset* of rank n. For $n \geq 1$, $B(n)$, $D(n)$, $\prod(n)$ are a few examples of graded posets.

In case of a graded poset $(P, \leq)$, there exists a unique rank function $\rho\colon P \longrightarrow \{0, 1, 2 \ldots, n\}$ defined as follows; $\rho(x) = 0$ if $x \in P$ is the minimal element and for $x, y \in P$, $\rho(y) = \rho(x) + 1$ if y covers x. The value of $x \in P$ under the rank function is called the *rank* of x.

Definition 3.1.7. Let $(P, \leq)$ and $(Q, \leq')$ be two finite posets. Let f be a bijective map from $(P, \leq)$ to $(Q, \leq')$ such that f and f^{-1} are order preserving, that is $a \leq b$ in P if and only if $f(a) \leq' f(b)$ in Q. Then, f is called an *order isomorphism* and the poset P is said to be isomorphic to Q, written as $P \backsimeq Q$.

Definition 3.1.8. Let $(P, \leq)$ and $(Q, \leq')$ be any two posets. Then the *direct product* $P \times Q$ of P and Q is a poset on the cartesian product of P and Q induced by the partial ordering $(a, b) \check{\leq} (\tilde{a}, \tilde{b})$ if $a \leq \tilde{a}$ in P and $b \leq' \tilde{b}$ in Q.

Example 3.1.9. Let $(P, \leq)$ and $(Q, \leq')$ be two posets as given in the following Hasse diagram (Figure 2).

Figure 2

The direct product $P \times Q$ of P and Q is a poset, whose Hasse diagram (Figure 3) is given below.

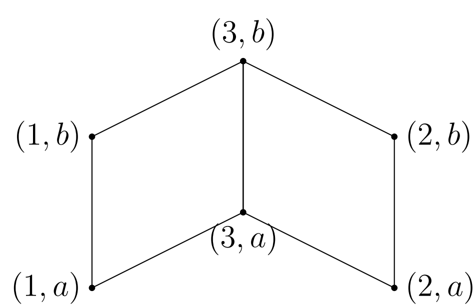


Figure 3

Example 3.1.10. The poset $B(n)$ is the direct product of n-copies of $C(1)$. We define a function $\phi\colon B(n) \longrightarrow \underbrace{C(1) \times \cdots \times C(1)}_{n-copies}$ as

$\phi(A) = \chi_A$, where χ_A is the characteristic function of A, $A \subseteq [n]$. Clearly, ϕ is a bijection. Let $A,\ B$ be two subsets in $B(n)$ such that $A \subseteq B$. Therefore, $\chi_A(i) = \chi_B(i)$ for all $i \in A$ and $\chi_A(i) = 0$, $\chi_B(i) = 1\ \forall\ i \in B - A$. Hence $\chi_A \leq \chi_B$ in $C(1) \times \cdots \times C(1)$. Thus, we see that ϕ is order preserving. Similarly, ϕ^{-1} is also order preserving. Hence, ϕ is an order isomorphism and we have $B(n) \backsimeq \underbrace{C(1) \times \cdots \times C(1)}_{n-copies}$.

Example 3.1.11. If $n = p_1^{e_1} p_2^{e_2} \dots p_t^{e_t};\ e_i \geq 1$ is the prime factorization of n, then the poset $D(n)$ can be expressed as the direct product $C(e_1) \times ... \times C(e_t)$. Let the function $\phi \colon D(n) \longrightarrow C(e_1) \times \cdots \times C(e_t)$ be defined as $\phi(d) = (\alpha_1, \alpha_2, \dots, \alpha_t)$, where $d \in D(n)$ is given by $d = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$. Clearly, ϕ is bijective. Let $d_1,\ d_2 \in D(n)$ with $d_1 \leq d_2$. If $d_1 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$ and $d_2 = p_1^{\beta_1} p_2^{\beta_2} \dots p_t^{\beta_t}$, then d_1 divides d_2 implies that $\alpha_i \leq \beta_i$ for $1 \leq i \leq t$. Thus we get that in the poset $C(e_1) \times \cdots \times C(e_t)$, $(\alpha_1, \alpha_2, \dots, \alpha_t) \leq (\beta_1, \beta_2, \dots, \beta_t)$. Hence ϕ is an order preserving map and so is ϕ^{-1}. Therefore, we write $D(n) \backsimeq C(e_1) \times \cdots \times C(e_t)$.

For a finite poset $(P,\ \leq)$, an element $\hat{0} \in P$ is called the *least element* of P if $\hat{0} \leq x\ \forall\ x \in P$. An element $\hat{1} \in P$ is called the *largest element* if $x \leq \hat{1}\ \forall\ x \in P$. We can obtain another poset $\widehat{P}$ from the given poset $(P,\ \leq)$ by associating $\hat{0}$ and $\hat{1}$ to P. If P is the poset given in Example 3.1.9, then the Hasse diagram of $\widehat{P}$ is given by

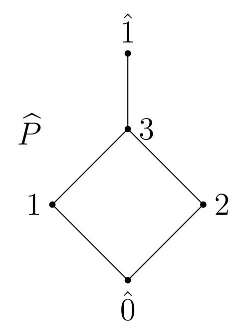


Figure 4

3.2 Incidence Algebra of a Finite Poset

Given a finite poset $(P, \leq)$, we consider a $\mathbb{C}$-algebra called the *incidence algebra*.

Definition 3.2.1. Let f be a $\mathbb{C}$-valued function defined on $P \times P$ such that $f(x, y) = 0$ whenever $x \nleq y$. Such a function $f\colon P \times P \longrightarrow \mathbb{C}$ is called an *incidence function*.

Let $I(P)$ be the set of all $\mathbb{C}$-valued incidence functions. Then $I(P)$ is a $\mathbb{C}$-vector space with respect to pointwise addition and pointwise scalar multiplication defined as

$$(f+g)(x,y) = f(x,y)+g(x,y) \text{ and } (\alpha f)(x,y) = \alpha f(x,y),\ \forall\, x, y \in P,$$

where $f,\ g \in I(P)$ and $\alpha \in \mathbb{C}$.

Definition 3.2.2. The set of all incidence functions, $I(P)$ forms a $\mathbb{C}$-algebra with a convolution product defined as

$$(f * g)(x,y) \ = \sum_{x \leq z \leq y} f(x,y)g(z,y), \qquad \forall\ f, g \in I(P).$$

This $\mathbb{C}$-algebra $I(P)$ is called the *incidence algebra* of P.

3.2.1 Properties of Incidence Algebra

The identity element of I(P) is the *Kronecker delta function* δ defined as

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

For $f \in I(P)$, we have $(f * \delta)(x, y) = \sum\limits_{x \leq z \leq y} f(x, y)\delta(z, y) = f(x, y)$. This implies that $f * \delta = f$. Similarly, $\delta * f = f$.

Proposition 3.2.3. *An incidence function $f \in I(P)$ is a unit in I(P) if and only if $f(x,x) \neq 0,\ \forall\ x \in P$.*

Proof. Let $f \in I(P)$ be a unit. Then there exists a function $g \in I(P)$ such that $f*g = g*f = \delta$. Thus, $(f*g)(x, x) = 1$ or $f(x, x)g(x, x) = 1$. Hence $f(x, x) \neq 0,\ \forall\ x \in P$. Conversely, assume that $f(x, x) \neq 0$, $\forall\ x \in P$. We shall define g recursively such that $f * g = \delta$. For any

$x \in P$, let $g(x,x) = \dfrac{1}{f(x,x)}$. If $x < y$ then $(f * g)(x,y) = 0$, implies that

$$f(x,x)g(x,y) + \sum_{x<z\leq y} f(x,z)g(z,y) = 0.$$

For $x < z \leq y$, we have $\ell\big([z,y]\big) < \ell\big([x,y]\big)$. Thus by induction, we may assume that $g(z,y)$ has already been defined. If y covers x, then

$$g(x,y) = -\ \frac{f(x,y)g(y,y)}{f(x,x)}.$$

Therefore, since $f(x,x) \neq 0$, we obtain

$$g(x,y) = -\ \frac{1}{f(x,x)} \sum_{x<z\leq y} f(x,z)g(z,y).$$

□

Proposition 3.2.4. *The incidence algebra $I(P)$ of a poset $(P,\ \leq)$ is commutative if and only if $(P,\ \leq)$ is an antichain.*

Proof. If $(P,\ \leq)$ is an antichain, then $f \in I(P)$ implies that $f(x,y) = 0$ if $x \neq y$. Now $(f * g)(x,y) = 0 = (g * f)(x,y)$ if $x \neq y$. For any $x \in P$,

$$(f * g)(x,x) = f(x,x)g(x,x) = g(x,x)f(x,x) = (g * f)(x,x).$$

Thus, $f * g = g * f$. Hence, $I(P)$ is commutative.
On the other hand, assume that $I(P)$ is commutative. Suppose the poset $(P,\ \leq)$ is not an antichain. Thus, P have distinct comparable elements. Therefore, we can choose a pair $(a,b) \in P \times P$ such that b covers a. Let

$$f(x,y) = \begin{cases} 1 & \text{if } (x,y) = (a,b), \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g(x,y) = \begin{cases} 1 & \text{if } x = y = b, \\ 0 & \text{otherwise.} \end{cases}$$

Now,

$$(f * g)(a,b) = f(a,a)g(a,b) + f(a,b)g(b,b) = 1,$$

and

$$(g * f)(a,b) = g(a,a)f(a,b) + g(a,b)f(b,b) = 0.$$

Hence $f * g \neq g * f$, a contradiction to $I(P)$ being commutative. □

Definition 3.2.5. Let $I(P)$ be the incidence algebra associated with a finite poset $(P, \leq)$. We define the *chain function*, $\eta \colon P \times P \longrightarrow \mathbb{C}$ as

$$\eta(x,y) = \begin{cases} 1 & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

We can see that $\eta^t(x,y) = \underbrace{(\eta * \cdots * \eta)}_{t-\text{times}}(x,y)$ gives the number of chains of length t from x to y in the poset $(P, \leq)$. Since $(P, \leq)$ is a finite poset, there exists a saturated chain with maximal length and therefore $\eta^m = 0$ for some m. This proves that the chain function $\eta \in I(P)$ is a nilpotent element. Thus we have $(\delta - \eta)^{-1} = \sum\limits_{t=0}^{\infty} \eta^t$.

3.2.2 Möbius Inversion Formula

For an incidence algebra $I(P)$ associated with a finite poset $(P, \leq)$ we define the *zeta function* $\zeta \colon P \times P \longrightarrow \mathbb{C}$ as

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{if } x \nleq y. \end{cases}$$

Clearly, $\zeta(x,x) \neq 0$ implies that it is invertible. The inverse of the zeta function ζ is called the *Möbius function* μ. That is, $\mu * \zeta = \zeta * \mu = \delta$. An recursive definition of the Möbius function is given as

$$\mu(x,x) = 1, \quad \forall\, x \in P,$$

and

$$\mu(x,y) = -\sum_{x \leq z < y} \mu(x,z).$$

Equivalently, we can define the Möbius function as $\mu(x,x) = 1$, $\forall\, x \in P$ and $\mu(x,y) = -\sum\limits_{x < z \leq y} \mu(z,y)$.

Lemma 3.2.6. *Suppose $(P, \leq_P)$ and $(Q, \leq_Q)$ are finite posets. Then for $x,\ x' \in P$ and $y,\ y' \in Q$ we have $\mu_{P\times Q}\big((x,y),(x',y')\big) = \mu_P(x,x')\mu_Q(y,y')$ where, $\mu_{P\times Q}$ is the Möbius function of the direct product $(P \times Q, \leq)$, μ_P and μ_Q are the Möbius functions of P and Q respectively.*

Proof. Let $x,\ x' \in P$ and $y,\ y' \in Q$. Since Möbius function is the inverse of zeta function, we have $\mu * \zeta = \delta$. Thus, the Möbius function of the direct product $P \times Q$ is given as

$$\sum_{(x,y)\leq(x',y')} \mu_{P\times Q}\Big((x,y),(x',y')\Big) = \begin{cases} 1 & \text{if } x = x' \text{ and } y = y' \\ 0 & \text{otherwise.} \end{cases} \qquad (1)$$

Now, $\sum\limits_{(x,y)\leq(x',y')} \mu_P(x,x')\mu_Q(y,y') = \sum\limits_{\substack{x\ \leq_P\ x' \\ y\ \leq_Q\ y'}} \mu_P(x,x')\mu_Q(y,y')$. Thus we get, $\sum\limits_{(x,y)\leq(x',y')} \mu_P(x,x')\mu_Q(y,y') = \Bigg(\sum\limits_{x\ \leq_P\ x'} \mu_P(x,x')\Bigg)\Bigg(\sum\limits_{y\ \leq\ y'} \mu_Q(y,y')\Bigg)$.
Hence,

$$\sum_{(x,y)\leq(x',y')} \mu_P(x,x')\mu_Q(y,y') = \begin{cases} 1 & \text{if } x = x' \text{ and } y = y' \\ 0 & \text{otherwise.} \end{cases} \qquad (2)$$

From (1) and (2), we get $\mu_{P\times Q} = (\mu_P)(\mu_Q)$. $\square$

Proposition 3.2.7. *Let $(P,\ \leq)$ be a finite poset and $f,\ g\colon\ P \longrightarrow \mathbb{C}$ be functions on P. Then, we have $g(x) = \sum\limits_{y\leq x} f(y),\ \forall\ x \in P$ if and only if $f(x) = \sum\limits_{y\leq x} g(y)\mu(y,x),\ \forall\ x \in P$. This is known as the Möbius inversion formula.*

Proof. The incidence algebra $I(P)$ of the finite poset $(P,\ \leq)$ acts on the vector space of $\mathbb{C}-$valued functions on P, $\mathbb{C}^P = \{f\colon P \longrightarrow \mathbb{C}\}$. For any $f \in \mathbb{C}^P,\ x \in P$ and $\theta \in I(P)$, we define $(f\theta)(x) = \sum\limits_{y\leq x} f(y)\theta(y,x)$. We observe that $(f\delta)(x) = \sum\limits_{y\leq x} f(y)\delta(y,x) = f(x)$ and for $\theta,\ \phi \in I(P)$, we need to prove that $\big((f\theta)\phi\big)(x) = f(\theta * \phi)$. From the definition, we can write

$$\big((f\theta)\phi\big)(x) = \sum_{y\leq x}(f\theta)(y)\phi(y,x) = \sum_{y\leq x}\left(\sum_{z\leq y} f(z)\theta(z,y)\right)\phi(y,x).$$

Hence, we get

$$\big((f\theta)\phi\big)(x) = \sum_{z\le y\le x} f(z)\theta(z,y)\phi(y,x) = \sum_{z\le x} f(z) \sum_{z\le y\le x} \theta(z,y)\phi(y,x).$$

By the definition of convolution product, we know that $\big((f\theta)\phi\big)(x) = \sum_{z\le x} f(z)(\theta * \phi)(z,x)$. Hence, $\big((f\theta)\phi\big)(x) = \big(f(\theta * \phi)\big)(x)$. This illustrates that $(f,\ \theta) \longmapsto f\theta$ is an action of $I(P)$ on $\mathbb{C}^P$. Now, $g(x) = \sum_{y\le x} f(y),\ \forall\ x \in P$ is equivalent to $g = f\zeta$. Also, $f(x) = \sum_{y\le x} g(y)\mu(y,x),\ \forall\ x \in P$ is equivalent to $f = g\mu$. Therefore, the Möbius inversion formula can be stated as $g = f\zeta \iff f = g\mu$, which follows at once from the action of $I(P)$ on $\mathbb{C}^P$. $\square$

Remark. On computing the Mӧbius inversion formula of the poset $B(n)$ ordered by inclusion, we can observe that the inclusion-exclusion principle is a special case of the Mӧbius inversion formula.

3.3 Order Complex of a Poset

To every finite poset one can associate a simplicial complex, called the *order complex* of the poset.

Definition 3.3.1. Let $(P,\ \le)$ be a finite poset. A simplicial complex $\mathbf{\Delta}(P)$ associated with P such that the i-dimensional face of $\mathbf{\Delta}(P)$ is a chain of length i in P is called an *order complex*.

Since subsets of a chain is also a chain, therefore $\mathbf{\Delta}(P)$ is indeed a simplicial complex. Also elements of P corresponds to vertices of $\mathbf{\Delta}(P)$.

Example 3.3.2. Let us look at some posets and order complexes associated with them (Figure 5).

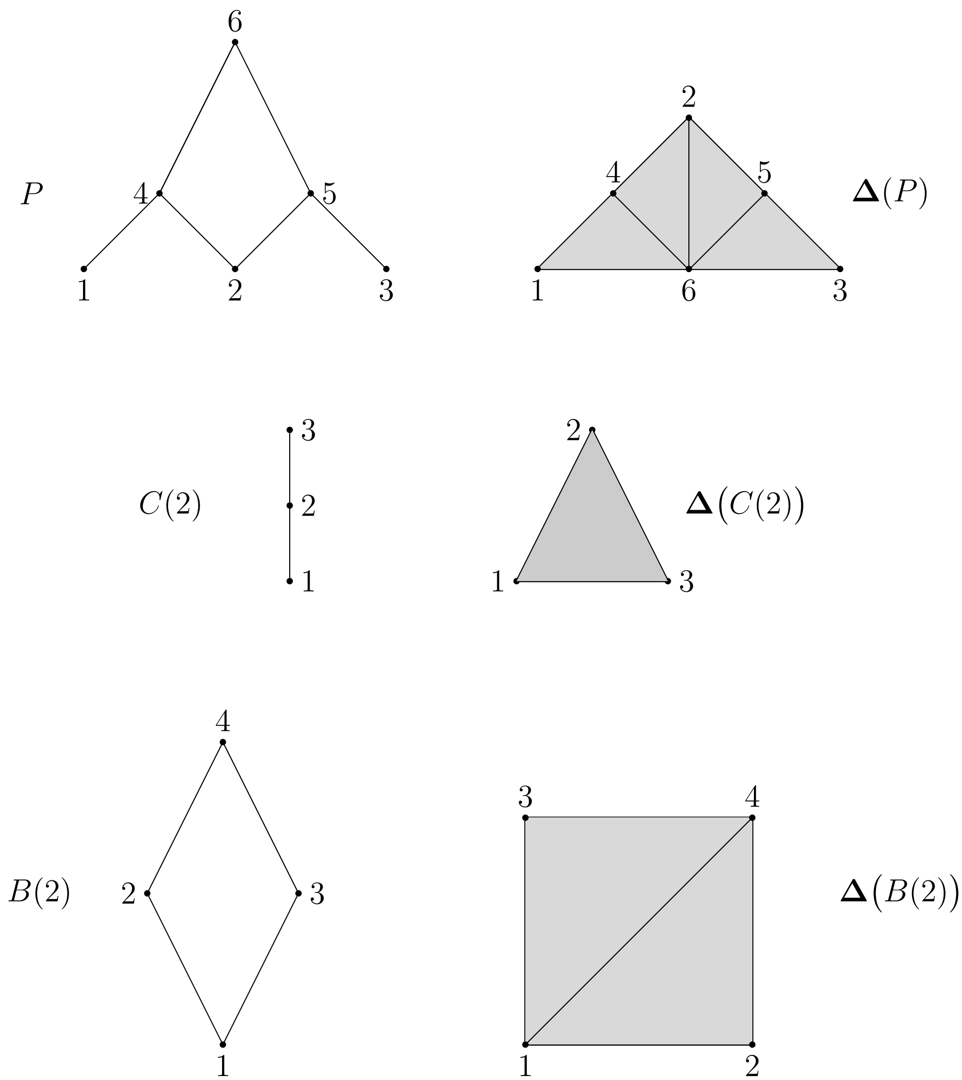


Figure 5

In the order complex $\mathbf{\Delta}(P)$ of a finite poset $(P, \leq)$, number of faces with dimension i is equal to the number of i-chains (chains of length i) in P.

Proposition 3.3.3 (Hall's Theorem)**.** *Let $(P, \leq)$ be a finite poset and $\widehat{P}$ be the poset obtained by adjoining $\hat{0}$ and $\hat{1}$ to P. Then,*

$$\mu_{\widehat{P}}(\hat{0},\hat{1}) = \sum_{i=0}^{|P|}(-1)^i C_i,$$

where C_i is the number of i-chains from $\hat{0}$ to $\hat{1}$ in P.

Proof. Möbius function is the inverse of zeta function and hence we have $\mu_{\widehat{P}}(\hat{0},\hat{1}) = \zeta_{\widehat{P}}^{-1}(\hat{0},\hat{1})$. We can write ζ as $\delta + (\zeta_{\widehat{P}} - \delta)$ and by using the fact that $(\zeta_{\widehat{P}} - \delta) = \eta_{\widehat{P}}$, we get $\mu_{\widehat{P}}(\hat{0},\hat{1}) = (\delta + \eta_{\widehat{P}})^{-1}(\hat{0},\hat{1})$. We know that $\eta_{\widehat{P}}$ is nilpotent and $\eta^i_{\widehat{P}}(\hat{0},\hat{1})$ gives the number of i-chains from $\hat{0}$ to $\hat{1}$, then we get

$$\mu_{\widehat{P}}(\hat{0},\hat{1}) = \sum_{i=0}^{\infty}(-1)^i \eta^i_{\widehat{P}}(\hat{0},\hat{1}) = \sum_{i=0}^{\infty}(-1)^i C_i.$$

□

Since C_i is the number of i-chains from $\hat{0}$ to $\hat{1}$ in the poset $\widehat{P}$, we have $C_0 = 0,\ C_1 = 1,\ C_2 = |P| = f_0(\Delta(P))$ and so on. In general,

$$C_{i+2} = f_i\big(\Delta(P)\big).$$

Thus, $\mu_{\widehat{P}}(\hat{0},\hat{1}) = -1 + f_0(\Delta(P)) - f_1(\Delta(P)) + \cdots + (-1)^{|P|-2} f_{|P|-2}(\Delta(P))$. By comparing this expression with that of reduced Euler characteristic of $\mathbf{\Delta}(P)$, we get

$$\mu_{\widehat{P}}(\hat{0},\hat{1}) = \widetilde{\chi}\big(\Delta(P)\big).$$

Definition 3.3.4. For a non-empty simplicial complex K with $\emptyset \in K$, we assign a poset $\mathcal{F}(K)$ of all faces of K ordered by inclusion. This poset $\mathcal{F}(K)$ is called the *face poset* of the simplicial complex K.

Example 3.3.5. Let $(P, \leq)$ be a poset whose Hasse diagram and order complex $\mathbf{\Delta}(P)$ are given as Figure 6.

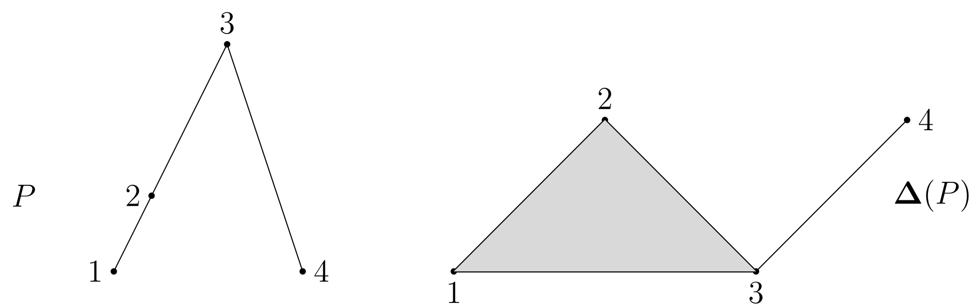


Figure 6

The face poset of the order complex $\mathbf{\Delta}(P)$ in Figure 6 is shown in the Hasse diagram (Figure 7).

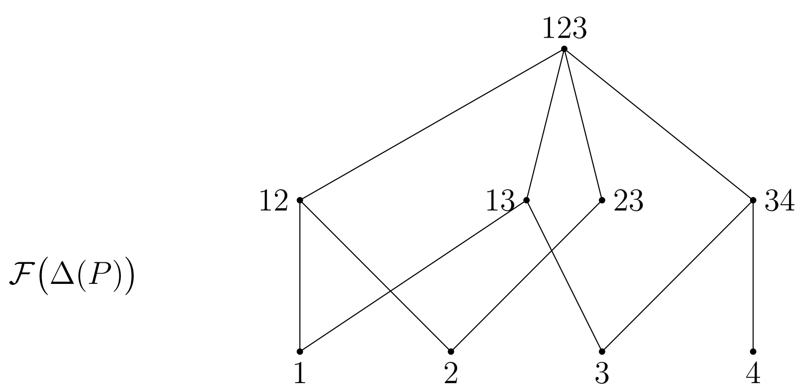


Figure 7

Let K be a simplicial complex with $\mathbf{f}$-vector $(f_0, f_1, \ldots, f_{d-1})$. The Mӧbius function μ of the poset obtained by attaching $\hat{1}$ to the face poset of K is given as

$$\mu(x,y) = \begin{cases} (-1)^{\ell(x,y)} & \text{if } x \leq y, \\ 0 & \text{otherwise,} \end{cases}$$

where $\ell(x,y)$ is the length of $[x,y]$.

Remark. *The order complex* $\mathbf{\Delta}\Big(\mathcal{F}\big(\mathbf{\Delta}(P)\big)\Big)$ *of the face poset* $\mathcal{F}\big(\mathbf{\Delta}(P)\big)$ *of* $\mathbf{\Delta}(P)$*, gives the first Barycentric subdivision of the simplicial complex* $\mathbf{\Delta}(P)$ *associated with a finite poset* P*.*

For example, consider the poset P given in Example 3.3.5. The order complex of the poset $\mathcal{F}\big(\mathbf{\Delta}(P)\big)$ given in Figure 7 is represented by the diagram (Figure 8).

$\mathbf{\Delta}\Big(\mathcal{F}\big(\mathbf{\Delta}(P)\big)\Big)$

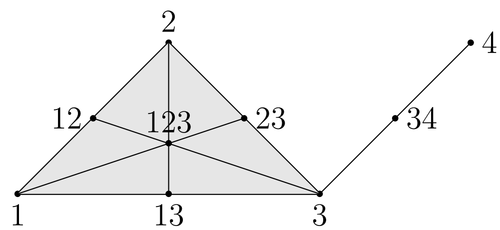


Figure 8

Clearly, Figure 8 represents the first Barycentric subdivision of the order complex $\mathbf{\Delta}(P)$ shown in Figure 6.

Chapter 4

Shellable Nonpure Complexes and Posets

This chapter draws ideas from one of the works of Björner and Wachs [2] in which they generalizes the concept of shellability of a simplicial complex by dropping the notion of purity. We define doubly indexed **f**-vectors and **h**-vectors for the purpose of nonpure simplicial complexes. Topological properties of shellable simplicial complexes indicate that the shellable simplicial complexes have the homotopy type of wedge of spheres of suitable dimensions. We also discuss the theory of lexicographic shellability by extending the technique from pure to nonpure posets.

4.1 Shellable Simplicial Complex

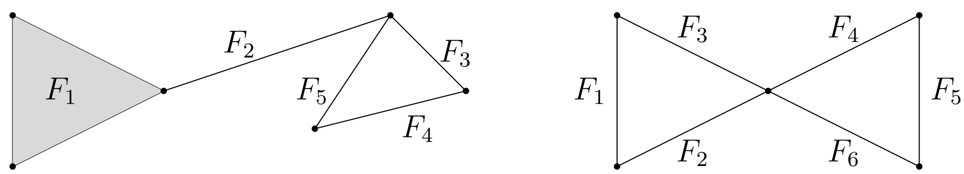
In this section, we familiarize the notion of shellability of a simplicial complex in a general perspective irrespective of whether the simplicial complex is pure or not.

Definition 4.1.1. Let A and B be two sets such that $A \subseteq B$. Then the interval $[A, B] = \{C \colon A \subseteq C \subseteq B\}$ is called a *Boolean interval.*

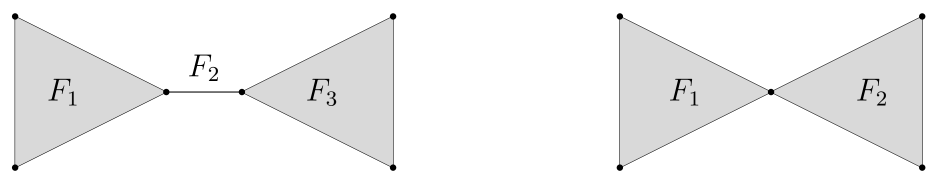
The Boolean interval, $[\emptyset, A] = \bar{A}$ is a simplicial complex on the vertex set A. In fact, $\bar{A} = \mathcal{P}(A)$ and is called a *simplex.*

Definition 4.1.2. A simplicial complex Δ defined on some (finite) vertex set V is said to be *shellable* if its facets can be arranged in a linear order $F_1, F_2, \ldots, F_t$ such that the subcomplex $\left(\bigcup_{i=1}^{k-1} \overline{F}_i\right) \bigcap \overline{F}_k$ is pure with dimension equal to $(\dim F_k - 1)$ for $2 \leq k \leq t$. A choice of linear ordering on facets of a shellable simplicial complex is called a *shelling*.

Some examples of shellable and nonshellable simplicial complexes are given below in Figure 1.



Shellable



Non shellable

Figure 1

Lemma 4.1.3. *Let Δ be a shellable simplicial complex. If $F_1, F_2, \ldots, F_t$ is a shelling of Δ, then the dimension of Δ is same as that of the first facet F_1 in the shelling.*

Proof. Let us assume that $\dim F_i < \dim F_k$, for all $1 \leq i < k$, where $k \leq t$. Since $F_i \nsubseteq F_k$ for $1 \leq i < k$, there exists at least one element in F_i that is not in F_k. Hence we have, $\dim(F_i \cap F_k) \leq \dim F_i - 1$. By our assumption, we conclude that $\dim(F_i \cap F_k) \leq \dim F_k - 2$. Therefore we get that, $\dim\left(\left(\bigcup_{i=1}^{k-1} \overline{F}_i\right) \bigcap \overline{F}_k\right) \leq \dim F_k - 2$. This contradicts the fact that Δ is shellable. $\square$

Shellability of a simplicial complex can also be verified using the following criteria.

Lemma 4.1.4. *For a simplicial complex Δ, the ordering $F_1, F_2, \ldots, F_t$ of facets is a shelling if and only if for all i and k such that $1 \leq i < k \leq t$ there exists a j with $1 \leq j < k$ and an element $x \in F_k - F_i$ such that $F_i \cap F_k \subseteq F_j \cap F_k = F_k - \{x\}$.*

Proof. Let $F_1, F_2, \ldots, F_t$ be a shelling of the simplicial complex Δ. By definition of shellability, $\widetilde{\Delta}_k = \left(\bigcup_{i=1}^{k-1} \overline{F}_i\right) \bigcap \overline{F}_k$ is a pure simplicial complex of dimension equal to $\dim F_k - 1$ for $2 \leq k \leq t$. Thus, for $1 \leq i < k \leq t$, we can see that $F_i \cap F_k$ is a face of $\widetilde{\Delta}_k$ and hence $F_i \cap F_k$ is contained in a maximal face, say $F_j \cap F_k$ of $\widetilde{\Delta}_k$. As $\widetilde{\Delta}_k$ is pure and is of dimension $\dim F_k - 1$, $F_j \cap F_k = F_k - \{x\}$ for some $x \in F_k - F_i$. This implies that there exists an element $x \in F_k$ such that $F_i \cap F_k \subseteq F_j \cap F_k = F_k - \{x\}$.
Now, in order to prove the converse let $F_1, F_2, \ldots, F_t$ be an ordering of facets of Δ such that $F_i \cap F_k \subseteq F_j \cap F_k = F_k - \{x\}$, for all $1 \leq i < k \leq t$ with $1 \leq j < k$. This implies that every face $F_i \cap F_k$ of $\widetilde{\Delta}_k = \left(\bigcup_{i=1}^{k-1} \overline{F}_i\right) \bigcap \overline{F}_k$ is contained in a maximal face of $\widetilde{\Delta}_k$ of the form $F_k - \{x\}$ for some $x \in F_k - F_i$. Therefore $\widetilde{\Delta}_k = \bigcup_{i=1}^{k-1} \overline{F}_i \cap \overline{F}_k$ is a pure simplicial complex with dimension $\dim F_k - 1$. Hence, the ordering $F_1, F_2, \ldots, F_t$ of facets of Δ is a shelling. $\square$

The concept of shellability can also be reformulated in terms of partitioning. For this, we need the following results.

Lemma 4.1.5. *Consider a simplicial complex Δ. Let F be a facet of Δ and $R \subseteq F$. Let Δ' be the subcomplex generated by facets of Δ other than F. Then the following conditions are equivalent.*

1. *The set $\overline{F} - \Delta'$ is equal to the Boolean interval $[R, F]$,*
2. *$\overline{F} \cap \Delta' = \bigcup_{x \in R} \overline{F - \{x\}}$.*

Proof. 1. $\implies$ 2. Given, $\overline{F} - \Delta' = [R, F]$. Thus, for some G in $\overline{F}$, G does not belong to Δ' if and only if $R \subseteq G$. By taking the contrapositive argument, we have that for $G \in \overline{F}$ and $G \in \Delta'$ if and only if

$R \nsubseteq G$. This implies that there exists $x \in R$ such that $x \notin G$. Thus we can conclude that $G \in \overline{F} \cap \Delta'$ if and only if $G \subseteq \overline{F - \{x\}}$, for some $x \in R$. Therefore, we have $\overline{F} \cap \Delta' = \bigcup_{x \in R} \overline{F - \{x\}}$.

2. $\implies$ *1.* Since we have $\overline{F} \cap \Delta' = \bigcup_{x \in R} \overline{F - \{x\}}$, the set $\overline{F} - \Delta'$ contains all the subsets of F that contains R. Clearly, $\overline{F} - \Delta' = [R, F]$. □

Definition 4.1.6. Let Δ be a shellable simplicial complex and let $F_1, F_2, \ldots, F_t$ be a shelling of Δ. For facet F_k in the shelling, we define a set $\mathcal{R}(F_k)$ as

$$\mathcal{R}(F_k) = \{x \in F_k \colon F_k - \{x\} \in \Delta_{k-1}\},$$

where $\Delta_{k-1} = \bigcup_{i=1}^{k-1} \overline{F}_i$. The set $\mathcal{R}(F_k)$ is called the *restriction* of the facet F_k.

Clearly, $\overline{F}_k \cap \Delta_{k-1} = \bigcup_{x \in \mathcal{R}(F_k)} \overline{F_k - \{x\}}$. Therefore by Lemma 4.1.5, we have $\overline{F}_k - \Delta_{k-1} = [\mathcal{R}(F_k), F_k]$. This indicates that $\mathcal{R}(F_k)$ is the unique minimum new face introduced in the k-th shelling step. In fact, we can say that a Boolean interval $[\mathcal{R}(F_k), F_k]$ has been added up each time a new facet F_k is added to the subcomplex Δ_{k-1}. This shows that, the simplicial complex Δ can be build inductively as a disjoint union of Boolean interval

$$\Delta = \coprod_{i=1}^{t} [\mathcal{R}(F_i), F_i].$$

This decomposition of the simplicial complex can be used to characterize shellability in a different perspective.

Proposition 4.1.7. *Let Δ be a simplicial complex and $F_1, F_2, \ldots, F_t$ be an ordering of facets of Δ. Consider a map $\mathcal{R} \colon \{F_1, F_2, \ldots, F_t\} \longrightarrow \Delta$. Then, the following conditions are equivalent.*

1. *The ordering $F_1, F_2, \ldots, F_t$ is a shelling of Δ and the map $\mathcal{R}$ is its restriction map.*
2. *The simplicial complex Δ can be decomposed as $\Delta = \coprod_{i=1}^{t} [\mathcal{R}(F_i), F_i]$, and $\mathcal{R}(F_i) \subseteq F_j$ implies $i \leq j$ for all i, j.*

Proof. *1.* $\implies$ *2.* The decomposition of Δ into disjoint union of Boolean intervals has been already shown. Since $\mathcal{R}(F_i)$ is the unique minimum new face introduced in the i-th shelling step, $\mathcal{R}(F_i) \subseteq F_j$ implies $i \le j$ for all i, j.

2. $\implies$ *1.* Since $\Delta_{k-1} = \coprod_{i=1}^{k-1} [\mathcal{R}(F_i), F_i], \Delta_k = \coprod_{i=1}^{k} [\mathcal{R}(F_i), F_i]$ and $\mathcal{R}(F_k) \nsubseteq F_i$ for $1 \le i \le k-1$, we see that $\overline{F}_k - \Delta_{k-1} = [\mathcal{R}(F_k), F_k]$. Thus by Lemma 4.1.5, $\Delta_{k-1} \cap \overline{F}_k = \bigcup_{x \in \mathcal{R}(F_k)} \overline{F_k - \{x\}}$. Hence the subcomplex $\Delta_{k-1} \cap \overline{F}_k$ is pure with dimension equal to $\dim F_k - 1$. Thus by definition, Δ is shellable and $\mathcal{R}$ is its restriction map. $\square$

For any shellable simplicial complex there is a shelling in which the facets are arranged in the order of decreasing dimension.

Proposition 4.1.8 (First rearrangement lemma)**.** *Let Δ be a shellable simplicial complex of dimension $d-1$ with $F_1, F_2, \ldots, F_t$ as a shelling of Δ and $\mathcal{R}$ be its restriction map. Let $F_{i_1}, F_{i_2}, \ldots, F_{i_t}$ be a rearrangement constructed by placing first all the facets of dimension $d-1$ in the induced order and then followed by lesser dimensional facets in order of decreasing dimension. This rearrangement is also a shelling of Δ and has the same restriction map $\mathcal{R}$ as the initial shelling.*

Proof. To prove that $F_{i_1}, F_{i_2}, \ldots, F_{i_t}$ is a shelling of Δ with same restriction map $\mathcal{R}$, it is enough to show that

$$\mathcal{R}(F_{i_j}) \subseteq F_{i_k} \implies j \le k.$$

Suppose to the contrary, there exists indices a and b such that $a < b$ with $|F_a| < |F_b|$ and $\mathcal{R}(F_a) \subseteq F_b$. Let b be the minimal such index. It is clear that $\mathcal{R}(F_a) \subsetneq F_b$, since $F_b \nsubseteq F_a$. Thus, $\mathcal{R}(F_a) \subseteq A \subseteq F_b$ for some set $A = F_b - \{x\}$. Since $F_{i_1}, F_{i_2}, \ldots, F_{i_t}$ is just a rearrangement of the shelling $F_1, F_2, \ldots, F_t$, we can say that $A \subseteq F_c$ for some $c < b$. Hence, $\mathcal{R}(F_a) \subseteq F_c \implies a < c$. Since the facets are rearranged according to the order of decreasing dimension, for $c < b$ we get $|F_c| \ge |F_b|$. Therefore,

$$a < c, \ |F_a| < |F_c| \ \text{ and } \mathcal{R}(F_a) \subseteq F_c.$$

But $c < b$, contradicts the minimality of b. $\square$

Proposition 4.1.9 (Second rearrangement lemma). *Let $F_1, F_2, \ldots, F_t$ be a shelling of Δ and $\mathcal{R}$ be its restriction map. Let $F_{i_1}, F_{i_2}, \ldots, F_{i_t}$ be a rearrangement of the given shelling such that the facets F with $\mathcal{R}(F) \neq F$ is placed first in the induced order and then all the other facets are considered in an arbitrary order. Then this rearrangement is also a shelling with restriction map $\mathcal{R}$.*

Proof. For the rearranged ordering $F_{i_1}, F_{i_2}, \ldots, F_{i_t}$, let $\mathcal{R}(F_{i_j}) \neq F_{i_j}$ for all $1 \leq j \leq a$ and $\mathcal{R}(F_{i_j}) = F_{i_j}$ if $j > a$. Suppose $\mathcal{R}(F_{i_j}) \subseteq F_{i_k}$. If $j > a$, then $\mathcal{R}(F_{i_j}) = F_{i_k}$ and this implies $j = k$. Suppose $j \leq a$. If $a < k$, then $j < k$. So suppose that $k \leq a$ also. Then $\mathcal{R}(F_{i_j}) \neq F_{i_j}$ and $\mathcal{R}(F_{i_k}) \neq F_{i_k}$. Now, $\mathcal{R}(F_{i_j}) \subseteq F_{i_k}$ implies that $i_j \leq i_k$. Hence $j \leq k$. Thus by Proposition 4.1.7, $F_{i_1}, F_{i_2}, \ldots, F_{i_t}$ is a shelling of Δ. $\square$

Let Δ be a simplicial complex and consider $0 \leq r \leq s \leq \dim \Delta$. If $r \leq \dim F \leq s$ for all the facets F of Δ, then Δ is said to be (r, s)*-pure*. Let us define

$$\Delta^{(r,s)} = \{A \in \Delta \colon\ \dim A \leq s,\ A \subseteq F \text{ for some facet } F \text{ with } \dim F \geq r\}.$$

Dimension of all the facets in $\Delta^{(r,s)}$ ranges from r to s. Hence, $\Delta^{(r,s)}$ is (r, s)-pure. The *s-skeleton* $\Delta^{(0,s)}$ of Δ is the set of all facets F of Δ with $\dim F \leq s$ and $\Delta^{(s,s)}$ is the subcomplex generated by all s-faces of Δ.

Theorem 4.1.10. *If a simplicial complex Δ is shellable, then $\Delta^{(r,s)}$ is also shellable, where $r \leq s$.*

Proof. Given a shellable simplicial complex, we can always have a shelling in which the facets appear in order of decreasing dimension. The facets with dimension less than r that appear towards the end of the shelling can be removed in order to get a shelling of $\Delta^{(r,d-1)}$, where $\dim \Delta = d-1$. Suppose $r < d-1$. We only have to show that $\Delta^{(r,d-2)}$ is shellable. Let

$$\Delta^{(r,d-1)} = \coprod_{i=1}^{t} [R_i, F_i]. \tag{4.1}$$

Let first k facets of the shelling be of dimension $d-1$. For $1 \leq r \leq k$, consider the elements of $F_i - R_i$ to be ordered as $x_1, x_2, \ldots, x_{g_i}$. Let $R_{i_j} = R_i \cap \{x_1, x_2, \ldots, x_{j-1}\}$ and $F_{i_j} = F_i \cap \{x_j\}$, for $1 \leq j \leq g_i$. Then, we can write $[R_i, F_i] = \{F_i\} \coprod_{j=1}^{g_i} [R_{i_j}, F_{i_j}]$. Thus, the simplicial complex $\Delta^{(r,d-2)}$ can be partitioned as

$$\Delta^{(r,d-2)} = \left(\coprod_{i=1}^{k} \coprod_{j=1}^{g_i} [R_{i_j}, F_{i_j}] \right) \coprod \left(\coprod_{i=k+1}^{t} [R_i, F_i] \right). \tag{4.2}$$

Let the facets F_{i_j} and F_i be ordered in the lexicographic order. In (4.1), $R_i \subseteq F_j \implies i \leq j$ and we can deduce the same for (4.2). This induces a shelling for $\Delta^{(r,d-2)}$. □

4.2 Enumeration of Faces of Non-pure Complexes

This section is about the face numbers and the **h**-triangle of non-pure simplicial complexes. A doubly indexed **f**-numbers $f_{i,j}$ and **h**-numbers $h_{i,j}$ are defined that refine the usual **f**-vectors and **h**-vectors for the non-pure case.

Let Δ be a $(d-1)$-dimensional simplicial complex. Let f_i be the number of i-dimensional faces of Δ. The numbers f_i are called the *face numbers* of Δ with $f_{-1} = 1$. The d-tuple, $\mathbf{f}(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ is called the **f**-*vector* of Δ. The $(d+1)$-tuple, $\mathbf{h}(\Delta) = (h_0, h_1, \ldots, h_d)$ is called the **h**-*vector* of Δ and is defined in general by

$$H(y) = F(y-1),$$

where $H(y) = \sum_{i=0}^{d} h_i y^{d-i}$ and $F(y) = \sum_{i=0}^{d} f_{i-1} y^{d-i}$ are polynomials. We have

$$F(y-1) = \sum_{i=0}^{d} f_{i-1}(y-1)^{d-i} = \sum_{i=0}^{d} f_{i-1} \sum_{j=i}^{d} (-1)^k \binom{d-i}{k} y^{d-i-k}.$$

By replacing $i+k$ with j and interchanging the summations we get

$$F(y-1) = \sum_{j=0}^{d} y^{d-j} \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1}.$$

We recall that **f**-vector and **h**-vector of Δ are related by the expression, $h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1}$. Therefore, we get

$$F(y-1) = \sum_{j=0}^{d} h_j y^{d-j} = H(y).$$

Hence, the functional relation $H(y) = F(y-1)$ can be used to define the **h**-vector of Δ.

Definition 4.2.1. Let Δ be a simplicial complex. For any face $A \in \Delta$, cardinality $|A|$ of A is called the *size* of A and $\delta(A)$ given by

$$\delta(A) = \max\big\{|F| \colon A \subseteq F \in \Delta\big\}$$

is called the *degree* of the face A.

Remark. In general, for any face $A \in \Delta$, $|A| \le \delta(A)$. For any facet $F \in \Delta$, degree of F is same as its size.

Definition 4.2.2. Let Δ be a $(d-1)$-dimensional simplicial complex. The doubly indexed **f**-*numbers* $f_{i,j}$ of Δ are defined as the number of faces of degree i and size j. The **h**-*numbers* $h_{i,j}$ are defined as

$$h_{i,j} = \sum_{k=0}^{j} (-1)^{j-k} \binom{i-k}{j-k} f_{i,k}.$$

The triangular integer arrays $\mathbf{f} = (f_{i,j})_{0 \le j \le i \le d}$ and $\mathbf{h} = (h_{i,j})_{0 \le j \le i \le d}$ given as

$$\begin{array}{llll} f_{0,0} & & & \\ f_{1,0} & f_{1,1} & & \\ f_{2,0} & f_{2,1} & f_{2,2} & \\ \vdots & & \ddots & \\ f_{d,0} & f_{d,1} & \cdots & f_{d,d} \end{array} \qquad \text{and} \qquad \begin{array}{llll} h_{0,0} & & & \\ h_{1,0} & h_{1,1} & & \\ h_{2,0} & h_{2,1} & h_{2,2} & \\ \vdots & & \ddots & \\ h_{d,0} & h_{d,1} & \cdots & h_{d,d} \end{array}$$

are called the **f**-*triangle* and **h**-*triangle* of Δ, respectively.

Clearly, the number $f_{i,j}$ gives the number of faces in $\Delta^{(i-1,i-1)} - \Delta^{(i,i)}$ with size j. The number of facets of Δ with different sizes can be obtained from the diagonal entries of the **f**-triangle. Moreover, the usual face numbers can be calculated from summing the column. That is, we have

$$f_{j-1} = \sum_{i \geq j} f_{i,j}.$$

For a pure simplicial complex Δ with dimension $d-1$, degree of all its faces is equal to d and therefore, $f_{i,j} = 0$, for all $0 \leq j \leq i \leq d-1$. Hence, only the last row of the **f**-triangle and **h**-triangle of Δ has non-zero entries for the case of pure simplicial complex. On re-indexing these entries, we can get the usual **f**-vector and the **h**-vector of Δ.

Proposition 4.2.3. *Consider a $(d-1)$-dimensional simplicial complex Δ. Let the two-variable polynomials $F(x,y)$ and $H(x,y)$ be defined as*

$$F(x,y) = \sum_{0 \leq j \leq i} f_{i,j} x^i y^{i-j} \quad \text{and} \quad H(x,y) = \sum_{0 \leq j \leq i} h_{i,j} x^i y^{i-j}.$$

1. *The doubly indexed **h**-numbers can be expressed in terms of these polynomials as $H(x,y) = F(x, y-1)$.*

2. *The fact that the column sums of **f**-triangle give the usual face numbers can be expressed as $F(y) = y^d F\left(\frac{1}{y}, y\right)$.*

Proof. 1. By definition, $F(x, y-1) = \sum\limits_{0 \leq j \leq i} f_{i,j} x^i (y-1)^{i-j}$. Using the binomial expansion of $(y-1)^{i-j}$, we get

$$F(x, y-1) = \sum_{j=0}^{i} f_{i,j} x^i \sum_{k=0}^{i-j} (-1)^k \binom{i-j}{k} y^{i-j-k}.$$

Let us put $i + j = m$, and we get

$$F(x, y-1) = \sum_{j=0}^{i} f_{i,j} x^i \sum_{m=j}^{i} (-1)^{(m-j)} \binom{i-j}{m-j} y^{i-m}.$$

On interchanging the summations, we get

$$F(x, y-1) = \sum_{m=0}^{i} x^i y^{i-m} \sum_{j=0}^{m} (-1)^{(m-j)} \binom{i-j}{m-j} f_{i,j}.$$

By definition, $h_{i,j} = \sum_{k=0}^{j} (-1)^{j-k} \binom{i-k}{j-k} f_{i,k}$. Hence the given equation can be written as

$$F(x, y-1) = \sum_{0 \le m \le i} h_{i,m} x^i y^{i-m} = H(x, y).$$

2. Using the definition of the two-variable polynomial $F(x, y)$, we can write

$$y^d F\left(\frac{1}{y}, y\right) = y^d \sum_{j=0}^{i} f_{i,j} \left(\frac{1}{y}\right)^i y^{i-j} = \sum_{0 \le j \le i} f_{i,j} y^{d-j}.$$

We can split the summation $\sum_{0 \le j \le i} f_{i,j} y^{d-j}$ as $\sum_{j=0}^{d} \left(\sum_{j \le i} f_{i,j} \right) y^{d-j}$. Since we have $f_{j-1} = \sum_{j=0}^{i} f_{i,j}$, we get

$$y^d F\left(\frac{1}{y}, y\right) = \sum_{j=0}^{d} f_{j-1} y^{d-j} = F(y).$$

□

The usual **h**-vector of a simplicial complex can be expressed in terms of the **h**-triangle using generating functions.

Proposition 4.2.4. *The usual* **h**-*vector of a simplicial complex can be expressed in terms of the* **h**-*triangle as* $H(y) = (y-1)^d H\left(\frac{1}{y}, y\right)$, *which is equivalent to*

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \sum_{s=j}^{d-i+j} h_{s,j} \binom{d-s}{i-j}, \quad i = 0, 1, \ldots, d.$$

Proof. Using the definition of **h**-vector $H(y) = F(y-1)$ and applying Proposition 4.2.3. to the definition, we get

$$H(y) = F(y-1) = (y-1)^d F\left(\frac{1}{y-1}, y-1\right) = (y-1)^d H\left(\frac{1}{y-1}, y\right).$$

Hence, we get $H(y) = (y-1)^d H\left(\frac{1}{y-1}, y\right)$. Using the definition of the polynomials $H(y)$ and $H\left(\frac{1}{y-1}, y\right)$, we get

$$\begin{aligned}\sum_{i=0}^{d} h_i y^{d-i} &= \sum_{0 \le j \le i \le d} h_{i,j}(y-1)^{d-i} y^{i-j} \\ &= \sum_{j=0}^{d} \sum_{i=j}^{d} h_{i,j} \sum_{k=0}^{d-i} \binom{d-i}{k} (-1)^{d-i-k} y^{k+i-j}.\end{aligned}$$

Let us set $k+i-j=t$, and we get

$$\begin{aligned}\sum_{i=0}^{d} h_i y^{d-i} &= \sum_{j=0}^{d} \sum_{i=j}^{d} h_{i,j} \sum_{t=i-j}^{d-i} \binom{d-i}{t-i+j} (-1)^{d-j-t} y^{t} \\ &= \sum_{t=0}^{d} \sum_{j=0}^{d-t} \sum_{i=j}^{j+t} h_{i,j} \binom{d-i}{t-i+j} (-1)^{d-j-t} y^{t}.\end{aligned}$$

Now, let us substitute $t = d-s$ in the above equation to get

$$\sum_{i=0}^{d} h_i y^{d-i} = \sum_{s=0}^{d} \left(\sum_{j=0}^{s} (-1)^{s-j} \sum_{i=j}^{d-s+j} h_{i,j} \binom{d-i}{s-j} \right) y^{d-s}.$$

On interchanging the summations, we get

$$\sum_{i=0}^{d} h_i y^{d-i} = \sum_{i=0}^{d} \left(\sum_{j=0}^{i} (-1)^{i-j} \sum_{s=j}^{d-i+j} h_{s,j} \binom{d-s}{i-j} \right) y^{d-i}.$$

On comparing the coefficients of y^{d-i}, we reach at the conclusion that

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \sum_{s=j}^{d-i+j} h_{s,j} \binom{d-s}{i-j}.$$

$\square$

Example 4.2.5. Let Δ be the simplicial complex generated by the facets $\{1,2,3\}$, $\{3,4,5\}$, $\{1,4\}$, $\{1,5\}$, $\{2,4\}$ and $\{2,5\}$. The **f**-triangle for Δ is

$$\begin{array}{llll} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 4 & \\ 1 & 5 & 6 & 2 \end{array}$$

By using the formula $h_{i,j} = \sum_{k=0}^{j} (-1)^{j-k} \binom{i-k}{j-k} f_{i,k}$, let us compute the **h**-triangle. Hnce, we have

$h_{0,0} = 0,$
$h_{1,0} = f_{1,0} = 0,$
$h_{1,1} = -f_{1,0} + f_{1,1} = 0,$
$h_{2,0} = \binom{2}{0} f_{2,0} = 0,$
$h_{2,1} = -\binom{2}{1} f_{2,0} + \binom{1}{0} f_{2,1} = 0,$
$h_{2,2} = \sum_{k=0}^{2} (-1)^{2-k} \binom{2-k}{2-k} f_{2,k} = 4,$
$h_{3,0} = f_{3,0} = 1,$
$h_{3,1} = -\binom{3}{1} f_{3,0} + \binom{2}{2} f_{3,1} = 2,$
$h_{3,2} = \sum_{k=0}^{2} (-1)^{2-k} \binom{3-k}{2-k} f_{3,k} = -1,$ and
$h_{3,3} = \sum_{k=0}^{3} (-1)^{3-k} \binom{3-k}{3-k} f_{3,k} = 0.$

Therefore, the **h**-triangle is

$$\begin{array}{llll} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 4 & \\ 1 & 2 & -1 & 0 \end{array}$$

We can say that the simplicial complex Δ in Example 4.2.5. is not shellable, because the pure simplicial complex $\Delta^{(2,2)}$ as shown in the diagram (Figure 2) is not shellable.

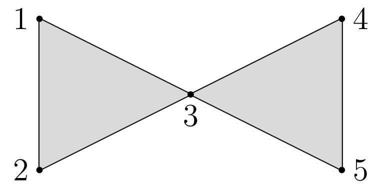


Figure 2

We will further prove that appearance of negative value in the **h**-triangle implies that the corresponding simplicial complex is not shellable.

Lemma 4.2.6. *Let Δ be a $(d-1)$-dimensional simplicial complex with $(f_{s,j})_{0\leq j\leq s\leq d}$ and $(h_{s,j})_{0\leq j\leq s\leq d}$ as the* **f***-triangle and the* **h***-triangle of Δ respectively. Then the following properties hold.*

1. *The value of $h_{d,0}$ is always 1 and $h_{s,0}=0$ for $0\leq s\leq d-1$.*

2. $\sum\limits_{j=0}^{s} h_{s,j} = f_{s,s}$.

3. *The reduced Euler characteristic of Δ is given in terms of the doubly indexed* **h***-numbers as* $\sum\limits_{j=0}^{d}(-1)^{j-1}h_{j,j}$.

4. *The* **h***-vector $(h^{'}_{s,j})_{0\leq j\leq s\leq c}$ of the $(c-1)$-skeleton $\Delta^{(0,c-1)}$ is given by*

$$h^{'}_{s,j} = \begin{cases} h_{c,j} + \sum\limits_{s=c+1}^{d}\sum\limits_{i=0}^{j}\binom{s-c-1+j-i}{j-i} h_{s,i} & \text{for } s=c, \\ h_{s,j} & \text{for all } 0\leq j\leq s<c. \end{cases}$$

5. *The* **h***-vector $(h^{''}_{s,j})_{0\leq j\leq s\leq c}$ of the subcomplex $\Delta^{(c-1,c-1)}$ generated by $(c-1)$-facets of Δ is given by*

$$h^{''}_{s,j} = \begin{cases} h^{'}_{c,j} & \text{for } s=c, \\ 0 & \text{for all } \ 0\leq j\leq s<c. \end{cases}$$

Proof. 1. The empty set $\emptyset$ is contained in all the facets F of Δ. Thus, we say that degree of $\emptyset$ is d, the cardinality of the facet with maximal dimension. Hence, we have $f_{i,0} = 0$ for $0 \leq i \leq d-1$ and $f_{d,0} = 1$. Therefore, we get

$$h_{s,0} = \begin{cases} 1 & \text{for } s = d, \\ 0 & \text{for all } \ 0 \leq s < d. \end{cases}$$

2. By the definition of doubly indexed **h**-number,

$$\sum_{j=0}^{s} h_{s,j} = \sum_{j=0}^{s} \sum_{k=0}^{j} (-1)^{j-k} \binom{s-k}{j-k} f_{s,k}.$$

On interchanging the summation and by equating $j-k$ with l, we get

$$\sum_{j=0}^{s} h_{s,j} = \sum_{k=0}^{s} \left(\sum_{l=0}^{s-k} (-1)^{l} \binom{s-k}{l} \right) f_{s,k} = \sum_{k=0}^{s} (1-1)^{s-k} f_{s,k} = f_{s,s}.$$

3. We have the relation $H(x,y) = F(x, y-1)$. Put $x = -1$ and $y = 0$, to get

$$H(-1,0) = F(-1,1) = \sum_{0 \leq j \leq i \leq d} (-1)^{j} f_{i,j}.$$

On equating $H(-1,0) = \sum_{0 \leq j \leq d} (-1)^{j} h_{j,j}$, we get

$$\sum_{0 \leq j \leq d} (-1)^{j} h_{j,j} = \sum_{j=0}^{d} (-1)^{j} \sum_{i=j}^{d} f_{i,j}.$$

Using the relation between the usual face number and $f_{i,j}$, we have

$$\sum_{0 \leq j \leq d} (-1)^{j} h_{j,j} = \sum_{j=0}^{d} (-1)^{j} f_{j-1} = -\widetilde{\chi}(\Delta).$$

Hence, we express the reduced Euler characteristics of Δ in terms of the **h**-vector as

$$\widetilde{\chi}(\Delta) = \sum_{j=0}^{d} (-1)^{j-1} h_{j,j}.$$

4. For the $(c-1)$-skeleton $\Delta^{(0,c-1)}$, the **f**-vector $(f'_{s,j})_{0\leq j\leq s\leq c}$ is given as

$$f'_{s,j} = \begin{cases} f_{s,j} & \text{for } s < c, \\ \sum\limits_{i\geq c} f_{i,j} & \text{for } \; s = c, \\ 0 & \text{for } \; s > c. \end{cases} \tag{4.3}$$

Using this, we compute the **h**-vector $(h'_{s,j})_{0\leq j\leq s\leq c}$. By using the definition of doubly indexed **h**-vector and (4.3), we have

$$h'_{s,j} = h_{s,j} \quad \text{for all} \;\; 0 \leq j \leq s < c.$$

For $s = c$, we have the following expression.

$$\begin{aligned} h'_{c,j} &= \sum_{k=0}^{j} (-1)^{j-k} \binom{c-k}{j-k} \left(\sum_{s=c}^{d} f_{s,j} \right) \\ &= \sum_{k=0}^{j} (-1)^{j-k} \binom{c-k}{j-k} f_{c,j} + \sum_{k=0}^{j} (-1)^{j-k} \binom{c-k}{j-k} \left(\sum_{s=c+1}^{d} f_{s,j} \right) \end{aligned}$$

By using the definition of $h_{c,j}$ and expressing $f_{s,j}$ in terms of **h**-vector, we get

$$h'_{c,j} = h_{c,j} + \sum_{s=c+1}^{d} \sum_{k=0}^{j} (-1)^{j-k} \binom{c-k}{j-k} \sum_{r=0}^{j} \binom{s-r}{j-r} h_{s,r}.$$

By interchanging the summation, we get

$$h'_{c,j} = h_{c,j} + \sum_{s=c+1}^{d} \left[\sum_{r=0}^{j} \left(\sum_{k=r}^{j} (-1)^{j-k} \binom{c-k}{j-k} \binom{s-r}{j-r} \right) \right] h_{s,r}. \tag{4.4}$$

Now, we need to show that $\binom{s-c-1+j-r}{j-r} = \sum\limits_{k=r}^{j} (-1)^{j-k} \binom{c-k}{j-k} \binom{s-r}{j-r}$. Consider $(1-x)^{s-c-1+j-r} = (1-x)^{s-r} \frac{1}{(1-x)^{c+1-j}}$. Using the binomial expansion, we get

$\sum\limits_{k=0}^{s-c-1+j-r} (-1)^k \binom{s-c-1+j-r}{k} x^k = \left[\sum\limits_{l=0}^{s-r} (-1)^l \binom{s-r}{l} x^l \right] \left[\sum\limits_{i=0}^{\infty} \binom{c+1-j+i-1}{i} x^i \right].$

Let $l = k - r$ and the right hand side becomes,

$$\left[\sum_{k=r}^{s} (-1)^{k-r} \binom{s-r}{k-r} x^{k-r} \right] \left[\sum_{i=0}^{\infty} \binom{c+1-j+i-1}{i} x^i \right].$$

On comparing the coefficients of x^{j-r} and summing the right hand side over $r \leq k \leq j$ (since for$k > j$, the quantity $\binom{c-k}{j-k}$ does not make any sense), we get

$$\binom{s-c-1+j-r}{j-r} = \sum_{k=r}^{j} \binom{c-k}{j-k}\binom{s-r}{j-r}. \tag{4.5}$$

From equations (4.4) and (4.5), we can write

$$h^{'}_{c,j} = h_{c,j} + \sum_{s=c+1}^{d} \sum_{i=0}^{j} \binom{s-c-1+j-i}{j-i} h_{s,i}.$$

5. For the subcomplex $\Delta^{(c-1,c-1)}$, the **f**-vector $(f^{''}_{s,j})_{0\leq j\leq s\leq c}$ is of the form

$$f^{''}_{s,j} = \begin{cases} \sum\limits_{i\geq c} f_{i,j} & \text{for } \ s = c, \\ 0 & \text{for otherwise.} \end{cases} \tag{4.6}$$

From equation (4.3), we have $f^{'}_{c,j} = \sum\limits_{i\geq c} f_{i,j}$. Therefore, we get that the **h**-vector $(h^{''}_{s,j})_{0\leq j\leq s\leq c}$ of $\Delta^{(c-1,c-1)}$ is of the form

$$h^{''}_{c,j} = h^{'}_{c,j}$$

and

$$h^{''}_{s,j} = 0 \text{ for all } 0 \leq j \leq s < c.$$

$\square$

For pure simplicial complexes, we have seen the Mac-Mullen characterization of **h**-vector. Similar combinatorial characterization of **h**-triangle is given as the following theorem.

Theorem 4.2.7. *Let Δ be a shellable simplicial complex of dimension $d-1$ with the restriction map $\mathcal{R}$. Let the $(h_{i,j})_{0\leq j\leq i\leq d}$ be the* **h***-triangle of Δ, then*

$$h_{i,j} = number\ of\ facets\ F\ with\ |F| = i\ and\ |\mathcal{R}(F)| = j.$$

Proof. Since Δ is a simplicial complex, we can write

$$\Delta^{(i-1,i-1)} - \Delta^{(i,i)} = \coprod_{|F|=i} [\mathcal{R}(F), F] \quad \forall\, 1 \leq i \leq d. \tag{4.7}$$

We know that $f_{i,j}$ is the number of size j faces in $\Delta^{(i-1,i-1)} - \Delta^{(i,i)}$ and let $\widetilde{h}_{i,j}$ denotes the number of facets F in Δ with $|F| = i$ and $|\mathcal{R}(F)| = j$. Therefore, we can write the equation (4.7) as

$$\sum_{j=0}^{i} f_{i,j} y^{i-j} = \sum_{j=0}^{i} \widetilde{h}_{i,j} \left(\sum_{k=j}^{i} \binom{i-j}{k-j} y^{i-k} \right).$$

Let us substitute $k - j$ with l in the above equation, then

$$\sum_{j=0}^{i} f_{i,j} y^{i-j} = \sum_{j=0}^{i} \widetilde{h}_{i,j} \left(\sum_{l=0}^{i-j} \binom{i-j}{l} y^{(i-j)-l} \right) = \sum_{j=0}^{i} \widetilde{h}_{i,j} (y+1)^{i-j}. \tag{4.8}$$

We have $F(x, y-1) = H(x, y)$, $\sum_{j=0}^{s} f_{s,j} x^i y^{s-j} = \sum_{j=0}^{s} h_{s,j} x^s (y+1)^{s-j}$. On comparing the coefficients of x^i, we get

$$\sum_{j=0}^{i} f_{i,j} y^{i-j} = \sum_{j=0}^{i} h_{i,j} (y+1)^{i-j}. \tag{4.9}$$

From equation (4.8) and (4.9), we can conclude that

$$h_{i,j} = \widetilde{h}_{i,j}.$$

□

This indicates that the **h**-triangle of a shellable complex has only non-negative entries.

4.3 Topological Properties

A pure shellable simplicial complex of dimension $(d-1)$ has the homotopy type of wedge of $(d-1)$-spheres. This section generalizes this fact for the non-pure shellable simplicial complex case. We consider the geometric realization (topological space) of the simplicial complex throughout the section whenever dealing with a simplicial complex.

Theorem 4.3.1. *Let Δ be a $(d-1)$ dimensional shellable simplicial complex. Then Δ has the homotopy type of a wedge of spheres, containing $h_{j,j}$ copies of the $(j-1)$-spheres for $1 \le j \le d$.*

Proof. Since Δ is a shellable simplicial complex, for a particular shelling of Δ, let us define

$$\tau = \big\{\text{Facets } F \text{ in the shelling such that } \mathcal{R}(F) = F\big\}.$$

Let $\Delta^* = \Delta - \tau$. By the second rearrangement lemma, we can conclude that the facets of Δ^* with induced ordering forms a shelling of Δ^* with the same restriction map $\mathcal{R}$ as Δ. Let us denote the k^{th} facet in the shelling of Δ^* as F_k and $\Delta^*_k = \bigcup_{i=0}^{k} \overline{F}_i$. We know that $\mathcal{R}(F_k)$ is the proper face of the facet F_k, which is not contained in any of the facets in Δ^*.

The simplex Δ^*_1 is contractible, then up to homotopy it is equivalent to a point. We construct the space Δ by attaching facets one by one to a simplex starting with Δ^*_1. On attaching F_2 to the simplex Δ^*_1, we observe that all the faces of F_2, except $\mathcal{R}(F_2)$, can be collapsed into a point. Hence, the new subcomplex is contractible. When we attach facets $F \in \tau$, then we see that all the proper faces of F are already collapsed to a point and therefore, this introduces a sphere attached to that point. Thus, each $(j-1)$-facet $F \in \tau$ is deformed to a $(j-1)$-sphere attached to the common point. Clearly, we see that there are $h_{j,j} = \Big|\{F \in \tau \colon |F| = j\}\Big|$ such spheres. $\square$

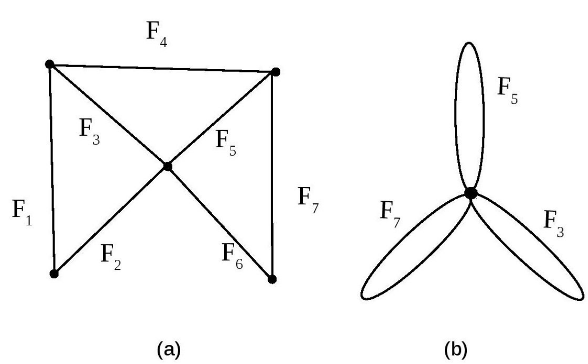


Figure 3

Example 4.3.2. A shellable simplicial complex and its homotopy type is shown in the diagram (Figure 3).

4.4 Lexicographically Shellable Posets

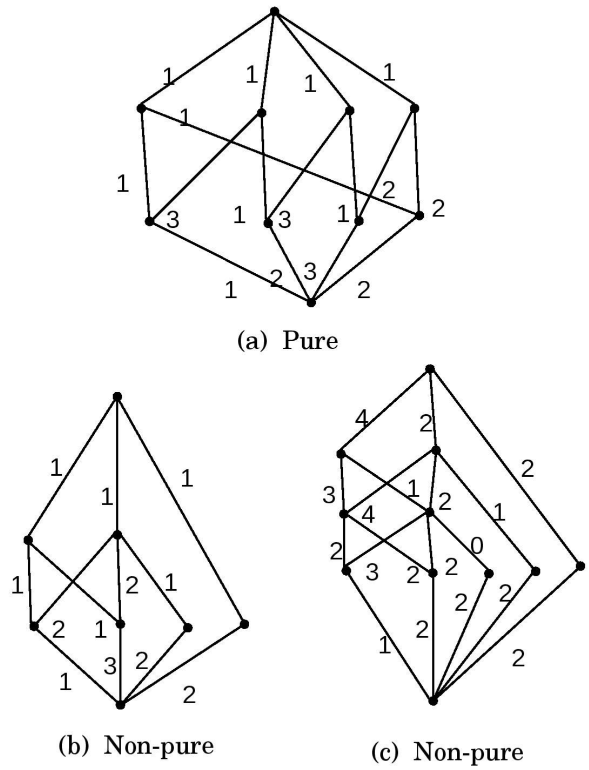
For any finite poset P, we can associate an order complex $\mathbf{\Delta}(P)$ such that the i-face of $\mathbf{\Delta}(P)$ is a chain of length i in P. Properties such as purity, shellability and topological properties of a poset can be understood by studying the order complex associated to it. The theory of lexicographic shellability deals with methods to study the order complexes through labeling the cover relations of a posets. This section generalizes the case to that of nonpure posets (that is, nongraded posets).

Let P be a bounded poset with a top element $\hat{1}$ and a bottom element $\hat{0}$ and we denote $\overline{P} = P - \{\hat{0}, \hat{1}\}$. For any poset P, we can adjoin new elements $\hat{0}$ and $\hat{1}$ such that $\hat{0} < x < \hat{1}$ for all $x \in P$ and form a new poset $\widehat{P} = P \cup \{\hat{0}, \hat{1}\}$. The notation $x \to y$ is used to denote that y covers x. The covering relation of any poset P is given by $\mathcal{E}(P) = \big\{(x, y) \in P \times P \colon x \to y\big\}$ and $\mathcal{M}(P)$ denotes the set of all maximal chains in P. If a chain $x_0 < x_1 < \cdots < x_k$ is maximal in the interval $[x_0, x_k]$, then it is called *unrefinable*.

For a bounded poset P, let $\mathcal{ME}(P)$ be the set of elements of the form $(m, x \to y) \in \mathcal{M}(P) \times \mathcal{E}(P)$ where m is any maximal chain of P and $x, y \in m$. Now, let us get familiarize with several types of cover relation labelings of a poset.

Definition 4.4.1. A map $\lambda \colon \mathcal{E}(P) \longrightarrow \Lambda$, where Λ is some poset, is called an *edge labeling* of poset P.

Examples of edge labeling of pure and non-pure posets are illustrated in Figure 4, where $\Lambda = [n]$ for some $n \geq 1$.



(a) Pure

(b) Non-pure

(c) Non-pure

Figure 4

Definition 4.4.2. A map $\lambda\colon \mathcal{ME}(P) \longrightarrow \Lambda$, where Λ is any poset, is the called *chain-edge labeling* of a poset P if it satisfies the following condition.

If any two maximal chains $m\colon \hat{0} = x_0 \to x_1 \to \cdots \to x_k = \hat{1}$ and $m'\colon \hat{0} = x'_0 \to x'_1 \to \cdots \to x'_n = \hat{1}$ agree along its first d edges, that is, $x_i = x'_i$ for $i = 0, 1, \ldots, d$, then $\lambda(m, x_{i-1} \to x_i) = \lambda(m', x'_{i-1} \to x'_i)$ for $i = 1, 2, \ldots, d$.

Given an edge labeling λ, we can naturally form a chain-edge labeling λ' by letting $\lambda'(m, x \to y) = \lambda(x \to y)$ for all maximal chains m with an edge $x \to y$.

An example of chain-edge labeling (not induced by edge labeling) is illustrated in Figure 5.

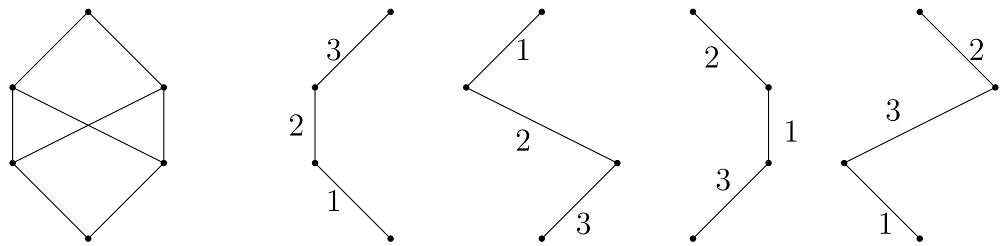


Figure 5

Let P be a bounded poset and λ be a chain-edge labeling of P. For each maximal chain $m\colon \hat{0} = x_0 \to x_1 \to \cdots \to x_n = \hat{1}$, we can associate an ordered string $\lambda(m) = \big(\lambda(m, x_0 \to x_1), \ldots, \lambda(m, x_{n-1} \to x_n)\big)$. The length of the maximal chain m decides the length of the above tuple $\lambda(m)$.

Definition 4.4.3. Let P be a bounded poset. Let $[x, y]$ be an interval of P with an unrefinable chain r from $\hat{0}$ to x. The pair $\big([x, y],\ r\big)$ is called a *rooted interval* with root r. This is denoted as $[x, y]_r$.

A maximal chain m of the interval $[x, y]$ is considered as the maximal chain of rooted interval $[x, y]_r$ and the chain $r \cup m$ obtained by adjoining m to the root r, becomes the maximal chain of the interval $[\hat{0}, y]$.

Let λ be a chain-edge labeling of the bounded poset P and $[x, y]_r$ be a rooted interval in P. Consider m to be a maximal chain in $[x, y]_r$. Let m' and m'' be maximal chains of P that contains the chain $r \cup m$ and let $\ell(r \cup m) = d$. The first d elements of $\lambda(m')$ and $\lambda(m'')$ are equal, thus a $\big(d - \ell(r)\big)$-tuple can be associated with the maximal chain m by deleting these first d entries. This is illustrated in the given diagram (Figure 6).

Let Λ^* be the set of all $\lambda_r(m)$ associated with maximal chains m of the bounded poset P. The set Λ^* is a poset under the *lexicographic partial ordering* defined as

$$(a_1, a_2, \ldots, a_p) \leq_L (b_1, b_2, \ldots, b_q)$$

if and only if either $a_i = b_i$ for $i = 1, 2, \ldots, p$ and $p \leq q$ or $a_i \neq b_i$ for some i and $a_i < b_i$ for the least such i.

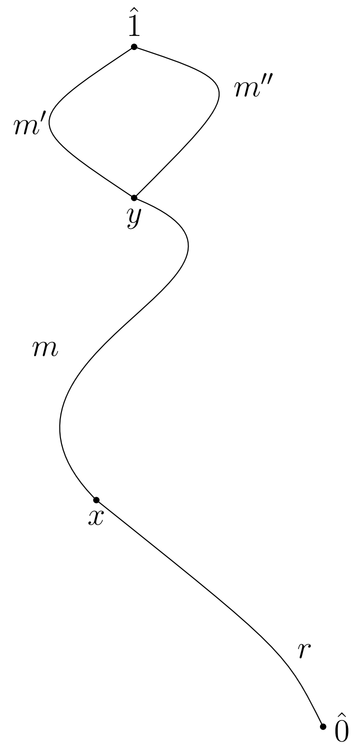


Figure 6

Definition 4.4.4. Let P be a bounded poset and $\lambda\colon \mathcal{ME}(P) \longrightarrow \Lambda$ be a chain-edge labeling of P. If there exists a unique maximal chain m in every rooted interval $[x,y]_r$ of P, such that the tuple $\lambda_r(m) = (a_1,\ldots,a_p)$ satisfies $a_1 < a_2 < \cdots < a_p$ in Λ, then the labeling λ is called the *chain-rising labeling* (*CR-labeling*). The unique maximal chain m is called the *rising chain* in $[x,y]$.

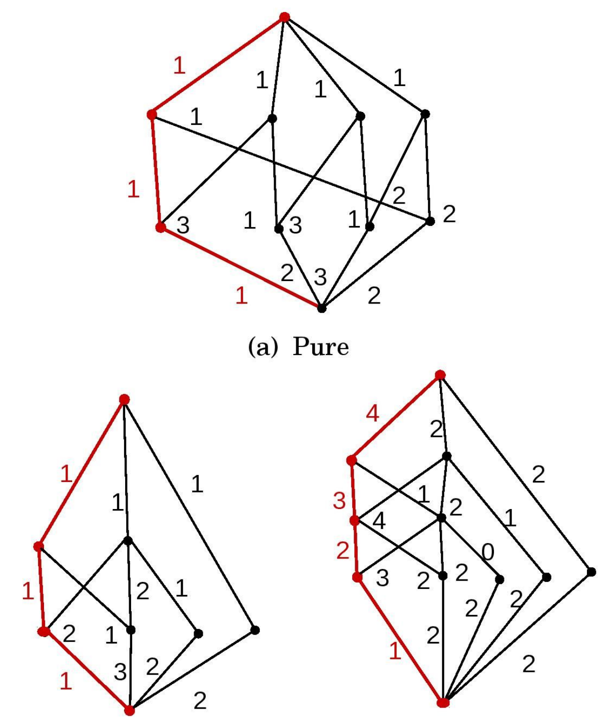
The concept of rising chain can be varied to form the *alternative CR-labeling*, where the rising chain is weakly increasing, that is, the tuple $\lambda_r(m) = (a_1,\ldots,a_p)$ is such that $a_1 \le a_2 \le \cdots \le a_p$.

Definition 4.4.5. Let P be a bounded poset and $\lambda\colon \mathcal{ME}(P) \longrightarrow \Lambda$ be a CR-labeling with the unique rising chain m for every rooted interval $[x,y]_r$. If the tuple $\lambda_r(m)$ is lexicographically strictly first than all $\lambda_r(m')$ for any other maximal chain m', then the labeling λ is called a *chain-lexicographic labeling* (*CL-labeling*).

A CR- or CL-labeling from an edge labeling $\lambda\colon \mathcal{E}(P) \longrightarrow \Lambda$, is called an *edge rising* (*ER-*) or *edge lexicographic* (*EL-*) *labeling* respectively.

Definition 4.4.6. A bounded poset that has an EL- or CL-labeling is called as *EL-* or *CL-shellable*.

Example 4.4.7. The diagram in Figure 7 (a) and (b) shows alternative EL-labeling. Figure 7 (c) is the Hasse diagram of a non-pure poset which has a standard EL-labeling with strictly increasing rising chain. The rising chains in $[\hat{0}, \hat{1}]$ of the corresponding posets have been highlighted in the diagram (Figure 7).



(a) Pure

(b) Non-pure

(c) Non-pure

Figure 7

Lemma 4.4.8. *Let $\lambda\colon \mathcal{ME}(P) \longrightarrow \Lambda$ be a CR-labeling. For every rooted interval $[x,y]_r$, let $a_1, a_2, \ldots, a_k$ be atoms of $[x,y]$ and a_1 belongs to the unique rising chain of $[x,y]_r$, then $\lambda(x \to a_1) < \lambda(x \to a_i)$ for $i = 2, \ldots, k$, if and only if λ is a CL-labeling. Here, $\lambda(x \to a_i)$ is the edge labeling induced by the chain-edge labeling λ and r is the root.*

Proof. Suppose $\lambda\colon \mathcal{ME}(P) \longrightarrow \Lambda$ be a CL-labeling. Then for every rooted interval $[x,y]_r$, there is a unique rising chain $m\colon x = x_0 \to x_1 \to .. \to x_n = y$. By definition, $\lambda_r(m) < \lambda_r(m')$ for all other maximal chains m' of $[x,y]_r$.

Let $a_1, a_2, \ldots, a_k$ be atoms of $[x,y]$ and a_1 belongs to m, that is, $x_1 = a_1$. Then we need to show that $\lambda(m, x \to a_1) < \lambda(m', x \to a_i)$ for $i = 2, \ldots, k$. If $\lambda(m', x \to a_1) < \lambda(m, x \to a_i)$, then the maximal chain m can not be lexicographically strictly first. Thus, we just have to verify that $\lambda(m, x \to a_1) = \lambda(m', x \to a_i)$ is not possible for any value of i. Without loss of generality, we consider the case for $i = 2$. Let us consider the rooted interval $[a_2, y]_{r'}$, where $r' = r \cup (x \to a_2)$ and let $\eta\colon a_2 = z_0 \to z_1 \to \cdots \to z_{n-1} = y$ be the unique rising chain in $[a_2, y]_{r'}$. If $\lambda(\eta, a_2 \to z_1) < \lambda(m, a_1 \to x_2)$, then $m'\colon x \to a_2 \to z_1 \to \cdots \to z_{n-1} = y$ is a maximal chain in $[x,y]_r$ such that $\lambda_r(m') < \lambda_r(m)$. This gives a contradiction. Thus,

$$\lambda(\eta, a_2 \to z_1) \geq \lambda(m, a_1 \to x_2) > \lambda(m, x \to a_1) = \lambda(m', x \to a_1).$$

This implies that m' is also a rising chain of $[x,y]_r$, contradicting the uniqueness of m. Thus we have, $\lambda(x \to a_i) \nleq \lambda(x \to a_1)$ or in other words, $\lambda(x \to a_1) < \lambda(x \to a_i)$ for $i = 1, 2, \ldots, k$.

Conversely, suppose that the given condition is true. Since λ is a CR-labeling, for every rooted interval $[x,y]_r$, there is a unique rising chain $m\colon x = x_0 \to x_1 \to \cdots \to x_n = y$ in $[x,y]_r$. Let the chain $m'\colon x = y_0 \to y_1 \to \cdots \to y_t = y$ be another maximal chain (different from m) in $[x,y]_r$. Without loss of generality, we may assume that $x_1 \neq y_1$. From the given condition, $\lambda(x \to x_1) < \lambda(x \to y_1)$. This implies that $\lambda_r(m) < \lambda_r(m')$. Hence, the given labeling λ is a CL-labeling on the poset P. □

Definition 4.4.9. Let P be a bounded poset and $\lambda\colon \mathcal{ME}(P) \longrightarrow \Lambda$ be a CR-labeling of P. Let $m\colon \hat{0} = x_0 \to x_1 \to \cdots \to x_{k+1} = \hat{1}$ be a maximal chain of P. If $\lambda(m, x_{i-1} \to x_i) \nless \lambda(m, x_i \to x_{i+1})$ in the ordered string $\lambda(m)$, then $\lambda(m)$ is said to have a *descent* at $i,\ 1 \leq i \leq k$. The set

$$\mathcal{D}(m) = \big\{i\colon \lambda(m, x_{i-1} \to x_i) \not< \lambda(m, x_i \to x_{i+1})\big\}$$

is called the *descent set* of m.

If for some maximal chain m, $\mathcal{D}(m) = \{1, 2, \ldots, k\}$, then the maximal chain m is said to be *falling*. The set $\mathcal{R}(m) = \{x_i \in m\colon i \in \mathcal{D}(m)\}$ is called the *restriction* of m.

For alternative EL-labelings with weakly increasing rising chains, the definition of descent set of a maximal chain will be altered to

$$\mathcal{D}(m) = \big\{i\colon \lambda(m, x_{i-1} \to x_i) \not\leq \lambda(m, x_i \to x_{i+1})\big\}.$$

Accordingly, the definition of restriction $\mathcal{R}(m)$ and that of a falling chain will be modified. In alternative EL-labeling the falling chains are strictly falling and that in the standard EL-labeling are weakly falling.

For a maximal chain m of P, let $\overline{m} = m - \big\{\hat{0}, \hat{1}\big\}$. Obviously, we can see that $\mathcal{R}(m) \subseteq \overline{m}$.

Proposition 4.4.10. *Let P be a bounded poset. A CR-labeling of P induces a Boolean interval partition as given below.*

$$\mathbf{\Delta}\big(\overline{P}\big) = \coprod_{m \in \mathcal{M}} \big[\mathcal{R}(m), \overline{m}\big],$$

where $\mathbf{\Delta}\big(\overline{P}\big)$ is the order complex associated with the poset $\overline{P} = P - \big\{\hat{0}, \hat{1}\big\}$.

Proof. Consider a chain $c\colon y_1 < y_2 < \cdots < y_e$ in $\overline{P}$. Let us construct a maximal chain containing c by the following means. Let m_1 be the unique rising chain of the interval $[\hat{0}, y_1]$ and m_2 be the rising chain of $[y_1, y_2]_{m_1}$. In similar manner, assume m_{i+1} to the rising chain of the rooted interval $[y_i, y_{i+1}]_{m_1 \cup m_2 \cup \cdots \cup m_i}$ and continue the process until we obtain a maximal chain $m = m_1 \cup m_2 \cup \cdots \cup m_{e+1}$. By construction we get, $\mathcal{R}(m) \subseteq c \subseteq \overline{m}$.

In order to prove the uniqueness, let m' be some maximal chain such that $\mathcal{R}(m') \subseteq c \subseteq \overline{m'}$. Since $\mathcal{R}(m') \subseteq c$, the descent set of $m' \cap [y_{i-1}, y_i]$ is empty for any $i = 1, 2, \ldots$ Thus, m' is uniquely determined. $\square$

If a bounded poset P has CR- or CL-labeling, then naturally all the intervals $[x, y]$ in P have such a labeling. This fact is stated in the following lemma.

Lemma 4.4.11. *Let* $\lambda\colon \mathcal{ME}(P) \longrightarrow \Lambda$ *be a CR-labeling (or CL-labeling) of a bounded poset* P *and let* $[x, y]_r$ *be a rooted interval in* P*. Then the labeling* $\lambda_r\colon \mathcal{ME}\big([x, y]\big) \longrightarrow \Lambda$ *is CR-labeling (or CL-labeling) of the interval* $[x, y]$*.*

Proof. Given a CR- or CL-labeling $\lambda\colon \mathcal{ME}(P) \longrightarrow \Lambda$, let m be the rising chain of P and $\lambda_r(m) = (a_1, a_2, \ldots, a_k)$ with $a_1 < a_2 < \cdots < a_k$. Consider the rooted interval $[x, y]_r$ with $l(r) = d$. Let m' be the maximal chain in $[x, y]$ such that $m' \subseteq m$. Let $\lambda_r\colon \mathcal{ME}([x, y]) \longrightarrow \Lambda$ be the labeling induced by λ such that $\lambda_r(x_{i-1} \to x_i) = \lambda(x_{i-1} \to x_i)$. Therefore, $\lambda_r(m') = (a_{d+1}, \ldots, a_j)$ is a of $\lambda(m)$. Hence, $a_{d+1} < \cdots < a_j$ implies that m' is the rising chain of the interval $[x, y]$. □

It is interesting to note that the Möbius function of a bounded poset P can be computed from a CR-labeling on P. We can choose an arbitrary root r. If the CR-labeling is induced from an EL-labeling or $x = \hat{0}$, then there is no need to choose the root r.

Proposition 4.4.12. *The Möbius function* $\mu(x, y)$ *of a bounded poset with CR-labeling is computed as the difference between the number of even length falling chains in and the number of odd length falling chains in the rooted interval* $[x, y]_r$*.*

Proof. Since we know that the CR-labeling is hereditary on intervals, it is enough to give the proof for $x = \hat{0}$ and $y = \hat{1}$. We have seen that CR-labeling induces a Boolean interval partition $\mathbf{\Delta}\big(\overline{P}\big) = \coprod_{m \in \mathcal{M}} \big[\mathcal{R}(m), \overline{m}\big]$. By the theorem of Phillip Hall (Proposition 3.3.3) we know that $\mu(x, y)$ is the number of odd length chains subtracted from the number of even length chains in the poset $\overline{P}$. If $\mathcal{R}(m) \neq \overline{m}$, then the Boolean interval $\big[\mathcal{R}(m), \overline{m}\big]$ does not contribute anything to the computation of $\mu(x, y)$. Whereas if the chain m is a falling chain then $\mathcal{R}(m) = \overline{m}$ and its contribution is $-(1)^{\ell(m)}$. □

Example 4.4.13. The poset given in Figure 4 (a) has only one falling chain whose length is 3, so $\mu\big(\hat{0}, \hat{1}\big) = -1$. Poset shown in Figure 4 (b) has two falling chains, one with a length of 2 and the other has length

3, therefore $\mu(\hat{0}, \hat{1}) = 0$. The poset in Figure 4 (c) has one falling chain of length 2 and another one with length 3, so $\mu(\hat{0}, \hat{1}) = 2$.

Theorem 4.4.14. *Let P be a bounded poset. If P is CL-shellable, then the order complex $\mathbf{\Delta}(\overline{P})$ is shellable. More precisely we can say that given a CL-labelling of P any ordering of the maximal chains of P that extends the lexicographic partial order of their labels is a shelling, whose restriction map equals the map $\mathcal{R}(m) = \{x_i \colon i \in \mathcal{D}(m)\}$.*

Proof. Given a CR-labeling λ of the poset P we have the Boolean interval partition $\mathbf{\Delta}(\overline{P}) = \coprod_{m \in \mathcal{M}} [\mathcal{R}(m), \overline{m}]$. In order to prove that the order complex $\mathbf{\Delta}(\overline{P})$ is shellable it is enough to prove that if $\mathcal{R}(m) \subseteq m'$ for $m \neq m'$, then $\lambda(m) < \lambda(m')$ in the lexicographic order.

Let $m \colon \hat{0} = x_0 \to x_1 \to x_2 \to \dots$ and $m' \colon \hat{0} = x'_0 \to x'_1 \to x'_2 \to \dots$ be any maximal chains in P and consider i be minimal such that $x_{i+1} \neq x'_{i+1}$. We consider $y \in P$ to be the minimal element such that $y \in m \cap m'$ and $x_i = x'_i < y$. Since $\mathcal{R}(m) \subseteq m'$, we can conclude that $\mathcal{R}(m) \subseteq m' \cap m$. Therefore there is no descent along the chain $m \cap [x_i, y]$. This implies that the chain $m \cap [x_i, y]$ is the rising chain of the rooted interval $[x_i, y]_{m \cap [\hat{0}, x_i]}$. By Lemma 4.4.8. we have $\lambda(m, x_i \to x_{i+1}) < \lambda(m', x'_i \to x'_{i+1})$, hence the order complex $\mathbf{\Delta}(\overline{P})$ is shellable. $\square$

The homology facets of a CL-shellable poset are the falling chains m with $\mathcal{R}(m) = m$. We generalize the topological properties of a lexicographically shellable poset as in the theorem stated below.

Theorem 4.4.15. *Let P be a CL-shellable poset. Then the order complex $\mathbf{\Delta}(\overline{P})$ has the homotopy type of wedge of spheres. For any fixed CL-labeling, the i^{th} homology group of $\mathbf{\Delta}(\overline{P})$,*

$$\tilde{\mathcal{H}}_i(\mathbf{\Delta}(\overline{P}),\ \mathbb{Z}) \cong \mathbb{Z}^{\textit{number of falling } (i+2)\textit{-chains}}.$$

Remark. From the above theorem, we conclude that i^{th} Betti number of the order complex $\mathbf{\Delta}(\overline{P})$ equals the number of falling chains of length $i+2$.

The approach of *recursive atom ordering* is an alternative way to study the lexicographic shellability. The elements of a poset that covers $\hat{0}$ are called the *atoms*.

Definition 4.4.16. Let P be a bounded poset. If the length of P equals one or if $\ell(P) > 1$ and P consists of atoms $a_1, a_2, \ldots, a_t$ ordered in such a way that

1. For all $j = 1, 2, \ldots, t$ the interval $[a_j, \hat{1}]$ admits a recursive atom ordering such for some $i < j$ such that the atoms in $[a_j, \hat{1}]$ that is in the interval $[a_i, \hat{1}]$ are placed first in the ordering.

2. For all $i < j$, if we have $a_i, a_j < y$ for some $y \in P$, then there is a $k < j$ and an atom z in $[a_j, \hat{1}]$ such that $a_k < z \leq y$.

Then the poset P is said to admit a *recursive atom ordering.*

A recursive atom ordering of the dual poset $(P^*, \leq)$ of P is called the *recursive coatom ordering* of P. It turns out that a bounded poset P admits recursive atom ordering if and only if P is CL-shellable. Thus we can produce an integer CL-labeling for the poset P. Hence we can conclude that as similar to the pure case, any poset P with a fixed CL-labeling $\mathcal{ME}(P) \longrightarrow \Lambda$ also admits an integer CL-labeling $\mathcal{ME}(P) \longrightarrow (\mathbb{Z}, \leq)$.

The weakly increasing rising chains in an alternative EL-labeling can be converted into a strictly increasing rising chain by considering the poset of labels as $\Lambda' = \Lambda \times \mathbb{Z}$ and relabel $m\colon\ \hat{0} = x_0 \to x_1 \to \cdots \to x_k = \hat{1}$ as

$$\lambda'(m,\ x_{i-1} \to x_i) = (\lambda(m,\ x_{i-1} \to x_i),\ i).$$

This relabeling converts a alternative EL-labeling to a strictly increasing CL-labeling. Furthermore, a strictly increasing CL-labeling can be converted to a weakly increasing CL-labeling by taking $\Lambda \times \mathbb{Z}$ as the label poset and then relabeling $m\colon\ \hat{0} = x_0 \to x_1 \to \cdots \to x_k = \hat{1}$ as

$$\lambda'(m,\ x_{i-1} \to x_i) = (\lambda(m,\ x_{i-1} \to x_i),\ -i).$$

During these conversions, the descent set $\mathcal{D}(m)$ and the restriction $\mathcal{R}(m)$ remains the same. These relabelings are illustrated in the diagram given below (Figure 8).

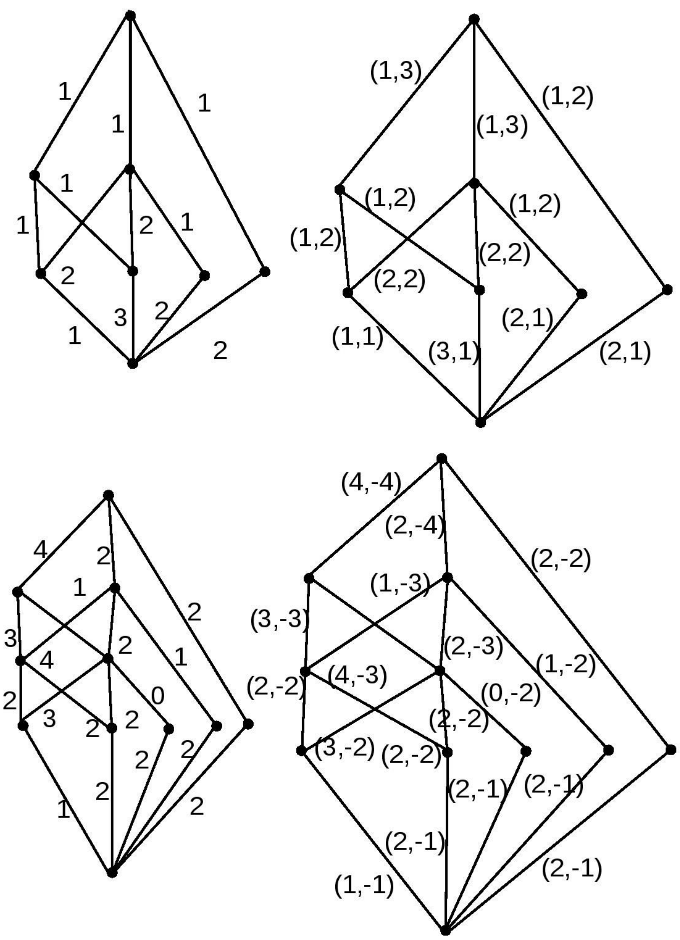

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Figure 8

Chapter 5

Partition Lattice

In this chapter we mainly discuss about a specific example of lexicographically shellable nonpure poset called the *k-equal partition lattice.* We shall have a brief glance through the n-partition lattice, which is pure shellable and then take up the nonpure case, the k-equal partition lattice and study its properties. Whenever we deal with the topological properties of a given lattice L, we usually consider $\overline{L} = L - \{\hat{0}, \hat{1}\}$. The Betti numbers for the k-equal partition lattice have been obtained following the work of Björner and Wachs[2].

5.1 Order Complex of Partition Lattice

Let $(P, \leq)$ be a poset and $z \in P$. The element z is said to be an *upper bound* of two elements x and y in P if we have $z \geq x$ and $z \geq y$. An upper bound $z \in P$ is called the *least upper bound* or the *supremum* of $x, y \in P$, if it is the minimum of all possible upper bounds of x and y. We denote the supremum of x and y as $x \vee y$. If $u \leq x$ and $u \leq y$ for some $u \in P$, then u is called the *lower bound* of x and y. The maximum of all the lower bounds of x and y is called the *greatest lower bound* or the *infimum* and is denoted as $x \wedge y$.

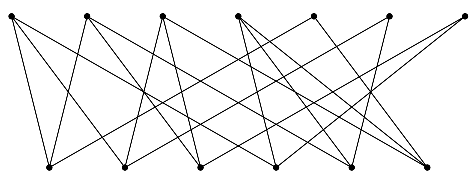
Definition 5.1.1. A poset L for which every pair of elements $x, y \in L$ has a supremum and an infimum is called as a *lattice.*

The set of all partitions of $[n]$ denoted by Π_n which is a poset with respect to the partial ordering defined by refinement is a lattice and is called as the *partition lattice.* Every maximal chain in the partition lattice Π_n is of length $n-1$. Therefore Π_n is a graded poset of rank $n-1$. Moreover, the order complex $\mathbf{\Delta}(\Pi_n)$ is a pure shellable simplicial complex.

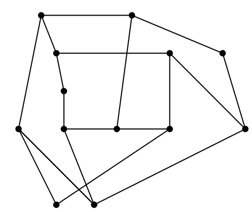
The following diagram (Figure 1) illustrate the poset $\overline{\Pi}_n$ and its order complex $\mathbf{\Delta}(\Pi_n)$ for $n = 3, 4$.

$\overline{\Pi}_3$

$\mathbf{\Delta}\big(\overline{\Pi}_3\big)$



$\overline{\Pi}_4$



$\mathbf{\Delta}\big(\overline{\Pi}_4\big)$

Figure 1

If $\pi = \{B_1, B_2, \ldots, B_t\} \in \Pi_n$, then the interval $[\pi, \hat{1}]$ is isomorphic to the partition lattice of the set $[t] \cong \{B_1, B_2, \ldots, B_t\}$. That is, we have $[\pi, \hat{1}] \cong \Pi_t$.

The order complex $\mathbf{\Delta}\big(\overline{\Pi}_n\big)$ is shellable and therefore it has the homotopy type of wedge of spheres. This can be stated as given below.

Theorem 5.1.2. *The order complex* $\mathbf{\Delta}\big(\overline{\Pi}_n\big)$ *of the poset* $\overline{\Pi}_n$ *is homotopic equivalent to a wedge of* $(n-1)!$ *spheres of dimension* $n-3$.

This theorem also reflects in the computation of the Möbius function of the partition lattice Π_n $\big($See [14]$\big)$.

Example 5.1.3. The order complex $\mathbf{\Delta}\big(\overline{\Pi}_4\big)$ in Figure 1 is homotopic equivalent to wedge of $3! = 6$ spheres. This is illustrated in the following diagram (Figure 2).

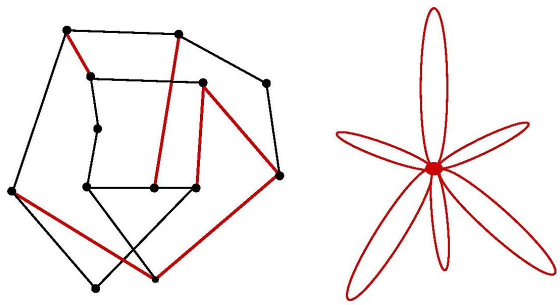


Figure 2

Proposition 5.1.4. *The Möbius function of the partition lattice* Π_n *of all partitions of the set* $[n]$ *is*

$$\mu\big(\hat{0},\hat{1}\big) = (-1)^{n-1}(n-1)!.$$

Let us compute the Möbius functions of the lattices Π_3 and Π_4 by hand and verify Proposition 5.1.4.

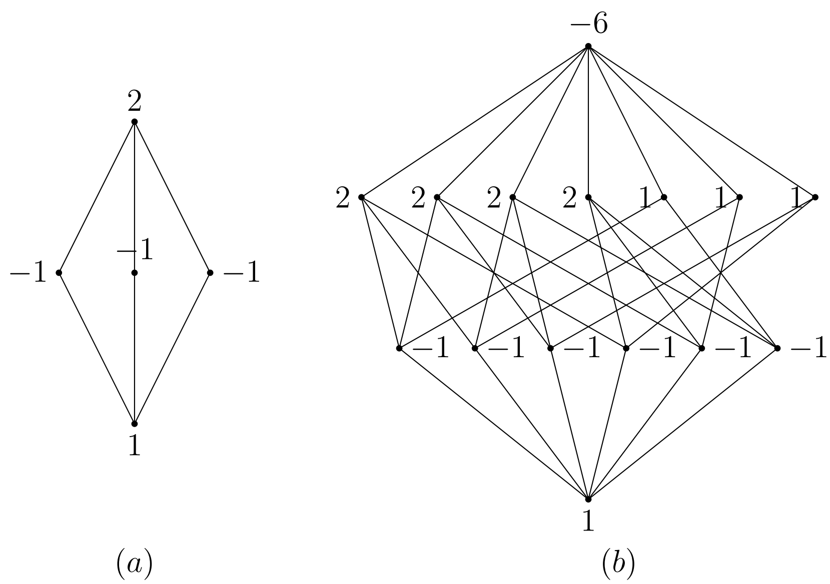


Figure 3

The Hasse diagram (Figure 3) indicates the Möbius function values of Π_3 $\big($Figure $3(a)\big)$ and Π_4 $\big($Figure $3(b)\big)$.

From the Möbius function values given in the diagram (Figure 3), we observe that for the partition lattice Π_3 (Figure 3(a)),

$$\mu(\hat{0}, \hat{1}) = 3 - 1 = 2 = (-1)^{3-1}(3-1)!.$$

For the partition lattice Π_4 in Figure 2(b),we get

$$\mu(\hat{0}, \hat{1}) = -11 + 6 - 1 = -6 = (-1)^{4-1}(4-1)!.$$

5.2 k-equal Partition Lattice

Let Π_n be the partition lattice on the set $[n]$ with refinement as the partial ordering. The *k-equal partition* is a sublattice of the pure partition lattice.

Definition 5.2.1. For $2 \leq k \leq n$, let $\Pi_{n,k}$ be a set of all partitions in Π_n having no blocks of sizes $2, 3, \ldots, k-1$. Under the induced ordering, the set $\Pi_{n,k}$ is a lattice and is called the *k-equal partition lattice*.

Remark. For $k = 2$, we have $\Pi_{n,k} = \Pi_n$. All the results of Π_n remains the same for $\Pi_{n,2}$.

Suppose σ covers π in $\Pi_{n,k}$. Let us consider the set of labels Λ to be a totally ordered set

$$\overline{1} < \overline{2} < \cdots < \overline{n} < 1 < 2 < \cdots < n.$$

We define an edge-labeling $\lambda\colon \mathcal{E}(\Pi_{n,k}) \longrightarrow \Lambda$ according to one of the following covering relations in $\Pi_{n,k}$.

1. If σ is obtained from π by merging singletons and creating a new k-block B, then $\lambda(\pi \to \sigma) = \max\ B$.

2. If σ is formed by merging a non-singleton block B of π with a singleton block $\{a\}$, then $\lambda(\pi \to \sigma) = a$.

3. If σ is obtained from π by merging any two non-singleton blocks B_1 and B_2, then $\lambda(\pi \to \sigma) = \overline{\max(B_1 \cup B_2)}$.

Theorem 5.2.2. *The edge labeling $\lambda\colon \mathcal{E}(\Pi_{n,k}) \longrightarrow \Lambda$ of $\Pi_{n,k}$ defined above is an EL-labeling.*

Proof. This can be proved in three steps. First of all we prove the theorem for an upper interval $[\pi, \hat{1}]$ then for the case where $\pi = \hat{0}$ and finally we prove it for a general interval $[\pi, \sigma]$ in $\Pi_{n,k}$.

Let $[\pi, \hat{1}]$ be an upper interval in $\Pi_{n,k}$. Suppose π has $p \geq 1$ non-singleton blocks $B_1, B_2, \ldots, B_p$ with $b_i = \max\ B_i$ and $b_1 < b_2 < \cdots < b_p$. Let $\{a_1\}, \{a_2\}, \ldots, \{a_q\}$ be the q singleton blocks in π ordered such that $a_1 < a_2 < \cdots < a_q$. Let us construct a maximal chain m of $[\pi, \hat{1}]$ with length $p + q - 1$ in the following manner. We start by merging the blocks B_1 and B_2 and form a new block $B_1 \cup B_2$. Then add the blocks $B_3, B_4, \ldots B_p$ one by one to the newly formed blocks. Once all the non-singleton blocks are merged, start merging the singletons $a_1 < a_2 < \cdots < a_q$ one by one in each successive steps. Thus we obtain a maximal chain m whose label is given by $\lambda(m) = (\bar{b}_2, \bar{b}_3, \ldots, \bar{b}_p, a_1, \ldots, a_q)$. Clearly we can observe that the labels are strictly increasing, $\bar{b}_2 < \bar{b}_3 < \cdots < \bar{b}_p < a_1 < \cdots < a_q$. Hence the maximal chain is the unique rising chain in $[\pi, \hat{1}]$ and by construction $\lambda(m)$ is lexicographically strictly first.

Now let us take $\pi = \hat{0}$, the case where the partition π has only singleton blocks. In this case we construct the rising chain by first forming the k-block $\{1, 2, \ldots, k\}$ and then adding the elements $k + 1, k + 2, \ldots$ one by one to the newly formed blocks. This gives us the unique rising chain m with the label $\lambda(m) = (k, k + 1, \ldots, n)$ which is lexicographically strictly first.

Now consider a general interval $[\pi.\sigma]$ in the k-equal partition lattice $\Pi_{n,k}$. Assume that the partition σ has r non-singleton blocks $C_1, C_2, \ldots, C_r$. Let us merge all the blocks of π contained in C_i together and keep all the other blocks of π as it is to construct a new partition π_i for $i = 1, 2, \ldots, r$. Each interval $[\pi, \pi_i]$ is isomorphic to an upper interval in the k-equal partition lattice on the set C_i. Thus the same mode of construction used in case of an upper interval can be considered to form a maximal chain m_i in the interval $[\pi, \pi_i]$ with a unique rising label $\lambda(m_i)$ that is lexicographically first in the interval $[\pi, \pi_i]$. Since the entries of $\lambda(m_i) \in C_i$ and that of $\lambda(m_j) \in C_j$, for $i \neq j$, we have that $\lambda(m_i) \cap \lambda(m_j) = \emptyset$.

Let the set $\{x_1, x_2, \ldots, x_t\}$ be the collection of all entries occurring in any $\lambda(m_i)$ for $i = 1, 2, \ldots, r$ such that $x_1 < x_2 < \cdots < x_t$. A maximal chain m in $[\pi, \sigma]$ can be constructed by building up the chains m_i in parallel so that we get the label $\lambda(m) = (x_1, x_2, \ldots, x_t)$. In the j^{th}

step of the construction we identify i such that $x_j \in \lambda(m_i)$ and merge all the blocks of m_i that contributes the label x_j in $\lambda(m_i)$. Thus we get a unique rising chain m that is lexicographically first in the interval $[\pi, \sigma]$. This shows that the labeling λ in $\Pi_{n,k}$ is an EL-labeling. $\square$

As the lattice $\Pi_{n,k}$ admits EL-labeling, we say that it is lexicographically shellable. Therefore it is homotopic equivalent to wedge of spheres.

Proposition 5.2.3. *The order complex $\mathbf{\Delta}_{n,k} = \mathbf{\Delta}(\overline{\Pi}_{n,k})$ has the homotopy type of wedge of spheres. Let $\tilde{\beta}^d_{n,k}$ be the rank of the reduced homology group $\tilde{\mathcal{H}}_d(\mathbf{\Delta_{n,k}})$. Then $\tilde{\beta}^d_{n,k}$ is nonzero if and only if $d = n-3-t(k-2)$ for some $t,\ 1 \leq t \leq \left\lfloor \frac{n}{k} \right\rfloor$ and*

$$\tilde{\beta}^{n-3-t(k-2)}_{n,k} = (t-1)! \sum_{0=i_0 \leq \cdots \leq i_t = n-tk} \prod_{j=0}^{t-1} \binom{n-jk-i_j-1}{k-1} (j+1)^{i_{j+1}-i_j} .$$

Proof. By Theorem 4.4.15, we know that the number $\tilde{\beta}^d_{n,k}$ is the number of falling chains of length $d+2$ in $\mathbf{\Delta}_{n,k}$. Let us construct a falling chain m in $\mathbf{\Delta}_{n,k}$. The falling chain m has a label $\lambda(m)$ such that all the unbarred labels comes first followed by the barred ones. Therefore the construction of m is carried out in two stages, the first stage where we create the k-blocks and merges the singletons and the second stage where the non-singleton blocks are combined.

Suppose that by the end of the first stage of construction t number of k-blocks have been created. This results in the availability of t non-singleton blocks and $n-tk$ singletons for the second stage. Total $t+n-tk$ steps are taken to finish the first stage, that is t steps to create the t k-blocks and then a $n-tk$ steps to merge the remaining singletons. In the second stage we merge the t available non-singletons in $t-1$ steps. Hence the length of the falling chain is exactly $n-1-t(k-2)$.

Let us count the number of falling chains of length $n-1-t(k-2)$ in $\mathbf{\Delta}_{n,k}$. First k-block created in the first stage of construction should include the entry n in it. This process is continued till the t^{th} k-block is formed. Then the remaining singletons are merged into the blocks one by one. Let $0 = i_0 \leq i_1 \leq \cdots \leq i_t = n-tk$. Suppose $i_{j+1} - i_j$ singletons are merged between the formation of $(j+1)^{\text{st}}$ k-block and $(j+2)^{\text{nd}}$ k-block, for $j = 0, 1, \ldots, t-2$ and with the $j = t-1$ step we create the t^{th} k-block. As mentioned earlier, the first k-block should carry the value n in it, then there are $\binom{n-1}{k-1}$ ways of creating it. We have $i_1 - i_0$ singletons left to merge in order to get the second k-block. Thus the largest i_1 singletons are also added to the first k-block. Therefore for $j = 0$ case, we can have $\binom{n-1}{k-1}$ initial segments of the falling chain. After the formation of the first k-block there will be $n-k-i_1$ singletons available and the largest among these singletons should be definitely considered when the second k-block is constructed. Hence there is a choice of $k-1$ elements from the available $n-k-i_1-1$ singletons. Once the second k-block is created, we take the top $i_2 - i_1$ elements and add it one by one to the two k-blocks formed. Thus these elements have two choices, either they can be merged into the first k-block or to the second k-block. This give rise to $\binom{n-k-i_1-1}{k-1}2^{i_2-i_1}$ possibilities. All the other blocks of the first stage can be created in the same manner. Therefore in general, the number of initial segments of the falling chain available after the first stage is

$$\prod_{j=0}^{t-1} \binom{n-jk-i_j-1}{k-1}(j+1)^{i_{j+1}-i_j} \ .$$

In the beginning of the second stage we have t blocks of size $\geq k$. Let it be $B_1, B_2, \ldots, B_t$ such that $\max B_1 < \max B_2 < \cdots < \max B_t$. In order to construct the falling chain we merge B_1 to one of the $t-1$ blocks, then this union is merged to the remaining $t-2$ blocks and so on. This process can be carried out in $(t-1)!$ number of ways. So, the total number of falling chains of length $n-1-t(k-2)$ is

$$(t-1)! \sum_{0=i_0\leq\cdots\leq i_t=n-tk} \prod_{j=0}^{t-1} \binom{n-jk-i_j-1}{k-1}(j+1)^{i_{j+1}-i_j} \ .$$

By equating $d+2 = n-1-t(k-2)$, we get $d = n-3-t(k-2)$ and we can conclude that

$$\tilde{\beta}_{n,k}^{n-3-t(k-2)} = (t-1)! \sum_{0=i_0\leq\cdots\leq i_t=n-tk} \prod_{j=0}^{t-1} \binom{n-jk-i_j-1}{k-1}(j+1)^{i_{j+1}-i_j} \ .$$

□

The Betti numbers $\tilde{\beta}_{n,k}^{n-3-t(k-2)}$ of $\mathbf{\Delta}_{n,k}$ can be computed by counting the falling chains in an alternative and simpler way, that is in terms of *standard tableaux* of hook shape.

Let $\mathcal{D}_{k,1^j}$, for $j \geq 0$ be the diagram constructed with k cells in the first row, the *arm* and one cell in each of the remaining j rows which is known as *leg* of the diagram. The construction $\mathcal{D}_{k,1^j}$ is known as a hook shaped *Ferrers diagram.*

Definition 5.2.4. For $j_1, j_2, \ldots, j_t \geq 0$ and $t \geq 1$, let $\mathcal{D}^k(j_1, j_2, \ldots, j_t)$ be the skew diagram consisting of $\mathcal{D}_{k,1^{j_i}}$, $i = 1, 2, \ldots, t$. $\mathcal{D}^k(j_1, j_2, \ldots, j_t)$ is constructed by joining the northeast corner of $\mathcal{D}_{k,1^{j_i}}$ to the southwest corner of $\mathcal{D}_{k,1^{j_{i+1}}}$ for $i = 1, 2, \ldots, t-1$. The new construction, $\mathcal{D}^k(j_1, j_2, \ldots, j_t)$ is called a *broken hook diagram* of type (n, k, t), where $n = j_1 + j_2 + \cdots + j_t + kt$ is the total number of cells in the diagram.

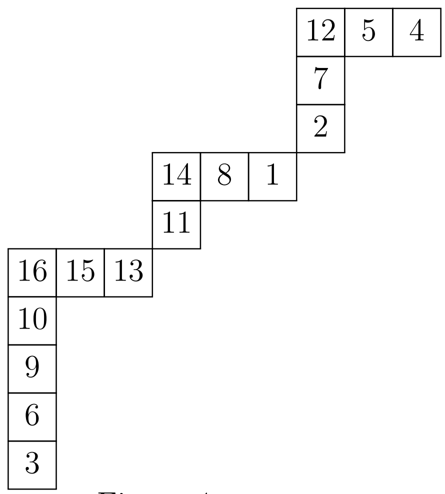


Figure 4

If the cells of a broken hook is filled with distinct numbers from $\{1, 2, \ldots, n\}$, then it is called a *tableau* of the broken hook diagram. A tableaux is said to be a *standard* tableau if the entries in each row from left to right and that from top to bottom along each column are arranged in the decreasing order. If the entry n is filled in the northwest corner cell of the leftmost hook of a standard tableaux $\mathcal{T}$, then $\mathcal{T}$ is said to be *left standard.* An example of a left standard tableau of $\mathcal{D}^3(4, 1, 2)$ is shown in the diagram (Figure 4).

Let $\mathcal{T}$ be a left standard tableau of the broken hook shape $\mathcal{D}^k(j_1, j_2, \ldots, j_t)$ of size n and let $\mathcal{T}_i$ be the hook tableau formed by restricting $\mathcal{T}$ to its i^{th} hook $\mathcal{D}_{k,1^{j_i}}$. For each left tableau $\mathcal{T}$ we will associate a falling chain $m_{\mathcal{T}}$ of length $n - 1 - t(k-2)$. This falling chain will be constructed from the top of the tableau to its bottom in two steps. In the first step we separate the $\mathcal{T}_i$ blocks from rest of the entries of $\mathcal{D}^k(j_1, j_2, \ldots, j_t)$ in a decreasing order of $i = t, t-1, \ldots, 2$. This construction gives rise a partition just below $\hat{1}$ with two blocks, one consisting the entries of $\mathcal{T}_t$. Then we have a partition with three blocks, one with the $\mathcal{T}_t$ entries and another with $\mathcal{T}_{t-1}$. As the construction of the falling chain reaches the end of first stage, we have t number of non-singleton blocks $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_t$.

In the second stage of construction either remove a singleton from the bottom of the leg of any one of the $\mathcal{T}_i$ or split the arm of any one of the $\mathcal{T}_i$ into k singletons if all the cells of its leg has been already peeled off. At each step of the second stage, the choice will be either the smallest entry on the bottom of a leg or the arm with the smallest maximum depending on the smallest value among the both.

The covering relations in the first stage has all labels equal to $\overline{n}$ and that in the second stage are unbarred and equals the entry in the cells that are peeled off from the bottom of the leg or the maximum of the decomposed arm. Thus we form a unique falling chain $m_{\mathcal{T}}$ in association with the given left standard tableau $\mathcal{T}$.

Proposition 5.2.5. *Let $\mathcal{T}_{n,k,t}$ be the set of all standard tableaux consisting the broken hook diagram of type (n, k, t), for $1 \leq t \leq \left\lfloor \frac{n}{k} \right\rfloor$. The Betti numbers of the order complex $\mathbf{\Delta}_{n,k} = \mathbf{\Delta}(\overline{\Pi}_{n,k})$ is given by*

$$\tilde{\beta}_{n,k}^{n-3-t(k-2)} = \sum_{\substack{j_1+j_2+\cdots+j_t=n, \\ j_i \geq k}} \binom{n-1}{j_1-1, j_2, \ldots, j_t} \prod_{i=1}^{t} \binom{j_i-1}{k-1}.$$

Proof. Since we can associate a falling chain of length $n-1-t(k-2)$ to a left standard tableau of broken hook diagram of type (n, k, t), we will just count the number of left standard tableaux of broken hook shape $\mathcal{D}^k(j_1-k, j_2-k, \ldots, j_t-k)$ in order to get the Betti number $\tilde{\beta}_{n,k}^{n-3-t(k-2)}$.

We know that in a left standard tableau, the entry n is fixed in the left most cell. Therefore we are left with numbers $1, 2, \ldots, n-1$ which has to be filled among the available t hooks. This can be carried out in $\binom{n-1}{j_1-1, j_2, \ldots, j_t}$ number of ways.

In each left standard tableau $\mathcal{T}_i$, the maximal entry is in the left most cell. Thus in order to fill the arms, we need to choose $k-1$ entries from the available $j_i - 1$ entries for each $\mathcal{T}_i$. Thus the arms of the tableau can be filled in $\prod_{i=1}^{t} \binom{j_i-1}{k-1}$ number of ways and each left standard tableau can be filled in $\binom{n-1}{j_1-1, j_2, \ldots, j_t} \prod_{i=1}^{t} \binom{j_i-1}{k-1}$ number of ways. Then the summand

$$\sum_{\substack{j_1+j_2+\cdots+j_t=n, \\ j_i \geq k}} \binom{n-1}{j_1-1, j_2, \ldots, j_t} \prod_{i=1}^{t} \binom{j_i-1}{k-1}$$

counts the total number of left standard tableaux of broken hook shape $\mathcal{D}^k(j_1-k, j_2-k, \ldots, j_t-k)$, which equals $\tilde{\beta}_{n,k}^{n-3-t(k-2)}$. $\square$

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