INTERACTING URN MODELS

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A dissertation submitted for the partial fulfillment of BS-MS dual degree in Science

Indian Institute of Science Education and Research Mohali
April 2018

Certificate of Examination

This is to certify that the dissertation titled **"Interacting Urn Models"** submitted by **Somya Singh** (Reg. No. MS13097) for the partial fulfillment of BS-MS dual degree program of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Neeraja Sahasrabudhe at the Indian Institute of Science Education and Research Mohali.
This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Neeraja Sahasrabudhe
(Thesis Supervisor).

Acknowledgement

First and foremost, I would like to thank my thesis supervisor Dr.Neeraja Sahasrabudhe, without whose help and supervision, this thesis would have never been possible. The discussions that I had with her has enhanced my capabilities at a professional level. I would like to thank Dr.Lingaraj Sahu to guide me in my summer and winter projects. Also I would like to thank my thesis committee members Prof.Kapil Hari Paranjape and Dr.Amit Kulshrestha for their valuable suggestions and criticism of my work.

I am highly grateful to the Deputy Librarian, Dr. Vishaki, Library information assistant Mr. Shameer K. K. for their promptness and assistance.

I would like to thank Suhani, Aatashi, Rhythm and all other friends for supporting and helping me during the entire period of the project. I am also thankful to my seniors for their useful advice and help. I would like to acknowledge the moral support and encouragement, that I have received from my parents and brother.

Finally, I am thankful to IISER Mohali for providing me infrastructure and computer centre for all the technical support. I would like to acknowledge DST INSPIRE, Government of India for the financial support.

Somya Singh

Abstract

We study interacting two-color urn models. We consider N interacting two-colour Pólya and/or Friedman urns. Each urn i assigns a weight vector $\tilde{p}_i$ to all the other urns. At each time step, all the urns are updated simultaneously according to the Friedman scheme (or the Pólya scheme) such that the reinforcement probabilities for a given color in urn i depend on the vector $\tilde{p}_i$ and the fraction of balls of that color across all the N urns. An interesting characteristic in study of the interacting urn models of these kinds is the possibility of synchronization (common limiting distribution of the fraction of balls of each color) of all the urns as $t \to \infty$. We obtain expressions for the rate of synchronization in our model. We also use stochastic approximation and stable convergence techniques to further study our model and prove fluctuation results.

Introduction

Random Processes is a study of randomly evolving systems with certain properties. Urn models are special case of random processes with reinforcement where at each discrete time step the state of the system is reinforced via a randomly evolving process. In recent times, randomly evolving systems that interact with each other have gained interest for research in Mathematics, Physics, Computer Science etc. In this thesis, we study a model of interacting Friedman and Pólya urns.

As mentioned above, urn models are an important class of Random reinforcement processes. The idea of urn model problems was first given by Jacob Bernoulli in 1713 in his famous book *Ars Conjectandi*(The Art of Conjecturing) [4]. He considered the problem of determining proportion of different coloured pebbles in the urn after drawing some pebbles from the urn. This problem is known as the **Inverse Probability Problem**. The classical Pólya urn model was proposed by G. Pólya in 1923 and is defined as follows: Consider an urn with balls of finite number of colors. At any give discrete time t, a ball is drawn from the urn (with replacement) uniformly at random and its colour is observed. Another ball of the same colour is then added to the urn. This reinforcement is carried out at every time-step t. Asymptotic properties of this urn process have been of interest in several areas including modeling epidemic spread. The simplest case is to study an urn with balls of two colours, namely, white and black. The most well-known result for two-colour Pólya urns establishes that the fraction of balls of either color converges to a random limit as $t \to \infty$. The distribution of the random limit is given by beta distribution with parameters given by the initial state of the system. Numerous studies have been done to generalize this simple model. One of the earliest known extensions of Pólya urns was given by Bernard Friedman in 1949. In a two-colour Friedman urn model at time t

the ball drawn from the urn (again, uniformly at random) is replaced along with α balls of the same colour and β balls of another color. A celebrated result states that as long as both α and β are strictly positive (and the urn has non-zero number of white and black balls in the beginning), the fraction of balls of each colour converges to a deterministic limit of 1/2.

In recent times, single urn models have been extended to understand the more complicated phenomenon of interacting or dependent random processes as many system that evolve at multiple nodes with certain dependence in the reinforcement step can be modelled using such interacting processes. In particular, dynamics of interacting urns has become a topic of interest for many researchers in Applied Mathematics as well as interdisciplinary areas like Network Science. Urn models are used in understanding epidemic spread, opinion dynamics etc. In [10], authors study network epidemics where the opinions are modelled as balls in the urns placed at each node and the opinion update depends on a ball is drawn from the "super urn" which consists of all the balls from the urn and its neighbours. In [7], pair-wise interaction of neighbours on a graph are taken into account. In [12] the interacting urns have a reinforcement scheme which is exponential. [9] takes into account random step sizes in stochastic approximation schemes for urn models. [18] is a P.h.D. dissertation on Pólya urn models with Countably infinite number of colors.

In this thesis, we study a model of N interacting two-colour urns such that the probability of adding α white balls (and β black balls) to each urn at time-step t depends on weighted fraction of white balls across all urns. In other words, we associate a non-zero weight vector $\widetilde{p}_i = (p_i^1, \ldots, p_i^N)$ to each urn U_i for $1 \leq i \leq N$. At any time t, α balls of white colour (β balls of black colour) are added to i^{th} urn U_i with probability given by the inner product of the weight vector of U_i and the vector of fraction of white balls in each urn at time t. The motivation for this model comes the interacting Pólya urn model of P. Dai Pra *et. al* [16], where an interacting system of Pólya urns is considered such that the weightage given to total fraction of white balls in the system at time t is p and that of i^{th} urn is $(1-p)$ for a fixed $0 \leq p \leq 1$. To some extent, our work in this thesis generalizes the results in [6],[16] and [17].
It is expected that in such interacting systems, each component synchronizes to the same "composition" as $t \to \infty$. The aim is to understand the asymp-

totic limits of fraction of balls of white colour by studying various properties of the interacting model. For single urn models as well as for interacting urns, several probabilistic techniques like the method of moments, the martingale method, stochastic approximation etc. are used to study the convergence and fluctuation properties. We discuss these methods throughout the thesis as and when required. For a detailed study of methods used in single urn models and the details on the connection of urn models to random walks see [15].

The thesis is organized as follows: In chapter 1, we discuss some basics of single urn models and results for limiting distributions of Friedman and Pólya urns using method of moments [8] and Exchangeability of Random Variables [14] respectively.In chapter 2, we introduce the model and compare it with the interacting urn model in [16] and [17]. We also obtain the rate of convergence for synchronization. Chapter 3 consists of fluctuation results obtained using the theory of stochastic approximation as well as ideas from stable convergence. Chapter 4 briefly discusses the embedding of urn model into a continuous time branching process introduced by K. B. Athreya and S. Karlin [2].

The appendix consists of some auxiliary results on solving difference equations and the notion of stable convergence. In the appendix, we also define and briefly discuss few other concepts mentioned in the thesis.

The results obtained in this thesis are an extension or a more general form of existing results on a class of interacting Pólya and Friedman urns. These models can be further extended by considering random reinforcements, dependence of interaction parameter or interaction matrix on time and/or dependence on the underlying graph structure. All of these are interesting problems and are possible directions to explore in future.

Contents

Chapter 1

Prerequisites

The simplest urn model consists of a single urn with balls of two colors. At a given time t, a ball is drawn out of the urn and depending on the color of this ball, some balls are added or subtracted from the urn. This process of drawing and adding balls to the urn is a **Stochastic process** [4.6.1] with ratio of balls of each color as the **State Space** and discrete time as **Parameter Space**. We will now discuss some general terminologies and notations used for urn models

1.0.1 Urn Scheme

Consider an urn with balls of k colors (or a k-color urn). Let C_t^i denote the number of balls of color i in the urn at time t and $C_t = (C_t^1, \dots C_t^k)$ denote the corresponding vector. Clearly, $\sum_{i=1}^{k} C_t^i$ is the total number of balls in the urn at time t. An urn scheme is defined as follows:

Definition 1.0.1 (Urn Scheme)**.** *An Urn Scheme is a $k \times k$ matrix $(a_{ij})_{1 \leq i,j \leq k}$ in which the entry a_{ij} denotes the number of balls of color j that are added to the urn, when a ball of color i is drawn out of the urn.*

The matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & . & . & a_{1k} \\ a_{21} & a_{22} & . & . & . & . & a_{2k} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{k1} & . & . & . & . & . & a_{kk} \end{bmatrix}$$

is also known as the reinforcement matrix of the urn model. Then, the reinforcement of the model is given by:

$$C_{t+1} = AC_t$$

Assume that each entry of this matrix is an integer. If any of the entries is negative, then modulus of that many balls has to removed from the urn. Throughout this thesis, we limit our discussion to two-color urns. Results obtained can be extended to urns with balls of finitely many different colors.

1.0.2 Tenability Conditions

Suppose we have a 2 color urn containing white and black balls with urn scheme

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

Then no matter which color is drawn out of the urn, a white ball will always be removed from the urn, which will ultimately lead to extinction of white balls from the urn. At this point the urn scheme cannot be further executed. Such an urn is called **Untenable**.

To ensure that every stochastic path is possible for the urn, entries of the urn scheme must follow certain **tenability conditions**.

We limit our discussion on tenability to 2-color models. Consider a 2×2 urn scheme for an urn containing black and white balls.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The tenability conditions of this urn can easily be calculated.

Tenability conditions for various urn models are discussed in detail in Chapter 2 of [14]. We reproduce a summary of that discussion below.

- The cases

  $$\begin{bmatrix} - & - \\ - & - \end{bmatrix} \begin{bmatrix} - & - \\ - & + \end{bmatrix} \begin{bmatrix} - & + \\ - & + \end{bmatrix} \begin{bmatrix} + & - \\ + & - \end{bmatrix}$$

  are not tenable under any conditions since they have negative columns.

- $\begin{bmatrix} - & - \\ + & + \end{bmatrix}$

  An urn is said to be in a "Critical State" if balls of one color are completely exempted from the urn. Let us say the urn has W_0 white balls and B_0 black balls. One of the "most critical path" is to draw a white ball whenever possible.For such a path W_0 must be a multiple of $|a|$.At this stage the Urn must contain at least one blue ball for the continuation of the process. This gives

$$B_0 - W_0\frac{b}{a} > 0$$

  The next critical stage is when number of white balls in the urn is again 0(Hence, W_0 must be a multiple of c). At this stage we must have non zero blue balls in the urn i.e.

$$B_0 - W_0\frac{b}{a} + 2d - c\frac{b}{a} > 0$$

  the urn must be tenable at each i^{th} critical stage. Hence,

$$B_0 - W_0\frac{b}{a} + (i-1)d - (i-1)c\frac{b}{a} > 0 \qquad \text{for all} \quad i \geq 1$$

  Or

$$B_0 - W_0\frac{b}{a} > (i-1)\left(c\frac{b}{a} - d\right) \qquad \text{for all} \quad i \geq 1$$

  Since i is arbitrary large and LHS is positive, we must have

$$\frac{cb}{a} \leq d$$

  Summarizing, urn of this case is tenable only if:

  1. W_0 and c are both multiples of $|a|$.
  2. $det(A) \leq 0$
  3. $det\begin{vmatrix} a & b \\ W_0 & B_0 \end{vmatrix} < 0$

  The case $\begin{bmatrix} + & + \\ - & - \end{bmatrix}$ is symmetrical to this case.

Similarly one can calculate tenability conditions for other cases. For detailed discussion see [14].

- $\begin{bmatrix} - & + \\ + & - \end{bmatrix}$

  1. W_0 and c are both multiples of $|a|$.
  2. B_0 and b are both multiples of $|d|$.
  3. Both b and c are positive.

- $\begin{bmatrix} - & + \\ + & + \end{bmatrix}$

  W_0 and c must be multiple of $|a|$ and if $b = 0$, B_0 must be positive.

- $\begin{bmatrix} + & + \\ - & + \end{bmatrix}$
  $c = 0$ and $W_0 = 0$

1.1 The Pólya Eggenberger Urn

G. Pólya and F. Eggenberger first studied 2×2 Urn scheme of the type

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

i.e. if a white ball is drawn out of the urn, we put another α white balls in the urn along with this ball and vice-versa if a black ball is drawn.

We will now look at the distribution of white balls in the urn as $t \to \infty$

Theorem 1. *(Theorem 3.2 in [14])*
Let W_0 and B_0 be the number of white and black balls respectively in Pólya urn at time $t = 0$. Let $\widetilde{W_n}$ be the number of times a white ball is drawn in n drawings from the urn.

$$\frac{\widetilde{W_n}}{n} \xrightarrow{\mathcal{D}} \beta\left(\frac{W_0}{s}, \frac{B_0}{s}\right)$$

Proof. (Sketch of the proof)

We can choose k white balls from any of the n time instants. At rest of the times,a black ball is chosen.

$$P(\widetilde{W_n} = k) = \binom{n}{k} \frac{\left\langle \frac{W_0}{s} \right\rangle_k \left\langle \frac{B_0}{s} \right\rangle_{n-k}}{\left\langle \frac{\tau_0}{s} \right\rangle_n}$$

By Stirling's approximation we have; as $x \to \infty$

$$\frac{\Gamma(x+r)}{\Gamma(x+s)} = x^{r-s} + O(x^{r-s-1})$$

So as $n \to \infty$

$$P\left(\frac{\widetilde{W_n}}{n} \leq x\right) \to \frac{\Gamma\left(\frac{\tau_0}{s}\right)}{\Gamma\left(\frac{W_0}{s}\right)\Gamma\left(\frac{B_0}{s}\right)} \int_0^x u^{\frac{W_0}{s}-1}(1-u)^{\frac{B_0}{s}-1} du$$

$$= \beta\left(\frac{W_0}{s}, \frac{B_0}{s}\right)$$

□

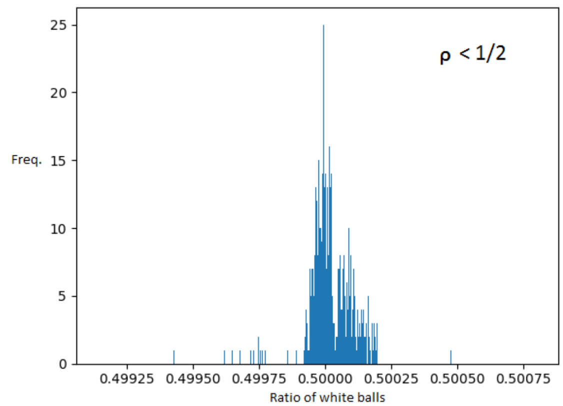
1.2 Bernard Friedman's Urn

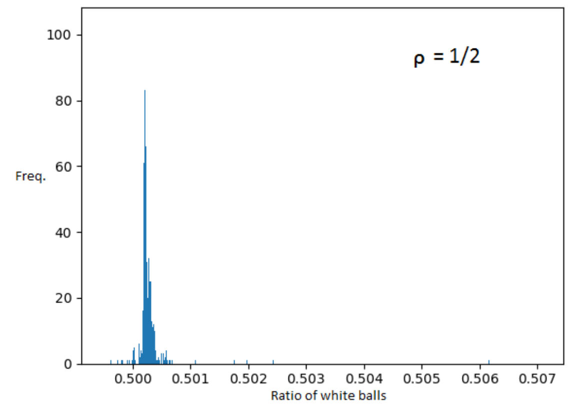
In 1949, Bernard Friedman extended the idea of Pólya urns to a model where black as well as white balls are added to the urn on drawing of a ball. Such Urns are called Friedman's urn [8]. At each time step, a ball is drawn uniformly at random and it is added back to the urn along with α balls of the same color and β balls of the other color. urn scheme for a Friedman urn is

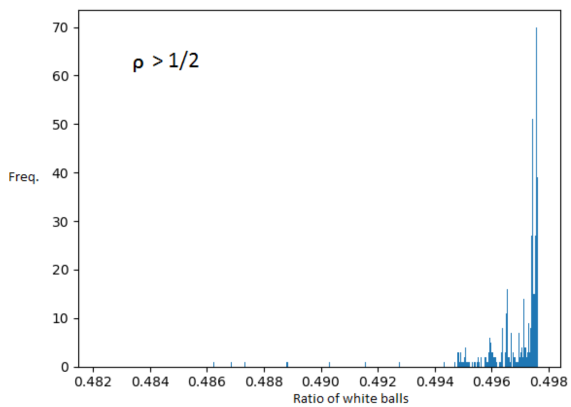
$$\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$$

Unlike Pólya urns, in Friedman's urn fraction of balls of either color approaches $\frac{1}{2}$ with probability 1.

In the simulations below, Friedman's urn is simulated 1000 times and a graph for frequency v/s Ratio of white balls is plotted. A sharp peak appears in all the cases because of asymptotic approach of ratio of white balls to one-half.







Here,

$$\rho = \frac{\alpha - \beta}{\alpha + \beta}$$

This choice of ρ will be clear as we discuss limit theorem for Friedman urns in the next section.

In 1964, in the paper titled "Bernard Friedman urn", David A. Freedman proved that in certain regimes (depending on ρ) the fluctuation of fraction of balls of either color around the limit $\frac{1}{2}$ is Gaussian. We reproduce the results and a sketch of the proof that uses method of moments as an important tool.

1.2.1 Method Of Moments and Limit Theorems for Friedman's Urn

Theorem 2. *(David A. Freedman [8],1965)*

- *For* $\rho > \frac{1}{2}$,
$$\lim_{n\to\infty} n^{-\rho}(W_n - B_n)$$
converges to a finite random variable with probability 1 and in r^{th} *mean* $(0 < r < \infty)$.

- *For* $\rho = \frac{1}{2}$ *the distribution of* $(n \log n)^{\frac{-1}{2}}(W_n - B_n)$ *converges to normal with mean 0 and variance* $(\alpha - \beta)^2$.

- *For* $\rho < \frac{1}{2}$ *the distribution of* $n^{\frac{-1}{2}}(W_n - B_n)$ *converges to normal with mean 0 and variance* $(1 - 2\rho)^{-1}(\alpha - \beta)^2$.

We will now illustrate how Method of Moments[8] and Difference Equations[4.4] is used to prove the theorem in case $\rho > \frac{1}{2}$

Notation:

$$\rho = \frac{\alpha - \beta}{\alpha + \beta} = \frac{\delta}{\sigma}$$
$$s = W_0 + B_0$$
$$a_n(j) = 1 + \frac{j\delta}{s + \sigma n}$$
$$x_n(k) = E[(W_n - B_n)^k]$$

Lemma 3. *(David A. Freedman ,1965, [8])*
For each nonnegative integer k,

$$lim_{n\to\infty} n^{-\rho k} E[(W_n - B_n)^k] = \mu(k)$$

with $0 \leq \mu(k) < \infty$. *If* k *is even, then* $\mu(k) > 0$.

Proof. The result is trivial for $k = 0$, and $\mu(0) = 1$.The proof follows by induction for even k.

$$E\{(W_{n+1} - B_{n+1})^{2k+2} | \mathcal{F}_n\}$$

$$= \left[\frac{W_n}{(s+\sigma_n)}\right](W_n - B_n + \delta)^{2k+2} + \left[\frac{B_n}{(s+\sigma n)}\right]\left(W_n - B_n - \delta\right)^{2k+2}$$

$$= \left[\frac{W_n}{(s+\sigma_n)}\right]\sum_{j=0}^{2k+2}\binom{2k+2}{j}\delta^j (W_n - B_n)^{2k+2-j}$$

$$+ \left[\frac{B_n}{(s+\sigma_n)}\right]\sum_{j=0}^{2k+2}\binom{2k+2}{j}(-\delta)^j (W_n - B_n)^{2k+2-j}$$

$$= a_n(2k+2)(W_n - B_n)^{2k+2}$$

$$+ \sum_{j=1}^{k}\left[\binom{2k+2}{2j}\delta^{2j} + \left[\frac{\binom{2k+2}{2j+1}}{(s+\sigma n)}\right]\delta^{2j+1}\right](W_n - B_n)^{2k+2-2j} + \delta^{2k+2}$$

So, we have

$$x_{n+1}(2k+2) = a_n(2k+2)x_n(2k+2) + b_n(2k+2)$$

with

$$b_n(2k+2)$$

$$= \sum_{j=1}^{k}\left[\binom{2k+2}{2j}\delta^{2j} + \left[\frac{\binom{2k+2}{2j+1}}{(s+\sigma n)}\right]\delta^{2j+1}\right]x_n^{2k+2-2j} + \delta^{2k+2}$$

Suppose the Lemma is true for even $k \le 2k$. Then $0 \le b_n(2k+2) = O(n^{2k\rho})$, and by Lemma 14,

$$\lim_{n\to\infty} x_n(2k+2)\prod_{\nu=0}^{n} a_\nu(2k+2)^{-1}$$

$$= x_0(2k+2) + \sum_{j=0}^{\infty} b_j(2k+2)\prod_{\nu=0}^{j} a_\nu(2k+2)^{-1}$$

which is positive and finite. By (4.5) $\prod_{\nu=0}^{n} a_\nu(2k+2) \sim n^{(2k+2)\rho}$, and hence the theorem holds for 2k+2. By induction, it holds for even k.

Also,

$$E\{(W_{n+1} - B_{n+1})^{2k+2}|\mathcal{F}_n\} = a_n(1)(W_n - B_n)$$

So,

$$x_{n+1}(1) = a_n(1)x_n(1) = x_0(1)\prod_{\nu=0}^{n} a_\nu(1)$$

Since $\prod_{\nu=0}^{n} a_\nu(1) \sim n^\rho$ by [4.6], the Lemma holds when $k = 1$. The proof for odd k can be similarly done by induction.

□

Now, for the proof of case $\rho > \frac{1}{2}$

Define

$$Z_n = (W_n - B_n)\prod_{\nu=0}^{n-1} a_\nu(1)^{-1}$$

Note that $Z_n : n \geq 0$ is a martingale. Also by the above Lemma, $sup_n E(Z_n) < \infty$. So, by Martingale Convergence Theorem, Z_n converges to a finite limit with probability 1.
Since $\prod_{\nu=0}^{n-1} a_\nu(1) \sim n^\rho$, $\lim_{n\to\infty} n^{-\rho}(W - n - B_n) = Z$ exists and is finite with probability 1.

1.3 Other Extensions

Other then Pólya and Friedman urns, there have been several attempts to generalize these urn models. See [14].

1.3.1 Bagchi-Pal Urns

Bagchi and Pal (1985) introduced a more general model with urn scheme:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

However, tenability conditions restrict the generality of this model by the following conditions:

1. $a + b = c + d = K$,Where K is some positive integer.

2. $b > 0$, $c > 0$

3. if $a < 0$, then a divides W_0 and c.

4. if $d < 0$, then d divides B_0 and b.

1.3.2 The Ehrenfest Urn

The Ehrenfest Urn scheme is

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

This urn is slightly different from the rest since number of balls in this urn does not change with time.
Ehrenfest urns are ideal for modelling a confined two particle two compartment system where removal and addition of a ball is analogous to movement of the ball to one of the two compartments of the system. In fact, the idea comes from the Ehrenfest model of diffusion in physics. It was first proposed by Tatiana and Paul Ehrenfest. Various other urn schemes have been studied,especially those which replicate stochastic processes of practical importance. For example, following two urn schemes have been studied in [11]:

1. $\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$

   Such an urn scheme is ideal for modelling a process of removal of defective(black) items from a box and replacing them with the non-defective(white) ones.

2. $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

   This model corresponds to mere removal of defective(black) items from the box.

1.3.3 O.K. Corral Urn Model

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

The process starts with $W_0 = B_0 = N$ and stops when either W_n or B_n is zero. The process was used to model famous gunfight at O.K. Corral Arizona, United States and was first introduced in 1998 by David Williams and Paul Mcllroy.

In the next chapter, we consider an extension of these single urn models to multiple dependent processes.

Chapter 2

Proposed Interacting Urn Model: Synchronization

In recent times, interacting urn models have garnered a lot of interest from researchers in Mathematics, Physics, Computer Science etc. It could emerge as an important way to model random processes that update or reinforce in a dependent manner.

A general set-up of interacting urn models is as follows:
Consider N urns such that the probability of adding α_r balls of color r to i^{th} urn at time t depends not only on the composition of i^{th} urn at time t but also on composition of all or a subset of N urns at that time t.

If $X_t^r(i)$ denotes number of balls of color r in i^{th} urn at time t.

$$X_{t+1}^r(i) = X_t^r(i) + Y_{t+1}^r(i)$$

where $Y_{t+1}^r = \alpha_r \quad \text{with probability} \quad f(X_t^r(1), X_t^r(2), .., X_t^r(N))$.

In this thesis, we study a special case of this general set-up.

2.1 Proposed Model

We consider N urns labelled $1, 2, \ldots, N$. We assume that all urns contain equal number of balls m at time $t = 0$. Each urn contains $a_i > 0$ white and $b_i > 0$ black balls ($a_i + b_i = m, \ \forall \ 1 \leq i \leq N$) at time $t = 0$. We consider a system where that all the urns are updated simultaneously at discrete time steps $t \geq 0$.

In this generalized model, we are looking at the interaction of N identical urns each reinforced with Friedman's urn scheme. By interaction we mean that the probability of a white ball drawn from urn i depends on the composition of i^{th} as well as the other $N-1$ urns.

Interaction between urns is governed by the following fixed $N \times N$ Matrix:

$$P = \begin{bmatrix} p_1^1 & p_1^2 & . & . & . & . & p_1^N \\ p_2^1 & p_2^2 & . & . & . & . & p_2^N \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ p_N^1 & p_N^2 & . & . & . & . & p_N^N \end{bmatrix}$$

where p_i^j = weightage of j^{th} urn when updating i^{th} urn and $\sum_{j=1}^{N} p_i^j \leq 1$. The matrix P is called the interaction matrix of the model.

We use the following notation:

- $X_t(i)$ = Number of white balls in urn i at time t.
- $Z_t(i)$ = Fraction of white balls in urn i at time t.
- $Y_t(i)$ = Number of white balls added to urn i at time t.

Note that at each time step a total number of $\alpha + \beta$ balls are added to each urn. Therefore, the total number of balls in each urn at any given time is same. More precisely, the total number of balls in each urn at time t is given by $N_t = m + (\alpha + \beta)t$. Hence $Z_t(i) = X_t(i)/N_t$.

Define:

1. n-dimensional vectors: $\tilde{p}_i = [p_i^1, p_i^2, ..., p_i^N]$, the weight-vector associated with i^{th} urn (note that this is independent of time), and $\widetilde{Z}_t = [Z_t(1), Z_t(2), ..., Z_t(N)]$, the vector of fraction of balls of white color in each urn at time t.

2. $\langle \tilde{p}_i, \widetilde{Z}_t \rangle = \sum_{j=1}^{N} p_i^j Z_t(j)$ is the usual inner-product.

The reinforcement depends on the random variables $\{Y_t(i)\}_{i=1}^N$ for $t \geq 1$. Let $\mathcal{F}_t$ denote the σ-field generated by $\{Y_t(1), \ldots, Y_t(N)\}$. The reinforcement scheme for the model is given by:

$$Y_{t+1}(i) = \begin{cases} \alpha & \text{with probability} \langle \tilde{p}_i, \widetilde{Z}_t \rangle \\ \beta & \text{with probability } 1 - \langle \tilde{p}_i, \widetilde{Z}_t \rangle \end{cases}$$

(2.1)

With the above reinforcement scheme we have:

$$X_{t+1}(i) = X_t(i) + Y_{t+1}(i)$$

In other words,

$$Z_{t+1}(i) = \frac{(\alpha+\beta)t + m}{(\alpha+\beta)(t+1) + m} Z_t(i) + \frac{1}{(\alpha+\beta)(t+1) + m} Y_{t+1}(i)$$

The corresponding n-dimensional recursion relation is:

$$\widetilde{Z_{t+1}} = \frac{(\alpha+\beta)t + m}{(\alpha+\beta)(t+1) + m} \widetilde{Z}_t + \frac{1}{(\alpha+\beta)(t+1) + m} \widetilde{Y_{t+1}} \qquad (2.2)$$

where, $\widetilde{Y_{t+1}} = [Y_{t+1}(1), Y_{t+1}(2), ..., Y_{t+1}(N)]$

Remark. *The model described above generalizes the models in [17] and [16]. In particular, taking $p_j^i = p/N$ for $i \neq j$ and $p_i^i = 1-p+p/N$ for $i = 1, \ldots, N$ gives the model considered in [16]. With the same substitution for p_i^j's and for $\alpha = 1$ and $\beta = 0$, we get the model in [17].*

2.2 $\mathcal{L}^2$- Synchronization

In this section, we study the synchronization of Friedman urns with interactions defined by the proposed model. By synchronization, we mean that the convergence of fraction of balls of white color in each urn to a common limit as $t \to \infty$. In case of Friedman urns, this limit is 1/2. We obtain the rates of convergence of:

- $Var(Z_t(i) - Z_t)$
- $Var(Z_t(i) - \overline{Z_t(i)})$ where $\overline{Z_t(i)} = \sum\limits_{j=1}^{N} p_i^j Z_t(j)$

Throughout this section we use the following notation from [16]:
For two positive sequences a_t, b_t, $a_t \sim b_t$ if

$$0 < \liminf_{t \to +\infty} \frac{a_t}{b_t} < \limsup_{t \to +\infty} \frac{a_t}{b_t} < +\infty$$

Theorem 4. *Let* $\rho = \frac{\alpha-\beta}{\alpha+\beta}$ *and* $Z_t = \frac{1}{N} \sum\limits_{i=1}^{N} Z_t(i)$

$$Var(Z_t(i) - Z_t) \sim \begin{cases} t^{2\left(1 - \left\{\left(1-\frac{1}{N}\right)p_i^i - \frac{1}{N}\sum\limits_{i \neq j} p_j^i\right\}\rho\right)} & \textit{for} \quad 0 \leq \left\{\left(1-\frac{1}{N}\right)p_i^i - \frac{1}{N}\sum\limits_{i \neq j} p_j^i\right\}\rho < \frac{1}{2} \\ t^{-1}\log t & \textit{for} \quad \left\{\left(1-\frac{1}{N}\right)p_i^i - \frac{1}{N}\sum\limits_{i \neq j} p_j^i\right\}\rho = \frac{1}{2} \\ t^{-1} & \textit{for} \quad \frac{1}{2} < \left\{\left(1-\frac{1}{N}\right)p_i^i - \frac{1}{N}\sum\limits_{i \neq j} p_j^i\right\}\rho < 1 \end{cases}$$

The idea of the proof is similar to the proof of Theorem 1 in [16].

Proof. We first compute $Var(Y_{t+1}(i)|\mathcal{F}_t)$

$$Var(Y_{t+1}(i)|\mathcal{F}_t) = E(Y_{t+1}(i)^2|\mathcal{F}_t) - \left(E(Y_{t+1}(i)|\mathcal{F}_t)\right)^2$$

$$= \alpha^2 \sum_{i=1}^{N} p_i^j Z_t(j) + \beta^2 \left(1 - \sum_{j=1}^{N} p_i^j Z_t(j)\right) - \left(\alpha \sum_{i=1}^{N} p_i^j Z_t(j) + \beta\left(1 - \sum_{j=1}^{N} p_i^j Z_t(j)\right)\right)^2$$

$$\left[\alpha^2 - \beta^2 - 2\left(\alpha - \beta\right)\beta\right]\sum_{i=1}^{N} p_i^j Z_t(j) - \left(\alpha - \beta\right)^2\left(\sum_{i=1}^{N} p_i^j Z_t(j)\right)^2$$

So, we have

$$Var(Y_{t+1}(i)|\mathcal{F}_t) = \left(\alpha - \beta\right)^2 \sum_{i=1}^{N} p_i^j Z_t(j)\left[1 - \sum_{i=1}^{N} p_i^j Z_t(j)\right] \tag{2.3}$$

Set $x_t = Var(Z_t(i) - Z_t)$

$$x_{t+1} = E\left[Var(Z_{t+1}(i) - Z_{t+1}|\mathcal{F}_t)\right] + Var[E(Z_{t+1}(i) - Z_{t+1}|\mathcal{F}_t)] \tag{2.4}$$

$$E\left[Var(Z_{t+1}(i) - Z_{t+1}|\mathcal{F}_t)\right] = E\Bigg[Var\left(\frac{(\alpha+\beta)t + m}{(\alpha+\beta)(t+1) + m}\right)(Z_t(i) - Z_t)|\,\mathcal{F}_t$$

$$+\frac{1}{(\alpha+\beta)(t+1)+m}\left(Y_{t+1}(i) - \frac{1}{N}\sum_{i=1}^{N} Y_{t+1}(j)\right)|\mathcal{F}_t\Bigg]$$

$$= E\left[Var\left\{\frac{1}{(\alpha+\beta)(t+1)+m}\left(Y_{t+1}(i) - \frac{1}{N}\sum_{i=1}^{N} Y_{t+1}(j)|\mathcal{F}_t\right)\right\}\right]$$

Since $Y_t(i)$'s are conditionally independent,

$$= \frac{1}{\left((\alpha+\beta)(t+1)+m\right)^2} E\left[Var\left(Y_{t+1}(i)|\mathcal{F}_t\right) - \frac{1}{N}\sum_{j=1}^{N} Var\left(Y_{t+1}(j)|\mathcal{F}_t\right)\right]$$

Putting equation (2.2) in the above equation we get

$$E\left[Var(Z_{t+1}(i) - Z_{t+1}|\mathcal{F}_t)\right] = \frac{(\alpha-\beta)^2}{\left((\alpha+\beta)(t+1)+m\right)^2} E\left[\overline{Z_t(i)}\left(1 - \overline{Z_t(i)}\right) - Z_t - \frac{1}{N}\sum_{j=1}^{N}\overline{Z_t(j)}^2\right]$$

Note that,

$$E\left[Var(Z_{t+1}(i) - Z_{t+1}|\mathcal{F}_t)\right] \sim \frac{1}{t^2} \tag{2.5}$$

Moving on to second term of equation (2.3),

$$Var[E(Z_{t+1}(i) - Z_{t+1}|\mathcal{F}_t)] =$$

$$\frac{1}{((\alpha+\beta)(t+1)+m)^2} Var\left[((\alpha+\beta)t+m)(Z_t(i)-Z_t)+E(Y_{t+1}(i)|\mathcal{F}_t)-\frac{1}{N}\sum_{j=1}^{N}E(Y_{t+1}(j)|\mathcal{F}_t)\right]$$

$$= \frac{1}{((\alpha+\beta)(t+1)+m)^2} Var\Bigg[((\alpha+\beta)t+m)(Z_t(i)-Z_t)+\Big\{(\alpha-\beta)\overline{Z_t(i)}+\beta\Big\}$$

$$-\frac{1}{N}\left\{(\alpha-\beta)\sum_{j=1}^{N}\overline{Z_t(i)}+\beta N\right\}\Bigg]$$

$$= \frac{\left((\alpha+\beta)t+m+\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j}p_j^i\right\}(\alpha-\beta)\right)^2}{((\alpha+\beta)(t+1)+m)^2} Var(Z_t(i)-Z_t)$$

$$+\frac{(\alpha-\beta)^2}{((\alpha+\beta)(t+1)+m)^2}\sum_{i\neq j} Var\left\{\frac{(p_i^i-1)}{N}+p_i^j\right\}Z_t(j)$$

$$+\frac{(\alpha-\beta)^2}{((\alpha+\beta)(t+1)+m)^2}\left(\frac{p_i^i-1}{N}\right)^2 Var(Z_t(i))+\frac{1}{((\alpha+\beta)(t+1)+m)^2}\text{Covariance terms}$$

Hence,

$$Var[E(Z_{t+1}(i)-Z_{t+1}|\mathcal{F}_t)] = \frac{\left((\alpha+\beta)t+m+\left\{\left(1-\frac{1}{N}\right)p_i^i-\sum_{i\neq j}p_j^i\right\}(\alpha-\beta)\right)^2}{((\alpha+\beta)(t+1)+m)^2} Var(Z_t(i)-Z_t) \tag{2.6}$$

$$+O\left(\frac{1}{t^2}\right)$$

Combining equations (2.3),(2.4) and (2.5) we get

$$x_{t+1} = \frac{\left((\alpha+\beta)t+m+\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j}p_j^i\right\}(\alpha-\beta)\right)^2}{((\alpha+\beta)(t+1)+m)^2} x_t+O\left(\frac{1}{t^2}\right) \tag{2.7}$$

So,

$$x_{t+1} = f(t)x_t+O\left(\frac{1}{t^2}\right)$$

$$f(t) = \frac{\left((\alpha+\beta)t + m + \left\{\left(1-\frac{1}{N}\right)p_i^i - \frac{1}{N}\sum_{i\neq j} p_j^i\right\}(\alpha-\beta)\right)^2}{((\alpha+\beta)(t+1)+m)^2} = 1$$

$$+\frac{\left(1-\left\{\left(1-\frac{1}{N}\right)p_i^i - \frac{1}{N}\sum_{i\neq j} p_j^i\right\}\rho\right)^2}{(t+1+m')^2} - \frac{2\left(1-\left\{\left(1-\frac{1}{N}\right)p_i^i - \frac{1}{N}\sum_{i\neq j} p_j^i\right\}\rho\right)}{(t+m'+1)^2}$$

$$\text{Where} \quad \rho = \frac{\alpha-\beta}{\alpha+\beta} \quad \text{and} \quad m' = \frac{m}{\alpha+\beta}$$

Note that

$$\lim_{t\to\infty} f(t) = 1 \quad \text{and} \quad 0 < f(t) < 1$$

Now set

$$\zeta_t := \frac{x_t}{\prod_{k=0}^{t-1} f(t)}$$

$$\zeta_{t+1} = \zeta_t + F(t)$$

Where,

$$F(t) := \frac{1}{\prod_{k=0}^{t} f(k)} O\left(\frac{1}{t^2}\right)$$

Since $\zeta_0 = x_0 = 0$, we get

$$\zeta_t = \sum_{i=0}^{t-1} F(i)$$

$$x_t = \left[\prod_{k=0}^{t-1} f(k)\right] \sum_{i=0}^{t-1} F(i)$$

$$\prod_{k=0}^{t-1} f(k) = \exp\left[\sum_{k=0}^{t-1} \log\left(1 - \frac{2\left(p_i^i\rho - 1\right)}{k+m'+1} + \frac{\left(1-\left\{\left(1-\frac{1}{N}\right)p_i^i - \frac{1}{N}\sum_{i\neq j} p_j^i\right\}\rho\right)^2}{(k+m'+1)^2}\right)\right]$$

$$= \exp\left[-2\left(1-\left\{\left(1-\frac{1}{N}\right)p_i^i - \frac{1}{N}\sum_{i\neq j} p_j^i\right\}\rho\right)\sum_{k=0}^{t-1}\frac{1}{k+m'+1} + O(1)\right]$$

$$= \exp\left[-2\left(1-\left\{\left(1-\frac{1}{N}\right)\left\{\left(1-\frac{1}{N}\right)p_i^i-\sum_{i\neq j}p_j^i\right\}-\frac{1}{N}\sum_{i\neq j}p_j^i\right\}\rho\right)\log(t+m')+O(1)\right]$$

$$\sim t^{-2\left(1-\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j}p_j^i\right\}\rho\right)}$$

Hence,

$$F(t)\sim t^{-2\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j}p_j^i\right\}\rho}$$

$$\zeta_t=\sum_{i=0}^{t-1}F(i)\sim\begin{cases}1 & \text{for}\quad 0\leq\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j}p_j^i\right\}\rho<\frac{1}{2}\\ \log t & \text{for}\quad \left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j}p_j^i\right\}\rho=\frac{1}{2}\\ t^{1-2\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j}p_j^i\right\}\rho} & \text{for}\quad \frac{1}{2}<\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j}p_j^i\right\}\rho<1\end{cases}$$

Hence,

$$x_t\sim\begin{cases}t^{2\left(1-\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j}p_j^i\right\}\rho\right)} & \text{for}\quad 0\leq\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j}p_j^i\right\}\rho<\frac{1}{2}\\ t^{-1}\log t & \text{for}\quad \left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j}p_j^i\right\}\rho=\frac{1}{2}\\ t^{-1} & \text{for}\quad \frac{1}{2}<\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j}p_j^i\right\}\rho<1\end{cases}$$

Note that if P is doubly stochastic, then

$$\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j}p_j^i\right\}=p_i^i-\frac{1}{N}$$

$\square$

Theorem 5. *Let* $\rho = \frac{\alpha-\beta}{\alpha+\beta}$ *and* $\overline{Z_t(i)} = \sum_{j=1}^{N} p_i^j Z_t(j)$

$$Var(Z_t(i) - \overline{Z_t(i)}) \sim \begin{cases} t^{-2\left(\frac{1+\left\{\left(1-\frac{1}{N}\right)p_i^i - \frac{1}{N}\sum_{i\neq j} p_j^i\right\}(\alpha-\beta)}{\alpha+\beta}+1\right)} & if \ \frac{2\left(1+\left\{\left(1-\frac{1}{N}\right)p_i^i - \frac{1}{N}\sum_{i\neq j} p_j^i\right\}(\alpha-\beta)\right)}{\alpha+\beta} < 0 \\ t^{-1}\log t & if \ \frac{2\left(1+\left\{\left(1-\frac{1}{N}\right)p_i^i - \frac{1}{N}\sum_{i\neq j} p_j^i\right\}(\alpha-\beta)\right)}{\alpha+\beta} = 0 \\ t^{-1} & if \frac{2\left(1+\left\{\left(1-\frac{1}{N}\right)p_i^i - \frac{1}{N}\sum_{i\neq j} p_j^i\right\}(\alpha-\beta)\right)}{\alpha+\beta} > 0 \end{cases}$$

Proof. The computation method in this proof is exactly same as that of the previous proof. Set $y_t = Var(Z_{t+1}(i) - \overline{Z_{t+1}(i)})$

$$y_{t+1} = E\left[Var(Z_{t+1}(i) - \overline{Z_{t+1}(i)}|\mathcal{F}_t)\right] + Var[E(Z_{t+1}(i) - \overline{Z_{t+1}(i)}|\mathcal{F}_t)]$$

$$E\left[Var(Z_{t+1}(i) - \overline{Z_{t+1}(i)}|\mathcal{F}_t)\right] = E\Bigg[Var\left(\frac{(\alpha+\beta)t+m}{(\alpha+\beta)(t+1)+m}\right)\left(Z_t(i) - \overline{Z_t(i)}\right)$$

$$+\frac{1}{(\alpha+\beta)(t+1)+m}\left(Y_{t+1}(i) - \sum_{i=1}^{N} p_i^j Y_{t+1}(j)\right)|\mathcal{F}_t\Bigg]$$

$$= E\left[Var\left\{\frac{1}{(\alpha+\beta)(t+1)+m}\left(Y_{t+1}(i) - \sum_{i=1}^{N} p_i^j Y_{t+1}(j)|\mathcal{F}_t\right)\right\}\right]$$

$$= \frac{(\alpha-\beta)^2}{((\alpha+\beta)(t+1)+m)^2} E\left[(1-p_i^i)\overline{Z_t(i)}\left(1-\overline{Z_t(i)}\right) + \sum_{i\neq j} p_i^j \overline{Z_t(j)}(1-\overline{Z_t(j)})\right]$$

So,

$$E\left[Var(Z_{t+1}(i) - \overline{Z_{t+1}(i)}|\mathcal{F}_t)\right] \sim \frac{1}{t^2}$$

Similarly,

$$Var[E(Z_{t+1}(i) - \overline{Z_{t+1}(i)}|\mathcal{F}_t)] =$$

$$\frac{1}{((\alpha+\beta)(t+1)+m)^2} Var\left[((\alpha+\beta)t+m)(Z_t(i)-\overline{Z_t(i)})+E(Y_{t+1}(i)|\mathcal{F}_t)-\sum_{j=1}^{N} p_i^j E(Y_{t+1}(j)|\mathcal{F}_t)\right]$$

$$= \frac{1}{((\alpha+\beta)(t+1)+m)^2} Var\left[((\alpha+\beta)t+m)(Z_t(i)-\overline{Z_t(i)})+(\alpha-\beta)\left\{\overline{Z_t(i)}+\sum_{j=1}^{N} p_i^j \overline{Z_t(j)}\right\}\right]$$

$$= \frac{\left\{(\alpha+\beta)t+m-\left(1+\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j} p_j^i\right\}(\alpha-\beta)\right)\right\}^2}{((\alpha+\beta)(t+1)+m)^2} Var(Z_t(i)-\overline{Z_t(i)})+O\left(\frac{1}{t^2}\right)$$

Hence, we have

$$y_{t+1} = \frac{\left\{(\alpha+\beta)t+m-\left(1+\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j} p_j^i\right\}(\alpha-\beta)\right)\right\}^2}{((\alpha+\beta)(t+1)+m)^2} y_t+O\left(\frac{1}{t^2}\right)$$

By following the same construction in the previous section and replacing ζ, $f(t)$ and $F(t)$ by ζ', $f(t)'$ and $F(t)'$ respectively, we get

$$y_{t+1} = f'(t)y_t + O\left(\frac{1}{t^2}\right)$$

where

$$f'(t) = \frac{\left\{(\alpha+\beta)t+m-\left(1+\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j} p_j^i\right\}(\alpha-\beta)\right)\right\}^2}{((\alpha+\beta)(t+1)+m)^2}$$

$$F'(t) \sim t^{\frac{2(1+p_i^i(\alpha-\beta))}{\alpha+\beta}}$$

$$\zeta_t' = \sum_{i=0}^{t-1} F'(i) \sim \begin{cases} 1 & \text{if } \frac{2\left(1+\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j} p_j^i\right\}(\alpha-\beta)\right)}{\alpha+\beta} < 0 \\ \log t & \text{if } \frac{2\left(1+\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j} p_j^i\right\}(\alpha-\beta)\right)}{\alpha+\beta} = 0 \\ t^{2\left(\frac{1+\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j} p_j^i\right\}(\alpha-\beta)}{\alpha+\beta}\right)+1} & \text{if} \frac{2\left(1+\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum_{i\neq j} p_j^i\right\}(\alpha-\beta)\right)}{\alpha+\beta} > 0 \end{cases}$$

So,

$$y_t \sim \begin{cases} t^{-2\left(\frac{1+\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum\limits_{i\neq j}p_j^i\right\}(\alpha-\beta)}{\alpha+\beta}+1\right)} & \text{if } \frac{2\left(1+\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum\limits_{i\neq j}p_j^i\right\}(\alpha-\beta)\right)}{\alpha+\beta} < 0 \\ t^{-1}\log t & \text{if } \frac{2\left(1+\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum\limits_{i\neq j}p_j^i\right\}(\alpha-\beta)\right)}{\alpha+\beta} = 0 \\ t^{-1} & \text{if} \frac{2\left(1+\left\{\left(1-\frac{1}{N}\right)p_i^i-\frac{1}{N}\sum\limits_{i\neq j}p_j^i\right\}(\alpha-\beta)\right)}{\alpha+\beta} > 0 \end{cases}$$

□

2.3 Almost Sure convergence Results

Again, taking inspiration from [16] and [17], we try to get an almost sure convergence result for the proposed model of interacting Pólya and Friedman urns by using the convergence theorem of Quasi-Martingales. We first define what are Quasi-Martingales

Definition 2.3.1. *A stochastic process A_t on probability space $(\Omega, \mathcal{F}, P)$ is a Quasi-Martingale if*

$$\sum_{t=0}^{+\infty} E[|E(A_{t+1}|\mathcal{F}_t) - A_t|] < \infty$$

Lemma 6. *Z_t is a Quasi-Martingale when P is a doubly stochastic matrix and $Z_t(i)$ is always a Quasi-Martingale.*

Proof. **For Z_t:**

$$\sum_{t=0}^{\infty} E|E[Z_{t+1}|\mathcal{F}_t]-Z_t| = \sum_{t=0}^{\infty} E\left|\frac{N_t}{N_{t+1}}Z_t+\frac{1}{N.N_{t+1}}(\alpha-\beta)\sum_{i=1}^{N}\langle\widetilde{p}_i,\widetilde{Z}_t\rangle+\frac{\beta}{N_{t+1}}-Z_t\right|$$

$$= \sum_{t=0}^{\infty} E\left|\frac{1}{N.N_{t+1}}(\alpha-\beta)\sum_{i=1}^{N}(\sum_{j=1}^{N}p_i^j)Z_t(j)+\frac{\beta}{N_{t+1}}-\frac{(\alpha+\beta)}{N_{t+1}}Z_t\right|$$

$$= \sum_{t=0}^{\infty}\frac{1}{N_{t+1}}E\left|\frac{\alpha}{N}\left(\sum_{j=1}^{N}(\sum_{i=1}^{N}p_i^j)-1\right)Z_t(j)-\frac{\beta}{N}\left(\sum_{j=1}^{N}(\sum_{i=1}^{N}p_i^j)+1\right)Z_t(j)+\beta\right|$$

(if in matrix P along with row sums one,all the columns also sum up to one i.e. P is doubly stochastic then)

$$= \sum_{t=0}^{\infty} \frac{1}{N_{t+1}} E\bigg|\beta - \frac{2\beta}{N}\bigg(\sum_{j=1}^{N} Z_t(j)\bigg)\bigg|$$

$$= \sum_{t=0}^{\infty} \frac{\beta}{2N_{t+1}} E\bigg|\frac{1}{2} - Z_t\bigg|$$

Since for Friedman's urn $Z_t \to \frac{1}{2}$ almost surely, Z_t is a quasi-Martingale if P is a doubly Stochastic Matrix.

For $Z_t(i)$**:**

$$\sum_{t=0}^{\infty} E\bigg|E(Z_{t+1}(i)|\mathcal{F}_t) - Z_t(i)\bigg|$$

$$= \sum_{t=0}^{\infty} E\bigg|\bigg(\frac{N_t - N_{t+1}}{N_{t+1}}\bigg)Z_t(i) + \frac{\beta}{N_{t+1}} + \frac{(\alpha-\beta)}{N_{t+1}}\overline{Z_t(i)}\bigg|$$

$$= \sum_{t=0}^{\infty} E\bigg|\frac{-(\alpha-\beta) - \beta + \beta}{N_{t+1}} Z_t(i) + \frac{\beta}{N_{t+1}} + \frac{(\alpha-\beta)}{N_{t+1}}\overline{Z_t(i)}|$$

$$= \sum_{t=0}^{\infty} E\bigg|\frac{-2\beta}{N_{t+1}}\bigg(Z_t(i) - \frac{1}{2}\bigg) + \frac{(\alpha-\beta)}{N_{t+1}}(\overline{Z_t(i)} - Z_t(i))\bigg|$$

Which converges (Using the fact that in Friedman urn fraction of balls of either color converges to one-half and theorem 5).

□

In fact, one can show that $\widetilde{Z}_t$ converges to $\gamma = \left(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right)$ almost surely using theory of stochastic approximation which we shall discuss in the next chapter.

Chapter 3

Proposed Interacting Urn Model: Fluctuations

In Chapter 1, Theorem 2 we discussed David A. Freedman's limit theorems for a single Friedman urn. Those results can be re-written to express fluctuations of the fraction of white (or black) balls around the limit 1/2. In this chapter, we prove similar fluctuation results for the proposed model of interacting urns (Friedman and/or Polya). We take two different approaches:

1. Using the theory of Stochastic Approximation (for details see [5]).
2. Using the method proposed by Giacomo Aletti, Irene Crimaldi and Andrea Ghiglietti in [1].

Similar results are obtained for the special case of $p_i^j = p/N \quad \text{for} \quad i \neq j$, $p_i^i = 1 - p + p/N$ for Pólya urns in [16] and for Friedman urns in [17].

3.1 Stochastic Approximation scheme for the Proposed model

We limit our discussion on Stochastic Approximation to the results relevant for the analysis of our model. For more details see [5].

A stochastic approximation scheme in R^d is given by :

$$x_{t+1} = x_t + a(t+1)[h(x_t) + M_{t+1}], t \geq 0 \tag{3.1}$$

with following assumptions:

A1. The map $h : R^d \to R^d$ is Lipschitz: $||h(x) - h(y)|| \leq L||x - y||$ for some $0 < L < \infty$.

A2. $\{a(t)\}$ are called step sizes satisfying $\sum_t a(t) = \infty$; $\sum_t a(t)^2 < \infty$

A3. $\{M_t\}$ is a martingale difference sequence with respect to the increasing family of σ-fields

$$\mathcal{F}_t := \sigma(x_m, M_m, m \leq t) = \sigma(x_0, M_1, .., M_t), n \geq 0$$

That is,

$$E[M_{t+1}|\mathcal{F}_t] = 0 \quad a.s. \quad t \geq 0$$

Furthermore, $\{M_t\}$ are square-integrable with

$$E[||M_{t+1}||^2|\mathcal{F}_t] \leq K(1 + ||x_t||^2) \quad a.s. \quad t \geq 0$$

for some constant $K > 0$.

A4. $sup_t||x_t|| < \infty \quad a.s.$

In this section we will try to get a stochastic scheme for our model. Equation (2.2) can be rewritten as

$$\widetilde{Z_{t+1}} = \widetilde{Z}_t - \frac{(\alpha+\beta)}{(\alpha+\beta)(t+1)+m}\widetilde{Z}_t + \frac{1}{(\alpha+\beta)(t+1)+m}\widetilde{Y_{t+1}}$$

$$+\frac{1}{(\alpha+\beta)(t+1)+m}E[\widetilde{Y_{t+1}}|\mathcal{F}_t] - \frac{1}{(\alpha+\beta)(t+1)+m}E[\widetilde{Y_{t+1}}|\mathcal{F}_t]$$

Where

$$E[\widetilde{Y_{t+1}}|\mathcal{F}_t] = \begin{bmatrix} E(Y_{t+1}(1)|\mathcal{F}_t) \\ E(Y_{t+1}(2)|\mathcal{F}_t) \\ . \\ . \\ E(Y_{t+1}(N)|\mathcal{F}_t) \end{bmatrix} = (\alpha - \beta) \begin{bmatrix} \langle \tilde{p_1}, \tilde{Z_t} \rangle \\ \langle \tilde{p_2}, \tilde{Z_t} \rangle \\ . \\ . \\ \langle \tilde{p_N}, \tilde{Z_t} \rangle \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}$$

($\mathcal{F}_t$ is the sigma algebra generated by $(\widetilde{Z_0}, \widetilde{Z_1}, ..., \widetilde{Z}_t)$)

Putting $\rho = \frac{\alpha-\beta}{\alpha+\beta}$ in above equation we get,

$$\widetilde{Z_{t+1}} = \widetilde{Z}_t + \frac{1}{(t+1) + \frac{m}{\alpha+\beta}} \left[\rho \begin{bmatrix} \langle \tilde{p_1}, \tilde{Z}_t \rangle \\ \langle \tilde{p_2}, \tilde{Z}_t \rangle \\ . \\ . \\ \langle \tilde{p_N}, \tilde{Z}_t \rangle \end{bmatrix} + \frac{\beta}{\alpha+\beta} \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix} - \widetilde{Z}_t \right] + \tag{3.2}$$

$$\frac{1}{(t+1) + \frac{m}{\alpha+\beta}} \left[\frac{\widetilde{Y_{t+1}} - E(\widetilde{Y_{t+1}} | \mathcal{F}_t)}{(\alpha+\beta)} \right]$$

Lemma 7. *Equation* (3.2) *is a stochastic approximation scheme with*

$$M_{t+1} = \frac{\widetilde{Y_{t+1}} - E[\widetilde{Y_{t+1}} | \mathcal{F}_t]}{(\alpha+\beta)}$$

$$h(\widetilde{Z}_t) = \left[\rho \begin{bmatrix} \langle \tilde{p_1}, \tilde{Z}_t \rangle \\ \langle \tilde{p_2}, \tilde{Z}_t \rangle \\ . \\ . \\ \langle \tilde{p_N}, \tilde{Z}_t \rangle \end{bmatrix} + \frac{\beta}{\alpha+\beta} \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix} - \widetilde{Z}_t \right]$$

$$a(t+1) = \frac{1}{(t+1) + \frac{m}{\alpha+\beta}}$$

Proof. 1. To show that $h(\widetilde{Z}_t)$ is Lipschitz we need to show that for any $t_1, t_2 \in N \cup \{0\}$, $||h(\widetilde{Z}_{t_1} - \widetilde{Z}_{t_2})|| \leq K||\widetilde{Z}_{t_1} - \widetilde{Z}_{t_2}||$ for some constant K.

$$||h(\widetilde{Z_{t_1}} - \widetilde{Z_{t_2}})|| = \rho \begin{bmatrix} \langle \tilde{p_1}, \tilde{Z}_{t_1} - \tilde{Z}_{t_2} \rangle \\ \langle \tilde{p_2}, \tilde{Z}_{t_1} - \tilde{Z}_{t_2} \rangle \\ . \\ . \\ \langle \tilde{p_N}, \tilde{Z}_{t_1} - \tilde{Z}_{t_1} \rangle \end{bmatrix} - (\widetilde{Z_{t_1}} - \widetilde{Z_{t_2}})$$

$$\leq ||\widetilde{Z_{t_1}} - \widetilde{Z_{t_1}}||$$

$$\leq \rho \begin{bmatrix} \langle \tilde{p_1}, \tilde{Z}_{t_1} - \tilde{Z}_{t_2} \rangle \\ \langle \tilde{p_2}, \tilde{Z}_{t_1} - \tilde{Z}_{t_2} \rangle \\ . \\ . \\ \langle \tilde{p_N}, \tilde{Z}_{t_1} - \tilde{Z}_{t_1} \rangle \end{bmatrix} + (\widetilde{Z_{t_1}} - \widetilde{Z_{t_2}})$$

$$\leq \rho N \sqrt{\left(\sum_{i=1}^{N}(Z_{t_1} - Z_{t_2})(i)\right)^2} + (\widetilde{Z_{t_1}} - \widetilde{Z_{t_2}})$$
$$\leq \rho N^2(\widetilde{Z_{t_1}} - \widetilde{Z_{t_2}}) + (\widetilde{Z_{t_1}} - \widetilde{Z_{t_2}})$$
$$= (\rho N^2 + 1)(\widetilde{Z_{t_1}} - \widetilde{Z_{t_2}})$$

Hence assumption A1. is satisfied

2. Note that step size of our model:

$$a(t) = \frac{1}{(t+1) + \frac{m}{\alpha+\beta}}$$

Satisfies assumption A2.

3.
$$E[M_{t+1}|\mathcal{F}_t] = E\left[\frac{\widetilde{Y_{t+1}} - E[\widetilde{Y_{t+1}}|\mathcal{F}_t]}{(\alpha+\beta)}\middle|\mathcal{F}_t\right] = 0$$

Also,
$$E[||M_{t+1}||^2|\mathcal{F}_t] = E[||(\widetilde{Y_{t+1}} - E[\widetilde{Y_{t+1}(i)}|\mathcal{F}_t])||^2|\mathcal{F}_t]$$
$$= \sum_{i=1}^{N} E[(Y_{t+1}(i) - E(Y_{t+1}(i)|\mathcal{F}_t))^2|\mathcal{F}_t]$$
$$= \sum_{i=1}^{N} E[(Y_{t+1}(i))^2|\mathcal{F}_t] + E[E(Y_{t+1}(i)|\mathcal{F}_t))^2|\mathcal{F}_t] - 2Y_{t+1}(i)E[E(Y_{t+1}(i)|\mathcal{F}_t)|\mathcal{F}_t]$$
$$= \sum_{i=1}^{N}[\alpha^2\langle\tilde{p}_i, \widetilde{Z}_t\rangle + \beta^2(1 - \langle\tilde{p}_i, \widetilde{Z}_t\rangle] + \sum_{i=1}^{N} E[(\alpha\langle\tilde{p}_i, \widetilde{Z}_t\rangle + \beta(1 - \langle\tilde{p}_i, \widetilde{Z}_t\rangle)^2|\mathcal{F}_t]$$
$$-2\sum_{i=1}^{N} E[Y_{t+1}(i)[\alpha\langle\tilde{p}_i, \widetilde{Z}_t\rangle + \beta(1 - \langle\tilde{p}_i, \widetilde{Z}_t\rangle)]|\mathcal{F}_t]$$
$$= \sum_{i=1}^{N}[\alpha^2\langle\tilde{p}_i, \widetilde{Z}_t\rangle + \beta^2(1 - \langle\tilde{p}_i, \widetilde{Z}_t\rangle + (\alpha\langle\tilde{p}_i, \widetilde{Z}_t\rangle + \beta(1 - \langle\tilde{p}_i, \widetilde{Z}_t\rangle))^2 - 2(\alpha\langle\tilde{p}_i, \widetilde{Z}_t\rangle + \beta(1 - \langle\tilde{p}_i, \widetilde{Z}_t\rangle))^2]$$
$$= \sum_{i=1}^{N}[\alpha^2\langle\tilde{p}_i, \widetilde{Z}_t\rangle + \beta^2(1 - \langle\tilde{p}_i, \widetilde{Z}_t\rangle) - (\alpha\langle\tilde{p}_i, \widetilde{Z}_t\rangle + \beta(1 - \langle\tilde{p}_i, \widetilde{Z}_t\rangle))^2$$

$$= \sum_{i=1}^{N} (\alpha - \beta)^2 \langle \tilde{p}_i, \widetilde{Z}_t \rangle [1 - \langle \tilde{p}_i, \widetilde{Z}_t \rangle]$$

$$\leq (\alpha - \beta)^2 \sum_{i=1}^{N} \left\{ \sum_{j=1}^{N} p_i^j Z_t(j)(1 - Z_t(j)) \right\}$$

$$\leq (\alpha - \beta)^2 N \sum_{i=1}^{N} Z_t(j)$$

$$\leq (\alpha - \beta)^2 N^2 [1 + \sum_{i=1}^{N} Z_t^2(j)]$$

Hence,

$$E[||M_{t+1}||^2 | \mathcal{F}_t] \leq (\alpha - \beta)^2 N^2 [1 + ||\widetilde{Z}_t||^2]$$

Therefore assumption A3. is satisfied

4. Since $0 \leq Z_t(i) \leq 1$ for all t and $1 \leq i \leq N$ we have

   $sup_t ||\widetilde{Z}_t|| < \infty$ a.s. and hence assumption A4. is satisfied.

$\square$

Theorem 8. *(theorem A.1 in [13]) For a general stochastic Approximation scheme given by*

$$x_{t+1} = x_t + a(t)[h(x_t) + M_{t+1}]$$

The set Θ^∞ of limiting values of h as $t \to \infty$ is a.s. a compact connected set, stable by the flow of

$$ODE_h \equiv \dot{x} = h(x)$$

Furthermore if $x^* \in \Theta^\infty$ is a uniformly stable equilibrium on Θ^∞ of ODE_h, then

$$x_t \to x \quad a.s. \quad as \quad t \to \infty$$

Theorem 9. *From Lemma 7 and Theorem 8, we conclude that $\widetilde{Z}_t \to \gamma \quad a.s$ where $\gamma = (\frac{1}{2}, ..., \frac{1}{2})$*

3.2 Fluctuation Theorems

Theory of Stochastic Approximation can also be used to understand fluctuations around the limit. Recently these methods were used in [13] to study urn models quite effectively.
The following result for our model is on the same lines as Theorem A.2 in [13].

Theorem 10. *For Our Proposed Model, assume that the function h is differentiable at γ and all the eigenvalues of $-Dh(\gamma)$ have positive real parts.Assume that for some $\delta > 0$,*

$sup_{t \geq t_0} E[||M_{t+1}||^{2+\delta}|\mathcal{F}_t] < \infty$ *a.s.*

$E[M_{t+1}M_{t+1}^t|\mathcal{F}_t] \to \Gamma$ *a.s. as* $n \to \infty$

where Γ is a deterministic symmetric definite positive matrix.

For step size $a(t) = \frac{1}{(t+1)+\frac{m}{\alpha+\beta}}$

(a) For$\rho < \frac{1}{2}$

$$\sqrt{n}\left(\widetilde{Z}_t - \gamma\right) \to \mathcal{N}\left(0, \frac{1}{2(\mathcal{R}e(\lambda_{min})) - 1}\Sigma\right) \quad as \quad n \to +\infty$$

λ_{min} *denotes the eigenvalue of* $-Dh(\gamma)$ *with the lowest real part*

$$\Sigma := \int_0^\infty e^{-(-Dh(\gamma)^t - \frac{I_d}{2})u}\Gamma e^{-(-Dh(\gamma) - \frac{I_d}{2})u} du$$

$$E[M_{t+1}M_{t+1}^T|\mathcal{F}_t] \xrightarrow[a.s.]{n\to\infty} \Gamma$$

(b) For $\rho = \frac{1}{2}$,

$$\sqrt{\frac{n}{\log n}}\left(\widetilde{Z}_t - \gamma\right) \to \mathcal{N}(0, \Sigma)$$

as $n \to \infty$

(c) For $\rho \in \left(0, \frac{1}{2}\right)$, $n^{\lambda_{min}}\left(\widetilde{Z}_t - \gamma\right)$ *a.s. converges as* $n \to \infty$ *towards a finite random variable.*

Proof.

$$h(\widetilde{Z}_t) = \begin{bmatrix} \rho \begin{bmatrix} \langle \tilde{p_1}, \tilde{Z}_t \rangle \\ \langle \tilde{p_2}, \tilde{Z}_t \rangle \\ . \\ . \\ \langle \tilde{p_N}, \tilde{Z}_t \rangle \end{bmatrix} + \frac{\beta}{\alpha+\beta} \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix} - \widetilde{Z}_t \end{bmatrix}$$

Considering $Z_t(i)$ and $Z_t(j)$ independent for all i, j jacobian of h is given by:

$$\begin{bmatrix} (\rho p_1^1 - 1) & \rho p_1^2 & . & . & . & . & \rho p_1^N \\ \rho p_2^1 & (\rho p_2^2 - 1) & . & . & . & . & \rho p_2^N \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \rho p_N^1 & . & . & . & . & . & (\rho p_2^2 - 1) \end{bmatrix}$$

The Jacobian can be written as $\rho S - I$ where S is a matrix with row sums one.

Hence the maximum eigenvalue for Dh would be $\rho - 1$ and minimum eigenvalue for $-Dh$ would be $1 - \rho$.

Since $\rho - 1$ is negative, real part of eigenvalues of Dh are negative which implies that real part of eigenvalues of $-Dh$ are all positive.

(a), (b) and (c) of Theorem holds respectively for $\rho < \frac{1}{2}$, $\rho = \frac{1}{2}$ and $\rho \in \left(0, \frac{1}{2}\right)$ $\square$

We explicitly calculate Γ, Σ and $E[M_{t+1}M_{t+1}^T|\mathcal{F}_t]$.

- Calculation of Γ

  We need to calculate $E[M_{t+1}M_{t+1}^T|\mathcal{F}_t]$

$$M_{t+1}M_{t+1}^T = \frac{1}{(\alpha+\beta)^2}[A - B][A^T - B^T]$$

Where $A = \begin{bmatrix} Y_{t+1}(1) \\ Y_{t+1}(2) \\ . \\ . \\ . \\ . \\ Y_{t+1}(N) \end{bmatrix}$

$$B = \begin{bmatrix} E[Y_{t+1}(1)|\mathcal{F}_t] \\ E[Y_{t+1}(2)|\mathcal{F}_t] \\ . \\ . \\ . \\ . \\ E[Y_{t+1}(N)|\mathcal{F}_t] \end{bmatrix} = (\alpha - \beta) \begin{bmatrix} \langle \widetilde{p_1}, \widetilde{Z_t} \rangle \\ . \\ . \\ . \\ . \\ \langle \widetilde{p_N}, \widetilde{Z_t} \rangle \end{bmatrix} + \beta \begin{bmatrix} 1 \\ . \\ . \\ . \\ . \\ 1 \end{bmatrix}$$

Note that $E[M_{t+1}M_{t+1}^T|\mathcal{F}_t]$ will be symmetric matrix

- Diagonal Elements of $E[M_{t+1}M_{t+1}^T|\mathcal{F}_t]$:

$$\Big((\alpha-\beta)\langle \widetilde{p_i}, \widetilde{Z_t} \rangle+\beta\Big)^2 + E[(Y_{t+1}(i))^2|\mathcal{F}_t] - 2\Big((\alpha-\beta)\langle \widetilde{p_i}, \widetilde{Z_t} \rangle+\beta\Big)\langle \widetilde{p_i}, \widetilde{Z_t} \rangle$$

$$= \Big((\alpha-\beta)\langle \widetilde{p_i}, \widetilde{Z_t} \rangle+\beta\Big)^2 + (\alpha^2-\beta^2)\langle \widetilde{p_i}, \widetilde{Z_t} \rangle+\beta^2 - 2\Big((\alpha-\beta)\langle \widetilde{p_i}, \widetilde{Z_t} \rangle+\beta\Big)\langle \widetilde{p_i}, \widetilde{Z_t} \rangle$$

Since our model is a friedman's urn model $Z_t(i) \to \frac{1}{2}$ a.s. as $t \to \infty$ for all i.

Hence $\langle \widetilde{p_i}, \widetilde{Z_t} \rangle \to \frac{1}{2}$ a.s. $t \to \infty$ for all i.

Hence the diagonal elements of Γ will be

$$\bigg(\frac{(\alpha-\beta)}{2} + \beta\bigg)^2 + \frac{(\alpha^2-\beta^2)}{2} + \beta^2 - 2\bigg(\frac{(\alpha-\beta)}{2} + \beta\bigg)\frac{1}{2}$$

$$= \frac{3}{4}(\alpha^2+\beta^2) - \frac{(\alpha+\beta-\alpha\beta)}{2}$$

- Off-Diagonal Elements of $E[M_{t+1}M_{t+1}^T|\mathcal{F}_t]$

$$\Big((\alpha-\beta)\langle\widetilde{p_i},\widetilde{Z_t}\rangle+\beta\Big)\Big((\alpha-\beta)\langle\widetilde{p_j},\widetilde{Z_t}\rangle+\beta\Big)+E[(Y_{t+1}(i))(Y_{t+1}(j))|\mathcal{F}_t]$$
$$-\Big((\alpha-\beta)\langle\widetilde{p_i},\widetilde{Z_t}\rangle+\beta\Big)\langle\widetilde{p_j},\widetilde{Z_t}\rangle-\Big((\alpha-\beta)\langle\widetilde{p_j},\widetilde{Z_t}\rangle+\beta\Big)\langle\widetilde{p_j},\widetilde{Z_t}\rangle$$
$$=\Big((\alpha-\beta)\langle\widetilde{p_i},\widetilde{Z_t}\rangle+\beta\Big)\Big((\alpha-\beta)\langle\widetilde{p_j},\widetilde{Z_t}\rangle+\beta\Big)+\alpha^2\langle\widetilde{p_i},\widetilde{Z_t}\rangle\langle\widetilde{p_j},\widetilde{Z_t}\rangle$$
$$+\beta^2\Big(1-\langle\widetilde{p_i},\widetilde{Z_t}\rangle\Big)\Big(1-\langle\widetilde{p_j},\widetilde{Z_t}\rangle\Big)+\alpha\beta\Big(\langle\widetilde{p_i},\widetilde{Z_t}\rangle+\langle\widetilde{p_j},\widetilde{Z_t}\rangle-2\langle\widetilde{p_i},\widetilde{Z_t}\rangle\langle\widetilde{p_j},\widetilde{Z_t}\rangle\Big)$$
$$-\Big((\alpha-\beta)\langle\widetilde{p_i},\widetilde{Z_t}\rangle+\beta\Big)\langle\widetilde{p_j},\widetilde{Z_t}\rangle-\Big((\alpha-\beta)\langle\widetilde{p_j},\widetilde{Z_t}\rangle+\beta\Big)\langle\widetilde{p_j},\widetilde{Z_t}\rangle$$

Which converges to

$$\left(\frac{(\alpha-\beta)}{2}+\beta\right)^2+\frac{\alpha^2}{4}+\frac{\beta^2}{4}-\left(\frac{(\alpha-\beta)}{2}+\beta\right)+\frac{\alpha\beta}{2}$$
$$=\frac{1}{2}(\alpha^2+\beta^2)+\alpha\beta-\frac{(\alpha+\beta)}{2}$$

- Calculation of Σ

$$\Sigma := lim_{T\to+\infty}\int_0^T e^{-(-Dh(x^*)^t-\frac{I_d}{2})u}\Gamma e^{-(-Dh(x^*)-\frac{I_d}{2})u}du$$
$$= lim_{T\to+\infty}\int_0^T e^u e^{(Dh(x^*)u)^t}\Gamma e^{(Dh(x^*))u}du$$

3.3 Stable Convergence Approach

In [1] authors study a model with interacting reinforced stochastic processes. Our model is a particular case of this model where these reinforced stochastic processes are urn models. We will discuss some similarities between our model and the model in the [1]. Their method can be adopted to obtain CLT of the fluctuation results for our model of interacting Pólya urns.

Following definition is given in [1]:

Definition 3.3.1. *A Reinforced Stochastic Process is a S-dimensional stochastic process, where S is a finite set such that $Y = [Y(x) : x \in S]$ has each component $Y(x) = (Y_n(x))_{n\geq 1}$ a stochastic process.*
Conditional probabilities for these Y_n's are given by:

$$P(Y_{n+1}(x) = 1|Z_0(x), Y_1(x), ..., Y_n(x)) = Z_n(x)$$

Where

$$Z_n(x) = (1 - r_{n-1})Z_{n-1}(x) + r_{n-1}Y_n(x) \tag{3.3}$$

with $0 \leq r_n \leq 1$ and $Z_0(x)$ are random variables.

Note that for our model, S is the Number of urns in the network i.e. N and $Y_n(x)$ gives the number of balls added to urn x at time n.

For our model of friedman urns:

$$\widetilde{Z}_{t+1} = \frac{(\alpha+\beta)t + m}{(\alpha+\beta)(t+1)+m}\widetilde{Z}_t + \frac{1}{(\alpha+\beta)(t+1)+m}\widetilde{Y}_{t+1} \tag{3.4}$$

Where

$$\widetilde{Z}_t = [Z_t(1), Z_t(2), ..., Z_t(N)]$$

And

$$\widetilde{Y}_t = [Y_t(1), Y_t(2), ..., Y_t(N)]$$

$Z_t(i)$ is the fraction of balls in i^{th} urn at time t and $Y_t(i)$ is the number of balls added to urn i at time t.

On comparing equation (3.3) and (3.4) we get that our model is a Reinforced Stochastic Process with

$$r_t = \frac{(\alpha+\beta)}{(\alpha+\beta)(t+1)+m}$$

In [1], The transpose of Matrix P from our model is referred as the weighted adjacency matrix.

Assumptions and Notations

The Matrix P is irreducible and Diagonalizable. The diagonalizability of P ensures that there exists a nonsingular matrix $\widetilde{U}$ such that $\widetilde{U}^T P^T (\widetilde{U}^T)^{-1}$ is diagonal with $\lambda_j \in Sp(P^T)$(where $Sp(M)$ for a matrix M is its spectrum) . Without loss of generality we may assume that norm of every column in $\widetilde{U}$ is 1

Define $\widetilde{V} = (\widetilde{U}^T)^{-1}$ Note that each column u_j of $\widetilde{U}$ is a left eigenvector of P^T corresponding to the eigenvalue $\lambda_j \in Sp(P^T)$ and each column v_j of $\widetilde{V}$ is a right eigenvector of P^T

From the definition of $\widetilde{V}$ we have

$$u_j^T v_j = 1 \quad \text{and} \quad u_h^T v_j = 0, \quad \forall h \neq j$$

Using Frobenius-Perron theorem and the fact that P is doubly stochastic, we have: The eigenvalue $\lambda_1 := 1$ of P^T has multiplicity 1,
$\lambda_{max} = 1$ and

$$u_1 = v_1 = N^{-1/2}\mathbf{1} \qquad \text{and}$$
$$[v_1]_j := v_{1,j} \in (0, +\infty) \qquad \forall j = 1, .., N$$

Other Notations

- U and V denote matrices whose columns are respectively left and right eigenvectors of P^T corresponding to $Sp(P^T)\backslash\{1\}$. Note that U and V are sub matrices of $\widetilde{U}$ and $\widetilde{V}$ respectively.
- λ^* is an eigenvalue of P^T such that

$$Re(\lambda^*) = max\left\{Re(\lambda_j) : \lambda_j \in Sp(P^T)\backslash\{1\}\right\}$$

Interacting Pólya Urns: Asymptotics

This section gives CLT convergence results for a simplified version of our model using stable convergence (Appendix-B).

Consider N interacting Pólya urns with updation scheme of our model given by interaction matrix P and urn scheme

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

where $\alpha > 0$

Following theorem for our model is on the same lines as Theorem 3.1 in [1]

Theorem 11. *There exists a random variable Z_∞ such that*

$$\widetilde{Z}_t \to Z_\infty \mathbf{1}$$

Proof. 1. As P is irreducible and diagonalizable, we have:

$$V^T u_1 = U^T v_1 = 0 \qquad V^T U = U^T V = I \qquad \textit{and} \qquad I = u_1 v_1^T + U V^T \tag{3.5}$$

Also, if D is the diagonal matrix whose elements are $\lambda_j \in Sp(P^T) \backslash \{1\}$ we have,

$$P = u_1 v_1^T + U D V^T \tag{3.6}$$

2. The dynamics for a Pólyaurn interacting network can be expressed as:

$$\widetilde{Z_{t+1}} = \frac{t+m}{t+m+1} \widetilde{Z}_t + \frac{1}{t+m+1} \widetilde{Y_{t+1}}$$

$$\widetilde{Z_{t+1}} - \widetilde{Z}_t = \frac{-1}{t+m+1} \widetilde{Z}_t + \frac{1}{t+m+1} \left[\widetilde{Y_{t+1}} - E[\widetilde{Y_{t+1}} | \mathcal{F}_t] \right] + \frac{1}{t+m+1} \alpha P \widetilde{Z}_t$$

Above equation can be written as:

$$\widetilde{Z_{t+1}} - \widetilde{Z}_t = \frac{-1}{t+m+1} (I - \alpha P) \widetilde{Z}_t + \frac{1}{t+m+1} \Delta M_{t+1} \tag{3.7}$$

Where $\Delta M_t = \widetilde{Y}_t - \alpha P \widetilde{Z}_t$ is a martingale difference sequence with respect to the filtration $\mathcal{F} := (\mathcal{F}_t)_t$. Now, if v_1 is a right eigenvector of P^T then it will also be a right eigenvector of αP^T.
So, $v_1^T \alpha P = (\alpha P^T v_1)^T = v_1^T$, which gives $v_1^T (I - \alpha P) = \mathbf{0}$.
Hence, from equation (3.7) we deduce that $(v_1^T Z_t)_t$ is a bounded real martingale.

3. Idea in [1] is to decompose $\widetilde{Z}_t$ into a vector, each of whose component is a martingale and a process which approaches to zero vector as $t \to \infty$.

$$\widetilde{Z}_t = Z_t \mathbf{1} + \hat{Z}_t$$

where $Z_t = N^{\frac{-1}{2}} v_1^T Z_t = N^{-1}\mathbf{1^T}\widetilde{Z}_t$ and $\hat{Z}_t = (I - N^{-1}\mathbf{11^T})\widetilde{Z}_t$

4. To show: $Z_t \xrightarrow{a.s.} Z_\infty$

   Note that $Z_t = N^{-1}\mathbf{1}^T Z_t = N^{-1}\sum_{i=1}^{N} Z_{n,i}$. Hence Z_t is bounded. Also from equation (3.7) we have:

$$Z_{t+1} - Z_t = N^{-1/2}\frac{1}{t+m+1}(v_1^T \Delta M_{t+1})$$

   Which gives $E[Z_{t+1} - Z_t|\mathcal{F}_t] = 0$ Hence, by **Martingale Convergence Theorem** we have

$$Z_t \xrightarrow{a.s.} Z_\infty$$

   Where Z_∞ is a finite random variable

5. To show: $\hat{Z}_t \xrightarrow{a.s.} \mathbf{0}$

   (a) $Pu_1 = (u_1^T P^T)^T = u_1$
   Which gives $(I - P^T)u_1 = u_1$

   Hence,

$$(I - P)\widetilde{Z}_t = (I - P)(u_1\sqrt{N}Z_t + \hat{Z}_t) = (I - P)\hat{Z}_t$$

   Using the above relation, equation (2.13) can be written as

$$\widetilde{Z_{t+1}} - \widetilde{Z}_t = -\frac{1}{t+m+1}(I - P)\hat{Z}_t + \frac{1}{t+m+1}\Delta M_{t+1}$$

   Multiplying both sides by UV^T

$$\hat{Z_{t+1}} - \hat{Z}_t = -\frac{1}{t+m+1}[UV^T - UV^T(u_1 v_1^T + UDV^T)]\hat{Z}_t + \frac{1}{t+m+1}UV^T\Delta M_{t+1}$$

$$= -\frac{1}{t+m+1}(UV^T - UDV^T)\hat{Z}_t + \frac{1}{t+m+1}UV^T\Delta M_{t+1}$$

$$= -\frac{1}{t+m+1}U(I - D)V^T\hat{Z}_t + \frac{1}{t+m+1}UV^T\Delta M_{t+1} \qquad (3.8)$$

(b) Define $Z_{V,t} = V^T \hat{Z}_t$
From equation (3.5) we have $\hat{Z}_t = UZ_{V,t}$, so it is enough to show that $Z_{V,t}$ converges almost surely to $\mathbf{0}$.

From equation (3.8) we have

$$Z_{V,t+1} = \left(I - \frac{1}{t+m+1} (I - D) \right) Z_{V,t} + \frac{1}{t+m+1} V^T \Delta M_{t+1}$$

So,

$$E[||Z_{V,t+1}||^2|\mathcal{F}_t] = E[\overline{Z}_{V,t+1}^T Z_{V,t+1}|\mathcal{F}_t]$$

$$= \overline{Z}_{V,t}^T \left(I - \frac{1}{t+m+1} (I - \bar{D}) \right) \left(I - \frac{1}{t+m+1} (I - D) \right) Z_{V,n} +$$

$$\frac{1}{(t+m+1)^2} E[\Delta M_{t+1}^T \overline{V}\overline{V}^T \Delta M_{t+1}|\mathcal{F}_t]$$

$$= \overline{Z}_{V,t}^T Z_{V,t} - \frac{1}{t+m+1} \overline{Z}_{V,t}^T \left(2I - \overline{D} - D \right) Z_{V,t} + \frac{1}{(t+m+1)^2} \xi_t$$

where $(\xi_t)_t$ is a suitable bounded sequence of $\mathcal{F}_t$-maesurable random variables.
Since $Re(\lambda_j) < 1$ for any $\lambda_j \in Sp(P^T) \setminus \{1\}$,the matrix $2I - (\overline{D} + D)$ is positive definite i.e. a symmetric matrix with all positive eigenvalues. So, we can write

$$E[||Z_{V,t}||^2|\mathcal{F}_t] \leq ||Z_{V,t}||^2 + O\left(\frac{1}{t^2}\right)$$

Hence $(||Z_{V,t}||^2)_t$ is a bounded positive almost submartingale and so it converges almost surely. In order to prove that the limit is zero, it is enough to prove that $E[||Z_{V,t}||^2]$ converges to zero.

$$E[Z_{V,t+1}||^2] = E\left[\overline{Z}_{V,t}^T \left(I - \frac{1}{t+m+1} (I - \overline{D}) \right) \left(I - \frac{1}{t+m+1} (I - D) \right) Z_{V,t} \right] +$$

$$\frac{1}{(t+m+1)^2} E[\Delta M_{t+1}^T \overline{V} V^T \Delta M_{t+1}]$$

$$\leq E\left[\overline{Z}_{V,t}^T \left(I - \frac{1}{t+m+1} (I - \overline{D}) \right) \left(I - \frac{1}{t+m+1} (i - D) \right) Z_{V,t} \right]$$

$$+C_1 \frac{1}{(t+m+1)^2}$$

For some constant $C_1 \geq 0$.

Also,

$$\left[\left(I - \frac{1}{t+m+1}\left(I - \overline{D}\right)\right)\left(I - \frac{1}{t+m+1}\left(I - D\right)\right)\right]_{jj} =$$

$$1 - 2\frac{1}{t+m+1}\left(1 - Re(\lambda_j)\right) + \frac{1}{(t+m+1)^2}|1-\lambda_j|^2$$

Set $a_j = 1 - Re(\lambda_j)$ and $a^* = min_j\{a_j\} = 1 - Re(\lambda)$.

We have

$$E\left[\overline{Z}_{V,t}^T\left(I - \frac{1}{t+m+1}\left(I - \overline{D}\right)\right)\left(I - \frac{1}{t+m+1}\left(I - D\right)\right)Z_{V,t}\right]$$

$$\leq \sum_{j=2}^{N}\left(1 - 2a_j\frac{1}{t+m+1}\right)E\left[\overline{Z}_{V,t}^j Z_{V,t}^j\right] + C_2\frac{1}{(t+m+1)^2}$$

$$\leq \left(1 - 2a^*\frac{1}{t+m+1}\right)E\left[||Z_{V,t}||^2\right] + C_2\frac{1}{(t+m+1)^2}$$

Set $x_t := E\left[||Z_{V,t}||^2\right]$, then above inequality can be written as

$$x_{t+1} \leq \left(1 - 2a^*\frac{1}{t+m+1}\right)x_t + (C_1 + C_2)\frac{1}{(t+m+1)^2}$$

Since we have $Re(\lambda^*) < 1$,above recursion implies that

$$\lim_t x_t = 0$$

which concludes the proof.

Following two convergence results for our model with interacting Pólya urns are on the same lines as Theorem 3.2 in [1] and Theorem 3.3 in [1] respectively.

Theorem 12. *The following holds for the Proposed model with Pólya urns:*

1. *if* $Re(\lambda^*) < \frac{1}{2}$, *then*

$$\sqrt{n}(\widetilde{Z}_t - Z_\infty \mathbf{1}) \to \mathcal{N}(\mathbf{0}, Z_\infty(1 - Z_\infty(\tilde{\Sigma} + \hat{\Sigma})) \qquad \textit{stably,}$$

where,

$$\tilde{\Sigma} = \frac{||v_1||^2}{N} \mathbf{1}\mathbf{1}^T$$

and

$$\hat{\Sigma} := U\hat{S}U^T$$

with

$$[\hat{S}]_{h,j} := \frac{v_h^T v_j}{1 - (\lambda_h + \lambda_j)} \qquad \textit{with} \quad 2 \leq h, j \leq N$$

2. *if* $Re(\lambda^*) = \frac{1}{2}$ *then,*

$$\frac{\sqrt{n}}{\sqrt{\ln(n)}}(\widetilde{Z}_t - Z_\infty \mathbf{1}) \to \mathcal{N}(\mathbf{0}, Z_\infty(1 - Z_\infty)\hat{\Sigma}^* \qquad \textit{stably}$$

where,

$$\hat{\Sigma}^* := U\hat{S}^*U^T$$

and

$$[\hat{S}^*]_{h,j} := \begin{cases} v_h^T v_j & \textit{if} \quad \lambda_h + \lambda_j = 0 \\ 0 & \textit{if} \quad \lambda_h + \lambda_j \neq 0 \end{cases}$$

with $2 \leq h, j \leq N$.

Theorem 13. *For any* $h, j \in \{1, 2.., N\}$, $h \neq j$, *we have:*

1. *if* $Re(\lambda^*) < \frac{1}{2}$, *then*

$$\sqrt{n}(Z_t(h) - Z_t(j)) \to \mathcal{N}(0, Z_\infty(1 - Z_\infty)\Sigma_{h,j}) \qquad \textit{stably,}$$

where,

$$\Sigma_{h,j} := [\hat{\Sigma}]_{h,h} + [\hat{\Sigma}]_{j,j} - 2[\hat{\Sigma}]_{h,j}$$

2. *if* $Re(\lambda^*) > \frac{1}{2}$, *then*

$$\frac{\sqrt{n}}{\sqrt{\ln(n)}}(Z_t(h) - Z_t(j)) \to \mathcal{N}(0, Z_\infty(1 - Z_\infty)\Sigma^*_{h,j}) \qquad \textit{stably}$$

where,

$$\Sigma^*_{h,j} := [\hat{\Sigma}^*]_{h,h} + [\hat{\Sigma}^*]_{j,j} - 2[\hat{\Sigma}^*]_{h,j}$$

□

We omit the proofs as they follow from the same argument as in [1].

Although, the same results can be obtained via stochastic approximation (as seen in section 3.1), we believe that understanding a new approach to obtain the same results would be useful.

Chapter 4

Branching Processes and Urn Models

In this chapter, we discuss the concept of embedding a suitable discrete time process into a continuous time Branching Process. Once we get a branching process that is stochastically same as an Urn Model, various results from branching processes and Markov chain theory can be obtained for the model. This process of Embedding also has applications in study of contagious diseases. This embedding was introduced by K.B. Athreya and Samuel Karlin in their paper [2].

We will first state the definition and construction of a multi-type branching process and Continuous Time Markov Branching Process and then discuss the concept of Embedding from the book [3].The idea is to find a suitable sequence of stopping times such that the continuous time process observed at these times is stochastically similar to a discrete urn process.

4.1 Multi-Type Branching Process

A multi-type branching process allows finite number of particle types in which every particle of every type can have any number of offsprings of any type.

For a r-type process, we need r generating functions, each in r variables.

Generating function for type i particle is defined as:

$$f^{(i)}(s_1, ..., s_r) = \sum_{j_1, j_2, ..., j_r \geq 0} p^{(i)}(j_1, .., j_r) s_1^{j_1} .... s_r^{j_r}$$

$$0 \leq s_\alpha \leq 1 \qquad \alpha = 1, ...r$$

$p^{(i)}(j_1, j_2, , , .j_r)$ = the probability that a type i parent produces j_1 particles of type 1,j_2 particles of type 2 and so on.
The generating function for the whole process is given by:

$$\vec{f}(\vec{s}) = (f^{(1)}(\vec{s}), ..., f^{(r)}(\vec{s}))$$

where $\vec{s} \in (0, 1)^r$.

Definition 4.1.1. *r-type branching process is a Markov chain $\{A_n; n = 0, 1, 2...\}$ on R_+^r with transition function*

$$P(\vec{i}, \vec{j}) = P\{Y_{n+1} = \vec{j} | Y_n = \vec{i}\}, \qquad \vec{i}, \vec{j} \in R_+^r$$

$$= \textit{coefficient} \quad \textit{of} \quad \vec{s}^{\vec{j}} \quad \textit{in} \quad [f(\vec{s})]^i$$

4.2 The Continuous time Multitype Branching Process

Let $A_j(t)$ = the number of type j particles existing at time t, and set

$$\mathbf{A}(t) = (A_1(t), ..., A_r(t));$$

Definition 4.2.1. *(See Chapter 5, [3])*
A stochastic process $\{A(t, \omega); t \geq 0\}$ on a probability space (Ω, F, P) is called a r-dimensional continuous time Markov branching process if:

1. *Its state space is Z_+^r*

2. *It is a stationary strong Markov process with respect to the fields*

$$F_t = \sigma\{A(s, \omega); s \leq t\};$$

3. The transition probabilities $P(\mathbf{i}, \mathbf{j}; t)$ satisfy

$$\sum_{j \in R_+^r} P(\mathbf{i}, \mathbf{j}; t) \vec{s}^{\mathbf{j}} = \prod_{k=1}^{r} \left[\sum_{j \in R_+^r} P(e_k, \mathbf{j}; t) \mathbf{s}^{\mathbf{j}} \right]^{i_k;}$$

for all $\mathbf{i} \in R_+^r$ and $\vec{s} \in (0,1)^r$.

The transition functions are determined by the parameters

$$\vec{a} = (a_1, ..., a_r) \in R_+^r,$$

$$\vec{0} \leq \mathbf{p}(j) = (p^{(1)}(\vec{j}), .., p^{(r)}(\vec{j})), \qquad \sum_{\vec{j} \in Z^+} p^i_{\vec{j}} = 1$$

Let

$$\vec{f}(\vec{s}) = (\vec{f}^{(1)}(\vec{s}), ..., \vec{f}^{(r)}(\vec{s})),$$

where

$$\vec{f}^i(\vec{s}) = \sum_{j \in Z_+^r} p^{(i)}(\vec{j}) \vec{s}^{\vec{j}}$$

Let

$$u^i(\vec{s}) = a_i[\vec{f}^i(\vec{s}) - s_i]$$

The function $\mathbf{u}(\vec{s})$ is called the infinitesimal generating function for the continuous time branching process.

Now, we describe the embedding of urn models into a multi-type continuous time branching process. Given an embedding, it is enough to describe the infinitesimal generating functions.

4.3 Embedding of Urn Model into continuous time Branching process

We first discuss embedding of a single Friedman Urn into a continuous time branching process. Let W_t and B_t denote number of white and black balls respectively in the Urn at time t.

Consider a two-type branching process $A(t) = (A_1(t), A_2(t))$ with infinitesimal generating functions

$$a_1 = a_2 = 1$$
$$h_1(s) = s_1^{\alpha+1} s_2^{\beta}$$
$$h_2(s) = s_1^{\beta} s_2^{\alpha+1}$$

This branching process is such that if any paticle dies, then it gives rise to $\alpha + 1$ particles of its own type and β particles of the other type.Let τ_t's be the times at which any particle dies. Then we have the following result:

Theorem 14. *(Theorem 1 in [2])*
The stochastic processes $\{(W_t, B_t); t = 0, 1, 2...\}$ *and* $\{A(\tau_t) = (A_1(\tau_t), A_2(\tau t)); t = 0, 1, 2...\}$ *are equivalent.*

Clearly, this can be extended to consider embedding of more than one independent Friedman urns. This is done in the following example.

Example:
Consider two independent Friedman Urns with Urn schemes

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_1 \end{bmatrix} \qquad \begin{bmatrix} \alpha_3 & \alpha_4 \\ \alpha_4 & \alpha_3 \end{bmatrix}$$

This set of independent Friedman urns can be embedded in a four-type branching process with infinitesimal generating functions:

$$a_1 = a_2 = a_3 = a_4 = 1$$
$$h_1(s) = s_1^{\alpha_1+1} s_3^{\alpha_3}$$
$$h_2(s) = s_2^{\alpha_2+1} s_3^{\alpha_4}$$
$$h_3(s) = s_3^{\alpha_1+1} s_1^{\alpha_3}$$
$$h_4(s) = s_4^{\alpha_1+1} s_2^{\alpha_3}$$

For our model, urns are not independent (their dependence is governed by the parameters $p_i^j{}'s$ and therefore a new approach is needed to understand interacting (or dependent) urn processes from the branching processes point of view.

Appendix

We state some of the definitions and results mentioned or used in the thesis.

4.4 Appendix-A (Difference Equations)

We need various results from Difference equations in order to understand the proof for theorem 2. The following results for Difference Equations have been taken from [8].
Let x_n, a_n, b_n be real numbers for $n \geq 0$ with

$$x_{n+1} = a_n x_n + b_n \tag{4.1}$$

Since above equation is a recurrence relation, we can write x_{n+1} in terms of x_0

$$x_{n+1} = x_0 \prod_{\nu=0}^{n} a_\nu + \sum_{j=0}^{n-1} b_j \prod_{\nu=j+1}^{n} a_\nu + b_n \tag{4.2}$$

$$= \left(\prod_{\nu=0}^{n} a_\nu \right) \left(x_0 + \sum_{j=0}^{n} b_j \prod_{\nu=0}^{j} a_\nu^{-1} \right) \tag{4.3}$$

When, $a_\nu \neq 0$ for $0 \leq \nu \leq n$.

Suppose $b > 0$, $c > 0$, a is real and

$$a_n = 1 + \frac{a}{(b + cn)} \qquad \text{for} \quad n \geq 0 \tag{4.4}$$

Then,

$$\prod_{\nu=0}^{n} a_\nu = \left[\frac{\Gamma\left(\frac{b}{c}\right)}{\Gamma\left(\frac{(a+b)}{c}\right)} \right] \left[\frac{\Gamma\left[\frac{a+b}{c} + n + 1\right]}{\Gamma\left[\frac{b}{c} + n + 1\right]} \right] \tag{4.5}$$

Hence,using Stirling approximation we have

$$\prod_{\nu=0}^{n} a_{\nu} = \left[\frac{\Gamma\left(\frac{b}{c}\right)}{\Gamma\left(\frac{(a+b)}{c}\right)} \right] n^{\frac{a}{c}} \tag{4.6}$$

Using equation (3.1) and (3.6) we can conclude

Lemma 15. *(Lemma 6.4 in [8])*
If $\{a_n\}$ is defined by (3.4) with $a > 0$; and $b_n = O(n^d)$ with $d < c^{-1}a - 1$; and $\{x_n\}$ satisfies (3.1):then $lim_{n\to\infty} x_n \prod_{\nu=0}^{n} a_{\nu}^{-1} = x_0 + \sum_{j=0}^{\infty} b_j \prod_{\nu=0}^{j} a_{\nu}^{-1}$, the series converging absolutely.

4.5 Appendix-B (Stable Convergence)

The following section is taken from the paper [1] (Appendix B: Stable convergence and its variants).

Definition 4.5.1. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let S be a polish space, endowed with its Borel $\sigma-$ field. a kernel on S, or a random probability measure on S, is a collection $K = \{K(\omega) : \omega \in \Omega\}$ of probability measures on the Borel $\sigma-$ field of S such that, for each bounded Borel real function f on S, the map*

$$\omega \mapsto Kf(\omega) = \int f(x)K(\omega)(dx)$$

is $\mathcal{A}-$ measurable.

Given a sub-σ-field $\mathcal{H}$ of $\mathcal{A}$, a kernel K is said to be $\mathcal{H}-$measurable if all the above random variables Kf are $\mathcal{H}-$measurable.

On $(\Omega, \mathcal{A}, P)$, let (Y_n) be a sequence of $S-$ valued random variables, let $\mathcal{H}$ be a sub-σ-field of $\mathcal{A}$ and let K be a $\mathcal{H}$- measurable kernel on S. Then we say that Y_n converges $\mathcal{H}-$ stably to K, and we write $Y_n \to K$ $\mathcal{H}$-stably, if

$$P(Y_n \in .|H) \xrightarrow{weakly} E[K(.)|H] \qquad \text{for all}$$

H $\in \mathcal{H}$ with $P(H) > 0$

4.6 Appendix-C (Auxiliary Results)

4.6.1 Stochastic Processes

A collection of random variables $\{X(t), t \in T\}$ defined on the probability space$(\Omega, \mathcal{F}, P)$ is called a stochastic process.

$$X : T \times \Omega \to R$$

$$X(t, \omega) = X_\omega(t)$$

Note that $X^{-1}\{(-\infty, x]\} \in \mathcal{F}, \quad \forall x \in R$

The set $\{t \in T\}$ is called the **parameter space(T)** of index set. Whereas the collection of all possible values of $X(t)$ for $t \in T$ is called the **state space(S)**.

4.6.2 Exchangeable Random Variables

A finite sequence of Random Variables $X_1, X_2, ..., X_k$ are said to be Exchangeable if

$$(X_1, X_2, ..., X_k) \overset{\mathcal{D}}{=} (X_{i_1}, X_{i_2}, ..., X_{i_k})$$

For any permutation $(i_1, ..., i_k)$ of $(1, 2..., k)$.

An infinite sequence of Random Variables is said to be Exchangeable if every finite collection of its variables is Exchangeable.

De Finetti's Theorem for Indicators

Theorem 16. *(Theorem 1.2 in [14])*
Let $X_1, X_2, ...$ be an infinitely Exchangeable sequence of indicators.Then, there is a distribution function $F(x)$ such that

$$P(X_1 = 1, ..., X_k = 1, X_{k+1} = 0, ..., X_n = 0) = \int_0^1 x^k(1-x)^{n-k} dF(x)$$

For each n and $0 \leq k \leq n$

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