Study of Riemannian Geometry

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Certificate of Examination

This is to certify that the dissertation titled **Study of Riemannian Geometry** submitted by **Jyoti Tanwar** (Reg. No. MS13067) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 19, 2018

Declaration

The work presented in this dissertation has been carried out by me under the guidance of **Dr.Chetan Tukaram Balwe** at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Jyoti Tanwar
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In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Chetan Tukaram Balwe
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Abstract

In this reading project, I have focus on two main theorems of Riemannian geometry, namely *Cartan* and *Rauch* theorems. These two theorems provide us two compare the geometrical properties of a given Riemannian manifold with the other one. I started with studying all the tools that are necessary for understanding these theorems. I thoroughly studied Riemannian manifolds, geodesics, connections, curvature and the most interesting Jacobi fields.

Contents

Chapter 1

Introduction

In this chapter, we will study some basic results from the theory of differentiable manifold, which will help us to study Riemannian Geometry more comfortably. Further, we have introduce the definition of Riemannian metric and a tool for differentiating vector fields called connection. At the end of this chapter, we will be able to conclude that for a Riemannian manifold, we have automatically a connection called *Levi - Civita* connection exists.

1.1 Basic theory of Differentiable Manifold

Definition1.1.1 A *Topological Manifold* of dimension *n* is a set M which is Hausdorff, second countable and is locally isomorphic to $\mathbb{R}^n$.

Definition1.1.2 A *Coordinate chart* on a topological manifold M is a pair (U, ϕ), where U is an open set in $\mathbb{R}^n$ and $\phi : U \longrightarrow \bar{U}$ is a homeomorphism (that is, continuous, bijective map with continuous inverse) from an open subset of M to openset $\bar{U}$ in $\mathbb{R}^n$. Two charts (U, ϕ) and (V, ψ) are *smoothly compatible* if the transition map $\psi \circ \phi^{-1}$ is a diffeomorphism. A collection of charts is said to be atlas $\mathcal{A}$ if the union of open sets in each chart covers M.

Definition1.1.3 A *smooth manifold* M of dimension n is a topological manifold with a maximal atlas containing of smoothly compatible charts.

Defintion1.1.4 Given a smooth manifold M and a smooth curve γ in M that starts at point p , that is, $\gamma(0) = p$, denote $\mathcal{D}(M)$ to be the set of all functions on M that are differentiable at p, we define *tangent vector to the curve* γ at $t = 0$ is a map $\gamma'(0) : \mathcal{D}(M) \to \mathbb{R}$ given by

$$\gamma'(0)f = \left.\frac{d(f \circ \gamma)}{dt}\right|_{t=o}$$

A tangent vector at a point p in M is the tangent vector of some curve γ : $(-\epsilon,\epsilon) \to M$ with $\gamma(0) = p$. We denote set of all tangent vectors to M at p by T_pM. Now this T_pM is n dimensional vector space.

Defintion1.1.5 A *vector field* X on a manifold M is a function that associates each point p in M to a vector in T_pM ,that is , $X(p) \in T_pM$. In other words, a vector field is a map $X : M \to TM$, where TM is the vector bundle or set of all tangent vectors at all point in M and if this map is smooth then the vector field X is said to be smooth. Denote $\mathcal{C}^\infty(U)$ to be the set of all real-valued functions on U that are smooth, where $U \subseteq M$ and let $f \in \mathcal{C}^\infty(U)$ then Xf will be the real valued function on M defined by

$$Xf(p) = X(p)f$$

Definition1.1.6 Let τ(M) denote the set of all vector fields on M and let X , $Y \in \tau(M)$, then the *Lie Bracket* $[X, Y]$ of X and Y is again a vector field and defined by

$$[X, Y]f = (XY - YX)f$$

We have the following properties of Lie bracket in the next proposition.

Proposition1.1.7 Let $X, Y, Z \in \tau(M)$, $f, g \in \mathcal{C}^\infty(U)$ and a, b be two real numbers then

1. $[X, Y] = -[Y, X]$
2. $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
3. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$
4. $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$

1.2 Riemannian Metrics

With the help of inner product we can do calculus on a Euclidean space, that is, we can find the angle between curves and the length of curves. Therefore, in order to perform geometry on an arbitrary smooth manifold we will need the concept of inner product.

Definition1.2.1: For a given differentiable manifold M, a *Riemannian Metric* $\langle , \rangle$ is an assignment of an inner product $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ to each $p \in M$. This inner product is required to be differentiable in the sense that if V and W are two differentiable vector fields on an open set U, then $\langle V, W\rangle : U \longrightarrow \mathbb{R}$ is a differentiable real valued function on M defined by

$$\langle V, W\rangle(p) = \langle V(p), W(p)\rangle$$

A differentiable Manifold with a Riemannian metric is called a **Riemannian Manifold**.

Example: ($\mathbb{R}^3$, $\langle , \rangle$) with usual dot product on tangent spaces is a Geometric Surface. If M is a surface in $\mathbb{R}^3$ then the dot product from $\mathbb{R}^3$ applied to tangent vectors on M furnishes an inner product and makes M into a Geometric surface.

Definition1.2.2: Given two Riemannian Manifolds M and N, an *isometry* is defined as a diffeomorphism $f : M \longrightarrow N$ (that is f is differentiable bijection with differentiable inverse) such that :

$$\langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_{f(p)}$$

$\forall p \in M$, u , $v \in T_pM$ and df is the differential of map f defined on tangent space of M to tangent space of N.

1.3 Connections

We do not have any natural method to differentiate vector fields. Connection acts an a rule for differentiating vector fields on a smooth manifold M. Let $\Gamma(M)$ denote the set of all vector fields of class $\mathcal{C}^\infty$ on M and $\mathcal{D}(M)$ is the ring of all real-valued functions of class $\mathcal{C}^\infty$ on M.

Definition 1.3.1 Suppose $\gamma : [a, b] \to M$ be any smooth curve on a Riemannian Manifold M, then the vector field along this curve is defined as the map $V : [a, b] \to TM$ such that $V(t) \in T_{\gamma(t)}M$.

Definition 1.3.2 For any X, Y, $Z \in \Gamma(M)$ and f , $g \in \mathcal{C}^\infty(M)$, an *Affine Connection* ∇ on M is a map $\nabla : \Gamma(M) \times \Gamma(M) \longrightarrow \Gamma(M)$ defined by $\nabla(X, Y) = \nabla_X Y$ such that :

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$
2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$
3. $\nabla_X (fY) = f\nabla_X Y + X(f)Y$

With the help of connection , we can find the directional derivative of one vector field in the direction of other vector field. In other words, it connects the tangent spaces of Riemannian Manifold.

Proposition 1.3.3 For a given smooth manifold M, with a connection ∇, with γ as a smooth curve on M and V as the vector field along the curve γ, we can associate a unique vector field $\dfrac{\mathrm{D}V}{\mathrm{d}t}$ along γ, called the *Covariant Derivative* of a vector field along γ such that :

1. $\dfrac{\mathrm{D}(V + W)}{\mathrm{d}t} = \dfrac{\mathrm{D}V}{\mathrm{d}t}$

2. $\dfrac{\mathrm{D}(fV)}{\mathrm{d}t} = \dfrac{\mathrm{d}f}{\mathrm{d}t}V + f\dfrac{\mathrm{D}V}{\mathrm{d}t}$

3. Let V is induced by a vector field $Y \in \Gamma(M)$, that is, $Y(\gamma(t)) = V(t)$, then $\dfrac{\mathrm{D}V}{\mathrm{d}t} = \nabla_{\frac{d\gamma}{dt}} Y$.

Definition 1.3.4 Given a smooth manifold M with an affine connection ∇, we say a vector field V along a curve $\gamma : [a,b] \to M$ is *Parallel* if its covariant derivative is zero, that is $\dfrac{DV}{dt} = 0 \; \forall \; t \in [a,b]$.

In the next proposition, it is shown that how one can move to one tangent vector in one vector space to another tangent vector in other tangent space without loosing any information.

Proposition1.3.5 Suppose a smooth manifold M with an affine connection ∇. Let $\gamma : [a,b] \to M$ be a smooth curve in this manifold M and $V_\circ \in T_{\gamma(t_\circ)}\mathrm{M}$, then there exists a unique parallel vector field along this curve γ, which will extend the given tangent vector $V_\circ$, that is, $V(t_\circ) = V_\circ$

1.4 Riemannian Connection

Definition 1.4.1 Suppose we have a Riemannian Manifold M with an affine connection ∇ and a smooth curve γ and let X and X' are vector field along γ, then this connection is said to be *compatible with the metric*, if

$$\langle X, X' \rangle = \text{Constant}$$

Proposition 1.4.2 Given a Riemannian manifold M and a connection ∇ on this manifold. Let $\gamma : I \longrightarrow M$ be a differentiable curve in M and suppose V and W are two vector fields along the curve γ, then ∇ is compatible with the Riemannian metric if and only if

$$\frac{d}{dt}\langle V, W \rangle = \langle \frac{DV}{dt}, W \rangle + \langle V, \frac{DW}{dt} \rangle, \qquad t \in I$$

Corollary1.4.3 For Riemannian manifold M, having connection ∇ the following result gives the condition of compatibility of a Riemannian metric and connection ∇, that is, ∇ is compatible with the metric if and only if $X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \; \forall \; X \; , \; Y \; , \; Z \in \Gamma(\mathrm{M})$

Definition1.4.4 For a given Riemannian Manifold M, an affine connection ∇ is said to be *symmetric* if

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

Theorem1.4.5 This theorem gives the guarantee of existence of connection on a Riemannian Manifold, that is, if M is a Riemannian Manifold, then there exists a unique affine connection ∇ which is symmetric as well as compatible with the Riemannian metric.

Chapter 2

Geodesics

2.1 Geodesics

In this chapter we will talk about geodesics and its properties and the exponential map. Geodesics are the generalization of straight lines from Euclidean space to arbitrary Riemannian manifold. These are the curve whose acceleration is zero.

Definition 2.1.1 A curve $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ on a surface is said to be *geodesic* at $t_0 \in I$ if $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$ at $t = t_0$

A curve γ is said to be geodesic, if γ is a geodesic at all points in I. Suppose the connection on M is Levi-Civita connection. This implies ∇ is compatible with the metric. Let $\gamma : I \longrightarrow M$ be a geodesic, then

$$\frac{d(||\gamma'(t)||}{dt} = \frac{d\langle\gamma'(t), \gamma'(t)\rangle}{dt}$$
$$= \langle\frac{D\gamma'(t)}{dt}, \gamma'(t)\rangle + \langle\gamma'(t), \frac{D\gamma'(t)}{dt}\rangle$$
$$= \langle 0, \gamma'(t)\rangle + \langle\gamma'(t), 0\rangle = 0$$

Hence, the length of tangent vector $\frac{d\gamma}{dt}$ is constant. The arc length

$$s(t) = \int_{t_0}^{t} |\gamma'(t)|dt = c(t - t_0)$$

γ is normalized when the value of $c = 1$. Let (U, x) be a coordinate chart around a point $\gamma(t_0)$. Then in U, we can write $\gamma(t)$ as follows

$$\gamma(t) = (x_1(t), x_2(t), ...x_n(t))$$

Now we know that γ is geodesic if and only if $\dfrac{D(\gamma'(t))}{dt} = 0$

$$\text{i.e iff} \sum_k \left(x_k'' + \sum_{i,j} \Gamma^k_{i,j} x_i' x_j' \right) \frac{\partial}{\partial x^k} = 0$$

if and only if

$$x_k'' + \sum_{i,j} \Gamma^k_{i,j} x_i' x_j' = 0$$

Theorem2.1.2 Let X $\in \Gamma$(M), $V \subseteq M$ be an open set in M, $p \in V$, then there exists an open set $V_\circ \subseteq V$, $p \in V_\circ$, a number $\delta > 0$, such that $\forall\ q \in V_\circ$, then there exists a $\mathcal{C}^\infty$ map ϕ : (-δ, δ) $\times$ $V_\circ \longrightarrow V$ defined as $\phi_t(q) = \phi(t, q)$ is called the flow of vector field X on V.

For each curve γ in M , we can define a unique curve in TM by $t \longmapsto (\gamma(t), \dfrac{d\gamma(t)}{dt})$ this implies $t \longmapsto (x_1(t), x_2(t), ...x_n(t), x_1'(t), x_2'(t), ...x_n'(t))$, we have

$$x_k'(t) = -\sum \Gamma^k_{ij} dx_i' dx_j'$$

Let $x_k' = y_k$, then we get

$$y_k'' = -\sum \Gamma^k_{ij} y_i y_j$$

where k = 1,2,3,........n.

It can be easily verified that the above equation is equivalent to the first order differential equation. It follows that $t \longmapsto (\gamma(t), \gamma'(t))$ satisfy the above equation and by existence and uniqueness theorem for first order differential equation, we get that geodesics exists on an arbitrary manifold.

Lemma 2.1.3 For a Riemannian Manifold M and a tangent bundle TM, there exists a unique vector field G on M, whose trajectory can be given by:

$$t \longmapsto (\gamma(t), \gamma'(t))$$

where γ is a geodesic on the manifold M of dimension n.

We call this vector field G as a *geodesic field* on tangent bundle TM and its flow as a *geodesic flow* on TM. Now by applying the theorem 2.1.2 on G, take a point (p, 0) $\in \mathcal{U} \subset TU$, with (p, 0) $\in \mathcal{U}$ and a $\mathcal{C}^\infty$ map ϕ : (-δ, δ) $\times$ $\mathcal{U} \longrightarrow TU$, such that $t \longmapsto \phi(t, q, v)$ is the unique trajectory of geodesic field G satisfying $\phi(0, q, v) = (q, v)$ $\forall$ $(q, v) \in \mathcal{U}$. We can choose $\mathcal{U} = \{(q, v) \in TU; q \in V$ and $v \in T_qM$ with $|v| < \epsilon_1\}$, where $V \subset U$ is a neighborhood of a point $p \in M$. Leting $\gamma = \pi \circ \phi$, where π is a projection, that is, $\pi : TM \longrightarrow M$, then we have the following result:

Proposition2.1.4 Let $p \in M$ be an arbitrary point, then there exists an open set $V \subset M$, a number $\delta > 0$, $\epsilon_1 > 0$ and a $\mathcal{C}^\infty$ map $\gamma :$ (-δ, δ) $\times U \longrightarrow M$ where $U = \{(q, v)$; $q \in V$, $v \in T_qM$, $|v| < \epsilon_1$, such that the curve $t\gamma(t, q, v)$, $t \in (-\epsilon, \epsilon)$ is the unique geodesic having velocity v and when $t = 0$ it passes through q, $\forall$ $q \in V$, $v \in T_qM$ with $|v| < \epsilon_1$

From the above proposition, we can talk about any geodesic from any point in a Riemannian manifold with a particular direction. We can increase the size of geodesic by decreasing its interval size.

Lemma 2.1.5 (**Homogenity of geodesics** : $\gamma(t, q, av) = \gamma(at, q, v)$

where $\gamma(t, q, v)$ is defined on (-ϵ, ϵ) and $\gamma(at, q, v)$ is defined on (-$\frac{\epsilon}{a}$, $\frac{\epsilon}{a}$) with a $\in \mathbb{R}$, $a > 0$

2.1.1 The Exponential Map

We know for any point $p \in M$ and an initial velocity $v \in T_pM$, we have a unique maximal geodesic γ_v. We have a map from tangent bundle to the set of geodesics in M. This implies it defines a map from the subset of tangent bundle to M itself, by sending the vector v to the point obtained by following γ_1 for time 1. In other words, the exponential map provides a map from tangent space of any point to the manifold itself.

Definition2.1.6: Let $p \in M$, and $\mathcal{U}$ as defined before, the map $\mathsf{exp} : \mathcal{U} \longrightarrow M$ defined by $\mathsf{exp}(q, v) = \gamma(1, q, v) = \gamma(|v|, q, \frac{v}{|v|})$ is called the exponential map on $\mathcal{U} \subset TU$.

It can be easily verified that exp map is differentiable. Let $B_\epsilon(0) \subset T_qM$ be an open ball of radius ϵ and centred at origin and now define $exp_q : B_\epsilon(0) \longrightarrow M$ as

$$exp_p(v) = exp(q, v)$$

.

Proposition 2.1.7: Given a point q $\in M$, there exists an open neighbourhood $B_\epsilon(0) \subset T_qM$ such that the exponential map at $\mathsf{exp}_q(v) : B_\epsilon(0) \longrightarrow M$ is a diffeomorphism onto its range. We call the image of $B_\epsilon(0)$ in M, a normal neighbourhood of $q \in M$.

2.1.2 Minimizing property of geodesics

A curve joining two points is said to be minimizing if its length is less than or equal to the length of all piecewise smooth curve joining the same points. Suppose the exponential map is defined on an open set V containing 0 in T_pM and assume that this map is a diffeomorphism , then the image U of this open set in M is said to be *normal neighborhood* of p. Now if the closure of $B_\epsilon(0)$ sits inside this open set V, then the image of $B_\epsilon(0)$ under exponential map is called the normal ball and denoted by $B_\epsilon(p)$.

Proposition 2.1.9 Suppose p be any point in M, and U be a normal neighbourhood of p in M and B be a normal ball centred at p. Let γ be a geodesic that starts at p and joins the points $\gamma(0)$ and $\gamma(1)$, that is, $\gamma : [0,1] \longrightarrow M$. Now if we have another curve $\alpha : [0,a] \longrightarrow M$ which is piecewise smooth and joins the same points $\gamma(0)$ and $\gamma(p)$, then $\mathcal{L}(\gamma) \leq \mathcal{L}(\alpha)$ and if both the curves have same length then $\gamma([0,1]) = \alpha([0,a])$

Theorem2.1.10:This theorem says that for any point p in the manifold M, there exists a neighborhood W of p such that when we define exponential map on each points in W, exp_q, where $q \in W$, then this map will be a diffeomorphism on $B_\epsilon(0)$ which is a subset of tangent space at q in M and its image will contain that neighborhood W.

Corollary: Let $\gamma : I \longrightarrow M$ be a piecewise smooth curve such that its parametre is proportional to arc length and length of γ is less than or equal to length of any other piecewise smooth curve with same end points, then γ is a geodesic.

Chapter 3

Curvature

3.1 Introduction

Now our main idea is to define a curvature for a manifold which matches with our intuition about curvature. The basic idea is that if we transport a tangent vector on a manifold M parallel to itself along a curve, then we get the same vector (that is vector with same direction and magnitude) on a flat or Euclidean surface $\mathbb{R}$, but if the surface is curved then the direction of vector will change and we get a new vector. Riemannian curvature tensor is a measure of failure of second covariant derivative to commute. Thus we defined curvature tensor as :

Definition 3.1: The *curvature tensor* R of a Riemannian manifold M is a correspondence which associates to every pair $(X,Y) \in \Gamma(M)$, a map $R(X,Y)$ where $R(X,Y) : \Gamma(M) \longrightarrow \Gamma(M)$ given by $R(X,Y)(Z) = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]}$

The next proposition tells the properties the properties of Riemannian curvature tensor.

Proposition 3.1.2: If X_1, Y_1, X, Y, Z and W are any vector fields on M, then :

1. R is bilinear in Γ(M) $\times$ Γ(M), i.e., $R(fX_1 + gX_2, Y_1) = fR(X_1, Y_1) + gR(X_2, Y_1)$, $R(X_1, fY_1 + gY_2) = fR(X_1, Y_1) + gR(X_1, Y_2)$,where $f, g \in \Gamma(M)$

2. For any two vector fields X, Y on M, the curvature operator $R(X,Y) : \Gamma(M) \longrightarrow \Gamma(M)$ is linear, i.e, $R(X,Y)(Z+W) = R(X,Y)Z + R(X,Y)W$ and $R(X,Y)fZ = fR(X,Y)Z$.

Notation: $\langle R(X,Y), Z, W\rangle = (X,Y,Z,W)$.

Proposition 3.1.3

1. $(X, Y, Z, T) + (Y, Z, X, T) + (Z, X, Y, T) = 0$
2. $(X, Y, Z, T) =$ - (Y, X, Z, T)
3. $(X, Y, Z, T) = (X, Y, T, Z)$
4. $(X, Y, Z, T) = (Z, T, X, Y)$

3.1.1 Sectional Curvature

In this section, the definition of sectional curvature and answer to the question - why it has given so much importance is introduced. Sectional Curvature depends on the two dimensional subspace σ_p of tangent space at T_pM at p, spanned by two linearly independent vectors. Sectional Curvature $K(\sigma_p)$ is of great importance because if we know $K(\sigma_p)$ for all σ, then we can find the Riemannian curvature tensor R completely. The basic idea behind the sectional curvature is to assign curvatures to the planes. Basically a sectional curvature of a plane in a tangent space is the gaussian curvature (product of principal curvatures) of the surface swept by by the geodesics with starting directions in the given plane.

Definition3.1.2: Let us consider a manifold M. let p be any point in the manifold and T_pM is the tangent space at p. Take two linearly independent vectors x and y in T_pM. Let σ_p be the subspace spanned by vectors x and y. Then the sectional curvature $K(\sigma_p)$ of the section σ_p is defined as :

$$K(\sigma_p) = \frac{(x, y, x, y)}{|x|^2|y|^2 - \langle x, y\rangle^2}$$

with $K(\sigma_p)$ is independent of choice of the vectors $x, y \in \sigma_p$.

Lemma3.1.2: Let V be a vector space of dimension $n \geq 2$, provided with an inner product $\langle , \rangle$. Let $R : V \times V \times V \longrightarrow V$ and $R' : V \times V \times V \longrightarrow V$ be two trilinear maps such that R and R' satisfy the four symmetric properties defined above. Define $K(\sigma)$ and $K'(\sigma)$ as follows:

$$K'(\sigma) = \frac{(x, y, x, y)'}{|x|^2|y|^2 - \langle x, y\rangle^2}$$

$$K(\sigma) = \frac{(x, y, x, y)}{|x|^2|y|^2 - \langle x, y\rangle^2}$$

where x and y are two linearly independent vectors in the subspace σ of V. Now if $K'(\sigma) = K(\sigma)$, $\forall\ \sigma$ then we have $R = R'$.

Chapter 4

Jacobi fields

4.1 Jacobi fields

In this chapter, we will introduced a relation between geodesics and curvature. Jacobi fields are the vector fields along geodesics that satisfy a certain differential equation. With the help of Jacobi fields, we can describe how fast the geodesics starting from a given point p and tangent to $\sigma \subset T_pM$ spread apart. In this chapter we will see that the spreading of geodesics depend on $K(\sigma)$. In order to understand the Jacobi fields, let us look first at exponential map.

Observation: Let M be a Riemannian manifold, p be any point in M. Let v be any vector in T_pM. Take a parametrized surface in M , that is , $f : [0, 1] \times (-\epsilon, \epsilon) \longrightarrow M$ given by

$f(t, s) = \exp_p tv(s)$

where $v(s)$ is a curve in T_pM with $v(0) = v$. Now

$$\frac{df}{ds}(1, 0) = d\exp_p)_{tv(0)}tv'(0))$$

Put $\dfrac{df}{ds}(1,\, 0) = J(t)$ and $\gamma(t) = \mathrm{dexp}_p)(tv)$.

Our aim is to examine $J(t)$ on $\gamma(t)$. Since γ is a geodesic , we have $\dfrac{D}{\partial t}\dfrac{\partial f}{\partial t} = 0$ for all (t, s). Therefore we get

$$\begin{aligned} 0 =& \frac{D}{\partial s}\ (\frac{D}{\partial t}\ \frac{\partial f}{\partial t}) = \frac{D}{\partial t}\ \frac{D}{\partial s}\ \frac{\partial f}{\partial t} - R(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t})\frac{\partial f}{\partial t} \\ &= \frac{D}{\partial t}\ \frac{D}{\partial t}\ \frac{\partial f}{\partial s} + R(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s})\ \frac{\partial f}{\partial t}. \end{aligned}$$

when we put $\dfrac{\partial f}{\partial s}(t, 0)$, then it shows that $J(t)$ satisfy the differential equation $\dfrac{D^2 J}{dt^2} + R(\gamma'(t),\, J(t))\ \gamma'(t) = 0$

With the help of above differential equation, we will define Jacobi fields along a geodesic :

Definition Let M be a Riemannian manifold and $\gamma:\ [0,a] \longrightarrow M$ be a geodesic in M. A vector field J along the geodesics γ is said to be Jacobi field if it satisfies the *Jacobi equation* :

$$\frac{D^2J}{dt^2} + R(\gamma'(t),\ J(t))\gamma'(t) = 0$$

for all t $\in$ [0, a]

Property of Jacobi fields which tells us that a Jacobi fields can be determined by its initial conditions.

Proposition 4.1.1 A Jacobi field J along a geodesic $\gamma : [0,a] \longrightarrow M$ can be determined by the initial conditions $J(0)$ and $J'(0)$.

Proof Choose an orthonormal basis $e_1, e_2, ...e_n$ in T_pM. Extend this basis to parallel orthonormal fields $e_1(t), e_2(t), ...e_n(t)$ along γ. Then we can write J as

$$J(t) = \sum f_i(t)e_i(t)$$

where $f_1, f_2, ...f_n$ are smooth functions. This implies

$$\frac{D^2}{dt^2}\ (J(\mathrm{t})) = \sum f_i''(t)e_i''(t)$$

and

$$R(\gamma', J)\gamma' = \sum \langle R(\gamma', J)\gamma', e_i\rangle e_i = \sum\nolimits_i \sum\nolimits_j f_j\langle R(\gamma', e_j)\gamma', e_i\rangle)e_i$$

hence the Jacobi equation becomes

$$f_i'' + \sum f_j\langle R(\gamma', e_j)\gamma', e_i\rangle) = 0 \text{ for all } i = 1, 2, ..., n$$

Since the above equation is linear second order differential equation, therefore with the given initial conditions $J(0)$ and $J'(0)$, there exists a smooth solution that is well defined on $[0, a]$.

From above equation we see that Jacobi equation is equal to linear second order differential equation .Therefore for a given geodesic γ, there exists exactly $2n$ Jacobi fields along γ.

Also we observe that if γ is a geodesic then γ' also satisfies the Jacobi equation, that is,

$$\frac{D^2J}{dt^2} + R(\gamma'(t),\ \gamma'(t))\ \gamma'(t) = 0$$

Similarly we can see that t$\gamma(t)$ is a Jacobi field and can vanish only when $t = 0$.

Earlier we construct a Jacobi field along a geodesic with help of exponential map. The next proposition will show that this is the only method to construct the Jacobi fields with $J(0) = 0$.

Proposition 4.1.2 Let $\gamma : [0, a] \longrightarrow M$ be a geodesic and let J be a Jacobi field along γ such that $J(0) = 0$. Define $\dfrac{DJ}{dt}(0) = w$ and $\gamma'(0) = v$ with $w \in T_{av}(T_{\gamma(0)}M)$. Now take a curve $v(s)$ in $T_{\gamma(0)}M$ such that $v(0) = av$ and $v'(0)$ =w. Put $f(t, s) = \exp_p(\frac{t}{a}v(s))$ and $p = \gamma(0)$. Define a Jacobi field $\bar{J}$ as $\bar{J}(t) = \dfrac{df}{ds}(t, 0)$. Then we have $\bar{J} = J$ on $[0, a]$.

Corollary 4.1.3 With a given geodesic γ : [0, a] $\longrightarrow M$, we can write a Jacobi field J along γ as

$$J(t) = (d\exp_p)_{t\gamma'(0)}(tJ'(0)), \quad \text{t} \in [0, a]$$

4.2 Relation between spreading of geodesics and curvature

Proposition 4.2.1 Let γ : [0, a] $\longrightarrow M$ be geodesic with $\gamma(0) = p$ in M and $\gamma'(0) = v$. Let $w \in T_v(T_pM)$ such that $|w| = 1$. Let J be a Jacobi field along γ. From previous corollary we write J as

$$J(\text{t}) = (\text{dexp}_p)_{tv}(tw)$$

Then the Taylor expansion of $|J(t)|^2$ about $t = 0$ is given as

$$|J(t)|^2 = t^2 - \tfrac{1}{3} \langle R(\text{v, w})\text{v, w}\rangle t^4 + R(\text{t}),$$

where $lim_{t\to 0} \frac{R(t)}{t^4} = 0$

Corollary 4.2.2 Let $\gamma : [0, l] \longrightarrow M$ be a geodesic parametrized by arc length and let $\langle v, w\rangle = 0$, then the sectional curvature $K(v, w)$ at point p with respect to the plane spanned by orthonormal vectors v and w is equal to the $\langle R(v, w)v, w\rangle$ because the basis vectors are orthonormal. Therefore the Taylor expansion of $|J|^2$ about $t = 0$ becomes

$$|J(t)|^2 = t^2 - \tfrac{1}{3} K(p, \sigma)t^4 + R(t)$$

and $|J(t)| = t - \frac{1}{6} K(p, \sigma)t^3 + \bar{R}(\text{t})$ $\; lim_{t\to 0} \frac{\bar{R}(t)}{t^3} = 0$

Observation From above relation we get that locally the geodesics spread apart less than the rays in TpM if $K_p(\sigma) > 0$ and more apart if $K_p(\sigma) < 0$. .

4.3 Conjugate points

In this section, the main aim is to introduced to relation between singularity of Jacobi fields and the exponential map.

Definition 4.3.1. Let $\gamma : [0,a] \longrightarrow M$ be a geodesic in M. Let $p = \gamma(0)$ and q $= \gamma(t_0)$ be any two points in M for $t_0 \in (o,a]$.Then q is said to be conjugate to p, if there exists a Jacobi field (not identically zero) along the geodesic γ which vanishes at p and q. The multiplicity of the conjugate point is the dimension of the space of such Jacobi fields.

Applying the above definition to $\gamma(0)$, we can say that $\gamma(t_\circ)$ is conjugate to $\gamma(0)$ if and only if $\gamma(0)$ is conjugate to $\gamma(t_\circ)$.

Lemma 4.3.2.Let $\gamma : [0,a] \longrightarrow M$ be a geodesic in M. Let $J_1(t), J_2(t), \ldots.., J_k(t)$ be the Jacobi fields along the geodesic γ such that $J_i(0) = 0$ $\forall$ i = 1,2,3,....,k. Then the set $\{J_1(t), J_2(t), \ldots.., J_K(t)\}$ will be linearly independent set if and only if the set $\{J_1'(0), J_2'(0), \ldots.., J_k'(0)\}$ is linearly independent.

Proof Let us assume that $J_1(t), J_2(t), \ldots.., J_k(t)$ are linearly independent. We have to prove that the set $\{J_1'(0), J_2'(0), \ldots.., J_k'(0)\}$ is linearly independent. Suppose to the contrary that $\{J_1'(0), J_2'(0), \ldots.., J_k'(0)\}$ is not linearly independent. Then

$$\lambda_1 J_1'(0) + \lambda_2 J_2'(0) + \lambda_3 J_3'(0) + \ldots\ldots\ldots + \lambda_k J_K'(0) = \text{O such that } \lambda_i \neq 0 \text{ for some i.}$$

Without loss of generality, assume that $\lambda_1 \neq 0$. Then we have

$$J_1'(0) = \left(\tfrac{-\lambda_2}{\lambda_1}\ J_2'(0) + \tfrac{-\lambda_3}{\lambda_1}\ J_3'(0) + \ldots\ldots\ldots + \tfrac{-\lambda_k}{\lambda_1}\ J_k'(0)\right)$$

$$= \left(\tfrac{-\lambda_2}{\lambda_1}\ J_2 + \tfrac{-\lambda_3}{\lambda_1}\ J_3 + \ldots\ldots\tfrac{-\lambda_k}{\lambda_1}\ J_k\right)'(0)$$

$$\text{Since } J_1(0) = \tfrac{\lambda_2}{\lambda_1}\ J_2 + \tfrac{\lambda_3}{\lambda_1}\ J_3 + \ldots\ldots\ \tfrac{\lambda_k}{\lambda_1}\ J_k(0)$$

From above two initial conditions we determined the Jacobi field J as

$$J = \tfrac{\lambda_2}{\lambda_1}\ J_2 + \tfrac{\lambda_3}{\lambda_1}\ J_3 + \ldots\ldots + \tfrac{\lambda_k}{\lambda_1}\ J_k.$$

But this will imply that $J_1, J_2, \ldots\ldots, J_k$ are linearly dependent which is a contradiction. Hence $J_1'(0), J_2'(0), \ldots\ldots, J_k'(0)$ are linearly independent.

For converse part, assume that $J_1, J_2, \ldots J_k$ are linearly dependent. Then we have

$$\lambda_1 J_1 + \lambda_2 J_2 + \ldots\ldots + \lambda_k J_k = 0 \text{ such that } \lambda_i \neq 0 \text{ for some } i.$$

Again assume that $\lambda_1 \neq 0$, therefore we have

$$J_1 = \frac{\lambda_2}{\lambda_1}\ J_2 + \frac{\lambda_3}{\lambda_1}\ J_3 + ... + \frac{\lambda_k}{\lambda_1}\ J_k$$

But this will imply

$$J_1'(0) = \left(\frac{-\lambda_2}{\lambda_1}\ J_2'(0) + \frac{-\lambda_3}{\lambda_1}\ J_3'(0) + ... + \frac{-\lambda_2}{\lambda_1}\ J_2'(0)\right)$$

which cannot be possible. Hence proved.

Observation If M is manifold of dimension n, then we know that dim $T_pM =$ dim $M = n$. From above theorem we conclude that for a given geodesic $\lambda : [0, a] \longrightarrow M$, there exists exactly n linearly independent Jacobi fields which vanishes at $\gamma(0)$. And further this also implies that the multiplicity of a conjugate point can never be greater than $n - 1$.

Definition 4.3.3 Let $p \in M$. Then the set $\{q$; q is a conjugate point to p $\forall$ geodesics γ such that $\gamma(0) = p$ $\}$ is called the *conjugate locus* of p and denoted by *C(p)*.

4.4 Conjugate points and the singularities of exponential map

Proposition 4.4.1 Let p and q be any two points in Riemannian manifold M. let $\gamma : [0, a] \longrightarrow M$ be a geodesic such that $\gamma(0) = p$ and $\gamma(t_\circ) = q$, where $t_\circ \in [0, a]$. Then q is conjugate to p if and only if $v_\circ = t_\circ \gamma'(0)$ is a critical point of $\exp_p$. Furthermore, the multiplicity of q as a conjugate point to p is equal to the dimension of the kernel of map $(d\exp)_{v_\circ}$

Proof Given $p = \gamma(0)$ is conjugate to $q = \gamma(t_0)$, this implies that the there exists a Jacobi field J such that $J(0) = J(t_0) = 0$. Let v and w are defined as $v = \gamma'(0)$ and $w = J'(0)$. Now we know that a Jacobi field can be written as

$$J(t) = (d\exp_p)_{tv}(tw), \quad t \in [0, a]$$

Now J is non zero if and only if $w \neq 0$. This implies q is conjugate to p along γ if and only if

$$0 = J(t_0) = (\mathrm{dexp}_p)_{t_0 v}\ (t_0 w) \quad w \neq 0$$

which happens only when $t_\circ v$ is a critical point of $\exp_p$.

4.5 Properties of Jacobi fields

Proposition 4.5.1. $\gamma : [0, a] \longrightarrow M$ be a geodesic and J be a Jacobi field along γ. Then we have

$$\langle J(t), \gamma'(t) \rangle = \langle J'(0), \gamma'(0) \rangle t + \langle J(0), \gamma'(0) \rangle$$

Proof The Jacobi equation can be written as :

$$J'' = - R(\gamma', J)\gamma' = 0$$

Taking inner product with γ' on both sides, we get

$$\langle J'', \gamma' \rangle = -\langle R(\gamma', J)\gamma', \gamma' \rangle$$

The left side of above equation is equal to $\langle J', \gamma' \rangle'$. Therefore we have

$$\langle J', \gamma' \rangle = \langle J'(0), \gamma'(0) \rangle$$

Also

$$\langle J, \gamma' \rangle' = \langle J', \gamma' \rangle = \langle J'(0), \gamma'(0) \rangle$$

Now integrate above equation

$$\langle J, \gamma' \rangle = \langle J'(0), \gamma'(0) \rangle \mathrm{t} + \langle J(0), \gamma'(0) \rangle$$

Hence proved.

Corollary 4.5.2 If for any t_1 and $t_2 \in [0, a]$ such that $t_1 \neq t_2$, we have $\langle J, \gamma' \rangle(t_1) = \langle J, \gamma' \rangle(t_2)$, then $\langle J, \gamma' \rangle$ will be independent of t. In particular, if $J(0) = J(a) = 0$ then $\langle J, \gamma' \rangle(t) \equiv 0$.

Corollary 4.5.3 If $J(0) = 0$ .Then $\langle J'(0), \gamma'(0) \rangle = 0$ if and only if $\langle J, \gamma' \rangle(t) \equiv 0$. In particular, we have the dimension of the space of Jacobi fields J with conditions $J(0) = 0$ and $\langle J, \gamma' \rangle(t) \equiv 0$ will be $n - 1$.

Proposition 4.5.4 Let $\gamma : [0, a] \longrightarrow M$ be a geodesic in M, such that $\gamma(0)$ is not conjugate to $\gamma(a)$. Let $V_1 \in T_{\gamma(0)}M$ and $V_2 \in T_{\gamma(a)}M$. Then there exists a unique Jacobi field J along γ which satisfy $J(0) = V_1$ and $J(a) = V_2$.

Proof Take a space $\mathcal{J} = \{ J ; J$ is Jacobi field which satisfy $J(0) = 0 \}$. Now define a map $\Theta : \mathcal{J} \longrightarrow T_{\gamma(a)}M$ by:

$$\Theta\ (J) = J(a),\ \ J \in \mathcal{J}$$

Since it is given that $\gamma(a)$ is not conjugate to $\gamma(0)$, therefore there does not exist any Jacobi field along γ which vanishes only at 0 and a. Let $\Theta\ (J_1) = \Theta\ (J_2)$.

$$\Rightarrow J_1(\text{a}) = J_2(\text{a}) \Rightarrow J_1 \text{ - } J_2(\text{a}) = 0$$

This cannot be possible because $\gamma(a)$ is not conjugate to $\gamma(0)$, therefore $J_1 = J_2$, $\Rightarrow \Theta$ is injective. Since Θ is linear and dim $\mathcal{J}$ = dim $T_{\gamma(a)}M$, this implies Θ is an isomorphism. Therefore, there exists a Jacobi field $\bar{J}$ in $\mathcal{J}$ such that $\bar{J}_1(0) = 0$ and $\bar{J}_1$ (a) = V_2

Similarly there exists a Jacobi field $\bar{J}_2$ in $\mathcal{J}$ such that $\bar{J}_2$(a) = 0 and $\bar{J}_2(0) = V_1$. Hence the require Jacobi field is $J = J_1 + J_2$, where uniqueness comes from isomorphism of Θ.

Corollary Let M be a Riemannian manifold of dimension n and $\gamma : [0, a] \longrightarrow M$ be a geodesic in M. Let $\mathcal{J}^{\perp}$ be the space of all Jacobi fields which satisfy $J(0) = 0$ and $J'(0) \perp \gamma'(0)$. Suppose $\{J_1, J_2, ..., J_{n-1}\}$ be the basis for $\mathcal{J}^{\perp}$. Let $\{\gamma'(t)\}^{\perp} \subset T_{\gamma(t)}M$ be the orthogonal complement of $\gamma'(t)$. Now if $\gamma(t)$, t $\in (0, a]$ is not conjugate to $\gamma(0)$, then we have $\{J_1(t), J_2(t), ..., J_{n-1}(t)\}$ as a basis for the orthogonal complement of $\gamma'(t)$.

Chapter 5

Spaces of Constant Curvature

Riemannian manifolds having constant sectional curvature are the most simple one. The advantage of spaces having constant curvature is that there exists large number of isometries for these spaces .Two examples of spaces having constant sectional curvature are spaces with $k \equiv 0$ called Euclidean space $\mathbb{R}^n$ and spaces with $K \equiv 1$ called the unit sphere $S^n \subset \mathbb{R}^{n+1}$. In this chapter we will discover a new space called Hyperbolic space with $k \equiv$ -1. In fact we will prove that these are the only simply connected manifolds with constant sectional curvature.

5.1 Theorem of Cartan

Set up for the theorem Let M and $\bar{M}$ be two Riemannian manifolds with same dimensions n and having curvature R and $\bar{R}$ respectively. Let $p \in M$ and $\bar{p} \in \bar{M}$ are two fix points in two manifolds and define a $i : T_pM \longrightarrow T_{\bar{p}}\bar{M}$ be an isometry. Choose a normal neighborhood $V \subset M$ of p such that $\exp_{\bar{p}}$ is defined at $i \circ \exp_p^{-1}(V)$. Define $f : V \longrightarrow \bar{M}$ by

$$f = \exp_{\bar{p}} \circ\, i \circ \exp_p^{-1}, \ \ q \in V$$

Take a unique unit speed geodesic γ which joins p and any point say q. That is , $\gamma : [0, a] \longrightarrow M$ with $\gamma(0) = p$ and $\gamma(a) = q$. Let P_t be the parallel transport transport along γ from $\gamma(0)$ to $\gamma(a)$.Consider a geodesic $\bar{\gamma} : [0, a] \longrightarrow \bar{M}$ such that $\bar{\gamma}(0) = \bar{p}$ and $\bar{\gamma}'(0) = i(\gamma'(0))$. Let $\bar{P}_t$ be the parallel transport transport along $\bar{\gamma}$. Now define a map $\phi_t : T_q(M) \longrightarrow T_{f(q)}(M)$ by

$$\phi_t(v) = \bar{P}_t \circ i \circ P_t^{-1}(v), \ \ v \in T_q(M)$$

Theorem 5.1.1 With the above set up and notations, suppose for all geodesics that starts at p and all $x, y, z, \in T_q(M)$ we have

$$\phi_t(R(x,y)z) = \bar{R}(\phi_t(x), \phi_t(y))\phi_t(z)$$

then $f : V \longrightarrow f(V)$ is local isometry.

Proof From the definition of local isometry, f : $V \longrightarrow \bar{M}$ is a local isometry if and only if :

$$\langle\ v, w\ \rangle_p = \langle\ df_q(v), df_q(w)\ \rangle_q, \quad \forall \text{ q} \in M \text{ and } v, w \in T_qM$$

holds. But we know

$$\langle\ v, w\ \rangle = \tfrac{1}{2}\left(||v||^2 + ||w||^2 - ||v-w||^2\right)$$

Therefore it is enough to prove that

$$||v||_q = ||df_q(v)||, \;\; q \in M \text{ and } v \in T_qM$$

For this take a vector q in V and a unit speed geodesic $\gamma : [0,l] \longrightarrow M$ with $\gamma(0) = p$ and $\gamma(\text{a}) = q$. Let $v \in T_qM$. By proposition in previous chapter, there exists a unique Jacobi field J which satisfy $J(0) = 0$ and $J(l) = v$. Since T_qM is a vector space so we can find an orthonormal basis for T_qM. Suppose $\{e_1, e_2, ..., e_n\}$ be the required basis such that $e_n = \gamma'(0)$. Take the parallel transport of these basis vectors, i.e, $e_i(t)$ be the parallel transport of e_i along the curve γ $\forall$ i = 1,2,3.....,n. Then we can write the Jacobi field as

$$J(t) = \sum_i f_i(t)e_i(t)$$

But Jacobi equation says that :

$$J'' + R(\gamma', J)\ \gamma' = 0$$

Therefore we get,

$$f_j'' + \sum_i \langle R(e_n, e_i)e_n, e_j\rangle\ f_i = 0$$

As discussed earlier, take another unit speed geodesic $\bar{\gamma'} : [0,l] \longrightarrow \bar{M}$ which satisfy $\bar{\gamma}(0) = \bar{p}$ and $\bar{\gamma'}(0) = i(\gamma'(0))$. Suppose $\bar{J}$ be the vector field along $\bar{\gamma}$ which is defined as

$$\bar{J}(t) = \phi_t(J(t)), \;\; t \in [0,l]$$

If we take $\bar{e_j}(t) = \phi_t(e_j(t))$. Then we get

$$\bar{J}(t) = \sum_i f_i(t)\bar{e_i}(t)$$

By Hypothesis,

$$\langle\ R(e_n, e_i)e_n, e_j\ \rangle = \langle\ \bar{R}(\bar{e_n}, \bar{e_i})\bar{e_n}, e_j\rangle$$

Therefore, we get

$$f_j'' + \sum_i \langle\ \bar{R}(\bar{e_n}, \bar{e_i})\bar{e_n}, \bar{e_j}\rangle\ f_i = 0$$

This implies $\bar{J}$ is also a Jacobi field along γ' with $\bar{J}(0) = 0$. We can write J and $\bar{J}$ as follows :

$$J(t) = (d\exp_p)_{t\gamma'(0)}(\mathrm{t}J'(0)),$$
$$\bar{J}(t) = (d\exp_{\bar{p}})_{t\bar{\gamma'}(0)}(\mathrm{t}\bar{J}'(0)),$$

As we know parallel transport is an isometry , this implies $|\bar{J(l)} = |J(l)$. Now it is enough to prove that

$$\bar{J(l)} = df_q(v) = df_q(J(l))$$

therefore, $\bar{J}(l) = (d\exp_{\bar{p}})_{l\bar{\gamma'}(0)}(li\bar{J}'(0)) = (d\exp_{\bar{p}})_{l\bar{\gamma'}(0)} \circ i \circ ((d\exp_p)_{l\gamma'(0)})^{-1}(J(l)) = df_q(J(l))$

5.2 Hyperbolic space

Definition Consider $H^n = \{(x_1, x_2, ...x_n)\ \mathbb{R}^n\ ;\ x_n > 0\}$. H^n is Riemannian manifold with metric defined as

$$g_{ij}(x_1, x_2, ...x_n) = \frac{\delta_{ij}}{x_n^2}$$

The pair (H^n, g_{ij}) is called the Hyperbolic space having dimension n.

Proposition The pair (H^n, g_{ij}) is *complete simply connected* Riemannian manifold with constant sectional curvature, $K \equiv$ -1.

Definition Space forms are the complete Riemannian manifold with constant sectional curvature.

Theorem Let M^n be a space form having curvature K Then the universal covering of M consists of covering metric is isomorphic to -

1. H^n, whenever $K \equiv$ -1,
2. $\mathbb{R}^n$, whenever $K \equiv 0$,
3. S^n, whenever $K \equiv 1$.

Chapter 6

Comparison Theorem

Now we will study about techniques which will help us to compare the geometry of a given Riemannian manifold M with another Manifold $\bar{M}$, which is simply connected and of constant curvature. Under some assumptions on the sectional curvature of a given Riemannian manifold M, we can conclude that M has certain geometrical properties as that of $\bar{M}$. Further we can talk about the topological properties of M and can compare with that of $\bar{M}$.

6.1 The Rauch Comparison Theorem

One of the technique for discovering geometrical properties of a given Riemannian Manifold M is Rauch's theorem. In this theorem, we usually compare the lengths in two Riemannian manifolds such that there is a relation between their curvatures. Basically Rauch Comparison theorem tells the dependence of spreading/converging of geodesics on Curvature. We will need *Index lemma* in order to prove the Rauch Comparison theorem.

Index Lemma Let M be a Riemannian manifold in which we have a geodesic $\gamma : [0, a] \longrightarrow M$. Suppose $\gamma(0)$ is not conjugate to $\gamma(t)$, for all t $\in (0, a]$. Suppose we have a Normal Jacobi field J, that is $\langle J, \gamma' \rangle = 0$. Let V be a piecewise smooth normal vector field. Define index $I_{t_0}(J, J) = \int_0^{t_0} \{\langle J', J' \rangle$ - $\langle R(\gamma', J)\gamma', J \rangle\}$dt. Similarly define $I_{t_0}(V, V)$. Now if $J(0) = V(0) = 0$ and $J(t_0 = V(t_0)$ for $t_0 \in (0, a]$ then

$$I_{t_0}(J, J) \leq I_{t_0}(V, V)$$

and equality occurs if and only if $V = J$ on $[0, t_0]$.

Proof Jacobi fields which are normal to γ' and satisfies $J(0) = 0$ forms a $(n-1)$ dimensional vector space. Choose a basis elements as $\{J_1, J_2, \ldots\ldots, J_{n-1}\}$ for this vector space of normal Jacobi fields. This implies $J = \sum_{i=1}^{n-1} \alpha_i J_i$, where the α_i are constants. Since we have assumed that $\gamma(0)$ has no conjugate points along γ, therefore there does not exist any Jacobi field which vanishes at any point on (0, a].. This implies $\{J_1(t), J_2(t),......,J_{n-1}(t)\}$ forms the basis elements of the orthogonal complement of $\gamma'(t)$ in $T_{\gamma'(t)}M$.As V is also normal to γ', therefore for t $\neq 0$, V can be written as

$$V(t) = \sum_{i=1}^{n-1} J_i(t)f_i(t)$$

where f_i are smooth real valued functions on $(0, a]$. Now $f_i : (0, a] \longrightarrow \mathbb{R}$. We will extend the domain of each f_i from $(0, a]$ to $[0, a]$.Since $J_i(0) = 0$, therefore by lemma , there exist differentiable functions $\phi_i : [0, a] \longrightarrow \mathbb{R}$ with $\phi_i(0) = J_i'(0)$ (this implies $\phi_i(0)$ are linearly independent) and $J_i(t) = \mathrm{t}\phi_i(t)$, t $\in [0, a]$. Therefore we can write

$$V(t) = \sum_{i=1}^{n-1} g_i\phi_i(t)$$

where g_i are piecewise differentiable functions on $[0, a]$ and $g_i(0) = 0$. Again apply the same lemma to g_i, there exist $h_i : [0, a] \longrightarrow \mathbb{R}$ such that $h_i(0) = g_i'(0)$ and $g_i(t) = \mathrm{t}h_i(t)$. This implies

$$V(t) = \sum_{i=1}^{n-1} th_i(t)\phi_i(t)$$

and hence $f_i = h_i$ and domain of f_i extended. We will show that

$$\langle V', V'\rangle - \langle R(\gamma', V)\gamma', V\rangle = \langle \sum_{i=1}^{n-1} f_i'J_i, \sum_{j=1}^{n-1} f_j'J_j\rangle + \frac{d}{dt}\langle \sum_{i=1}^{n-1} f_iJ_i, \sum_{j=1}^{n-1} f_jJ_j'\rangle$$

Now $R(\gamma', V)\gamma' = R(\gamma', \sum_{i=1}^{n-1} f_iJ_i)\gamma' = \sum_{i=1}^{n-1} R(\gamma', J_i)\gamma' = -\sum_{i=1}^{n-1} f_iJ_i''$

where the last equality comes from the fact that each J_i is a Jacobi field. We have $\langle V', V'\rangle - \langle R(\gamma', V)\gamma', V\rangle = \langle \sum_{i=1}^{n-1} f_i'J_i + \sum_{i=1}^{n-1} f_iJ_i', \rangle + \sum_{j=1}^{n-1} f_j'J_j + \sum_{j=1}^{n-1} f_jJ_j'\rangle - (-\sum_{i=1}^{n-1} f_iJ_i'')$

$$= \langle \sum_{i=1}^{n-1} f_i'J_i + \sum_{j=1}^{n-1} f_j'J_j\rangle + \langle \sum_{i=1}^{n-1} f_i'J_i, \sum_{j=1}^{n-1} f_jJ_j'\rangle + \langle \sum_{i=1}^{n-1} f_iJ_i', \sum_{j=1}^{n-1} f_j'J_j\rangle + \langle \sum_{i=1}^{n-1} f_iJ_i', \sum_{j=1}^{n-1} f_jJ_j'\rangle$$
$$+ \langle \sum_{i=1}^{n-1} f_iJ_i'', \sum_{j=1}^{n-1} f_jJ_j\rangle.$$

the second term of R.H.S of equation 6.1 can be written as

$$\frac{d}{dt}\langle\sum_{i=1}^{n-1} f_iJ_i,\sum_{j=1}^{n-1} f_jJ_j'\rangle = \langle\sum_{i=1}^{n-1} f_i'J_i, +\sum_{i=1}^{n-1} f_iJ_i',\sum_{j=1}^{n-1} f_jJ_j'\rangle + \langle\sum_{i=1}^{n-1} f_iJ_i,\sum_{j=1}^{n-1} f_j'J_j' + \sum_{j=1}^{n-1} f_jJ_j''\rangle$$
$$= \langle\sum_{i=1}^{n-1} f_i'J_i,\sum_{j=1}^{n-1} f_jJ_j'\rangle + \sum_{i=1}^{n-1} f_iJ_i',\sum_{j=1}^{n-1} f_jJ_j'\rangle + \langle\sum_{i=1}^{n-1} f_iJ_i,\sum_{j=1}^{n-1} f_jJ_j''\rangle + \langle\sum_{i=1}^{n-1} f_iJ_i,\sum_{j=1}^{n-1} f_j'J_j'\rangle.$$

Hence to prove $6.a$ we have to show that

$$\langle\sum_{i=1}^{n-1} f_iJ_i',\sum_{j=1}^{n-1} f_j'J_j\rangle = \langle\sum_{i=1}^{n-1} f_iJ_i,\sum_{j=1}^{n-1} f_j'J_j'\rangle.$$

Now in order to prove above equation let $h(t) = \langle J_i, J_j\rangle$ - $\langle J_i, J_j'\rangle$. After calculations, we get $h'(t) = 0$ and hence the required equation is proved. Applying $6.a$ to V and J, we get $I_{t_0}(\ V,V) = I_{t_0}(\ J,J) + \int_0^{t_0}\left|\sum_{i=1}^{n-1} f_i'J_i\right|^2 dt$ and hence $I_{t_0}(\ J,J) \leq I_{t_0}(\ V,V)$. Equality holds when f_i is constant and as $f_i(t_0) = \alpha_i$ hence $f_i(t) = \alpha_i$ that is $V = J$.

Theorem of Rauch Consider we have two Riemannian manifolds M_1 and M_2 of dimensions n_1 and n_2 respectively. Let $\gamma_1 : [0,a] \longrightarrow M_1$ and $\gamma_2 : [0,a] \longrightarrow M_2$ be two geodesics with same velocity. Let J_1 and J_2 be two Jacobi fields along γ_1 and γ_2 respectively such that the following conditions are satisfied by J_1 and J_2

1. $J_1(0) = J_2(0) = 0$
2. $\langle J_1'(0), \gamma_1'(0)\rangle = \langle J_2'(0), \gamma_2'(0)\rangle$
3. $|J_1'(0)| = |J_2'(0)|$

Further assume that γ_2 does not have any conjugate point on $(0,a]$. Now if for all $v \in T_{\gamma_1(t)}M_1$ and $u \in T_{\gamma_2(t)}M_2$, $K_2(u,\gamma_2'(t)) \geq K_2(v,\gamma_1'(t))$, then $|J_2| \leq |J_1|$.

Proof Suppose J_1 and J_2 are normal Jacobi fields. Define $f_1(t) = |J_1(t)|^2$ and $f_2(t) = |J_2(t)|^2$. This implies $f(t) = \frac{f_1}{f_2}$ is well defined in $(0,a]$. Now $\lim_{t\to 0} f(x) = 1$. Therefore in order to prove that $|J_2| \leq |J_1|$, it is enough to prove that $\dfrac{df}{dt} \geq 0$ or equivalently $f_1'f_2 \geq f_1f_2'$. Let t_0 be any point in $(0,a]$ such that $f_1(t_0) = 0$. Then $f_1'(t_0) = 2\langle J_1'(t_0),\ J_1(t_0)\rangle = 0$. Trivially inequality is proved. Let us assume that $f_1(t_0) \neq 0$. Take $U_1(t) = \frac{1}{\sqrt{f_1(t_0)}}J_1(t_0)$ and $U_2(t)$ $= \frac{1}{\sqrt{f_2(t_0)}}J_2(t_0)$. Then

$$\frac{f_1'(t_0)}{f_1(t_0)} = \frac{d}{dt}\langle U_1(t), U_1(t)\rangle = 2\int_0^{t_0}\{\langle U_1', U_1'\rangle - \langle U, R(\gamma_1', U_1)\gamma'\rangle\}dt = 2I_{t_0}(\ U_1, U_1)$$

Similarly $\frac{f_2'(t_0)}{f_2(t_0)} = 2I_{t_0}(\ U_2,U_2)$. Take $E = \frac{\gamma_1'(t)}{|\gamma_1'(t)|}$ and and $\bar{E} = U_1(t)$ and extend this to an orthonormal basis $\{E_i\}$ .Similarly extend this $F = \frac{\gamma_2'(t)}{|\gamma_2'(t)|}$ and and $\bar{F} = U_2(t)$. Let $V(t)$

$= \sum_{i=1}^{n_1} f_i(t)\{E_i\}$ and and $\bar{V}(t) = \sum_{i=1}^{n_2} f_i(t)\{F_i\}$. Clearly $\langle V(t), V(t)\rangle = \langle\ \bar{V}(t), \bar{V}(t)\rangle$ and $V' = \bar{V}'$. Since there is a restriction on curvature, this implies $I_{t_0}(\ \bar{V}, \bar{V}) \leq I_{t_0}(\ V, V)$. From Index Lemma we get $I_{t_0}(\ U_2, U_2) \leq I_{t_0}(\ \bar{V}, \bar{V}) \leq I_{t_0}(\ U_1, U_2)$ and hence proved.

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