Noether Charge in f(R) Theories of Gravity

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CERTIFICATE OF EXAMINATION

This is to certify that the dissertation titled "Noether Charges in $f(\mathrm{R})$ Gravity" submitted by Ms. Sonali Maurya (Reg. No. MS13090) for the partial fulfillment of BS-MS dual degree programme of the Indian Institute of Science Education and Research, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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"Listen to the mustn’ts, child.
Listen to the don’ts.
Listen to the shouldn’ts, the impossibles, the won’ts.
Listen to the never haves, then listen close to me..
Anything can happen, child.
Anything can be"

-SHEL SILVERSTEIN

Abstract

We have studied Lagrangian formulation through Variational principle in Lagrangian depending on the first derivative of generalized coordinates and then for Lagrangian depending on higher order derivatives of generalized coordinates.We looked at The Theorem of Ostrogradsky,which gives explanation as to why higher order derivative theories are unstable. We have studied some aspects of $f(\mathrm{R})$ theories starting with the evaluation of field equations, first in Einstein Gravity and then in $f(\mathrm{R})$ Gravity via metric formalism.We have also discussed Gibbons-York-Hawking term ,which is required for action to be well posed, for both Hilbert-Einstein Action and $f(\mathrm{R})$ action.Further, we have studied an equivalent scalar representation and explored spherically symmetric solutions in $f(\mathrm{R})$ theories via Noether symmetry approach.

Dedication and acknowledgements

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Author's declaration

I declare that this work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgment of collaborative research and discussions. This thesis is a bonafide record of work done by me, and all sources listed within have been detailed in the bibliography.

LIST OF SYMBOLS

$\mu, \nu ..$ ≡Space-time indices of a vector or tensor.

i,j…≡Spatial indices of a vector or tensor

S≡Action

L≡Lagrangian

$\mathbf{L}_X$ ≡Lie derivative with respect to X

H≡ Hamiltonian

$g^{\mu\nu}$ ≡ Metric

G≡ Gravitational Constant

c≡ Speed of Light

$\kappa \equiv \frac{8\pi G}{c^4}$

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CHAPTER 1

INTRODUCTION

The need for a modified theory of gravity has its motivation coming from both High-energy Physics as well as Cosmology. GR(General Relativity) is not renormalizable and hence cannot be conventionally quantized, and thus we need to modify it. The energy budget of the Universe is 4% ordinary baryonic matter,20% dark matter and 76% dark energy.These last two material are not only dark , most of their properties are also unknown. Also, the Einstein Gravity is not able to explain the late time acceleration without including dark energy. One way to explain these observations is to modify gravity. f(R) theories come about by generalizing the Lagrangian of Hilbert-Einstein Action:

$$S = \int d^4x\sqrt{-g}\,R \tag{1.1}$$

to a more general function of R, Ricci scalar,

$$S = \int d^4x\sqrt{-g}\,f(R) \tag{1.2}$$

In subsequent chapters, we have studied Lagrangian formulation through Variational principle in Lagrangian depending on the first derivative of generalized coordinates and then for Lagrangian depending on higher order derivatives of generalized coordinates.We looked at the Theorem of Ostrogradsky , which explains as to why higher order derivative theories are unstable. We have studied some aspects of f(R) theories starting with the evaluation of field equations, first in Einstein Gravity and then in f(R) Gravity via metric formalism.We have also discussed Gibbons-York-Hawking boundary term , which is required for action to be well posed, for both Hilbert-Einstein Action and f(R) action.Further, we have studied an equivalent scalar representation and explored spherically symmetric solutions in f(R) theories via Noether symmetry approach and looked at the importance of boundary terms in these theories.For future research, we can study the dynamics of $f(R) = R^2$, i.e, a scalar field in the presence of the cosmological constant.

CHAPTER 2

LAGRANGIAN MECHANICS

Lagrangian is a quantity which describes the dynamics of a system, here dynamics refers to the motion of the system, and it does so by using the concept of Configuration space.The structure of a configuration space is that of a (differentiable) manifold.A manifold in simple terms is something that 'locally' looks like a n-dimensional Euclidean space, $\mathbb{R}^n$.The Lagrangian is defined on a configuration space, also called a manifold and a function on its tangent bundle, also referred to as Lagrangian function. To make the idea of manifold and tangent bundle more clear one can take the case of Newtonian potential system, where the configuration space is Euclidean, and the Lagrangian function is the difference between kinetic and potential energy [8], though for gravity we need to look for a general Lagrangian.

2.1 Variational Principle

In order to derive the equation of motion from a given Lagrangian one makes use of Hamilton's Principle of Least Action, which states that the motion of the system from time t_1 to time t_2 is such that the line integral, also called the action, has a stationary value for the actual path of the motion.The action **S** is defined as[2],

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt \tag{2.1}$$

For the action to be stationary its variation should be zero, which implies that δS should be zero,

$$\delta S = \int_{t_1}^{t_2} \delta L(q, \dot{q}, t)\, dt = 0 \tag{2.2}$$

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt \tag{2.3}$$

The variation in the second term, in the above equation, can be further simplified by integrating by parts:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \delta \dot{q} dt = \frac{\partial L}{\partial \dot{q}} \delta q \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) \delta q dt \tag{2.4}$$

and putting in the boundary conditions :$\delta q(t_1) = \delta q(t_2) = 0$, i.e, the endpoints are fixed, we obtain the following equation which is the so-called Euler-Lagrange equation or equations of motion:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) = 0 \tag{2.5}$$

2.1.1 Higher order Lagrangian

The Euler-Lagrange equation can be obtained for Lagrangian containing higher order derivatives in a similar manner using variational principle, except that it will be endowed with more boundary conditions.Now, considering a system whose Lagrangian depends non-degenerately upon $\ddot{q}$ which means that $\frac{\partial L}{\partial \ddot{q}}$ depends on $\dot{q}$, $L(q, \dot{q}, \ddot{q})$[4].The action is given as:

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, \ddot{q}) dt \tag{2.6}$$

and the corresponding change in action is :

$$\delta S = \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial \ddot{q}} \delta \ddot{q}\Big) dt \tag{2.7}$$

again simplifying the above equation by integrating by parts :

$$\begin{aligned} \int_{t_1}^{t_2} \frac{\partial L}{\partial \ddot{q}} \delta \ddot{q} dt &= \frac{\partial L}{\partial \ddot{q}} \delta \dot{q} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\Big(\frac{\partial L}{\partial \ddot{q}}\Big) \delta \dot{q} dt \\ &= \frac{\partial L}{\partial \ddot{q}} \delta \dot{q} \Big|_{t_1}^{t_2} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \ddot{q}}\Big) \delta q \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{d^2}{dt^2}\Big(\frac{\partial L}{\partial \ddot{q}}\Big) \delta q dt \end{aligned} \tag{2.8}$$

and putting in the boundary conditions, $\delta q(t_1) = \delta q(t_2) = 0$ and $\delta \dot{q}(t_1) = \delta \dot{q}(t_2) = 0$, i.e, both the generalized coordinates and its first derivative, are fixed at the boundaries, we obtain the following Euler-Lagrange equation:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) + \frac{d^2}{dt^2}\Big(\frac{\partial L}{\partial \ddot{q}}\Big) = 0 \tag{2.9}$$

2.2 Lagrangian formulation for Fields

A field, in essence, is a set of numbers at every point in space-time.Action for fields is written keeping in mind that now time is not the only independent variable. An integral over space is also taken into account; by doing this, space and time are treated on the same footing, this integrand has dimensions of Lagrangian density, but here simply the term Lagrangian is used for the sake of brevity.The Lagrangian depends upon fields which we denote by ϕ and it's higher order derivatives with respect to space-time like $L(\phi, \partial_\mu \phi)$, the corresponding action is[3]

$$S = \int L(\phi, \partial_\mu \phi) d^4 x \tag{2.10}$$

The quantity $\partial_\mu \phi$ denotes the first derivative of field ϕ with respect to space-time, written in Einstein summation convention, here $\partial_\mu \equiv \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.Now the variation in the action is given as:

$$\delta S = \int \Big[\frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi)\Big] d^4 x \tag{2.11}$$

Now converting the second term into a surface integral and keeping any variation in field zero at boundaries, the following equations are obtained, respectively.

$$\delta S = \int \Big\{\Big[\frac{\partial L}{\partial \phi} - \partial_\mu\Big(\frac{\partial L}{\partial(\partial_\mu \phi)}\Big)\Big]\delta\phi + \partial_\mu\Big(\frac{\partial L}{\partial(\partial_\mu \phi)}\delta\phi\Big)\Big\} d^4 x \tag{2.12}$$

$$\partial_\mu\Big(\frac{\partial L}{\partial(\partial_\mu \phi)}\Big) - \frac{\partial L}{\partial \phi} = 0 \tag{2.13}$$

The above equation is the Euler-Lagrange equation for the fields.Now, considering a case of a system for which Lagrangian has a dependence on second-order derivative of the field i.e, $L(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi)$, the action is given as:

$$S = \int L(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi) d^4 x \tag{2.14}$$

The variation in action is ,

$$\delta S = \int \Big[\frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) + \frac{\partial L}{\partial_\mu \partial_\nu \phi}\delta(\partial_\mu \partial_\nu \phi)\Big] d^4 x \tag{2.15}$$

Again integrating by parts and discarding the following terms as they will vanish on the boundary,

$$\Big[\frac{\partial L}{\partial(\partial_\mu \phi)} - \partial_\beta\Big(\frac{\partial L}{\partial(\partial_\mu \partial_\beta \phi)}\Big)\Big]\partial^\nu \phi + \frac{\partial L}{\partial(\partial_\mu \partial_\beta \phi)}\partial_\beta(\partial^\nu \phi) \tag{2.16}$$

we get the following Euler-Lagrange equation,

$$\frac{\partial L}{\partial \phi} - \partial_\mu\Big(\frac{\partial L}{\partial(\partial_\mu \phi)}\Big) + \partial_\nu \partial_\mu\Big(\frac{\partial L}{\partial(\partial_\mu \partial_\nu \phi)}\Big) = 0 \tag{2.17}$$

Here both $\delta\phi$ and $\delta(\partial\phi)$ are fixed on the boundary.

2.3 Noether's theorem

Noether's theorem states that each continuous symmetry of the system can be associated with a conserved quantity.These are not associated with discrete symmetry because they are static in nature.

Noether's theorem can be used to derive conserved quantities from the Lagrangian, one can do it for Lagrangian of discrete particles and fields. Considering the Lagrangian L, for a point particle, of form $L = L(q, \dot{q}, t)$ and taking an infinitesimal change in the coordinate q i.e, $q \Rightarrow q + \delta q$, the Lagrangian can change at most by time derivative of some constant quantity K, which is, in this case, of the form K(q,t) i.e, depending on q and t only, thus we get the following equations after introducing infinitesimal variation in the coordinates q,

$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} = \frac{dK}{dt} \tag{2.18}$$

$$\Big(\frac{\partial L}{\partial q} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big)\Big)\delta q + \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\delta q\Big) - \frac{dK}{dt} = 0 \tag{2.19}$$

Now the first quantity in the bracket with δq is nothing but the Euler-Lagrange equation which is zero for the classical system and thus whenever the equations of motion hold we have,

$$\frac{dJ}{dt} = 0 \tag{2.20}$$

where J is the conserved quantity which is given as,

$$J = \frac{dL}{d\dot{q}}\delta q - K \tag{2.21}$$

On closer look, it turns out that, one of the quantities in the above equation is nothing but, the boundary term that we were earlier discarding while calculating Equation of Motion.On the same line Noether's theorem can be used to calculate conserved quantities for Lagrangian of fields which are of the form, $L = L(\phi, \partial_\mu \phi)$, and introducing infinitesimal variation in field $\phi \Rightarrow \phi + \delta\phi$,

$$\delta L = \frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial(\partial^\mu \phi)}\delta(\partial^\mu \phi \tag{2.22}$$

The Lagrangian in case of fields can change almost by a divergence of some term.So we have,

$$\Big[\frac{\partial L}{\partial \phi} - \partial_\mu\Big(\frac{\partial L}{\partial(\partial_\mu \phi)}\Big)\Big] + \partial_\mu\Big(\frac{\partial L}{\partial(\partial_\mu \phi)}\delta\phi\Big) - \partial_\mu K^\mu = 0 \tag{2.23}$$

The first part of the equation is the Euler-Lagrange equation.

$$\partial_\mu j^\mu = 0 \tag{2.24}$$

$$j^\mu = \frac{\partial L}{\partial(\partial_\mu \phi)}\delta\phi - \partial_\mu K^\mu \tag{2.25}$$

Now looking at one particular example of conserved quantity which is Stress-energy tensor which is obtained by introducing Space-Time translations i.e,

$$\phi(x) \Rightarrow \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi \tag{2.26}$$

variation in the Lagrangian is,

$$L' = L + a^{\mu}\partial_{\mu}L = L + a^{\nu}\partial_{\mu}(\delta^{\mu}_{\nu}L) \tag{2.27}$$

The stress-energy tensor is given as

$$T^{\mu\nu} = \frac{\partial L}{\partial(\partial_{\mu}\phi)}\partial^{\nu}\phi - g^{\mu\nu}L \tag{2.28}$$

The stress-energy tensor on generalizing for the Lagrangian depending on the second derivative of ϕ, $L = L(\phi, \partial_{\mu}\phi, \partial_{\mu\nu}\phi)$, is given as

$$T^{\mu\nu} = \Big[\frac{\partial L}{\partial(\partial_{\mu}\phi)} - \partial_{\beta}\Big(\frac{\partial L}{\partial(\partial_{\mu}\partial_{\beta}\phi)}\Big)\partial^{\nu}\phi\Big] + \frac{\partial L}{\partial(\partial_{\mu}\partial_{\beta}\phi)}\partial_{\beta}(\partial^{\nu}\phi) - g^{\mu\nu}L \tag{2.29}$$

Chapter 3

The Theorem of Ostrogradsky

The theorem of Ostrogradsky shows that for the Lagrangian which depends on higher time derivatives of generalized coordinates have a linear instability in the Hamiltonian associated with them.Here we look at the usual Hamiltonian construction in context of single, one dimensional point particle with Lagrangian depending on no higher than first time derivatives and then review the Ostrogradsky's construction for Hamiltonian for Lagrangian involving second time derivative and also as to why this construction at all.The result is then generalized to Lagrangian depending on N derivatives [4,7] In the usual case of $L = L(q, \dot{q})$, The Euler-Lagrangian equation is:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \tag{3.1}$$

We are assuming here that the Lagrangian is nondegenerate, which means that $\frac{\partial L}{\partial \dot{q}}$ depends upon $\dot{q}$ and, this gives us the advantage of writing the laws of physics in the form that Newton assumed,

$$\ddot{q} = F(q, \dot{q}) \Longrightarrow q(t) = Q(t, q_\circ, \dot{q}_\circ) \tag{3.2}$$

And the solutions to above equation depends upon two initial values namely, $q_\circ = q(0)$ and $\dot{q}_\circ = \dot{q}(0)$, so we need two canonical coordinates Q and P, which are taken to be:

$$Q \equiv q \;\text{ and }\; P \equiv \frac{\partial L}{\partial \dot{q}} \tag{3.3}$$

The assumption of nondegeneracy allows us to invert the phase space transformation to solve for $\dot{q}$ in terms of Q and P, i.e, there exits a function v(Q,P), such that,

$$\left.\frac{\partial L}{\partial \dot{q}}\right|_{\substack{q=Q \\ \dot{q}=v}} = P \tag{3.4}$$

The Hamiltonian is obtained by the Legendre transformation on $\dot{q}$,

$$H(Q, P) \equiv P\dot{q} - L \tag{3.5}$$

The canonical evolution equations reproduce the Euler -Lagrange equations, and this is what is meant by the statement that, "the Hamiltonian generates time evolution."

(3.6) $$\dot{Q} \equiv \frac{\partial H}{\partial P} = v + P\frac{\partial v}{\partial P} - \frac{\partial L}{\partial \dot{q}}\frac{\partial v}{\partial P} = v$$

(3.7) $$\dot{P} \equiv -\frac{\partial H}{\partial Q} = -P\frac{\partial v}{\partial Q} + \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}}\frac{\partial v}{\partial P} = \frac{\partial L}{\partial q}$$

Now consider a Lagrangian, $L = L(q, \dot{q}, \ddot{q})$, which depends nondegenerately upon $\ddot{q}$, the Euler-Lagrange equation is,

(3.8) $$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0$$

Following the outlines of Hamiltonian construction of the previous section, if the same is done for the Hamiltonian construction of the Lagrangian of form, $L = L(q, \dot{q}, \ddot{q})$, we will have the following Legendre transformation on q and $\dot{q}$,

(3.9) $$H = p_1\dot{q} + p_2\ddot{q} - L(q, \dot{q}, \ddot{q})$$

here Hamiltonian $H = H(p_1, p_2, \dot{q}, \ddot{q})$ thus,

(3.10) $$dH = \frac{\partial H}{\partial p_1}dp_1 + \frac{\partial H}{\partial p_2}dp_2 + \frac{\partial H}{\partial \dot{q}}d\dot{q} + \frac{\partial H}{\partial \ddot{q}}d\ddot{q}$$

(3.11) $$dH = p_1 d\dot{q} + \dot{q}dp_1 + \ddot{q}dp_2 - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial \dot{q}}d\dot{q} - \frac{\partial L}{\partial \ddot{q}}d\ddot{q}$$

comparing above two equations, we get following relations,

(3.12) $$\dot{q} = \frac{\partial H}{\partial p_1}\ , \quad \ddot{q} = \frac{\partial H}{\partial p_2}$$

(3.13) $$p_1 - \frac{\partial L}{\partial \dot{q}} = \frac{\partial H}{\partial \dot{q}}\ , \quad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}$$

(3.14) $$p_2 = \frac{\partial L}{\partial \ddot{q}}$$

thus we have five relations and only four initial conditions which implies there is some redundancy in some information and thus we need to modify our construction.

Since the Lagrangian is nondegenerate in $\ddot{q}$, one can have following form,

(3.15) $$\ddddot{q} = F(q, \dot{q}, \ddot{q}, \dddot{q}) \implies q(t) = Q(q_\circ, \dot{q}_\circ, \ddot{q}_\circ, \dddot{q}_\circ)$$

and since the solution depends upon four initial values we need four canonical coordinates and the Ostrogradsky's choices for these are,

$$Q_1 \equiv q \quad , \quad P_1 \equiv \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}, \tag{3.16}$$

$$Q_2 \equiv \dot{q} \quad , \quad P_2 \equiv \frac{\partial L}{\partial \ddot{q}} \tag{3.17}$$

Again due to the assumption of nondegeneracy phase space transformation can be inverted to solve for $\ddot{q}$ in terms of Q_1 , Q_2 *and* P_2, thus there exits a function $a(Q_1, Q_2, P_2)$ such that,

$$\left.\frac{\partial L}{\partial \ddot{q}}\right|_{\substack{q=Q_1 \\ \dot{q}=Q_2 \\ \ddot{q}=a}} = P_2 \tag{3.18}$$

and the Hamiltonian is,

$$H(Q_1, Q_2, P_1, P_2) \equiv \sum_{i=1}^{2} P_i q^{(i)} - L = P_1 Q_2 + P_2 a(Q_1, Q_2, P_2) - L\Big(Q_1, Q_2, a(Q_1, Q_2, P_2)\Big) \tag{3.19}$$

They also generate time evolution equations,

$$\dot{Q}_1 = \frac{\partial H}{\partial P_1} = Q_2, \tag{3.20}$$

$$\dot{Q}_2 = \frac{\partial H}{\partial P_2} = a + P_2 \frac{\partial a}{\partial P_2} - \frac{\partial L}{\partial \ddot{q}}\frac{\partial a}{\partial P_2} = a, \tag{3.21}$$

$$\dot{P}_2 = -\frac{\partial H}{\partial Q_2} = -P_1 - P_2 \frac{\partial a}{\partial Q_2} + \frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial \ddot{q}}\frac{\partial a}{\partial Q_2} = -P_1 + \frac{\partial L}{\partial \dot{q}}, \tag{3.22}$$

$$\dot{P}_1 = -\frac{\partial H}{\partial Q_1} = -P_2 \frac{\partial a}{\partial Q_1} + \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \ddot{q}}\frac{\partial a}{\partial Q_1} = \frac{\partial L}{\partial q} \tag{3.23}$$

Thus Ostrogradsky's choices indeed reproduce time evolution equations.For the Lagrangian depending on n times derivative, the choices for 2N phase space coordinates are,

$$Q_i \equiv q^{(i-1)} \quad and \quad P_i \equiv \sum_{j=i}^{N} \Big(-\frac{d}{dt}\Big)^{j-i} \frac{\partial L}{\partial q^j} \tag{3.24}$$

We are assuming Non-degeneracy which means that we can solve for $q^{(N)}$ in terms of P_N and the Q_i's, and that there exists a function $A(Q_1, ..., Q_N, P_N)$ such that,

$$\left.\frac{\partial L}{\partial q^{(N)}}\right|_{\substack{q^{(i-1)}=Q_i \\ q^{(N)}=A}} = P_N \tag{3.25}$$

and the generalized Hamiltonian is and evolution equations are,

$$H \equiv \sum_{i=1}^{N} P_i q^{(i)} - L \tag{3.26}$$

$$H = P_1 Q_2 + P_2 Q_3 + ... + P_{N-1} Q_N + P_N A - L(Q_1 ...., Q_N, A) \tag{3.27}$$

$$\dot{Q}_i \equiv \frac{\partial H}{\partial P_i} \;\; and \;\; \dot{P}_i \equiv -\frac{\partial H}{\partial Q_i} \tag{3.28}$$

It can be seen from equation (3.27) that the Hamiltonian is only bounded from below with respect to P_N, and thus is unstable over half of the classical phase space.

CHAPTER 4

LAGRANGIAN FORMULATION

In this chapter we derive field equations,first in General Relativity and then in f(R) theories.We have also discussed the Gibbons-Hawking-York ,GHY,term which is needed in order for action to remain well posed.

4.1 Field equations in GR

The Einstein field equation can be obtained from the Hilbert-Einstein action by using variational principle[13],

$$S_{EH} = \int d^4x\sqrt{-g}R \tag{4.1}$$

Here $L_G = \sqrt{-g}R$ is the Einstein Lagrangian , R is Ricci scalar and g is the determinant of the metric.Ricci scalar,R can be further written as contraction of Ricci tensor and metric,$R = g^{ab}R_{ab}$.The variation in action is given as:

$$\delta S_{EH} = \int d^4x + \left[\sqrt{-g}R_{ab}\delta g^{ab} + R\delta(\sqrt{-g}) + \sqrt{-g}g^{ab}\delta R_{ab}\right] \tag{4.2}$$

$$I \qquad II \qquad III$$

Thus we need to calculate the variation in each of the above terms,which are calculated separately here, the first term is $\sqrt{-g}R_{ab}\delta g^{ab}$ for which we need to evaluate δg^{ab}, which is given as follows:

$$g_{ad} \rightarrow g_{ad} + \delta g_{ad} \tag{4.3}$$

$$g^{ad} \rightarrow g^{ad} + \delta g^{ad} \tag{4.4}$$

Now,

$$\delta^a_c = g^{ad} g_{dc} \tag{4.5}$$

the variation in δ is,

$$\begin{aligned} &= (g^{ad} + \delta g^{ad})(g_{dc} + \delta g_{dc}) \\ &= \delta^a_c + \delta g_{dc} g^{ad} + \delta g^{ad} g_{dc} + O(\delta^2) \end{aligned} \tag{4.6}$$

and since δ^a_c is constant tensor it implies that,

$$\delta g_{dc} g^{ad} + \delta g^{ad} g_{dc} = 0 \tag{4.7}$$

contracting above term with g^{cb}

$$\delta^b_d \delta g^{ad} + g^{cb} g^{ad} \delta g_{dc} = 0 \tag{4.8}$$

$$\delta g^{ab} = -g^{cb} g^{ad} \delta g_{dc} \tag{4.9}$$

Now for variation in II term,

$$\delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}\, g_{\mu\nu} \delta g^{\mu\nu} \tag{4.10}$$

Now we need to calculate variation in Ricci tensor is,

$$\delta R_{ab} = \nabla_c(\delta \Gamma^c_{ab}) - \nabla_b(\delta \Gamma^c_{ac}) \tag{4.11}$$

$$\int \sqrt{-g}\, g^{ab} \delta R_{ab} d^4x = \int d^4x \sqrt{-g}\, g^{ab} [\nabla_c(\delta \Gamma^c_{ab}) - \nabla_b(\delta \Gamma^c_{ac}) \tag{4.12}$$

$$\nabla_c(\sqrt{-g}\, g^{ab} \delta \Gamma^c_{ab}) = \nabla_c(\sqrt{-g}\, g^{ab}) \delta \Gamma^c_{ab} + \sqrt{-g}\, g^{ab} \nabla_c(\delta \Gamma^c_{ab}) \tag{4.13}$$

$$\nabla_c(\sqrt{-g}\, g^{ab}) = 0 \tag{4.14}$$

Thus we are left with

$$\nabla_c(\sqrt{-g}\, g^{ab} \delta \Gamma^c_{ab}) = \sqrt{-g}\, g^{ab} \nabla_c(\delta \Gamma^c_{ab}) \;, \tag{4.15}$$

Similarly,

$$\nabla_b(\sqrt{-g}\, g^{ab} \delta \Gamma^c_{ac}) = \sqrt{-g}\, g^{ab} \nabla_c(\delta \Gamma^c_{ac}) \tag{4.16}$$

Thus the integration becomes

$$\int d^4x [\nabla_c(\sqrt{-g}\, g^{ab} \delta \Gamma^c_{ab}) - \nabla_b(\sqrt{-g}\, g^{ab} \delta \Gamma^c_{ac})] \tag{4.17}$$

Now using the identity,

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}V^\mu) \tag{4.18}$$

we have,

$$\int \partial_c(\sqrt{-g}g^{ab}\delta\Gamma^c_{ab} - \sqrt{-g}g^{ac}\delta\Gamma^b_{ab})d\Omega \tag{4.19}$$

Now above term is converted into a surface integral by the divergence theorem ,which vanishes as all the variations on the surface are taken to be zero.Though there is one subtle point here which we will discuss later, as for now there is no contribution from this term.The final integral is,

$$\delta S_{EH} = \int -\sqrt{-g}[R^{cd} - \frac{1}{2}Rg^{cd}]\delta g_{cd}d^4x \tag{4.20}$$

$$G^{cd} = R^{cd} - \frac{1}{2}Rg^{cd} \tag{4.21}$$

where G^{cd} is the Einstein tensor and equating variation equal to zero , we get the Einstein field equation for vacuum,

$$G^{cd} = 0 \tag{4.22}$$

In the presence of other field which can be described by an appropriate Lagrangian density L_M,the matter Lagrangian, the action becomes[13]:

$$S = \int (L_G + \kappa L_M)d^4x \tag{4.23}$$

L_G is Lagrangian for GR and $\kappa = 8\pi G/c^4$and on variation with respect to metric g_{ab} :

$$\frac{\delta L_G}{\delta g_{ab}} = -\sqrt{-g}G^{ab} \tag{4.24}$$

and

$$\frac{\delta L_M}{\delta g_{ab}} = -\sqrt{-g}T^{ab} \tag{4.25}$$

where T^{ab} is stress-energy tenor for the fields present.The field equations in presence of matter fields becomes:

$$G^{ab} = \kappa T^{ab} \tag{4.26}$$

4.1.1 The GHY term

The Ricci tensor is obtained by contracting the Riemann tensor with metric,

$$R_{\mu\nu} = g^{\lambda\beta} R_{\lambda\nu\beta\mu} = R^{\lambda}_{\mu\lambda\mu} \tag{4.27}$$

and the Riemann tensor has the following structure,

$$R^{\lambda}_{\mu\beta\nu} = \partial_\beta \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\beta} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\lambda}_{\sigma\beta} - \Gamma^{\sigma}_{\mu\beta}\Gamma^{\lambda}_{\sigma\nu} \tag{4.28}$$

And in the Connection Γ,has the following structure,which depends on the first derivative of the metric,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right] \ , \tag{4.29}$$

And by looking at the structure of equations above it's easy to see that the Ricci tensor and the Riemann tensor depends on the second derivative of the metric and thus while calculating the variation of action involves a term depending on the second derivative of the metric and in order for, $\int \partial_c (\sqrt{-g} g^{ab} \delta \Gamma^{c}_{ab} - \sqrt{-g} g^{ac} \delta \Gamma^{b}_{ab}) d\Omega$ term, to vanish on the boundary one also needs to put the variation in the first derivative of g i.e,$\partial_\mu g$ zero on the boundary,which is not traditionally put to zero and neither we have a particular reason to do so.Thus in order for Action to remain well posed ,a boundary term was added to Einstein-Hilbert action by Gibbons,Hawking and York which is the so called GHY term in order to cancel out the boundary term that comes while evaluating the variation in Ricci tensor.The new action thus becomes[5,12],

$$S_{EH} + S_{GHY} = \int_{\Omega} d^4x \sqrt{-g} R + 2 \int_{\partial\Omega} d^3 y \epsilon \sqrt{h} K \tag{4.30}$$

Here,

- h is the determinant of the induced metric h_{ab} on the boundary.
- K is the trace of the extrinsic curvature.
- ϵ is -1 for spacelike surfaces and +1 for the timelike surfaces.
- y^a are the coordinates on the boundary.

This extra boundary term leaves the Einstein Field equation invariant.We first need to define certain terms before giving the full calculation.The induced metric h_{ab} is like the metric tensor, on the hyper-surface in the y^a coordinates.A hyper-surface for 4D ,Space-Time manifold is a 3D submanifold ,which can have timelike,spacelike or null.We define a unit normal to the surface which is given as,

$$n^{\alpha} n_{\alpha} \equiv \epsilon \begin{cases} +1 & \Sigma \ is \ timelike \\ -1 & \Sigma \ is \ spacelike \end{cases}$$

Now,

$$e^{\alpha}_{a} = \left(\frac{\partial x^{\alpha}}{\partial y^{a}}\right)_{\partial M} \quad a = 1,2,3 \tag{4.31}$$

all these three are tangential to the hyper-surface and the induced metric is defined as,

$$h_{ab} = g_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b} \tag{4.32}$$

we also have,

$$n_{\alpha} e^{\alpha}_{a} = 0 \tag{4.33}$$

Now we define transverse metric as,

$$h_{\alpha\beta} = g_{\alpha\beta} - \epsilon n_{\alpha} n_{\beta} \tag{4.34}$$

and the inverse of the h_{ab} is h^{ab} and satisfies the following relation,

$$h^{\alpha\beta} = h^{ab} e^{\alpha}_{a} e^{\beta}_{b} \tag{4.35}$$

Now we will look at variation in Einstein-Hilbert action once again and assume that the $\delta g_{\alpha\beta}\big|_{\partial M} = 0$.

$$\delta S_{EH} = \int_{M} \left[\sqrt{-g} R_{\alpha\beta} \delta g^{\alpha\beta} + R\delta(\sqrt{-g}) + \sqrt{-g}\, g^{\alpha\beta} \delta R_{\alpha\beta}\right] d^4 x \tag{4.36}$$

The first two terms give us the usual Einstein Field equation,

$$\delta S_{EH} = \int_{M} \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R\right) d^4 x + \int_{M} (\sqrt{-g}\, g^{\alpha\beta} \delta R_{\alpha\beta}) d^4 x \tag{4.37}$$

Now we are going to evaluate the last term in the above equation on the boundary for which we will make use of the following identities,

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta} \delta g^{\alpha\beta} \tag{4.38}$$

$$\delta R_{\alpha\beta} \equiv \nabla_{\mu}(\delta\Gamma^{\mu}_{\alpha\beta}) - \nabla_{\beta}(\delta\Gamma^{\mu}_{\alpha\nu}) \tag{4.39}$$

Now let,

$$g^{\alpha\beta} \delta R_{\alpha\beta} = \delta V^{\mu}_{;\mu} \tag{4.40}$$

$$\delta V^{\mu} = g^{\alpha\beta} \delta\Gamma^{\mu}_{\alpha\beta} - g^{\alpha\mu} \delta\Gamma^{\beta}_{\alpha\beta} \tag{4.41}$$

We will use ';' where ever we have a Covariant derivative and ',' for the partial derivative .Now using the Stokes theorem in the following form,

$$\begin{aligned} \int_{\delta M} A^{\mu}_{;\mu} \sqrt{-g}\, d^4 x &= \int_{\delta M} (\sqrt{-g} A)_{;\mu} d^4 x \\ &= \oint_{\partial M} A^{\mu} d\Sigma_{\mu} \\ &= \oint_{\partial M} A^{\mu} n_{\mu} \sqrt{|h|}\, d^3 y \end{aligned} \tag{4.42}$$

Let $A^{\mu} = \delta V^{\mu}$,thus

$$\int_{\delta M} V^{\mu}_{;\mu}\sqrt{-g}\, d^4x = \oint_{\partial M} V^{\mu} n_{\mu}\sqrt{|h|}\, d^3y \tag{4.43}$$

Now we need to evaluate $\delta V^{\mu} n_{\mu}\big|_{\partial M}$,assuming that$\delta g_{\alpha\beta}\big|_{\partial M} = 0$,

$$\begin{aligned} \delta\Gamma^{\mu}_{\alpha\beta}\Big|_{\partial M} &= \frac{1}{2}\delta g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} + g_{\alpha\beta,\mu}) + \frac{1}{2}g^{\mu\nu}(\delta g_{\nu\alpha,\beta} + \delta g_{\nu\beta,\alpha} + \delta g_{\alpha\beta,\mu})S \\ &= \frac{1}{2}g^{\mu\nu}(\delta g_{\nu\alpha,\beta} + \delta g_{\nu\beta,\alpha} + \delta g_{\alpha\beta,\mu}) \end{aligned} \tag{4.44}$$

contracting the above equation with $g^{\alpha\beta}$ we get,

$$g^{\alpha\beta}\,\delta\Gamma^{\mu}_{\alpha\beta}\Big|_{\partial M} = \frac{1}{2}g^{\alpha\beta}g^{\mu\nu}(\delta g_{\nu\alpha,\beta} + \delta g_{\nu\beta,\alpha} + \delta g_{\alpha\beta,\mu}) \tag{4.45}$$

and similarly the term $g^{\alpha\mu}\,\delta\Gamma^{\beta}_{\alpha\beta}\Big|_{\partial M}$ is

$$g^{\alpha\mu}\,\delta\Gamma^{\beta}_{\alpha\beta}\Big|_{\partial M} = \frac{1}{2}g^{\alpha\mu}g^{\beta\nu}(\delta g_{\nu\alpha,\beta} + \delta g_{\nu\beta,\alpha} + \delta g_{\alpha\beta,\mu}) \tag{4.46}$$

Now swappingα and ν and using the symmetric property of the metric and then subtracting equation 7 from 6 we obtain,

$$\delta V^{\mu} = g^{\mu\nu}g^{\alpha\beta}(\delta g_{\nu\beta,\alpha} - \delta g_{\alpha\beta,\nu}) \tag{4.47}$$

Now we need to evaluate $\delta V^{\mu} n_{\mu}\big|_{\mathbf{\partial M}}$,

$$\begin{aligned} \delta V^{\mu} n_{\mu}\big|_{\mathbf{\partial M}} &= n_{\mu}g^{\mu\nu}g^{\alpha\beta}(\delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}) \\ &= n^{\nu}g^{\alpha\beta}(\delta g_{\nu\beta,\alpha} - \delta g_{\alpha\beta,\nu}) \end{aligned} \tag{4.48}$$

replacing ν with μ

$$\begin{aligned} \delta V^{\mu} n_{\mu}\big|_{\mathbf{\partial M}} &= n^{\mu}g^{\alpha\beta}(\delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}) \\ &= n^{\mu}(h^{\alpha\beta} + \epsilon n^{\alpha}n^{\beta})(\delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}) \\ &= n^{\mu}(h^{\alpha\beta}(\delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}) \\ &= n^{\mu}(h^{ab}e^{\alpha}_{a}e^{\beta}_{b}\delta g_{\mu\beta,\alpha} + h^{\alpha\beta}\delta g_{\alpha\beta,mu} \\ &= -n^{\mu}h^{\alpha\beta}\delta g_{\alpha\beta,\mu} \end{aligned} \tag{4.49}$$

And thus finally the variation in the Einstein Hilbert Action is,

$$\delta S_{EH} = \int_M \Big(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\Big)d^4x - \oint_{\partial M}\epsilon n^{\mu}h^{\alpha\beta}\sqrt{|h|}\delta g_{\alpha\beta,\mu}d^3y \tag{4.50}$$

The variation in the GHY term is given as,

$$\delta S_{GHY} = 2\int_{\partial M} d^3y\epsilon\sqrt{h}\delta K \tag{4.51}$$

Here K is the trace of the extrinsic curvature and it is given as covariant derivative of unit normal,

$$
\begin{aligned}
K &= n^{\alpha}_{;\alpha} \\
&= g^{\alpha\beta} n_{\alpha;\beta} \\
&= (h^{\alpha\beta} + \epsilon n^{\alpha} n^{\beta}) n_{\alpha;\beta} \\
&= h^{\alpha\beta} n_{\alpha;\beta} \\
&= h^{\alpha\beta}(n_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta} n_{\gamma})
\end{aligned} \tag{4.52}
$$

The variation in K is

$$
\begin{aligned}
\delta K &= -h^{\alpha\beta}\delta\Gamma^{\gamma}_{\alpha\beta} n_{\gamma} \\
&= -h^{\alpha\beta} n^{\mu} g_{\mu\gamma}[\frac{1}{2} g^{\gamma\sigma}(\delta g_{\sigma\alpha,\beta} + \delta g_{\sigma,\beta,\alpha} - \delta g_{\alpha\beta,\sigma})] \\
&= h^{\alpha\beta} n^{\mu} \delta g_{\alpha\beta,\mu}
\end{aligned} \tag{4.53}
$$

Thus finally we get,

$$
\delta S_{GHY} = 2\oint_{\partial M} d^3 y \epsilon \sqrt{h} h^{\alpha\beta} n^{\mu} \delta g_{\alpha\beta,\mu} \tag{4.54}
$$

which cancels out the boundary term in the Einstein -Hilbert action i.e; in the equation(3.46) and thus maintaining the action well posed.

4.2 Field equations for f(R) Theory

f(R) theory is a modified theory of gravity in which we replace Ricci scalar,R, by some function of R i.e; f(R) in the Lagrangian for GR which takes the following form [1,5,12],

$$L = \sqrt{-g}\, f(R) \tag{4.55}$$

The action is given as:

$$S = \int d^4x \sqrt{-g} f(R) \tag{4.56}$$

Now the variation of Action w.r.t metric is ,

$$\begin{aligned} \delta S &= \int [\delta(\sqrt{-g}) f(R) + \sqrt{-g}\,\delta f(R)]\, d^4x \\ &= \int \Big[-\frac{1}{2}\sqrt{-g}\, g_{\mu\nu} f(R) \delta g^{\mu\nu} + \sqrt{-g}\, f'(R)\delta R \Big] d^4x \end{aligned} \tag{4.57}$$

Here we have taken Taylor series expansion of f(R) upto first order,

$$f(R+\delta R) = f(R) + f'(R)\delta R \tag{4.58}$$

Now we need to re-evaluate expression for δR,

$$\begin{aligned} \delta R &= \delta g^{\mu\nu} R_{\mu\nu} + \nabla_\sigma \Big(g^{\mu\nu}(\delta\Gamma^{\sigma}_{\mu\nu}) - g^{\mu\sigma}(\delta\Gamma^{\gamma}_{\mu\gamma}) \Big) \\ &= \delta g^{\mu\nu} R_{\mu\nu} + g_{\mu\nu}\nabla_\sigma\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\sigma\nabla_\gamma(\delta g^{\gamma\sigma}) \\ &= \delta g^{\mu\nu} R_{\mu\nu} + g_{\mu\nu}\Box(\delta g^{\mu\nu}) - \nabla_\mu\nabla_\nu(\delta g^{\mu\nu}) \end{aligned} \tag{4.59}$$

Here we have infer equation (3.40),(3.41) and (3.42) and defined $\Box \equiv \nabla_\sigma\nabla^\sigma$.Putting the final variation for R in equation(3.53):

$$\delta S = \int d^4x \sqrt{-g} \Big[f'(R) \Big(\delta g^{\mu\nu} R_{\mu\nu} + g_{\mu\nu}\Box(\delta g^{\mu\nu}) - \nabla_\mu\nabla_\nu(\delta g^{\mu\nu}) \Big) - \frac{1}{2}\sqrt{-g}\, g_{\mu\nu} f(R)\delta g^{\mu\nu} \Big] \tag{4.60}$$

We need to further evaluate the following terms and convert them into boundary terms so that we can discard them later:

$$\int d^4x \sqrt{-g}\, f'(R) g_{\mu\nu}\Box(\delta g^{\mu\nu}) \ , \ \int d^4x \sqrt{-g}\, f'(R)\nabla_\mu\nabla_\nu(\delta g^{\mu\nu}) \tag{4.61}$$

Introducing the following two quantities:

$$A_\alpha = f'(R) g_{\mu\nu}\nabla_\alpha(\delta g^{\mu\nu}) - \delta g^{\mu\nu} g_{\mu\nu}\nabla_\alpha f'(R) \tag{4.62}$$

and

$$B^\beta = f'(R)\nabla_\gamma(\delta g^{\beta\gamma}) - \delta g^{\beta\gamma}\nabla_\gamma f'(R) \tag{4.63}$$

Now taking the covariant derivative of(3.58) and applying the compatibility condition $\nabla^{\alpha} g_{\mu\nu} = 0$

$$\begin{aligned} \nabla^{\alpha} A_{\alpha} &= \nabla^{\alpha}(f'(R) g_{\mu\nu} \nabla_{\alpha}(\delta g^{\mu\nu})) - \nabla^{\alpha}(\delta g^{\mu\nu} g_{\mu\nu} \nabla_{\alpha} f'(R)) \\ &= f'(R) g_{\mu\nu} \Box(\delta g^{\mu\nu}) - \delta g^{\mu\nu} g_{\mu\nu} \Box f'(R) \end{aligned} \tag{4.64}$$

Similarly,taking the covariant derivative of equation(3.59)

$$\begin{aligned} \nabla_{\beta} B^{\beta} &= \nabla_{\beta}(f'(R) \nabla_{\gamma}(\delta g^{\beta\gamma})) - \nabla_{\beta}(\delta g^{\beta\gamma} \nabla_{\gamma} f'(R)) \\ &= f'(R) \nabla_{\sigma} \nabla_{\beta}(\delta g^{\sigma\beta}) - \delta g^{\sigma\beta} \nabla_{\sigma} \nabla_{\beta} f'(R) \end{aligned} \tag{4.65}$$

substituting equation(3.60) and (3.61) in equation (3.56) one obtains the following form:

$$\begin{aligned} \delta S = &\int d^4x \sqrt{-g} \Big[f'(R) \delta g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} \sqrt{-g}\, g_{\mu\nu} f(R) \delta g^{\mu\nu} + \delta g^{\mu\nu} g_{\mu\nu} \Box f'(R) + \delta g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} f'(R) \Big] \\ &+ \int d^4x \sqrt{-g} (\nabla^{\alpha} A_{\alpha} + \nabla_{\beta} B^{\beta}) \end{aligned} \tag{4.66}$$

Using the Stokes theorem(3.38) and re-writing the last two terms as surface derivatives ,the above action becomes:

$$\begin{aligned} \delta S = &\int d^4x \sqrt{-g} \Big[f'(R) \delta g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} \sqrt{-g}\, g_{\mu\nu} f(R) \delta g^{\mu\nu} + \delta g^{\mu\nu} g_{\mu\nu} \Box f'(R) + \delta g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} f'(R) \Big] \\ &+ \oint d^3y \epsilon \sqrt{|h|}\, n^{\alpha} A_{\alpha} + + \oint d^3y \epsilon \sqrt{|h|}\, n_{\beta} B^{\beta} \end{aligned} \tag{4.67}$$

where h,n,ϵ has their usual meanings as defined before in section(3.1.1).Now we need to evaluate A_{α} and B^{β} at the boundary.Since have fix the variation of the metric at the boundary, i.e, $\delta g_{\mu\nu} = 0$,we obtain the following expression for $A_{\alpha} n^{\alpha}$ and $B^{\beta} n_{\beta}$:

$$\begin{aligned} A_{\alpha} n^{\alpha} \big|_{\partial \mathbf{M}} &= -f'(R) n^{\alpha} (\epsilon n^{\mu} n^{\nu} + h^{\mu\nu}) \partial_{\alpha}(\delta g_{\mu\nu}) \\ &= -f'(R) n^{\alpha} h^{\mu\nu} \partial_{\alpha}(\delta g_{\mu\nu}) \end{aligned} \tag{4.68}$$

$$\begin{aligned} B^{\beta} n_{\beta} \Big|_{\partial \mathbf{M}} &= -f'(R) n_{\beta} (h^{\beta\nu} + \epsilon n^{\beta} n^{\mu})(h^{\gamma\mu} + \epsilon n^{\gamma} n^{\mu}) \partial_{\gamma}(\delta g_{\mu\nu}) \\ &= -f'(R) n^{\mu} h^{\gamma\nu} \partial_{\gamma}(\delta g_{\mu\nu}) \\ &= 0 \end{aligned} \tag{4.69}$$

finally , the equation (3.63) becomes:

$$\begin{aligned} \delta S = &\int d^4x \sqrt{-g} \Big[f'(R) \delta g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} \sqrt{-g}\, g_{\mu\nu} f(R) \delta g^{\mu\nu} + \delta g^{\mu\nu} g_{\mu\nu} \Box f'(R) + \delta g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} f'(R) + \Big] \\ &- \oint_{\partial \Omega} d^3y \epsilon f'(R) n^{\alpha} h^{\mu\nu} \partial_{\alpha}(\delta g_{\mu\nu}) \end{aligned} \tag{4.70}$$

Now in order to kill the surface derivatives in equation(3.66) one needs to take into account the GHY term for f(R),which is given as:

$$S^{f}_{GHY} = -2 \oint_{\partial \Omega} d^3y \epsilon f'(R) \sqrt{|h|} K \tag{4.71}$$

now the variation of above term is:

$$\delta S^{f}_{GHY} = 2\oint_{\partial\Omega} d^3y\epsilon\sqrt{|h|}f''(R)\delta RK + \oint_{\partial\Omega} d^3y\epsilon f'(R)n^{\alpha}h^{\mu\nu}\partial_{\alpha}(\delta g_{\mu\nu}) \tag{4.72}$$

where the second term cancels the surface term in the equation(3.66) and for the first term to vanish we need to impose that $\delta R = 0$. Thus finally the field equations for f(R) is:

$$f'(R)R_{\mu\nu} - \nabla_{\mu}\nabla_{\nu}f'(R) + g_{\mu\nu}\Box f'(R) - \frac{1}{2}g_{\mu\nu}f(R) = 0 \tag{4.73}$$

Now if we equate f(R)=R , $\frac{f(R)}{dR}$ = 1 and one recovers Einstein's field equations for vacuum,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{4.74}$$

Chapter 5

Equivalent scalar representation

The general Lagrangian of f(R) theory is conformally equivalent to general relativity plus a scalar field matter source and a self-interacting potential of a particular form.We derive what form this potential has and also what are some general f(R) theories corresponding to a particular potential. The Action for general relativity plus an ordinary scalar field is,

$$\begin{aligned} S &= \int d^4x\sqrt{-g}[R - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)] \\ &= \int d^4x\sqrt{-g}[R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)] \end{aligned} \tag{5.1}$$

On varying the above action with respect to metric we have

$$\delta S = \int d^4\left[R\delta(\sqrt{-g}) + \sqrt{-g}\delta R - \frac{1}{2}\delta(\sqrt{-g})g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}\delta(g^{\mu\nu})\sqrt{-g}\nabla_\mu\phi\nabla_\nu\phi - \delta(\sqrt{-g})V(\phi)\right] \tag{5.2}$$

Substituting value of $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta(g^{\mu\nu})$, we get following scalar field equations:

$$G_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - g_{\mu\nu}V(\phi) \tag{5.3}$$

Taking the following conformal transformation [6]:

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}\ ,\ \ f'(R) = \Omega^2 \tag{5.4}$$

where Ω is a smooth, non-vanishing function of space-time,is a point independent rescaling of the metric and is called a conformal factor. It preserves the causal structure of the manifold, by preserving angle between vectors.

The field equation for f(R) Lagrangian is:

$$f'(R)(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) - \frac{1}{2}(f(R) - Rf'(R))g_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box f'(R) = 0 \tag{5.5}$$

Here $f'(R) \equiv \frac{df}{dR}$.Under the above confromal transformation the Ricci scalar and the Ricci tensor has the following form for D=4, where D is the dimension[6]:

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{3}{2}\frac{1}{\Omega^4}\nabla_\mu\Omega^2\nabla_\nu\Omega^2 - \frac{1}{\Omega^2}\nabla_\mu\nabla_\nu\Omega^2 - \frac{1}{2}g_{\mu\nu}\frac{1}{\Omega^2}\Box\Omega^2 \tag{5.6}$$

$$\tilde{R} = \frac{1}{\Omega^2}(R + \frac{3}{2}\frac{1}{\Omega^4}\nabla_\alpha\Omega^2\nabla^\alpha\Omega^2 - 3\frac{1}{\Omega^2}\Box\Omega^2) \tag{5.7}$$

Replacing Ω^2 with f' in the equation (4.6) and (4.7) and rewriting them, they take the following form:

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{3}{2}\frac{1}{f'^2}\nabla_\mu f'\nabla_\nu f' - \frac{1}{f'}\nabla_\mu\nabla_\nu f' - \frac{1}{2}g_{\mu\nu}\frac{1}{f'}\Box f' \tag{5.8}$$

$$\tilde{R} = \frac{1}{f'}(R + \frac{3}{2}\frac{1}{f'^2}\nabla_\alpha f'\nabla^\alpha f' - 3\frac{1}{f'}\Box f') \tag{5.9}$$

The rescaled Einstein tensor is:

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \frac{3}{2}\frac{1}{f'^2}\nabla_\mu f'\nabla_\nu f' - \frac{1}{f'}\nabla_\mu\nabla_\nu f' - \frac{3}{4}\frac{1}{f'^2}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha f'\nabla_\beta f' + \frac{1}{f'}g_{\mu\nu}\Box f' \tag{5.10}$$

Now substituting value of $R - \frac{1}{2}Rg_{\mu\nu}$ from equation (4.5) in equation (4.10) we get,

$$\tilde{G}_{\mu\nu} = -\frac{3}{2}\frac{1}{f'^2}\nabla_\mu f'\nabla_\nu f' - \frac{3}{4}\frac{1}{f'^2}\tilde{g}_{\mu\nu}\tilde{g^{\alpha\beta}}\nabla_\alpha f'\nabla_\beta f' + \frac{1}{2}\frac{1}{f'^2}(f - Rf')\tilde{g}_{\mu\nu} \tag{5.11}$$

Introducing the scalar field ϕ,

$$\phi = \sqrt{\frac{2}{3}}\, ln(f'(R)) \tag{5.12}$$

The conformally transformed equations are:

$$\tilde{G}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}\tilde{g}_{\mu\nu}(\nabla_\alpha\phi\nabla^\alpha\phi) - \tilde{g}_{\mu\nu}V(\phi) \tag{5.13}$$

which are the field equations for general relativity plus a scalar field matter source and a self interacting potential of the form $V(\phi) = \frac{1}{2}\frac{1}{f'^2}(f - Rf')$. Now for various form of f(R) and we have its corresponding form of potentials[9]. For $f(R) = R$ one finds the potential to be, $V(\phi) = 0$ and for $V(\phi) = C$, constant one finds $f(R) = \frac{1}{8C}R^2$, and for $V(\phi) = \lambda\phi^n$ where$n = 2, 3, 4$ and putting $p = \Omega^2$, we get $V(p) = \mu(lnp)^n$, where $\mu \equiv (\frac{3}{2})^{n/2}\lambda$ and corresponding $f[r(p)] = 2\mu p^2(lnp + n)(lnp)^{n-1}$, where $r(p) = 2\mu p(2lnp + n)(lnp)^{n-1}$.

Chapter 6

Spherically symmetric solution

In this chapter we look at the spherically symmetric solution in f(R) gravity via Noether Symmetry Approach.We look at a particular class of f(R) theories of form, R^s, where s is a real number and R is Ricci scalar.[10,11]

6.1 Noether Symmetry Approach

If we have the following Euler-Lagrange equation :

$$\partial_\mu \frac{\partial L}{\partial(\partial_\mu q^j)} - \frac{\partial L}{\partial q^j} = 0 \tag{6.1}$$

and on contracting the above equation by $\alpha^j(q)$, a variable,

$$\alpha^j \Big(\partial_\mu \frac{\partial L}{\partial(\partial_\mu q^j)} - \frac{\partial L}{\partial q^j}\Big) = 0 \tag{6.2}$$

we can further convert the first term into a surface term :

$$\alpha^j \partial_\mu \frac{\partial L}{\partial(\partial_\mu q^j)} = \partial_\mu \Big(\alpha^j \frac{\partial L}{\partial(\partial_\mu q^j)}\Big) - \partial_\mu \alpha^j \frac{\partial L}{\partial(\partial_\mu q^j)} \tag{6.3}$$

We get the following equation by substituting (5.3)in (5.2),

$$\partial_\mu \Big(\alpha^j \frac{\partial L}{\partial(\partial_\mu q^j)}\Big) - \partial_\mu \alpha^j \frac{\partial L}{\partial(\partial_\mu q^j)} - \frac{\partial L}{\partial q^j} = 0 \tag{6.4}$$

Defining vector X to be,

$$X = \alpha^j \frac{\partial}{\partial q^j} + (\partial_\mu \alpha^j) \frac{\partial}{\partial(\partial_\mu q^j)} \tag{6.5}$$

Equation (5.4) takes the form,

$$\partial_\mu\Big(\alpha^j \frac{\partial L}{\partial(\partial_\mu q^j)}\Big) = \mathbf{L}_X L \tag{6.6}$$

where $\mathbf{L}_X$ is the Lie derivative of L w.r.t vector X, if $\mathbf{L}_X L = 0$, the term on L.H.S. is conserved.

$$\alpha^j \frac{\partial L}{\partial(\partial_\mu q^j)} = J^\mu \tag{6.7}$$

where J^μ is a constant.

6.2 Point-Like Lagrangian

One can find the spherically symmetric solutions for the f(R) theory by formulating a point-like Lagrangian from the following action:

$$S = \int d^4x\sqrt{-g}\,f(R) \tag{6.8}$$

by putting spherically symmetric constraints, for which the line element is given as:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + M(r)(d\theta + sin^2\theta d\phi) \tag{6.9}$$

Now we want to recast the action in(5.8) to the action of the form:

$$S = \int dr L(A, A', B, B', M, M', R, R') \tag{6.10}$$

where the configuration space is described by $\mathbb{Q} = (A, B, M, R)$.In order to recast the action we will make use the concept of Lagrange Multiplier,

$$S = \int d^4x\sqrt{-g}\Big(f(R) - \lambda(R - \bar{R})\Big) \tag{6.11}$$

here λ is the Lagrange Multiplier and on varying the action(5.11) with respect to δR one finds that,

$$\lambda = f_R \ \ , \ \ f_R \equiv \frac{df}{dR} \ \ , \ \ f(R) \equiv f$$

and $\bar{R}$ is the Ricci scalar calculated from (5.9), which has the following form:

$$\bar{R} = R^* + \frac{A''}{AB} + \frac{2M''}{BM} \tag{6.12}$$

$$R^* = \frac{A'M'}{ABM} - \frac{A'^2}{2A^2B} - \frac{M'^2}{2BM^2} - \frac{A'B'}{2AB^2} - \frac{B'M'}{B^2M} - \frac{2}{M} \tag{6.13}$$

here prime donates the derivative with respect to the radial coordinate.Now after substituting for $\bar{R}$ and λ the action becomes:

$$\begin{aligned} S &= \int drM\sqrt{A}\sqrt{B}\Big(f - f_R(R - R^*)\Big) \\ &= \int drM\sqrt{A}\sqrt{B}\Big(f - f_R(R - \bar{R} - f_R(\frac{A''}{AB} + \frac{2M''}{BM}))\Big) \\ &= \int drM\sqrt{A}\sqrt{B}\Big(f - f_R(R - \bar{R} - \Big(\frac{f_R M}{\sqrt{A}\sqrt{B}}\Big)' A' - 2\Big(\frac{\sqrt{A}}{\sqrt{B}}f_R\Big)' M'\Big) \end{aligned} \tag{6.14}$$

where the second term differs from third by a divergence term, which is discarded as we want only first order derivative terms in the Lagrangian, which is given as:

$$\left(\frac{f_R M A'}{\sqrt{A}\sqrt{B}}\right)' = \left(\frac{f_R M}{\sqrt{A}\sqrt{B}}\right)' A' + \frac{f_R M A''}{\sqrt{A}\sqrt{B}} \tag{6.15}$$

$$\left(\frac{\sqrt{A}}{\sqrt{B}} f_R M'\right)' = \left(\frac{\sqrt{A}}{\sqrt{B}} f_R\right)' M' + \frac{\sqrt{A}}{\sqrt{B}} f_R M'' \tag{6.16}$$

Now expanding the term $\left(\frac{f_R M}{\sqrt{A}\sqrt{B}}\right)' A'$:

$$\left(\frac{f_R M}{\sqrt{A}\sqrt{B}}\right)' A' = \frac{f_{RR} R' M A'}{\sqrt{A}\sqrt{B}} + \frac{f_R M' A'}{\sqrt{A}\sqrt{B}} + \frac{f_R M A'^2}{2A\sqrt{A}\sqrt{B}} - \frac{f_R M B' A'}{2B\sqrt{A}\sqrt{B}} \tag{6.17}$$

Expanding the term $\left(\frac{\sqrt{A}}{\sqrt{B}} f_R\right)' M'$:

$$\left(\frac{\sqrt{A}}{\sqrt{B}} f_R\right)' M' = \frac{2\sqrt{A} f_{RR} R' M'}{\sqrt{B}} + \frac{f_R M' A'}{\sqrt{A}\sqrt{B}} - \frac{f_R M' B' \sqrt{A}}{B\sqrt{B}} \tag{6.18}$$

Substituting these in the action(5.14) we get :

(6.19)

$$S = \int dr M\sqrt{A}\sqrt{B}\Big(-\frac{\sqrt{A} f_R M'^2}{2M\sqrt{B}} - \frac{f_R}{\sqrt{A}\sqrt{B}} A'M' - \frac{M f_{RR}}{\sqrt{A}\sqrt{B}} A'R' - 2\frac{\sqrt{A} f_{RR}}{\sqrt{B}} R'M' - \sqrt{A}\sqrt{B}[(2+MR)f_R - Mf]\Big)$$

The point-like Lagrangian is :

$$L = -\frac{\sqrt{A} f_R M'^2}{2M\sqrt{B}} - \frac{f_R}{\sqrt{A}\sqrt{B}} A'M' - \frac{M f_{RR}}{\sqrt{A}\sqrt{B}} A'R' - 2\frac{\sqrt{A} f_{RR}}{\sqrt{B}} R'M' - \sqrt{A}\sqrt{B}[(2+MR)f_R - Mf] \tag{6.20}$$

Since the Lagrangian(5.20) is independent of B the Euler-Lagrange equation for B gives:

$$\frac{\partial L}{\partial B} = 0 \tag{6.21}$$

and by evaluating the above equation and equating it equal to 0

(6.22)

$$\frac{\partial L}{\partial B} = \frac{\sqrt{A} f_R M'^2 \sqrt{B}}{4M} + \frac{f_R A' M' \sqrt{B}}{2\sqrt{A}} + \frac{M f_{RR} A' R' \sqrt{B}}{2\sqrt{A}} + \sqrt{A} f_{RR} R' M' \sqrt{B} - \frac{\sqrt{A}}{2\sqrt{B}}[(2+MR)f_R - Mf]$$

the value of B is calculated to be:

$$B = \frac{2M^2 f_{RR} A' R' + 2M f_R A' M' + 4AM f_{RR} M' R' + A f_R M'^2}{2M[(2+MR)f_R - Mf]} \tag{6.23}$$

substituting this value back to equation (5.20), the Lagrangian is simplified to:

$$L = \frac{[(2+MR)f_R - Mf]}{M}[2M^2 f_{RR} A' R' + 2MM'(f_R A' + 2A f_{RR} R') + A f_R M'^2] \tag{6.24}$$

and the equation(5.24) is the general point-like Lagrangian for f(R).For f(R)=R, the above Lagrangian is that of GR[10]:

$$L_{GR} = 4M'RA' + \frac{2ARM'^2}{M} \tag{6.25}$$

and value of B is:

$$B_{GR} = \frac{M'^2}{4M} + \frac{A'M'}{2A} \tag{6.26}$$

Looking at the Lagrangian for GR we find that degree of freedoms are further reduced to only two i.e, on A and M only.Evaluating the equation of motion for A

$$\frac{\partial L}{\partial A} = \frac{2RM'^2}{M} \quad , \quad \frac{\partial L}{\partial A'} = 4M'R \quad , \quad \frac{\partial}{\partial r}\frac{\partial L}{\partial A'} = 4M''R + 4M'R' \tag{6.27}$$

$$\frac{2RM'^2}{M} = 4M''R + 4M'R' \tag{6.28}$$

Similarly Equation motion for M gives:

$$4A''M^2 + 4A'M'M + 4AM''M - 2AM'^2 = 0 \tag{6.29}$$

and Equation motion for R gives:

$$\frac{2A'}{A} = -\frac{M'}{M} \tag{6.30}$$

If one takes the following value of A,B and M

$$A = c_1 - \frac{c_2}{r + c_3} \quad , \quad B = \frac{c_1 c_4}{A} \quad , \quad M = c_4(r + c_3) \tag{6.31}$$

Differential equations of (5.28),(5.29) and(5.30) are satisfied.For $c_2 = 2GM$, $c_1 = 1$, $c_3 = 0$ and $c_4 = 1$, one obtains the Schwarzschild solution.The quantity $\mathbf{L}_X L$ for L_{GR} is:

$$\alpha_1 \frac{\partial L}{\partial A} + \partial_r \alpha_1 . \frac{\partial L}{\partial A'} + \alpha_2 \frac{\partial L}{\partial M} + \partial_r \alpha_2 . \frac{\partial L}{\partial M'} = 0 \tag{6.32}$$

The values of α for which above equation is satisfied are:

$$\alpha_{GR} \equiv (-kA, kM) \tag{6.33}$$

where k is constant of integration.The conserved charge or constant of motion is obtained from boundary term is

$$\begin{aligned} \Sigma_\circ &= \alpha . \nabla_{q'} L \\ &= \frac{2GM}{c^2} \end{aligned} \tag{6.34}$$

which is Schwarzschild radius. Similarly for general $f(R) = R^s$ the value of $\alpha s'$ are:

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) = ((3-2s)kA, -kM, kR) \tag{6.35}$$

and the corresponding constant of motion is

$$\Sigma_\circ = 2skMR^{2s-3}[2s + (s-1)MR][(s-2)RA' - (2s^2 - 3s + 1)AR'] \tag{6.36}$$

For $f(R) = R^2$ i.e, scalar field in the presence of the cosmological constant, the Lagrangian in (6.24) takes the following form:

$$L = \frac{(4R + MR^2)}{M}[4M^2A'R' + 4MM'RA' + 8MM'AR' + 2M'^2AR] \tag{6.37}$$

$f_R = 2R$ and $f_{RR} = 2$. The value B (6.23) is:

$$B = \frac{4M^2A'R' + 4MRA'M' + 8MAM'R' + 2ARM'^2}{2M(4R + MR^2)} \tag{6.38}$$

the corresponding value of $\alpha s'$ (6.35) are:

$$\alpha \equiv (-kA, -kM, kR) \tag{6.39}$$

and the constant of motion has the following form:

$$\Sigma_\circ = 4kMR(4 + 3MR)(2RA' - 3AR') \tag{6.40}$$

Chapter 7

Results and Conclusions

We obtained the field equations for f(R) theories and looked at the role played by GHY term to keep the action well posed.We studied the equivalence between f(R) theories and the scalar field theory.We looked at the constant of motions obtained, for spherically symmetric case, which are obtained from Boundary terms.The results from each chapter are summarized below:

- In chapter 2, we got the equation for Lagrangian depending on Higher order derivative terms and also a structure for Noether Charge.
- In chapter 3, we studied Theorem of Ostrogradsky and argued why we need Ostrogradsky construction .
- In chapter 4, we derived field equations for GR and f(R) and studied importance of GHY term for both of them respectively.
- In chapter 5, we studied the equivalence between f(R) theories and the scalar field theory.
- In chapter 6, we looked at Spherically symmetric solutions of f(R) via Noether symmetry approach.

We learned why the Lagrangian for a physical theory contains terms upto first order derivative only and why Lagrangian containing higher order derivative terms are unstable . We also learned the importance of boundary terms in theories such as f(R) and GR, and that they should not be taken for granted.For future research we would like to study $f(R) = R^2$ i.e, a scalar field in the presence of cosmological constant and its dynamics in more detail.

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